PROBABILISTIC ROBOTICS:
RECURSIVE STATE ESTIMATION

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1

Let X_n the random variable indicating the state of the range sensor at the date n. More precisely,

$$X_n = \begin{cases} 0 & \text{if the sensor is faulty,} \\ 1 & \text{if the sensor is working correctly.} \end{cases}$$

Assuming the state of the sensor does not change, we have that

$$\begin{aligned} \forall n \in \mathbb{N}, \quad X_n &= X_0 \\ P(X_0 = 0) = p &= 0.01 \\ P(X_0 = 1) = 1 - p &= 0.99 \end{aligned}$$

Let $Z_n \in [0, 3]$ the real valued random variable giving the range measured by the sensor. By the problem statement,

$$\forall n \in \mathbb{N}, \quad Z_n = z_n \in [0, 1]$$

$$\begin{aligned} P(X_n = 0 \mid z_{1:n}) &= \eta P(z_n \mid X_t = 0, z_{1:n-1}) \times P(X_n = 0 \mid z_{1:n-1}) \\ &= \eta P(z_n \mid X_t = 0, z_{1:n-1}) \times P(X_{n-1} = 0 \mid z_{1:n-1}) && \text{(the update step is trivial} \\ & && \text{since the state does not change)} \\ &= \eta \times 1 \times P(X_{n-1} = 0 \mid z_{1:n-1}) \\ &= \eta P(X_0 = 0) && \text{(recurrence)} \\ &= \eta p \end{aligned}$$

$$\begin{aligned} P(X_n = 1 \mid z_{1:n}) &= \eta P(z_n \mid X_t = 1, z_{1:n-1}) \times P(X_n = 1 \mid z_{1:n-1}) \\ &= \eta P(z_n \mid X_t = 1, z_{1:n-1}) \times P(X_{n-1} = 1 \mid z_{1:n-1}) \\ &= \eta \times \frac{1}{3} \times P(X_{n-1} = 1 \mid z_{1:n-1}) \\ &= \eta \times \frac{1}{3^n} \times P(X_0 = 1) && \text{(recurrence)} \\ &= \eta \times \frac{1}{3^n} \times (1 - p) \end{aligned}$$

Note that the value of the normalizer η changes along the recurrence although it is not reflected in the notation. We can compute the normalizer:

$$\eta = \frac{1}{p + \frac{1}{3^n}(1 - p)}$$

The posterior probability of sensor fault at date n is therefore:

$$\boxed{P(X_n = 0 \mid z_{1:n}) = \frac{p}{p + \frac{1}{3^n}(1 - p)}}$$

It comes as no surprise this increases to 1 as the number of observation goes to infinity. Let's compute some numerical values:

n	$P(X_n = 0 \mid z_{1:n})$
1	0.0294
2	0.0833
3	0.2143
4	0.4500
5	0.7105
6	0.8804
7	0.9567
8	0.9851
9	0.9950
10	0.9983

2

Let X_t be the random variable which is the wheather of day t. We define $(\text{"}sunny\text{"}, \text{"}cloudy\text{"}, \text{"}rainy\text{"}) = (1, 2, 3)$.

2.1.

$$\begin{aligned}
&P(X_1 = 1 \cap X_2 = 2 \cap X_3 = 2 \cap X_4 = 3) \\
&= P(X_4 = 3 \mid X_1 = 1 \cap X_2 = 2 \cap X_3 = 2) \times P(X_1 = 1 \cap X_2 = 2 \cap X_3 = 2) \\
&= P(X_4 = 3 \mid X_3 = 2) \times P(X_3 = 2 \mid X_1 = 1 \cap X_2 = 2) \times P(X_1 = 1 \cap X_2 = 2) \\
&= P(X_4 = 3 \mid X_3 = 2) \times P(X_3 = 2 \mid X_2 = 2) \times P(X_2 = 2 \mid X_1 = 1) \times P(X_1 = 1) \\
&= 0.2 \times 0.4 \times 0.2 \times 1 \\
&= 0.016
\end{aligned}$$

2.2.

2.3.

