Deciphering Wisdom of Crowds from Their Influenced Binary Decisions

Weiyun Chen
Institute of Automation
Chinese Academy of Sciences
Beijing, China
weiyun.chen@ia.ac.cn

Xin Li
Department of Information Systems
City University of Hong Kong
Hong Kong, China
xin.li@cityu.edu.hk

Abstract—**The wisdom of crowds has been recognized as an effective decision making mechanism by aggregating information from different individuals to derive an overall decision. However, in this information aggregation process, individuals may be influenced by various factors and provide biased claims (or individual level decisions), especially when such claims are related to their economic benefits. In this research, we investigate crowd's claims in binary decisions under explicit constant influence and aim to understand their real but hidden belief (distribution) on the decision. Particularly, we take fixed odds betting on binary events as a representative scenario in this study. We model the relationship between event probability and crowds' belief distribution as a linear combination of Beta distributions. Taking a Maximization Likelihood Estimation (MLE) paradigm, we estimate the parameters of this distribution based on observed crowds' bets. In this process, we model individual betting decisions under the influence of odds using prospect theory. We apply the framework on a real world dataset of Olympic Games outcome betting. After identifying betting participants' hidden belief distribution, we also found that crowds' belief tend to tilt to the high probability side of the event (if there is no outside influence), which partially explains why the wisdom of crowds can make decision marking easier. We believe this paper contributes to the literature of crowd intelligence and can help generating more accurate digestions of the wisdom of crowds.**

Keywords-wisdom of crowds; collective belief; fixed odds betting;

I. INTRODUCTION

In business decision making, it is a common practice to aggregate the opinions of multiple people to predict the outcome of events. In classic experimental or application situations, these opinions usually can be truthfully induced as quantitative beliefs by incentive schemes, such as proper scoring rules. It is noticed that the aggregated beliefs of crowds often outperform domain experts' projections in predicting the true value. Such wisdom of crowds is observed in various applications, such as in public issues, financial markets, and sport games [1].

For properly aggregating the different or contradicting beliefs held by different individuals in crowds, various quantitative approaches are employed to explore the wisdom hidden in the collective beliefs, such as market based aggregation or prediction markets (PMs) [2], weighted averaging method [3] and Bayesian method etc. Market based aggregation assume the amount of wagered money represent the decision weight of individuals. Weighted averaging method usually takes the past performance of individuals as weight. These methods try to explore the right way of combining the useful information from the crowds with noise to predict the true value. The performance of these methods indicates that the wisdom hidden in the crowds often reflects the true value or its related probability in an appropriate manner. As a general result in these explorations of collective intelligence, it is often found that the simply averaged collective belief is a good indicator of the true state of world or its probability. For example, in his famous livestock show Francis Galton ask 787 participants to wager on the actual weight of an ox to win prize. He found that the average guess of the participants is 1197 pounds and the actual weight was 1198 pounds. In a more complex market based beliefs aggregation situation, e.g. PMs, market prices are often found to be a good indicator of probability of uncertain future event. Some scientists try to make an interpretation for this remarkable performance of PMs' price as indicator of future event predictions. They find the PMs' prices are closely correlated with the expectation of collective beliefs if they model the individual's choice behavior under risk in Expected Utility Theory (EUT) [17]. Thus empirical evidences and theoretical conclusion from field of PMs also suggest that expectation of collective belief is closely correlated with probability uncertain event.

