Math Document Template

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Download all python codes from

```
svn co https://github.com/SiddharthPh/
    Summer2020/trunk/geometry/LinearAlgebra/
    codes
```

and latex-tikz codes from

```
svn co https://github.com/SiddharthPh/
    Summer2020/trunk/geometry/LinearAlgebra/
    figs
```

1 Triangle Exercise.

1.0.1 Problem:

1. Draw the graphs of the equations

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{1.0.1.1}$$

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} - 12 = 0 \tag{1.0.1.2}$$

Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.

1.1 Solution

1. Let

$$\mathbf{x} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{1.1.1.1}$$

Substituting in (1.0.1.1),

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = -1 \tag{1.1.1.2}$$

$$\implies a = -1 \tag{1.1.1.3}$$

Simiarly, substituting

$$\mathbf{x} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \tag{1.1.1.4}$$

in (1.0.1.1),

$$b = 1 \tag{1.1.1.5}$$

The intercepts on the x and y-axis from above are

$$\mathbf{A} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.1.1.6}$$

A is the x-intercept of the line and is the point where it meets x-axis.
Using the above method, the intercepts on x and y-axis for the equation .(1.0.1.2) are

$$\mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{1.1.1.7}$$

C is the x-intercept of the line and is the point where it meets x-axis.
The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 12 \end{pmatrix} \tag{1.1.1.8}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & -1 & -1 \\ 3 & 2 & 12 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2 - 3R_1}{5}} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \end{pmatrix} \tag{1.1.1.9}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \tag{1.1.1.10}$$

$$\implies \mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

2. And the vertices of triangle (1.1.2) formed due to the intersection of lines and x-axis are:

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{1.1.2.1}$$

$$\mathbf{B} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.1.2.2}$$

$$\mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.2.3}$$

Where **B** and **C** are X-intercepts of line (1.0.1.1) and (1.0.1.2) respectively (from (1.1.1) and (1.1.1)). The equivalent python

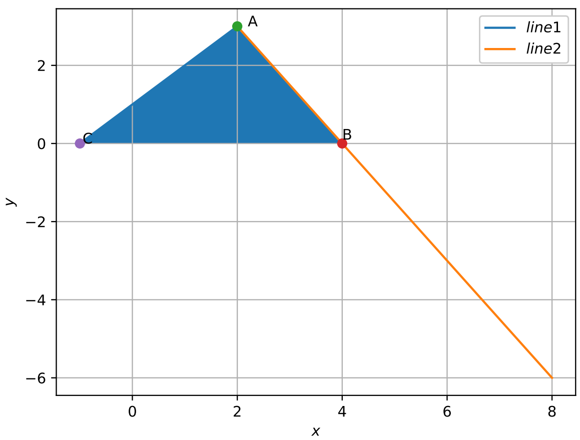

Fig. 1.1.2: Shaded Triangle

code for figure (1.1.2) is

```
codes/triangle/shaded.py
```

2 Quadrilateral Exercise

2.1 Problem

The angles of quadrilateral are in the ratio 3 : 5 : 9 : 13. Find all the angles of the quadrilateral.

2.2 Solution

1. Let the measure of angles $\angle A, \angle B, \angle C, \angle D$ of a quadrilateral are 3x, 5x, 9x and 13x respectively, where x is a real number.
   Using angle sum property, the sum of interior angles of a quadrilateral is 360 degree.

$$3x + 5x + 9x + 13x = 360^\circ \tag{2.2.1.1}$$
$$30x = 360^\circ \tag{2.2.1.2}$$
$$x = 12^\circ \tag{2.2.1.3}$$

From the above calculations,

$$\angle A = 3x = 3(12) = 36^\circ \tag{2.2.1.4}$$
$$\angle B = 5x = 5(12) = 60^\circ \tag{2.2.1.5}$$
$$\angle C = 9x = 9(12) = 108^\circ \tag{2.2.1.6}$$
$$\angle D = 13x = 13(12) = 156^\circ \tag{2.2.1.7}$$

3 Line Exercises

3.1 Point and Vector Exercise

3.1.1 Problem:

1. Find the distance between the following pairs of points
   a)
   $$\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{3.1.1.1.1}$$
   b)
   $$\begin{pmatrix} -5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{3.1.1.1.2}$$
   c)
   $$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} -1 \\ b \end{pmatrix} \tag{3.1.1.1.3}$$