2.4. Let

$$\begin{aligned}
Y_t &= \begin{bmatrix} P(X_t = 1) \\ P(X_t = 2) \\ P(X_t = 3) \end{bmatrix} \\
A &= \begin{bmatrix} P(X_{t+1} = 1 \mid X_t = 1) & P(X_{t+1} = 1 \mid X_t = 2) & P(X_{t+1} = 1 \mid X_t = 3) \\ P(X_{t+1} = 2 \mid X_t = 1) & P(X_{t+1} = 2 \mid X_t = 2) & P(X_{t+1} = 2 \mid X_t = 3) \\ P(X_{t+1} = 3 \mid X_t = 1) & P(X_{t+1} = 3 \mid X_t = 2) & P(X_{t+1} = 3 \mid X_t = 3) \end{bmatrix} \\
&= \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix}
\end{aligned}$$

We have

$$Y_t = AY_{t-1}$$

so that

$$\forall t \in \mathbb{N}, \quad Y_t = A^n Y_0$$

The matrix A is diagonalizable

$$A = PDP^{-1}$$

with

$$P = \begin{bmatrix} 9 & \sqrt{2}-1 & -\sqrt{2}-1 \\ 4 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{14} & \frac{1}{14} & \frac{1}{14} \\ \frac{-\sqrt{2}(\sqrt{2}-4)}{56} & \frac{-\sqrt{2}(\sqrt{2}+10)}{56} & \frac{\sqrt{2}(13\sqrt{2}+4)}{56} \\ \frac{-\sqrt{2}(\sqrt{2}+4)}{56} & \frac{-\sqrt{2}(\sqrt{2}-10)}{56} & \frac{\sqrt{2}(13\sqrt{2}-4)}{56} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1+\sqrt{2}}{5} & 0 \\ 0 & 0 & \frac{1-\sqrt{2}}{5} \end{bmatrix}$$

Since

$$\lim_{n\to\infty} D^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we have

$$\begin{aligned} \lim_{n\to\infty} Y^n &= P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} X_0 \\ &= \begin{bmatrix} \frac{9}{14} \\ \frac{2}{7} \\ \frac{1}{14} \end{bmatrix} \end{aligned}$$

whatever the initial distribution X_0 is.

The stationary distribution is

$$\Pi = \begin{bmatrix} \frac{9}{14} \\ \frac{2}{7} \\ \frac{1}{14} \end{bmatrix}$$

2.5. Let's compute the conditional probability distribution of X_{t-1} given X_t:

$$\begin{aligned} \forall (i,j) \in [\![1,3]\!]^2, \quad P(X_{t-1} = i \mid X_t = j) &= \eta P(X_t = j \mid X_{t-1} = i) \times P(X_{t-1} = i) \\ &= \frac{\overbrace{P(X_t = j \mid X_{t-1} = i)}^{\text{do not depend on t}} \times \overbrace{P(X_{t-1} = i)}^{\text{depends on t}}}{\underbrace{P(X_t = j)}_{\text{depends on t}}} \end{aligned}$$

We can consider the limit distribution as t goes to infinity:

$$\begin{aligned} m_{i,j} &= \frac{P(X_t = j \mid X_{t-1} = i) \times \Pi[i]}{\Pi[j]} \\ &= \frac{a_{j,i} \times \Pi[i]}{\Pi[j]} \end{aligned}$$

This gives the following transition matrix, rounded to 2 decimal place:

$$M = \begin{bmatrix} 0.80 & 0.45 & 0 \\ 0.18 & 0.40 & 0.80 \\ 0.02 & 0.15 & 0.20 \end{bmatrix}$$

2.6. The markov property states the probability law of future state conditioned on the current state does not depend on any other variables: the current state should be sufficient to compute the future stochastic evolution. This implies the state transition function can not depend on the season. Therefore, in order to restore the Markov property, we could incorporate the season into the state variable.

3

Let $(Z_t)_{t \in \mathbb{N}^*}$ denote the sequence of random variables measuring the wheather of the day.

$$\forall t \in \mathbb{N}^*, \quad Z_t \in [\![1, 3]\!]$$

3.1. We know for a fact that day 1 is sunny, and we do the following observations from day 2 to day 5:

t	z_t
2	2
3	2
4	3
5	1

We compute the distributions for $(X_t)_{t \in [\![2,5]\!]}$, being given the observations from the past until the current point of time:

$$\begin{aligned}
P(X_2 = 1 \mid z_2) &= \eta P(Z_2 = 2 \mid X_2 = 1) \times P(X_2 = 1) \\
&= \eta \times 0.4 \times 0.8 \\
&= 0.32\eta \\
P(X_2 = 2 \mid z_2) &= \eta P(Z_2 = 2 \mid X_2 = 2) \times P(X_2 = 2) \\
&= \eta \times 0.7 \times 0.2 \\
&= 0.14\eta \\
P(X_2 = 3 \mid z_2) &= \eta P(Z_2 = 2 \mid X_2 = 3) \times P(X_2 = 3) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\eta &= \frac{1}{0.32 + 0.14} \\
&= \frac{1}{0.46}
\end{aligned}$$

i	$P(X_2 = i \mid z_2)$
1	0.70
2	0.30
3	0

$$
\begin{aligned}
P(X_3 = 1 \mid z_{2:3}) &= \eta P(Z_3 = 2 \mid X_3 = 1, z_2) \times P(X_3 = 1 \mid z_2) \\
&= \eta P(Z_3 = 2 \mid X_3 = 1) \times \sum_{i=1}^{3} P(X_3 = 1 \mid X_2 = i, z_2) \times P(X_2 = i \mid z_2) \\
&= \eta P(Z_3 = 2 \mid X_3 = 1) \times \sum_{i=1}^{3} P(X_3 = 1 \mid X_2 = i) \times P(X_2 = i \mid z_2) \\
&= \eta \times 0.4 \times 0.6783 \\
&= 0.271\eta \\
P(X_3 = 2 \mid z_{2:3}) &= \eta P(Z_3 = 2 \mid X_3 = 2, z_2) \times P(X_3 = 2 \mid z_2) \\
&= \eta P(Z_3 = 2 \mid X_3 = 2) \times \sum_{i=1}^{3} P(X_3 = 2 \mid X_2 = i, z_2) \times P(X_2 = i \mid z_2) \\
&= \eta P(Z_3 = 2 \mid X_3 = 2) \times \sum_{i=1}^{3} P(X_3 = 2 \mid X_2 = i) \times P(X_2 = i \mid z_2) \\
&= \eta \times 0.7 \times 0.2609 \\
&= 0.183\eta \\
P(X_3 = 3 \mid z_2) &= \eta P(Z_2 = 2 \mid X_3 = 3) \times P(X_3 = 3 \mid z_2) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\eta &= \frac{1}{0.183 + 0.271} \\
&= \frac{1}{0.4539}
\end{aligned}
$$

i	$P(X_3 = i \mid z_{2:3})$
1	0.60
2	0.40
3	0

$$
\begin{aligned}
P(X_4 = 1 \mid z_{2:4}) &= \eta P(Z_4 = 3 \mid X_4 = 1, z_{2:3}) \times P(X_4 = 1 \mid z_{2:3}) \\
&= 0 \\
P(X_4 = 2 \mid z_{2:4}) &= \eta P(Z_4 = 3 \mid X_4 = 2, z_{2:3}) \times P(X_4 = 2 \mid z_{2:3}) \\
&= 0 \\
P(X_4 = 3 \mid z_{2:4}) &= 1
\end{aligned}
$$

i	$P(X_4 = i \mid z_{2:4})$
1	0
2	0
3	1

$$\begin{aligned}
P(X_5 = 1 \mid z_{2:5}) &= \eta P(Z_5 = 1 \mid X_5 = 1, z_{2:4}) \times P(X_5 = 1 \mid z_{2:4}) \\
&= \eta P(Z_5 = 1 \mid X_5 = 1) \times \sum_{i=1}^{3} P(X_5 = 1 \mid X_4 = i, z_{2:4}) \times P(X_4 = i \mid z_{2:4}) \\
&= \eta P(Z_5 = 1 \mid X_5 = 1) \times P(X_5 = 1 \mid X_4 = 3) \times 1 \\
&= \eta \times 0.6 \times 0.2 \\
&= 0.12\eta \\
P(X_5 = 2 \mid z_{2:5}) &= \eta P(Z_5 = 1 \mid X_5 = 2, z_{2:4}) \times P(X_5 = 2 \mid z_{2:4}) \\
&= \eta P(Z_5 = 1 \mid X_5 = 2) \times \sum_{i=1}^{3} P(X_5 = 2 \mid X_4 = i, z_{2:4}) \times P(X_4 = i \mid z_{2:4}) \\
&= \eta P(Z_5 = 1 \mid X_5 = 2) \times P(X_5 = 2 \mid X_4 = 3) \times 1 \\
&= \eta \times 0.3 \times 0.6 \\
&= 0.18\eta \\
P(X_5 = 3 \mid z_{2:5}) &= 0
\end{aligned}$$

