Quantifying Dismantlement

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Abstract

We propose a novel measure to quantify dismantlement of a fragmented network. The existing measure of dismantlement used to study percolation, influence maximisation, etc. is usually the size of the largest (or second largest[1]) component of the network. We modify the measure of uniformity used to prove the Szemeredi's Regularity Lemma to obtain the proposed measure. The proposed measure incorporates the notion that the measure of dismantlement increases as the number of disconnected components increase and decreases as the variance of sizes of these components increases.

1. Introduction

Sometimes, a fragmented and inactive network is more desirable than a functioning one. Consider, for example, the need to eliminate bacteria by disrupting their molecular network or by vaccinating a few individuals in a population to break up the contact network through which a pathogen spreads[2]. This enables us to ask the question, to what degree has a network been fragmented? It is known that the removal of a few highly connected nodes, or hubs, can break up a complex network into many disconnected components[3]. It is an important problem in network science to find these nodes. It also makes it necessary to quantify dismantlement to verify the effect these nodes have on the network upon removal, which is done by measuring the size of the largest component[4]. It effectively quantifies dismantlement in the context of this problem because the amount of dismantlement of a network increases according to the size of the largest component in most networks when the influential nodes are removed, one by one. It can be easily observed that there exist networks where the size of the largest component cannot effectively determine the amount of dismantlement. For example, a

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network of size $20k$, with disconnected components of sizes $\{10k, 10k\}$ and $\{10k, 3k, 2k, 2k, k, k, k\}$ (where k is any positive integer) appear to be equally dismantled.

Almost 30 years ago, in the course of the proof of a major result on the Ramsey properties of arithmetic progressions, Szemeredi developed a graph theoretical tool, the regularity lemma, whose fundamental importance has been realized more and more in recent years. The lemma says that all graphs can be approximated by random graphs in the following sense: every graph can be partitioned, into a bounded number of equal parts, so that most of its edges run between different parts and the edges between any two parts are distributed fairly uniformly. Just as we would expect it if they had been generated at random[5].

2. Proposed Measure

In the proof for Szemeredi's Regularity Lemma [5], given a graph $G = (V, E)$ and $n = |V|$ with a partition $P = \{V_0, V_1, .., V_k\}$, the uniformity of the partition is given by

$$q(P) = \sum_{i<j} \frac{|V_i||V_j|d^2(V_i, V_j)}{n^2}$$

where, d is the edge density between any two sub-graphs, given by

$$d(A, B) = \frac{||A, B||}{|A||B|}$$

where $||A, B||$ is the number of edges between the two sub-graphs. In the context of a disconnected graph, we consider the partition to be that of the disconnected components and we have $d(A, B) = 0$ for all the pairs of disconnected components A and B by definition. Since we are interested in disconnected networks we set the value of $d(A, B) = 1$ for all A and B. We also change the denominator to correctly normalise the measure for a disconnected graph.

$$max(\sum_{i<j} |V_i||V_j|) = \frac{n(n-1)}{2}$$

Therefore for a graph $G = (V, E)$ with disconnected components given by $P = \{V_0, V_1, .., V_k\}$, we obtain the following,

$$q(P) = \frac{2}{n(n-1)} \sum_{i<j} |V_i||V_j|$$

For a connected graph, the measure of dismantlement is zero and for a completely dismantled graph with no edges we obtain 1, the maximum possible value of dismantlement.

For a given partition $P = \{V_0, V_1, .., V_k\}$ the amount of dismantlement must necessarily increase when the number of components is increased by further breaking down the connected components. Suppose we obtain a partition $P^{'} = \{V_0^a, V_0^b, V_1, .., V_k\}$ by breaking down V_0, then $q(P)$ is strictly less than $q(P^{'})$. In fact,

$$q(P^{'}) - q(P) = \frac{2}{n(n-1)} |V_0^a||V_0^b| > 0$$

Furthermore, for a graph of size n and k disconnected components, the amount of dismantlement must increase as the variance of the sizes of components must decreases, as components of uniform size are desirable. The amount of dismantlement in a disconnected network must increase as the number of disconnected components increases. The proposed measure incorporates this notion because for any two disconnected components the product of the sizes of these components is maximised when the sizes are equal to each other. The product decreases as the the difference between the sizes increases.

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