3.1.2 Solution:

1. The distance between two vectors is given by
$$\|\mathbf{A} - \mathbf{B}\| \tag{3.1.2.1.1}$$
   a) The distance between $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is
   $$\left\| \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\| = 2.828 \text{ (From (3.1.2.1.1))}$$
   b) The distance between $\mathbf{A} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ is
   $$\left\| \begin{pmatrix} -5 \\ 7 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\| = 5.656 \text{ (From (3.1.2.1.1))}$$
   c) The distance between $\mathbf{A} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 \\ b \end{pmatrix}$ is
   $$\left\| \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} -1 \\ b \end{pmatrix} \right\| = a + 1 \text{ (From (3.1.2.1.1))}$$

3.2 Point on a line

3.2.1 Problem:
Find the coordinates of the point which divides the join of

$$\begin{pmatrix} -1 \\ 7 \end{pmatrix}, = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \tag{1.1}$$

in the ratio 2 : 3.

3.2.2 Solution:

1. $\mathbf{A} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$
   Then $\mathbf{C}$ that divides $\mathbf{A}, \mathbf{B}$ in the ratio $k : 1$ is
$$\mathbf{C} = \frac{k\mathbf{A} + \mathbf{B}}{k+1} \tag{3.2.2.1.1}$$

For the given problem k=2 : 3
Using the equation 3.2.2.1.1, the desired point is

$$\mathbf{C} = \frac{\frac{2}{3}\begin{pmatrix}-1\\7\end{pmatrix}+\begin{pmatrix}4\\-3\end{pmatrix}}{\frac{2}{3}+1} \tag{3.2.2.1.2}$$

$$\therefore \mathbf{C} = \begin{pmatrix}2\\1\end{pmatrix} \tag{3.2.2.1.3}$$

The following code plots the figure **??**

```
codes/line/section.py
```

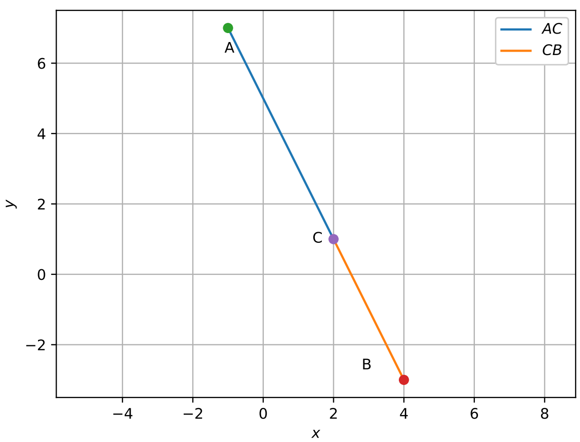

Fig. 3.2.2.1

3.3 *Lines and Planes*

3.3.1 *Problem:*

1. Verify whether the following are zeroes of the polynomial, indicated against them.
   a) $p(x) = 3x+1, x = \frac{1}{3}$
   b) $p(x) = 5x-\pi, x = \frac{4}{5}$
   c) $p(x) = 5lx+m, x = -\frac{m}{l}$
   d) $p(x) = 2x+1, x = \frac{1}{2}$

3.3.2 *Solution:*

1. Let

$$y = 3x+1 \implies \begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} \quad = -1 \tag{3.3.2.1.1}$$

Thus,

$$y = 0 \tag{3.3.2.1.2}$$

$$\implies 3x+1 = 0 \tag{3.3.2.1.3}$$

$$\text{or, } x = -\frac{1}{3} \tag{3.3.2.1.4}$$

Hence $\mathbf{x} = \frac{1}{3}$ is not a zero. This is verified in Fig. 3.3.2.1.

