Problem 8.5.19

Pothukuchi Siddhartha

***Abstract*—This a document explaining a question on the concept of cyclic quadrilateral.**

Download all python codes from

```
svn co https://github.com/SiddharthPh/
    Summer2020/trunk/geometry/Circle/codes
```

and latex-tikz codes from

```
svn co https://github.com/SiddharthPh/
    Summer2020/trunk/geometry/Circle/figs
```

1 Problem

Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadri- lateral is cyclic.

2 Construction

2.1 Constructing QuadrilateralABCD

1. Constructing quadrilateral $ABCD$:
Solution: The design parameters for constructing the quadrilateral ABCD are given in the Table. 2.1.1.

Input Values	
Parameters	Values
a	9
b	6.324
c	4.472
d	5
e	9.219

TABLE 2.1.1: Parameters for Quadrilateral ABCD

$$BC = a = 4.5, CD = b = 5.5, ADc = 4, \tag{2.1.1.1}$$

$$AB = d = 6, BD = e = 7 \tag{2.1.1.2}$$

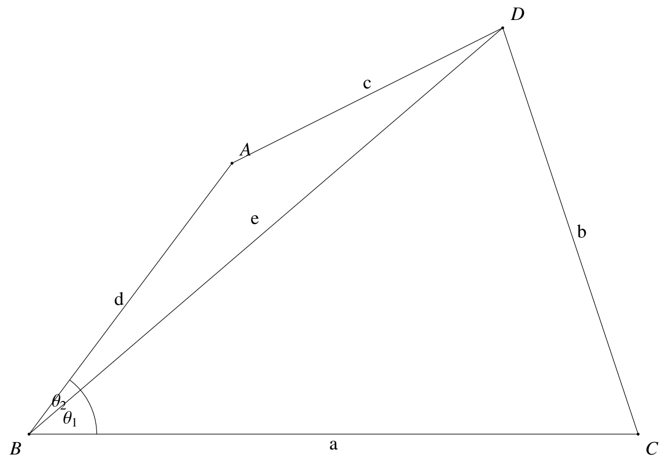

Fig. 2.1.1: QuadrilateralABCD by Latex-Tikz

Solution: The angles θ_1 and θ_2 in Fig. **??** are calculated using the cosine formula as

$$\cos\theta_1 = \frac{a^2 + e^2 - b^2}{2ae} \tag{2.1.1.3}$$

$$\cos\theta_2 = \frac{d^2 + e^2 - c^2}{2de} \tag{2.1.1.4}$$

The coordinates are then obtained as

$$\mathbf{A} = d\begin{pmatrix}\cos(\theta_1+\theta_2)\\ \sin(\theta_1+\theta_2)\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}a\\0\end{pmatrix}, \mathbf{D} = e\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix} \tag{2.1.1.5}$$

2. The values of A,B,C,D are shown in the Table .2.2.5
3. Draw Fig. 2.2.6.
Solution: The following Python code generates Fig. 2.2.6

```
codes/quad1.py
```

and the equivalent latex-tikz code is

```
figs/quad1.tex
```

Input Values	
Parameters	Values
A	$\begin{pmatrix}3\\4\end{pmatrix}$
B	$\begin{pmatrix}0\\0\end{pmatrix}$
C	$\begin{pmatrix}9\\0\end{pmatrix}$
D	$\begin{pmatrix}7\\6\end{pmatrix}$