Since the expectation of collective beliefs is an aggregated indicator of individual private information of the crowds and this indicator is often found to be closely correlated with the true state of the world or its probability, one problem is how the true state of the world or its probability is hidden in the distribution of the collective belief. In his book The Difference, Page shows that besides they should have enough expertise on the problem to analyze the crowd should show enough diversity to cover different aspects of information related to the decision (where conflict information may be cancelled out) [4]. But he does not analyze how the different beliefs among the crowds are distributed and the canceled out by each other. These findings provide limited meaningful insights into how this wisdom is distributed in crowds. Although some bottom-up models [5-7] in social influence and social learning provide a set of useful tools to understand some phenomenon in wisdom of crowds, they mainly base their arguments on too much simplified individual cognitive models and provide limited insight into this form of collective intelligence. In order to get deeper understanding of wisdom of crowds, it is necessary to develop models to estimate how the beliefs of crowds correlate with the true value or its probability. In this paper, we aim to know how the collective belief distributions are shaped by uncertain future events probabilities and why the probabilities of the uncertain events are often the mean belief of the crowds. Particularly, we take fixed odds betting on sports games as a representative context where the crowd provides influence claims and study this problem. In section 2 we explain the nature of fixed odds betting and people's influenced claims in this context. In section 3, we formulize our framework of crowd belief estimation in fixed odds betting context. In section 4, we apply the model on a real world dataset on 2008 Olympic Games betting, and report the estimated belief distribution and its characteristics. In section 5, we conclude this paper.

II. Wisdom of Crowds under Influence: Fixed Odds Betting

Fixed odds betting is a classic mechanism used in sports gambling [8], where participants bet on the outcomes of sports game and are rewarded proportionally to the amount betted. In many cases, fixed odds betting is conducted on events with binary outcomes, such as whether a team will win a game. The reward odds are usually set several days before the event, which remain unchanged during the process of betting [9, 10]. Through fixed odds betting, the money or betters staked on each side reflect crowds' claim on the sports game outcome. This mechanism can be employed in prediction market setting to address more general prediction problems. However, in this mechanism, participants will bet on their highest reward rather than solely their belief. If the odds of a participant's preferred side is much less than non-preferred side, s/he would stake on the other side for profit. The revealed betting results are obviously biased by the odds set up [11]. Figure 1 shows the influencing factors on the revealed results in this process.

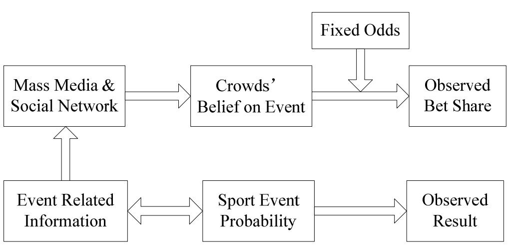

Figure 1: Information flow in Fixed Odds Sport Betting

Since the external influence (i.e., odds) does not change during betting scope and the decision problem is binary, fixed odds betting on binary decisions provides us an unique setting to study wisdom of crowds under influence. In this problem, odds, game results and final bet share on each side are observed. Each individual make decision according to odds and their inner belief on the event result. According to some existing findings in cognitive psychology, we can consider their beliefs on sport event are statistically determined by access frequency of event related information [12-14] which propagated by mass media and social network. Since the event probability is coupled with its related information, we can assume the crowds' belief depends on event probability. We denote this map from event probability to crowds' belief as $p \rightarrow f(x)$ where p denote the event probability and $f(x)$ denote the distribution of crowds belief. Our observed bet share implies the crowds' belief under the influence of odds and thus also implies the event probability. Although the mean belief of crowds is a good indicator of future events, in order to get more detailed understanding of the wisdom of crowds, our objective is to estimate hidden crowd belief distribution from data. Since both crowds' belief and event probabilities are not observable, our estimation need to be made based on observed bet share, odds, and the actual result of an event.

III. A Model on Crowd Belief Estimation

Let's consider binary betting with two results, A or B. The odds on each side's bettor are o_A and o_B respectively. Without loss of generality, we have o_A, o_B $\in$ $(0,+\infty)$ and $1/(1+o_A)+1/(1+o_B) > 1$, [9] so that bettors can't gain through betting on both sides. Each bettor would make betting choice based on their perceived utility of the two events:

$$\begin{cases} \text{if} \quad U(x_A, o_A) > U(x_B, o_B) & \text{then choose A} \\ \text{if} \quad U(x_A, o_A) < U(x_B, o_B) & \text{then choose B} \end{cases} \tag{1}$$