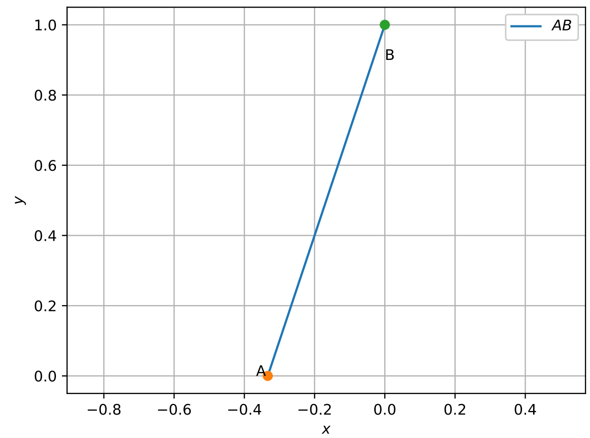

Fig. 3.3.2.1

2. Let

$$y = 5x-\pi \implies \begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} \quad = \pi \tag{3.3.2.2.1}$$

Thus,

$$y = 0 \tag{3.3.2.2.2}$$

$$\implies 5x-\pi = 0 \tag{3.3.2.2.3}$$

$$\text{or, } x = \frac{\pi}{5} \tag{3.3.2.2.4}$$

Hence $\mathbf{x} = \frac{4}{5}$ is not a zero. This is verified in Fig. 3.3.2.2.

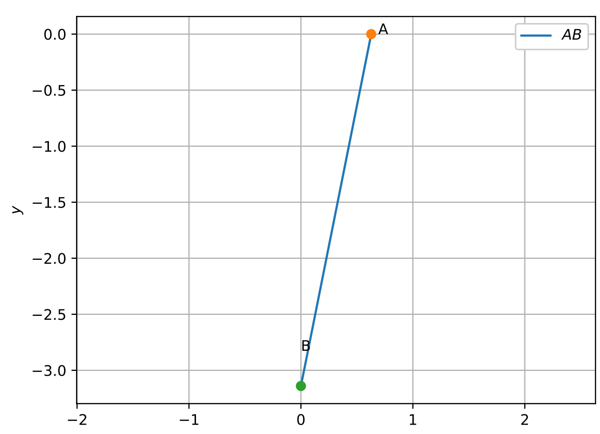

Fig. 3.3.2.2

3. Let

$$y = 5lx + m \implies \begin{pmatrix}5l & -1\end{pmatrix}\mathbf{x} = -m \tag{3.3.2.3.1}$$

Thus,

$$y = 0 \tag{3.3.2.3.2}$$
$$\implies 5lx + m = 0 \tag{3.3.2.3.3}$$
$$\text{or, } x = -\frac{m}{5l} \tag{3.3.2.3.4}$$

Hence $\mathbf{x} = -\frac{m}{l}$ is not a zero. This is verified in Fig. 3.3.2.3.

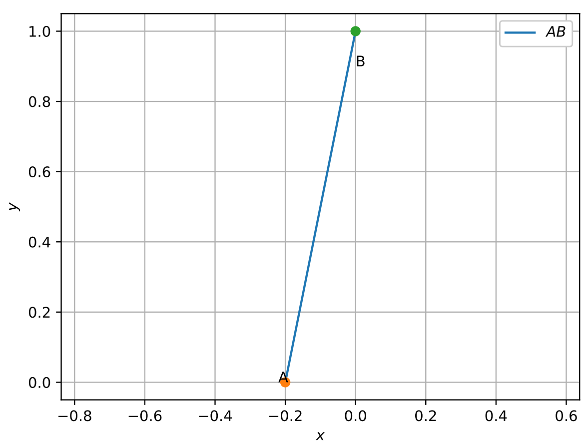

Fig. 3.3.2.3

4. Let

$$y = 2x + 1 \implies \begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -1 \tag{3.3.2.4.1}$$

Thus,

$$y = 0 \tag{3.3.2.4.2}$$
$$\implies 2x + 1 = 0 \tag{3.3.2.4.3}$$
$$\text{or, } x = -\frac{1}{2} \tag{3.3.2.4.4}$$

Hence $\mathbf{x} = \frac{1}{2}$ is not a zero. This is verified in Fig. 3.3.2.4.