$$\begin{aligned}
\eta &= \frac{1}{0.18 + 0.12} \\
&= \frac{1}{0.30}
\end{aligned}$$

i	$P(X_5 = i \mid z_{2:5})$
1	0.40
2	0.60
3	0

3.2. We know for a fact that day 1 is sunny, and we do the following observations from day 2 to day 5:

t	z_t
2	1
3	1
4	3

We compute the distributions for $(X_t)_{t \in [\![2,5]\!]}$, being given the observations from the past until the current point of time:

$$\begin{aligned}
P(X_2 = 1 \mid z_2) &= \eta P(Z_2 = 1 \mid X_2 = 1) \times P(X_2 = 1) \\
&= \eta \times 0.6 \times 0.8 \\
&= 0.48\eta \\
P(X_2 = 2 \mid z_2) &= \eta P(Z_2 = 1 \mid X_2 = 2) \times P(X_2 = 2) \\
&= \eta \times 0.3 \times 0.2 \\
&= 0.06\eta \\
P(X_2 = 3 \mid z_2) &= \eta P(Z_2 = 1 \mid X_2 = 3) \times P(X_2 = 3) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\eta &= \frac{1}{0.48 + 0.06} \\
&= \frac{1}{0.54}
\end{aligned}$$

i	$P(X_2 = i \mid z_2)$
1	0.89
2	0.11
3	0

$$\begin{aligned}
P(X_3 = 1 \mid z_{2:3}) &= \eta P(Z_3 = 1 \mid X_3 = 1, z_2) \times P(X_3 = 1 \mid z_2) \\
&= \eta P(Z_3 = 1 \mid X_3 = 1) \times \sum_{i=1}^{3} P(X_3 = 1 \mid X_2 = i, z_2) \times P(X_2 = i \mid z_2) \\
&= \eta P(Z_3 = 1 \mid X_3 = 1) \times \sum_{i=1}^{3} P(X_3 = 1 \mid X_2 = i) \times P(X_2 = i \mid z_2) \\
&= \eta \times 0.6 \times 0.7560 \\
&= 0.4536\eta \\
P(X_3 = 2 \mid z_{2:3}) &= \eta P(Z_3 = 1 \mid X_3 = 2, z_2) \times P(X_3 = 2 \mid z_2) \\
&= \eta P(Z_3 = 1 \mid X_3 = 2) \times \sum_{i=1}^{3} P(X_3 = 2 \mid X_2 = i, z_2) \times P(X_2 = i \mid z_2) \\
&= \eta P(Z_3 = 1 \mid X_3 = 2) \times \sum_{i=1}^{3} P(X_3 = 2 \mid X_2 = i) \times P(X_2 = i \mid z_2) \\
&= \eta \times 0.3 \times 0.222 \\
&= 0.0666\eta \\
P(X_3 = 3 \mid z_{2:3}) &= \eta P(Z_3 = 1 \mid X_3 = 3, z_2) \times P(X_3 = 3 \mid z_2) \\
&= \eta P(Z_3 = 1 \mid X_3 = 3) \times \sum_{i=1}^{3} P(X_3 = 3 \mid X_2 = i, z_2) \times P(X_2 = i \mid z_2) \\
&= \eta P(Z_3 = 1 \mid X_3 = 3) \times \sum_{i=1}^{3} P(X_3 = 3 \mid X_2 = i) \times P(X_2 = i \mid z_2) \\
&= 0 \\
\eta &= \frac{1}{0.4536 + 0.0666} \\
&= \frac{1}{0.5202}
\end{aligned}$$

i	$P(X_3 = i \mid z_{2:3})$
1	0.87
2	0.13
3	0

$$\begin{aligned}
P(X_4 = 1 \mid z_{2:4}) &= \eta P(Z_4 = 3 \mid X_4 = 1, z_{2:3}) \times P(X_4 = 1 \mid z_{2:3}) \\
&= 0 \\
P(X_4 = 2 \mid z_{2:4}) &= \eta P(Z_4 = 3 \mid X_4 = 2, z_{2:3}) \times P(X_4 = 2 \mid z_{2:3}) \\
&= 0 \\
P(X_4 = 3 \mid z_{2:4}) &= 1
\end{aligned}$$

i	$P(X_4 = i \mid z_{2:4})$
1	0
2	0
3	1

We will now compute the distribution of random variables $(X_t)_{t\in[\![2,4]\!]}$, conditionned on both past and future observations of the sensor. Let's define the transition matrix

