Intro to Game Theory

1 Packet Routing, continued

Recall from last lecture that we have:

$$\mathbb{E}[X_{e,t}] \leq \frac{c}{r}$$

$$k = (1+\delta)\frac{c}{r}$$

$$\Pr[X_{e,t} > k] < \frac{\exp(\delta)}{(1+\delta)^{(1+\delta)}}^{\frac{c}{r}}$$

We want δ and r such that:

$$\Pr[X_{e,t} > k] = \frac{1}{m^2(d+r)}$$

We choose $\delta = 1$ (sort of arbitrarily) so that:

$$\frac{1}{4}\exp(\frac{c}{r}) < \frac{1}{m^2(d+r)}$$

Let $r = \frac{qc}{\log(Nm)}$ for some constant q. For some small enough q:

$$\frac{1}{4}\exp(\frac{\log(Nm)}{q}) = \frac{1}{(Nm)^3}$$

$r \leq c \leq N, d \leq m$ so:

$$\frac{1}{(Nm)^3} \leq \frac{1}{m^2(d+r)}$$

As desired. This result means that, with probability $\geq 1 - \frac{1}{m}$, each meta-step of length k is long enough to route all the packets across the edge they want to cross. The total number of steps is:

$$\mathcal{O}((d+r)\frac{2c}{r}) = \mathcal{O}(c + \frac{dc}{r})$$

$$= \mathcal{O}(c + d\log(Nm))$$

2 Intro to Game Theory

Game theory is a mathematical framework for reasoning about outcomes that emerge when two or more entities with (usually) at least partially conflicting goals or preferences interact. The related field of **mechanism design** is similar to algorithm design but with incentive constraints. In other words, it deals with designing the rules of a game such that the outcomes of selfish behavior are not too detrimental to the desired behavior.

2.1 Two-player Zero Sum Games

Given a matrix $A = (a_{r,c})$: $a_{r,c}$ is the payoff to the row player if he plays r and the column player plays c. The column player must get $-a_{r,c}$ so the rewards sum to zero. If the row player moves first, he gets $\max_r \min_c a_{r,c}$, and if the column player moves first, she gets $\min_c \max_r -a_{r,c}$.

Proposition 1. $\max_r \min_c a_{r,c} \leq \min_c \max_r a_{r,c}$. *In other words, going first is never an advantage and is sometimes a disadvantage.*

Proof. Let $\hat{c} = \arg\min_c \max_r a_{r,c}$. Then:

$$\max_r \min_c a_{r,c} \leq \max_r a_{r,\hat{c}} = \min_c \max_r a_{r,c}$$

□

2.2 General Bimatrix Games

Given $n \times m$ matrics A and B, if the row player plays r and the column player plays c, then the row player gets $a_{r,c}$ and the column player gets $b_{r,c}$. Players then choose strategies to optimize their own reward. This model generalizes to k players with k tensors $A_1, \ldots, A_k$ in k dimensions such that $a^{(i)}_{(s_1,\ldots,s_k)}$ gives the payoff of player i when the players play the strategy vector $(s_1, \ldots, s_k)$.

Definition 2. An **equilibrium** is a strategy vector such that, given fixed strategies of everyone else, no player individually has incentive to change their strategy.

Notice that, for example, rock-paper-scissors does not have a pure equilibrium where everyone only chooses one strategy, but there is an equilibrium where everyone randomly chooses a play with uniform probability. We expand our definition:

Definition 3. A **mixed equilibrium** is a distribution of strategies for each player such that no player can increase their expected reward by changing their distribution.

Theorem 4. *(Nash): In games of finitely many players and finitely many pure strategies, there is always at least one mixed equilibrium, and it is called the **Nash equilibrium**.*

The proof for this theorem is nonconstructive and is based on Brouwer's Fixed Point Theorem. A natural question is, can we always construct such an equilibrium?

Theorem 5. *(Daskalakis, Goldberg, Papadimitriu, Chen, Deng): Finding a Nash equilibrium in a game with $k \geq 2$ players is **PPAD**-complete.*