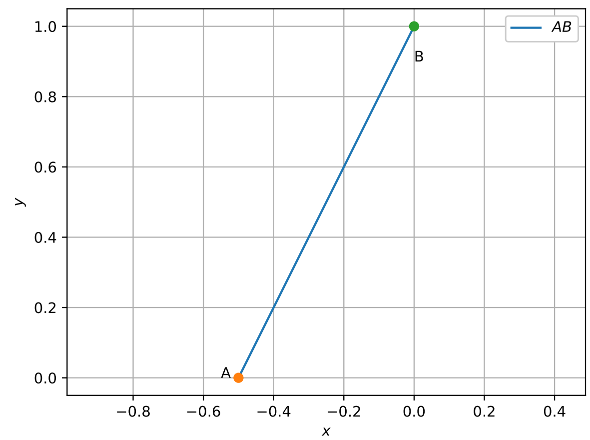

Fig. 3.3.2.4

3.4 Motion in a Plane

3.4.1 Problem:

1. Rain is falling vertically with a speed of 35 ms^{-1} after sometime with a speed of 12 ms^{-1} . Winds starts blowing in east to west direction. In which direction should a boy waiting at a bus stop hold his umbrella ?

3.4.2 Solution:

1. Let the boy be at point $\mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}$

   Initially, it was initially raining downward at speed of 35 ms^{-1} and when there is wind, it is raining downward at speed of 12 ms^{-1}.
   Let the boy be at origin.
   Let $\mathbf{u}$ be the initial rain vector = $\begin{pmatrix}0\\35\end{pmatrix}$

   Let $\mathbf{v}$ be the final rain vector = $\begin{pmatrix}l\\12\end{pmatrix}$(Where l is real number, and speed of rain in vertical direction changed to $12ms^{-1}$)
   $\|\mathbf{u}\| = \|\mathbf{v}\| = 35ms^{-1}$(As the speed of rain remains constant) The angle θ with the vertical, at which it is raining is calculated by:

$$\mathbf{u}^T\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \tag{3.4.2.1.1}$$
$$\frac{12}{35} = \cos\theta \tag{3.4.2.1.2}$$
$$\theta = 69.96 \tag{3.4.2.1.3}$$

   $\therefore$ Boy has to hold his umbrella at angle of 20.04° with the ground towards east.
   The code for the diagramatic(3.4.2.1) representation of the solution is

```
codes/line/speed.py
```

3.5 Matrix Exercise

3.5.1 Problem:

1. In the matrix A=$\begin{pmatrix}2 & 5 & 19 & -7\\ 35 & -2 & \frac{5}{2} & 12\\ \sqrt{3} & 1 & -5 & 17\end{pmatrix}$,write

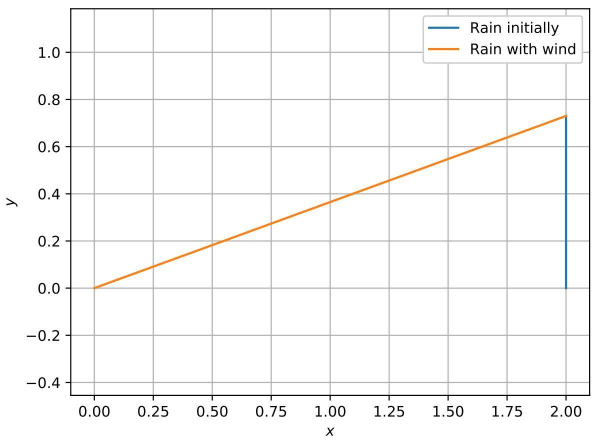

Fig. 3.4.2.1

a) The order of the matrix
b) The number of elements
c) Write the elements $a_{31}, a_{21}, a_{33}, a_{24}, a_{23}$.

3.5.2 Solution:

1. a) The order of matrix for above problem is 3x4.
   b) The number of elements=12
   c) The elements are

$$a_{31} = \sqrt{3} \tag{3.5.2.1.1}$$
$$a_{21} = 35 \tag{3.5.2.1.2}$$
$$a_{33} = -5 \tag{3.5.2.1.3}$$
$$a_{24} = 12 \tag{3.5.2.1.4}$$
$$a_{23} = \frac{5}{2} \tag{3.5.2.1.5}$$

The python implementation for the above exaample is given in

codes/line/matrix.py

3.6 Determinents

3.6.1 Problem: $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

3.6.2 Solution:

1. The determinent of a matrix 2x2 matrix is given by:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \tag{3.6.2.1.1}$$
$$Det = a_{11}a_{22} - a_{12}a_{21} \tag{3.6.2.1.2}$$

$\therefore$ Det = 18.