Where $U(\cdot)$ is the utility function given odds o_A, o_B and participants' believed event probability x_A, x_B. Here, we design the utility function based on the prospect theory [15] as:

$$\begin{aligned} & U(x,o) = w^+(x)v(o) + w^-(1-x)v(-1) \\ & w^+(x) = x^\gamma \Big/ \left[x^\gamma + (1-x)^\gamma\right]^{1/\gamma}, \; w^-(x) = x^\tau \Big/ \left[x^\tau + (1-x)^\tau\right]^{1/\tau} \\ & v(y) = \begin{cases} y^\alpha & \text{if } y \ge 0 \\ -\rho(-y)^\beta & \text{if y} < 0 \end{cases} \end{aligned} \tag{2}$$

Where $w^+()$ and $w^-()$ are probability weighting functions. Here we choose a commonly used one, while there are many alternatives existed. $v()$ is the valuation function for gain or loss. In the case of unit betting, the gain can be o_A and o_B and the loss can be -1. Although individual behavior can be heterogeneous, in this paper, we assume all bettors have consistent utility function for the sake of simplicity. The parameters of utility function can be estimated empirically. Particularly, according to previous literature [15], we assign α and β as 0.88, ρ as 2.25, γ as 0.61 and τ as 0.69 in this research.

By specifying utility function, participants' outcomes are purely based on individuals' projected event probability and odds. Since odds are given for each game, the collective behavior of bettors depends on their (unobserved) belief distribution, which is affected by actual event probability and can be denoted as $p \rightarrow f(x)$. Without loss of generalizability, we model the relationship between the probability of event p_{iA}, p_{iB} and crowds' belief distribution $f_{iA}(x)$, $f_{iB}(x)$ in each different games as a linear combination of Beta distributions (see Appendix for detail):

$$f_{i\mathrm{A}} = \sum_{l=1}^{D} \lambda_l Beta(\sum_{h=1}^{Z} u_{lh} p_{i\mathrm{A}}^h, \sum_{h=1}^{Z} u_{lh} (1-p_{i\mathrm{A}})^h)$$
$$f_{i\mathrm{B}} = \sum_{l=1}^{D} \lambda_l Beta(\sum_{h=1}^{Z} u_{lh} p_{i\mathrm{B}}^h, \sum_{h=1}^{Z} u_{lh} (1-p_{i\mathrm{B}})^h) \tag{3}$$
$$\sum_{l=1}^{D} \lambda_l = 1,\ \lambda_l > 0$$

Where Beta() represent Beta probability density function and i represent the index of games; λ_l is the coefficients of linear combination; D and Z are the order of Beta function and the polynomial. u_{lh} is the weight of polynomial components of event probability.

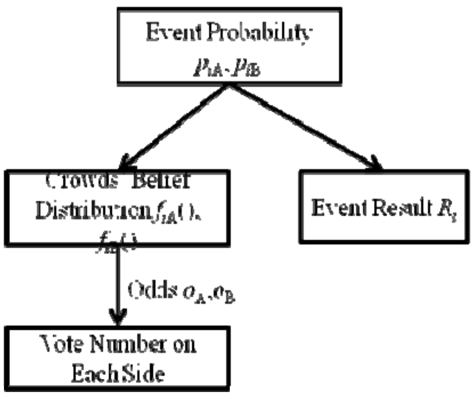

Figure 2. Probabilistic Model for Crowd Intelligence under Influence.

With these specifications, as shown in Figure 2, we can model the relationship between event probability, individual belief, individual claims, and observable event results as a probabilistic model. Since the relationship between individual belief and individual decision is deterministic (after controlling odds), it is presented as a solid line.