$$
\begin{aligned}
C &= \begin{bmatrix} P(Z_t = 1 \mid X_t = 1) & P(Z_t = 1 \mid X_t = 2) & P(Z_t = 1 \mid X_t = 3) \\ P(Z_t = 2 \mid X_t = 1) & P(Z_t = 2 \mid X_t = 2) & P(Z_t = 2 \mid X_t = 3) \\ P(Z_t = 3 \mid X_t = 1) & P(Z_t = 3 \mid X_t = 2) & P(Z_t = 3 \mid X_t = 3) \end{bmatrix} \\
&= \begin{bmatrix} 0.6 & 0.3 & 0 \\ 0.4 & 0.7 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

The sensor gives us the measurement:

t	z_t
2	1
3	1
4	3

$$
\begin{aligned}
&P(X_2 = 1 \mid z_{2:4}) \\
&= \eta P(Z_2 = 1 \mid X_2 = 1, z_{3:4}) \times P(X_2 = 1 \mid z_{3:4}) \\
&= \eta P(Z_2 = 1 \mid X_2 = 1) \times P(Z_3 = 1 \mid X_2 = 1, z_4) \times P(X_2 = 1 \mid z_4) \\
&= \eta P(Z_2 = 1 \mid X_2 = 1) \times (\sum_{i=1}^{3} P(Z_3 = 1 \mid X_3 = i, X_2 = 1, z_4) \times P(X_3 = i \mid X_2 = 1, z_4)) \\
&\quad \times P(X_2 = 1 \mid z_4) \\
&= \eta P(Z_2 = 1 \mid X_2 = 1) \times (\sum_{i=1}^{3} P(Z_3 = 1 \mid X_3 = i) \times P(X_3 = i \mid X_2 = 1)) \\
&\quad \times P(Z_4 = 3 \mid X_2 = 1) \times P(X_2 = 1) \\
&= \eta P(Z_2 = 1 \mid X_2 = 1) \times \underbrace{(\sum_{i=1}^{3} P(Z_3 = 1 \mid X_3 = i) \times P(X_3 = i \mid X_2 = 1))}_{①} \\
&\quad \times \underbrace{(\sum_{j=1}^{3} P(z_4 = 3 \mid X_4 = j, X_2 = 1) \times P(X_4 = j \mid X_2 = 1))}_{②} \times P(X_2 = 1)
\end{aligned}
$$

$$
\begin{aligned}
① &= 0.6 \times 0.8 + 0.3 \times 0.2 \\
&= 0.54 \\
② &= \sum_{j=1}^{3} P(Z_4 = 3 \mid X_4 = j) \times (\sum_{k=1}^{3} P(X_4 = j \mid X_3 = k) \times P(X_3 = k \mid X_2 = 1)) \\
&= \sum_{j=1}^{3} \sum_{k=1}^{3} \underbrace{P(Z_4 = 3 \mid X_4 = j)}_{\delta_{j,3}} \times P(X_4 = j \mid X_3 = k) \times P(X_3 = k \mid X_2 = 1) \\
&= \sum_{k=1}^{3} P(X_4 = 3 \mid X_3 = k) \times P(X_3 = k \mid X_2 = 1) \\
&= 0.2 \times 0.2 + 0.2 \times 0 + 0 \\
&= 0.04 \\
P(X_2 = 1 \mid z_{2:4}) &= \eta \times 0.6 \times 0.54 \times 0.04 \times 0.8 \\
&= 0.0104\eta
\end{aligned}
$$

We observe that

$$\begin{aligned}
① &= P(Z_3 = 1 \mid X_2 = 1, z_4) \\
&= (CA)_{1,1} \\
② &= P(Z_4 = 3 \mid X_2 = 1) \\
&= (CAA)_{3,1}
\end{aligned}$$

$$CA = \begin{bmatrix} 0.54 & 0.36 & 0.30 \\ 0.46 & 0.44 & 0.50 \\ 0 & 0.20 & 0.20 \end{bmatrix}$$
$$CAA = \begin{bmatrix} 0.5040 & 0.42 & 0.3840 \\ 0.456 & 0.46 & 456 \\ 0.04 & 0.12 & 0.16 \end{bmatrix}$$