3.7 Linear inequation

3.7.1 Problem: Solve x ≥ 3, y ≥ 2 graphically.

3.7.2 Solution:

1. Solve the following system of linear inequalities graphically.

$$\begin{aligned} x &\geq 3 \\ y &\geq 2 \end{aligned} \tag{3.7.2.1.1}$$

Let $u_1 \geq 0, u_2 \geq 0$. This may be expressed as

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \geq \mathbf{0} \tag{3.7.2.1.2}$$

(3.7.2.1.1) can then be expressed as

$$\begin{aligned} x &\geq 5 \\ y &\geq 2 \end{aligned} \tag{3.7.2.1.3}$$
$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} \geq \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{3.7.2.1.4}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \mathbf{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{3.7.2.1.5}$$
$$\text{or, } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \mathbf{u} \tag{3.7.2.1.6}$$

resulting in

$$\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathbf{u} \tag{3.7.2.1.7}$$
$$\text{or, } \mathbf{x} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{u} \tag{3.7.2.1.8}$$

after obtaining the inverse. Fig. 3.7.2.1 generated using the following python code shows the region satisfying (3.7.2.1.1)

codes/line/line_eq.py

4 Circle Example

4.1 Problem

Find the equation of a circle with centre $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and radius 4.

4.2 Exercise

1. The input values for the question are given in the table (5.2.1) The **O** is the centre of

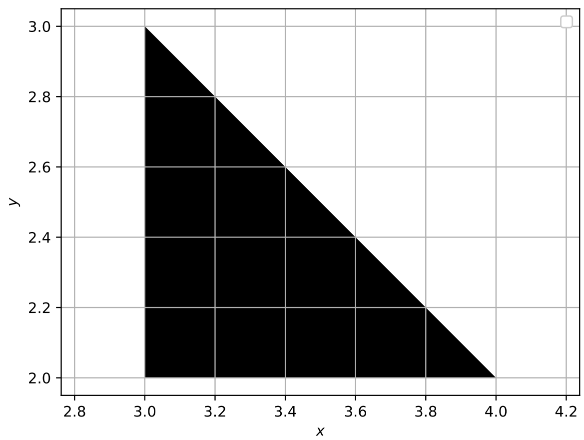

Fig. 3.7.2.1

Input Values	
O(centre)	$\begin{pmatrix}-3\\2\end{pmatrix}$
r(radius)	4

TABLE 4.2.1: Input Values

circle=$\begin{pmatrix}-3\\2\end{pmatrix}$, let r be the radius of the circle=4. Let $\mathbf{x}$ satisfy the circle equation

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x}+\begin{pmatrix}6 & -4\end{pmatrix}\mathbf{x} \tag{4.2.1.1}$$

The python code for the figure (5.2.1) is

```
codes/circle/circle1.py
```

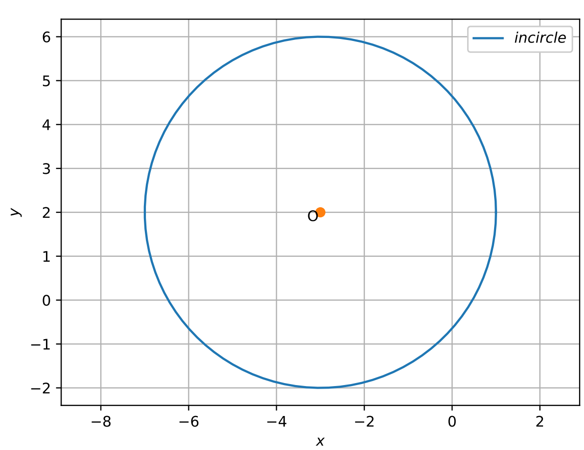

Fig. 4.2.1: Circle using python

5 Circle Exercise

5.1 Problem

Find the coordinates of point **A**, where AB is the diameter of circle whose centre is $(2, -3)$ and $\mathbf{B} = \begin{pmatrix}1\\4\end{pmatrix}$

5.2 Solution

1. The input values for the question are given in the table (5.2.1) The **A** is at the end of diameter, so the centre(**O**) is the midpoint of **AB**.