Suppose we have N i.i.d. fixed odds sport betting games, we will have observable data set $\{o_{i\mathrm{A}}, o_{i\mathrm{B}}, R_i, m_i, k_i, s_{i\mathrm{A}}, s_{i\mathrm{B}}\}$, $i \in \{1,\cdots, N\}$, where $o_{i\mathrm{A}}$, $o_{i\mathrm{B}}$ are the odds for event A and B in game i respectively; R_i represent ith game's result which take value 1 when A happens and take value 0 when B happens; mi denote the number of participants in game i and k_i denote the number of betting on side A; $s_{i\mathrm{A}}$ and $s_{i\mathrm{B}}$ denote the final bet share on each side $s_{i\mathrm{A}}= k_i/m_i$. Besides, we will also have unobservable event probability $p_{i\mathrm{A}}$ and $p_{i\mathrm{B}}$, $i \in \{1,\cdots, N\}$and unknown parameters in set:

$$\Theta = \{u_{lh}, \lambda_l, D, Z \mid \sum_{h=1}^{Z} u_{lh} x^h > 0 \text{ for } x \in (0,1),\ \sum_{l=1}^{D} \lambda_l = 1,\ \lambda_l > 0; \ h = 1,\dots,Z; l = 1,\dots,D\} \tag{4}$$

We seek to estimate the optimal $\theta \in \Theta$ based on observable data using MLE. Here we first simplify the relationship between individual belief and individual claim/decision. Since the utility function is an increasing continuous function to individual belief, there exist a threshold $c \in (0,1)$ satisfying $U(c, o_{\mathrm{A}}) = U(1-c, o_{\mathrm{B}})$. Here we name this threshold critical belief, which differentiates bettors into two groups. The ones with a belief higher than this value will have a higher utility on A and make a decision claim on A. This critical belief can be derived numerically from (2) if odds are given. After this, the corresponding distribution of crowds' claims/decisions is binomial. It is easily proven that for any $\theta \in \Theta$ there exist a list of $p_{i\mathrm{A}}$ and $p_{i\mathrm{B}}$, $i \in \{1,\cdots, N\}$which can maximize the observed bet number on each side when:

$$\begin{aligned} \Pr_i(\text{vote A}) &= \int_{c_i}^{1} f_{i\mathrm{A}}(x)dx \\ &= \int_{c_i}^{1} \left\{ \sum_{l=1}^{D} \lambda_l Beta(\sum_{h=1}^{Z} u_{lh} p_{iA}^h, \sum_{h=1}^{Z} u_{lh} (1-p_{iA})^h) \right\} dx \\ &= k_i / m_i = s_{i\mathrm{A}} \end{aligned} \tag{5}$$

Where Pri(vote A) denote the probability of voting on A for an independently sampled individual from the belief distribution $f_{i\mathrm{A}}(x)$. Because the $\theta \in \Theta$ is arbitrarily given, there are numerous θ and corresponding $p_{i\mathrm{A}}$ and $p_{i\mathrm{B}}$, $i \in \{1,\cdots, N\}$ which can maximize the likelihood of the observed bet number data. Therefore, under the constraint of maximizing the observed bet number $p_{i\mathrm{A}}$ is a function of θ which can be denoted as:

$$p_{iA} = G(c_i, s_{i\mathrm{A}}; \theta) \tag{6}$$

Where G() can be considered as an solution of $p_{i\mathrm{A}}$ from equation (5). If the parameters θ is known, the most possible $p_{i\mathrm{A}}$ and $p_{i\mathrm{B}}$, $i \in \{1,\cdots, N\}$ that can generate individual claims should fit this function. With these transformations, we convert the MLE problem to one that is based on observable observed game results R_i, $i \in \{1,\cdots, N\}$ because the estimated $p_{i\mathrm{A}}$ and $p_{i\mathrm{B}}$ from (6) should also maximize the game results. The log likelihood function of this part can be written as:

$$\begin{aligned} ll &= \log(\mathrm{Pro}(R_1,\dots,R_N \mid \theta)) \\ &= \log\left(\prod_{i=1}^{N} p_{i\mathrm{A}}{}^{R_i} (1-p_{i\mathrm{A}})^{1-R_i} \right) \\ &= \log\left(\prod_{i=1}^{N} G(c_i, s_{i\mathrm{A}}; \theta)^{R_i} (1-G(c_i, s_{i\mathrm{A}}; \theta))^{1-R_i} \right) \end{aligned} \tag{7}$$

In other words, we are looking for a set of optimal belief distribution parameters $\theta \in \Theta$ from which we can derive event probability $p_{i\mathrm{A}}$ (corresponding to individual claims) that can fit the final event results the best. In this research, to reduce computational complexity, we fix D and Z and estimate other belief distribution parameters u_{lh} and λ_l through maximizing ll in (7). Then, by adapting D and Z, we can get a series of optimal parameters u_{lh} and λ_l, which is when we need to decide what D and Z to use.