Then,

$$\begin{aligned}
& P(X_2 = 2 \mid z_{2:4}) \\
&= \eta P(Z_2 = 1 \mid X_2 = 2) \times P(Z_3 = 1 \mid X_2 = 2, z_4) \times P(Z_4 = 3 \mid X_2 = 2) \times P(X_2 = 2) \\
&= \eta P(Z_2 = 1 \mid X_2 = 2) \times (CA)_{1,2} \times (CAA)_{3,2} \times P(X_2 = 2) \\
&= \eta 0.3 \times 0.36 \times 0.12 \times 0.2 \\
&= 0.0026\eta \\
& P(X_2 = 3 \mid z_{2:4}) \\
&= \eta P(Z_2 = 1 \mid X_2 = 3) \times P(Z_3 = 1 \mid X_2 = 3, z_4) \times P(Z_4 = 3 \mid X_2 = 3) \times P(X_2 = 2) \\
&= 0 \\
\eta &= \frac{1}{0.0104 + 0.0026} \\
&= \frac{1}{0.013}
\end{aligned}$$

i	$P(X_2 = i \mid z_{2:4})$
1	0.8
2	0.2
3	0

$$
\begin{aligned}
P(X_3 = 1 \mid z_{2:4}) &= \frac{P(X_3 = 1 \cap Z_4 = 3 \mid z_{2:3})}{P(Z_4 = 3 \mid z_{2:3})} \\
&= \eta P(Z_4 \mid X_3 = 1, z_{2:3}) \times P(X_3 = 1 \mid z_{2:3}) \\
&= \eta \sum_{i=1}^{3} P(Z_4 = 3 \cap X_4 = i \mid X_3 = 1, z_{2:3}) \times P(X_3 = 1 \mid z_{2:3}) \\
&= \eta \sum_{i=1}^{3} P(X_4 = i \mid X_3 = 1, z_{2:3}) \times P(Z_4 = 3 \mid X_4 = i, z_{2:3}) \times P(X_3 = 1 \mid z_{2:3}) \\
&= \eta \sum_{i=1}^{3} P(X_4 = i \mid X_3 = 1) \times \underbrace{P(Z_4 = 3 \mid X_4 = i)}_{\delta_{3,i}} \times P(X_3 = 1 \mid z_{2:3}) \\
&= \eta P(X_4 = 3 \mid X_3 = 1) \times P(Z_4 = 3 \mid X_4 = 3) \times P(X_3 = 1 \mid z_{2:3}) \\
&= 0 \\
P(X_3 = 2 \mid z_{2:4}) &= \eta P(X_4 = 3 \mid X_3 = 2) \times P(Z_4 = 3 \mid X_4 = 3) \times P(X_3 = 2 \mid z_{2:3}) \\
&= \eta \times 0.2 \times 1 \times 0.21 \\
&= 0.042\eta \\
P(X_3 = 3 \mid z_{2:4}) &= \eta P(X_4 = 3 \mid X_3 = 3) \times P(Z_4 = 3 \mid X_4 = 3) \times P(X_3 = 3 \mid z_{2:3}) \\
&= 0
\end{aligned}
$$

i	$P(X_3 = i \mid z_{2:4})$
1	0
2	1
3	0

For day 4 this is trivial:

i	$P(X_4 = i \mid z_{2:4})$
1	0
2	0
3	1

4

4.1. Using the notation of the book for probability density functions,

$$
\begin{aligned}
p(x) &= \frac{1}{\sqrt{2\pi}\sigma_{init}} e^{-\frac{(x - x_{init})^2}{2\sigma_{init}^2}} \\
p(z \mid x) &= \frac{1}{\sqrt{2\pi}\sigma_{GPS}} e^{-\frac{(z - x)^2}{2\sigma_{GPS}^2}}
\end{aligned}
$$

4.2. We have the following property for conditional probability density functions:

$$
p(x \mid z) = \frac{p(z \mid x) \times p(x)}{p(z)}
$$

We first compute the probability density of Z:

$$\begin{aligned}
p(z) &= \int_{-\infty}^{\infty} p(z \mid x) \times p(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_{GPS}} e^{-\frac{(z-x)^2}{2\sigma_{GPS}^2}} \frac{1}{\sqrt{2\pi}\sigma_{init}} e^{-\frac{(x-x_{init})^2}{2\sigma_{init}^2}}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_{GPS}} e^{-\frac{(z-x)^2}{2\sigma_{GPS}^2}} \frac{1}{\sqrt{2\pi}\times 9\sigma_{GPS}} e^{-\frac{(x-x_{init})^2}{2\times 3\sigma_{GPS}^2}}\,\mathrm{d}x \\
&= \frac{1}{9\times 2\pi\sigma_{GPS}^2} \int_{-\infty}^{\infty} e^{-\frac{(z-x)^2}{2\sigma_{GPS}^2}-\frac{(x-x_{init})^2}{2\times 3\sigma_{GPS}^2}}\,\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
\frac{(z-x)^2}{2\sigma_{GPS}^2} + \frac{(x-x_{init})^2}{2\times 9\sigma_{GPS}^2} &= \frac{9(z^2-2xz+x^2)+x^2-2xx_{init}+x_{init}^2}{18\sigma_{GPS}^2} \\
&= \frac{9z^2+10x^2-2x(9z+x_{init})+x_{init}^2}{18\sigma_{GPS}^2} \\
&= \underbrace{\frac{9z^2+x_{init}^2-\frac{(9z+x_{init})^2}{10}}{18\sigma_{GPS}^2}}_{\text{does not depend on x}} + \frac{(x-\frac{(9z+x_{init})^2}{10})}{2\times\frac{9}{10}\sigma_{GPS}^2}
\end{aligned}$$

Thus,

$$\begin{aligned}
p(z) &= \frac{1}{3\times 2\pi\sigma_{GPS}^2} e^{\frac{-9z^2-x_{init}^2+\frac{(9z+x_{init})^2}{10}}{18\sigma_{GPS}^2}} \times \sqrt{2\pi}\sqrt{\frac{9}{10}}\sigma_{GPS} \\
&= \frac{1}{\sqrt{2\pi}\sqrt{10}\sigma_{GPS}} e^{\frac{-\frac{9}{10}z^2-\frac{9}{10}x_{init}^2+2\times\frac{9}{10}x_{init}}{2\times 9\sigma_{GPS}^2}} \\
&= \frac{1}{\sqrt{2\pi}\sqrt{10}\sigma_{GPS}} e^{-\frac{(z-x_{init})^2}{2\times 10\sigma_{GPS}^2}}
\end{aligned}$$

This shows Z has a normal distribution $Z \hookrightarrow \mathcal{N}(x_{init}, 10\sigma_{GPS}^2)$. It follows that

$$\begin{aligned}
p(x \mid z) &= \frac{\frac{1}{3\times 2\pi\sigma_{GPS}^2} e^{-\frac{(z-x_{init})^2}{2\times 10\sigma_{GPS}^2}} e^{-\frac{(x-\frac{(9z+x_{init})^2}{10})}{2\times\frac{9}{10}\sigma_{GPS}^2}}}{\frac{1}{\sqrt{2\pi}\sqrt{10}\sigma_{GPS}} e^{-\frac{(z-x_{init})^2}{2\times 10\sigma_{GPS}^2}}} \\
&= \frac{1}{\sqrt{2\pi}\frac{3}{\sqrt{10}}\sigma_{GPS}} e^{-\frac{(x-\frac{(9z+x_{init})^2}{10})}{2\times\frac{9}{10}\sigma_{GPS}^2}}
\end{aligned}$$

From which we conclude

$$\boxed{\text{The distribution of random variable } X \text{ conditioned on } (Z=z) \text{ is } \mathcal{N}(\tfrac{9z+x_{init}}{10}, \tfrac{9}{10}\sigma_{GPS}^2).}$$

5

Being sloppy on technical details,

$$\begin{aligned}
p(x \mid z, y) &= \frac{p(x, y \mid z)}{p(y \mid z)} \\
&= \frac{p(x \mid z) \times p(y \mid z)}{p(y \mid z)} \\
&= p(x \mid z)
\end{aligned}$$

6

Classical derivation.

$$\begin{aligned}
E[X - E[X]]^2 &= \int_\Omega (X - E[X])^2 \,\mathrm{d}P \\
&= \int_\Omega (X^2 - 2XE[X] + (E[X])^2) \,\mathrm{d}P \\
&= \int_\Omega X^2 \,\mathrm{d}P - 2E[X] \int_\Omega X \,\mathrm{d}P + (E[X])^2 \int_\Omega 1 \,\mathrm{d}P \\
&= E[X^2] - 2(E[X])^2 + (E[X])^2 \\
E[X - E[X]]^2 &= E[X^2] - (E[X])^2
\end{aligned}$$