Input values	
Parameters	Values
O	$\begin{pmatrix}2\\-3\end{pmatrix}$
A	$\begin{pmatrix}1\\4\end{pmatrix}$

TABLE 5.2.1: Input Values

$$\mathbf{O} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{5.2.1.1}$$

$$\mathbf{A} = 2\mathbf{O} - \mathbf{B} \tag{5.2.1.2}$$

$$\therefore \mathbf{A} = \begin{pmatrix}3\\-10\end{pmatrix} \tag{5.2.1.3}$$

The python code for the figure (5.2.1) is

```
codes/circle/circle.py
```

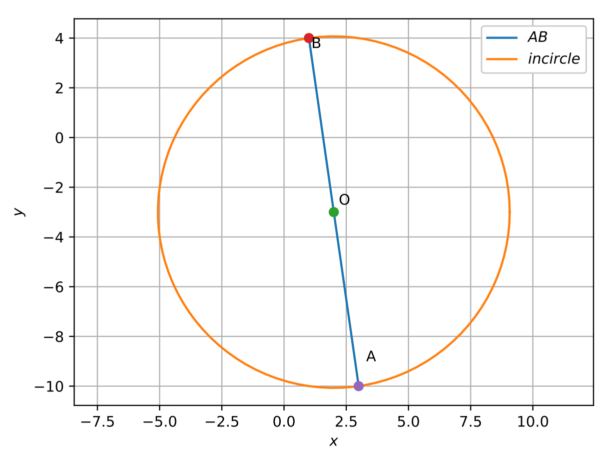

Fig. 5.2.1: Circle using python

6 Conics Exercise

6.1 Problem

1. Verify whether the following are zeroes of the polynomial, indicated against them.
   a) $p(x) = x^2 - 1, x = 1, -1$
   b) $p(x) = (x+1)(x-2), x = -1, 2$
   c) $p(x) = x^2, x = 0.$
   d) $p(x) = 3x^2 - 1, x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}.$

6.2 Solution

1. **Proof** For a general polynomial equation of degree 2,
   $p(x,y) \implies Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
   The vector form is
   $$\mathbf{x}^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \tag{6.2.0.1.1}$$
2. For eq: $y = x^2 - 1$
   Vector form is given by
   $$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} - 1 = 0 \tag{6.2.0.2.1}$$
   (From the equation 6.2.0.1.1.)
   Thus,
   $$y = 0 \tag{6.2.0.2.2}$$
   $$\implies x^2 - 1 = 0 \tag{6.2.0.2.3}$$
   $$x = +1, -1 \tag{6.2.0.2.4}$$
   Hence +1,-1 are zeros, which can be verified from the figure 6.2.0.2 The python code for the figure 6.2.0.2 is
   ```
   codes/conics/parab1.py
   ```

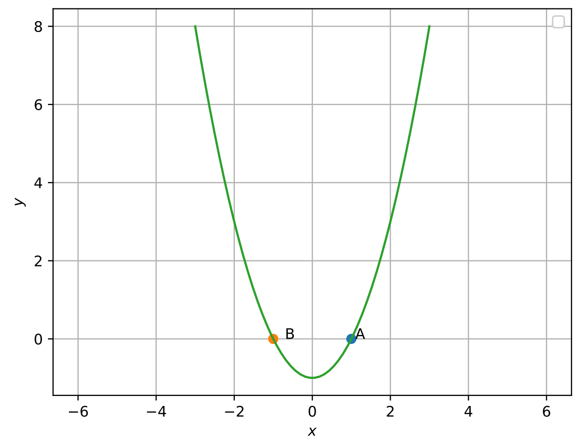

Fig. 6.2.0.2: Parabola 1

3. For eq: $y = (x+1)(x-2)$
   Equation can be represented as
   $y = x^2 - x - 2$
   Vector form is given by
   $$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{6.2.0.3.1}$$
   (From the equation 6.2.0.1.1.)
   Thus,
   $$y = 0 \tag{6.2.0.3.2}$$
   $$\implies (x+1)(x-2) = 0 \tag{6.2.0.3.3}$$
   $$x = -1, +2 \tag{6.2.0.3.4}$$
   Hence -1,+2 are zeros, which can be verified from the figure 6.2.0.3 The python code for the figure 6.2.0.3 is
   ```
   codes/conics/parab2.py
   ```