In general larger D and Z generally produce higher likelihood value of the model, which at the same time reduces the generalizability of the model. To tradeoff between model complexity and the likelihood value on the data, we employ

AICc criteria [16] to determine the appropriate D and Z. AICc value of a model is computed by:

$$AICc = 2t + 2t(t+1)/(N-t-1) - 2ll \tag{8}$$

Where t denotes the number of parameters, N denotes the sample size, and ll denotes the model's log likelihood value on data. In our framework, t equals to D*(Z+1) if D>1 and t equals Z if D=1. After calculating candidate models' AICc values, we can combine these models by weighting them according to their AICc. The weight of each component is computed as:

$$weight_j = \exp((AICc_{\min} - AICc_j)/2) \tag{9}$$

IV. Empirical Study

We collect a real world dataset from an online fixed odd betting Website on 2008 Olympic Game results. The Website attracted more than 170,000 participants. We collect 167 binary betting games. In these games, the number of bettors ranges from 108 to 2635. Average bettor per game is 564. The standard deviation is 516.

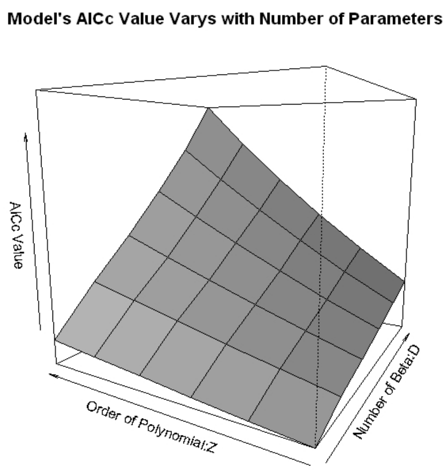

Figure 3: AICc Value of Candidate Models

To derive the parameters for belief distribution f_{iA} and f_{iB}, We first tune the parameters D and Z for its component Beta distributions. For the sake of simplicity, we restrict them to be in the range of 1 to 6, which provides 36 combinations of candidate component distributions. We estimate the maximized likelihood value on the dataset for each model and calculate their AICc values as shown in Figure 3. D=1 and Z=1 has the highest weight. While D and Z increase, the candidate models' AICc values increase and their weight decrease exponentially. Thus, we choose three component distributions with weight larger than 0.1 to build our model, which are distributions with (D=1, Z=1), (D=1, Z=2), and (D=1, Z=3) respectively. For the 167 games we have, we estimate the event probability, coefficients of the linear combinations of the components, and weights of the Beta distribution parameters. Since the estimated Beta distribution parameters for the third component is similar to that of the second one, we further combine the third component with the second one, which provides us the belief distribution function as :

$$\begin{aligned} f_{iA}(x) = 0.67 * Beta(5.94 * p_{iA}, 5.94 * p_{iB}) + \\ 0.33 * Beta(5.69 * p_{iA} + 0.16 * p_{iA}^2, 5.69 * p_{iB} + 0.16 * p_{iB}^2) \end{aligned} \tag{10}$$

