Edmonds-Karp Theorem and Hall's Theorem

1 Edmonds-Karp Theorem

1.1 Dinic's Theorem

Theorem 1. *The widest-path Ford-Fulkerson implementation (always augmenting P with largest minimum residual capacity) runs in $\mathcal{O}(m \log C^*)$ flow augmentations.*

There are several ways to implement Dinic's Theorem: find the widest path in $\mathcal{O}(m \log m)$ using binary search on the bottleneck, plus BFS; modify Bellman-Ford for $\mathcal{O}(mn)$ runtime; or modify Dijkstra's Algorithm for $\mathcal{O}(m + n \log n)$. However, Dinic's Theorem is not as interesting or as powerful as Edmond's Karp, so we'll study that one instead.

1.2 Edmonds-Karp Theorem

Lemma 2. *If P, Q are augmenting paths used in iterations i, i', Q pushes flow in the opposite direction of P on at least one edge, and for all times $j \in \{i+1, \ldots i'-1\}$ the path P_j does not push flow opposite to either P or Q on any edge, then $|Q| > |P|$.*

Proof. Without loss of generality, let $P = \{(1,2), (2,3), \ldots, (k-1,k)\}$. Then, Q is some other path with backward edges (not necessarily in the same order as P, or on the same nodes). So, $Q = \{Q_0, (v_1+1, v_1), Q_1, (v_2+1, v_2), \ldots, (v_r+1, v_r), Q_r\}$.

Because no P_j pushed flow opposite to Q, all Q_i were available when P was a shortest path. Thus $|Q_i| \geq v_{i+1} + 1 - v_i$ (Q_i goes from v_i to $v_{i+1}+1$). Total length of Q:

$$|Q| \geq r + \sum_{i=0}^{r} |Q_i|$$

$$\geq r + \sum_{i=0}^{r} (v_{i+1} + 1 - v_i)$$

$$= 2r + 1 + \sum_{i=0}^{r} (v_{i+1} - v_i)$$

$$= 2r + 1 + v_{r+1} - v_0$$

$$= 2r + k$$

$$> k = |P|$$

$\square$

Theorem 3. *If we always choose a shortest (s, t) path found with BFS, Ford-Fulkerson terminates in $\mathcal{O}(mn)$ iterations. Notice this is strongly polynomial – there is no dependence on c_e values.*

Proof. Key idea: the length of the shortest (s, t) path in $\mathcal{G}_f$ is always (1) non-decreasing, and (2) strictly increases "often enough".

(1). In each step, the length of the path either stays the same or increases.

(i). If P_i, P_{i+1} use no edge in opposite directions, then P_{i+1} was available in iteration i, so $|P_i| \leq |P_{i+1}|$.

(ii). If P_{i+1} uses an edge in the opposite direction from P_i, then $|P_i| < |P_{i+1}|$ by Lemma 2.

(2). If P_i, P_j saturate e or set the flow to 0 on e (with $j > 1$), then $|P_j| > |P_i|$. Thus, there exists some $i' \in \{i+1, \ldots, j-1\}$ such that $P_{i'}$ pushes flow on some e opposite to P_i. Let $i \leq k \leq l \leq i'$ be such that P_k and P_l push flow in opposite directions on at least some edge and $|l - k|$ is minimized. (We know k and l must exist because i, i' are candidates). By Lemma 2, the distance from s to t strictly increases from k to l, and by (1) it never decreases. Then, we have a strict increase from i to j.

Runtime: After every $2m+1$ rounds, some e got saturated twice or set to 0 twice by the pigeonhole principle. Thus, the distance from s to t increases at least once every $2m$ iteration, and can increase at most n times. So, there will be $\mathcal{O}(mn)$ flow augmentations. □

2 Hall's Theorem

Consider a bipartite graph $\mathcal{G}(X, Y, E)$.

Definition 4. $\Gamma(A) :=$ the **neighbors** of A (i.e., the set of all nodes incident to any node in A).

Lemma 5. *If there is a set $A \subseteq X$ or $A \subseteq Y$ with $|\Gamma(A)| < |A|$,$\mathcal{G}$ has no perfect matching.*

Proof. Trivial. □

Theorem 6. *$\mathcal{G}$ has a perfect matching if and only if there is no $A \subseteq X$ or $A \subseteq Y$ with $|\Gamma(A)| < |A|$.*

Notice that this statement (Hall's Theorem) is the converse of Lemma 5, so we don't get it for free.

Proof. We get the "only if" part of the theorem from Lemma 5, but we still have to prove the "if" case: If $\mathcal{G}$ has no perfect matching, then there is a set $A \subseteq X$ or $A \subseteq Y$ with $|\Gamma(A)| < |A|$.

Let $n = |X| = |Y|$. Because $\mathcal{G}$ has no perfect matching, the max-flow in the flow-graph $\mathcal{G}'$ is $v(f^*) \leq n - 1$. By the Max-Flow Min-Cut Theorem, there is a vertex set $S \in \mathcal{G}'$ with $c(S, \bar{S}) \leq n - 1$. Furthermore, there can be no edge $e = (x, y) \in \mathcal{G}$ with $X \in S$, $Y \in \bar{S}$ because $c_e = \infty$ in $\mathcal{G}'$.

Define $A = S \cap X$. Then:

$$n - 1 \geq c(S, \bar{S})$$

$$= |X \setminus S| + |Y \cap S|$$

$$= n - |A| + |\Gamma(A)| \text{ because } A \text{ cannot have neighbors outside } S$$

$$\implies |\Gamma(A)| \leq |A| - 1 < |A|$$

□

3 Edge-Disjoint Paths

Given a graph $\mathcal{G}$ and source/sink $s, t \in V$, find as many edge-disjoint (s,t) paths as possible. Reduction to MFMC: Give each edge e $c_e = 1$, then find the maximum integer (s,t) flow. From there, decompose the max-flow f^* into disjoint paths.

Lemma 7. *Let f be any acyclic (s,t) flow; then there are (s,t) paths $P_1, P_2, \ldots, P_k$ with $k \leq m$ and $a_1, a_2, \ldots, a_k$ such that for all edges e:*

$$\sum_{i:e \in P_i} a_i = f_e \text{ and if } f \in \mathbb{N}, \text{ all } a_i \in \mathbb{N}$$

The (P_i, a_i) can be computed in polynomial time.

Proof. By induction on the number of edges with $f_e > 0$.

If that number is zero, the $f = 0$. Otherwise, we start from s and follow and outgoing edge e with $f_e > 0$. Whenever we reach a node v, it has positive incoming flow, so by conservation, it has positive outgoing flow. Thus, we follow an outgoing edge with $f_e > 0$.

Because f is acyclic, no node repeats, so in at most n steps, we reach t. This defines an (s,t) path P with $f_e > 0 \; \forall e \in P$.

Define $\mathcal{E} := \min_{e \in P} f_e$. We add the pair $(P, \mathcal{E})$ to the decomposition and set $f'_e = f_e - \mathcal{E} \; \forall e \in P$. f' is still a flow by the augmentation lemma, and has one fewer edge with positive flow. Thus, we apply the inductive hypothesis to f', and add $(P', \mathcal{E}')$ to the decomposition. If all $f_e \in \mathbb{N}$, then all $\mathcal{E} \in \mathbb{N}$. $\square$