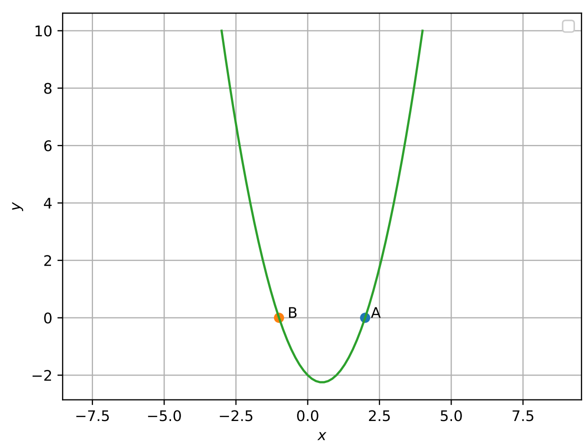

Fig. 6.2.0.3: Parabola 2

4. For eq: $y = x^2$
   Vector form is given by
   $$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{6.2.0.4.1}$$
   (From the equation 6.2.0.1.1.)
   Thus,
   $$y = 0 \tag{6.2.0.4.2}$$
   $$\implies x^2 = 0 \tag{6.2.0.4.3}$$
   $$x = 0 \tag{6.2.0.4.4}$$

Hence 0 is the zero, which can be verified from the figure 6.2.0.4 The python code for the figure 6.2.0.4 is

```
codes/conics/parab3.py
```

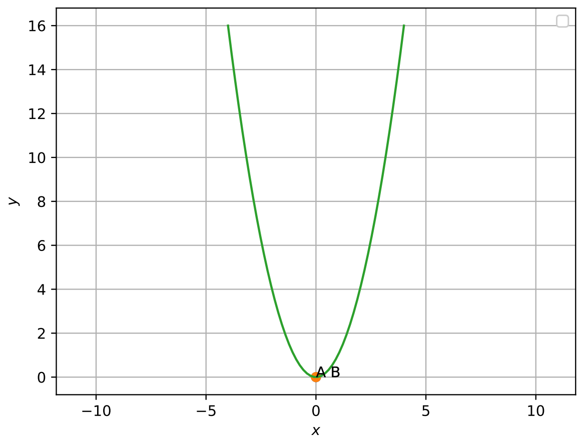

Fig. 6.2.0.4: Parabola 3

5. For eq: $y = 3x^2 - 1$
Vector form is given by

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} - 1 = 0 \tag{6.2.0.5.1}$$

(From the equation 6.2.0.1.1.)
Thus,

$$y = 0 \tag{6.2.0.5.2}$$

$$\implies 3x^2 - 1 = 0 \tag{6.2.0.5.3}$$

$$x = +\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \tag{6.2.0.5.4}$$

Hence $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$ are the zeros, which can be verified from the figure 6.2.0.5 The python code for the figure 6.2.0.5 is

```
codes/conics/parab4.py
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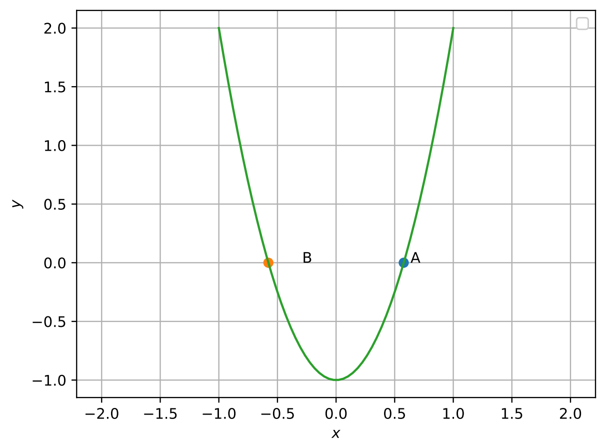

Fig. 6.2.0.5: Parabola 4