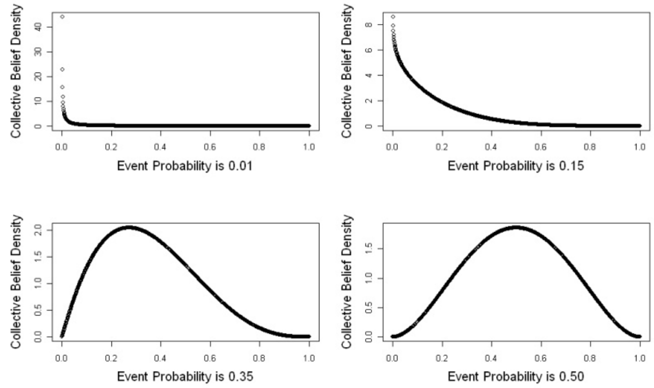

Figure 4: Belief Distribution varies with Event Probability

Figure 4 plots the shapes of belief distributions according to different event probability. If event probability is small, say 0.01, most of the crowds won't believe the event would happen. If event probability increases to 0.15, while most of the crowds still do not believe it would happen, their belief distribution will tilt right a little and show some characteristics of a power law distribution. If the event probability is close to 0.5, as shown in the two figures in the lower part of Figure 4, the belief distribution will gradually change to a form close to normal distribution.

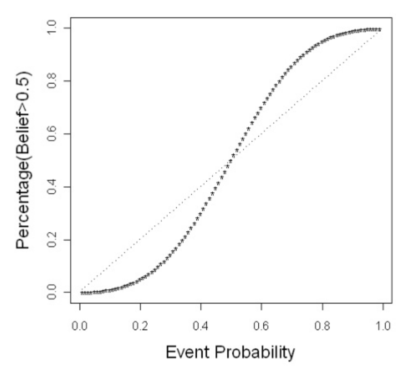

Figure 5: User Choice Distribution without Influence according to Event Probability.

From formula (10), if we consider people's decision is made based on belief threshold 0.5, we can plot user decision distribution according to event probability (as in Figure 5). It shows that the crowds would amplify actual probability for event to happen (or not happen). If event probability is less than 0.5, even less percentage of people will make an argument that the event would happen. However, if the event probability is larger than 0.5, there will be more percentage of people who would argue the event would happen. Thus the wisdom of crowd can make it easier for decision makers to make binary decisions. Furthermore, we can also derive the mean belief of the crowds (without influence) based belief distribution. It turns out that this value is very similar to event probability. That is consistent with empirical and theoretical findings in Prediction Market (PM) research [17], where user decisions are on a continuous spectrum of contract price. In such as setting the price of contract PMs converge to event probability [2].

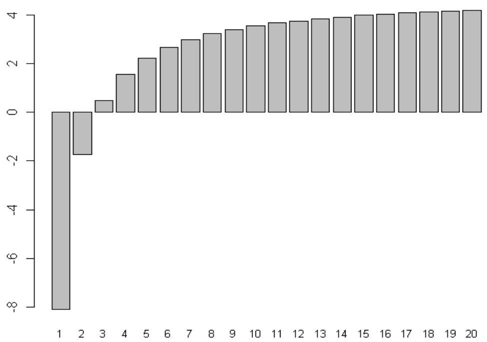

Figure 6: Expected scores increase with the number of averaging

Based on formula (10), we conduct a series of computational experiments [18, 19] to check whether the accuracy of average estimates of crowds increases with the individual number, which have been reported in [3] based on an interesting data set from an online experiment. In our computational experiment, we first fix the event probability and thus compute the corresponding crowd's belief distribution according to (10). In the second step, we simulate the event result according to the predesigned event probability and sample individual belief from the corresponding crowd's belief distribution. Then we average these sampled individual belief with different individual number as the wisdom of crowds to predict the simulated event results. Proper quadratic scoring rule is used to evaluate the performance of these predictions.

Figure 6 shows how the scores gained by average estimates of crowds increase with the individual number in the averaging process. We can see from Figure 6 that the expected individual score is about -8. And the expected score of average belief from three arbitrarily selected individuals have dramatically increased to about 0.5. The expected score increase slower when the number of individuals increases. This result is very similar with what we observe from the reported data set.

V. CONCLUSIONS

In this paper, we propose a framework to estimate crowds' hidden belief distribution from their influenced binary decision marking results. We model the belief distribution as a linear combination of Beta distributions whose parameters are based on (hidden) event probability. We estimate the parameter of the model in a MLE paradigm by considering odds' effect under prospect theory. From an Olympics Game betting dataset, we found that crowds tend to amplify event probability on making binary decisions, which may make decision making easier.

Although our research can still be improved in different ways, such as to incorporate larger dataset for validation, it shows the potentials to extract consumer (hidden) belief form their binary decisions under influence. Since influenced decisions are ubiquitous in our life, the uncover of hidden distributions can make our business decision easier and more accurate. In future, we will improve the framework to handle more complicated decision scenarios than binary decisions. We will also study scenarios with dynamic influences. Our ultimate goal is to build a comprehensive framework that can decipher the true belief of crowd in influenced decisions to support business decision making.

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APPENDIX

First, we assume the unknown belief distribution function can be represented as linear combination of finite Beta distributions because any probability density function can be approximated by a sum of series Beta distribution.

$$f_{iA}(x) = \sum_{l=1}^{D} \lambda_{iAl} \cdot Beta(\alpha_{iAl}, \beta_{iAl}) \quad \text{(A.1)}$$

$$f_{iB}(x) = \sum_{l=1}^{D} \lambda_{iBl} \cdot Beta(\alpha_{iBl}, \beta_{iBl}) \quad \text{(A.2)}$$

Where the number of Beta functions is D. Assume $\lambda_{iAl}, \lambda_{iBl}$ is constant. Allow $\alpha_{iAl}, \beta_{iAl}$ is a function of p_{iA} and $\alpha_{iBl}, \beta_{iBl}$ is a function of p_{iB}. We can represent this in form of following:

$$\alpha_{iAl} = \alpha_{Al}(p_{iA}),\ \beta_{iAl} = \beta'_{Al}(p_{iA}) \quad \text{(A.3)}$$

$$\alpha_{iBl} = \alpha'_{Bl}(p_{iB}),\ \beta_{iBl} = \beta_{Bl}(p_{iB}) \quad \text{(A.4)}$$

Where $\alpha_{Al}(\cdot), \alpha'_{Al}(\cdot), \beta_{Al}(\cdot), \beta'_{Bl}(\cdot)$ represent unknown function from. Because $p_{iA} + p_{iB} = 1$, (A.3) and (A.4) can be rewrite as:

$$\alpha_{iAl} = \alpha_{Al}(p_{iA}),\ \beta_{iAl} = \beta_{Al}(p_{iB}) \quad \text{(A.5)}$$

$$\alpha_{iBl} = \alpha_{Bl}(p_{iA}),\ \beta_{iBl} = \beta_{Bl}(p_{iB}) \quad \text{(A.6)}$$

Where $\alpha_{Al}(\cdot), \alpha_{Bl}(\cdot), \beta_{Al}(\cdot), \beta_{Bl}(\cdot)$ represent unknown function from. Because event A and B are equivalent, function α_{Al}, β_{Al} should be the same as α_{Bl}, β_{Bl} respectively. Thus we can step further represent (A.5) and (A.6) as following:

$$\alpha_{iAl} = \alpha_{l}(p_{iA}),\ \beta_{iAl} = \beta_{l}(p_{iB}) \quad \text{(A.7)}$$

$$\alpha_{iBl} = \alpha_{l}(p_{iA}),\ \beta_{iBl} = \beta_{l}(p_{iB}) \quad \text{(A.8)}$$

Because any function can be approximated by polynomial, we can represent these unknown functions $\alpha_l(\cdot)$, $\beta_l(\cdot)$ as form of polynomials. Furthermore, when the event probability equal to 0 or 1, we can assume all of the individuals believe it is impossible or true. That means $\alpha_l(\cdot)$, $\beta_l(\cdot)$ are polynomials without constant item. Thus, we have:

$$\alpha_{iAl} = \sum_{h=1}^{Q} u_{lh} p_{iA}^{h},\ \beta_{iAl} = \sum_{h=1}^{Q} u_{lh} (1 - p_{iA})^{h} \quad \text{(A.9)}$$

Therefore we get (4).