TABLE 2.1.2: Vertices A,B,C,D

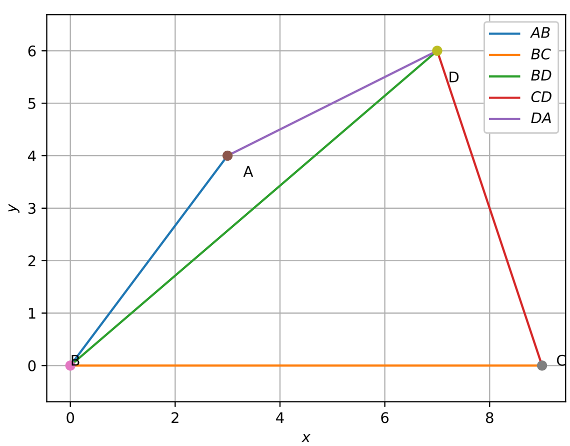

Fig. 2.1.3: QuadrilateralABCD generated using python

2.2 Cyclic QuadrilateralEFGH using angular bisectors.

1. The Figure of the quadriletral as obtained in the question looks like Fig. 2.2.1. with angles $\angle A, \angle C$ and $\angle B$ and$\angle D$ and sides a, b and c and d.
2. The design parameters for construction are: **Solution:** See Table. 2.2.2.
3. **Proof**: Finding angular bisector using unit vectors.
   Solution: : Let the angle between AB and BC be θ and between **R** and **BC** be α.

$$\mathbf{R} = \frac{\mathbf{A}-\mathbf{B}}{\|A-B\|} + \frac{\mathbf{C}-\mathbf{B}}{\|C-B\|} \tag{2.2.3.1}$$

$$\mathbf{R}\cdot\mathbf{BC} = \|R\|\,\|BC\|\cos\theta \tag{2.2.3.2}$$

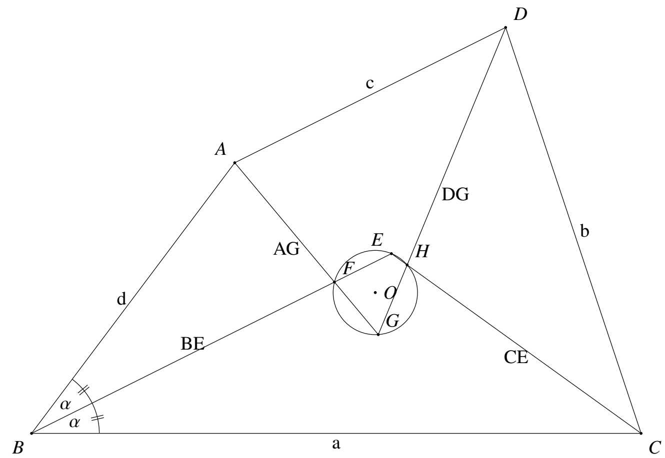

Fig. 2.2.1: Quadrilateraal by Latex-Tikz

Input Values	
Parameters	Values
A	$\begin{pmatrix}3\\4\end{pmatrix}$
B	$\begin{pmatrix}0\\0\end{pmatrix}$
C	$\begin{pmatrix}9\\0\end{pmatrix}$
D	$\begin{pmatrix}7\\6\end{pmatrix}$

TABLE 2.2.2: Quadrilateral ABCD

The resulting equation after simplifying is,

$$\cos\theta + 1 = \sqrt{2+2\cos\theta}\cos\alpha \tag{2.2.3.3}$$

By squaring on both sides

$$(\cos\theta+1)^2 = 2+2\cos\theta(\cos\alpha)^2 \tag{2.2.3.4}$$

$$\cos\theta = 2\cos^2\alpha - 1 \tag{2.2.3.5}$$

The above equation is the formula of $\cos 2\theta$ $\therefore \alpha = \frac{\theta}{2}$

4. Find the angular bisectors of each angle in Fig. 2.2.1
   Solution: From the given information, the line equation of acute angular bisector of $\angle B$ in vector form is

$$\mathbf{L1} = \mathbf{B} + s(\mathbf{R1}) \tag{2.2.4.1}$$

$$\mathbf{L1} = \begin{pmatrix}0\\0\end{pmatrix} + s\begin{pmatrix}1.6\\0.8\end{pmatrix} \tag{2.2.4.2}$$

Where **R1**(from .(2.2.3)) is the direction ration of the line **L1** obtained by the formula

$$\mathbf{R1} = \frac{\mathbf{A-B}}{||A-B||} + \frac{\mathbf{C-B}}{||C-B||}$$

Vector form of angular bisector of $\underline{/C}$ is

$$\mathbf{L2} = \mathbf{C} + t(\mathbf{R2}) \tag{2.2.4.3}$$

$$\mathbf{L2} = \begin{pmatrix} 9 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1.316 \\ 0.948 \end{pmatrix} \tag{2.2.4.4}$$

Where **R2** is the d.r of the line **L2** obtained by the formula

$$\mathbf{R2} = \frac{\mathbf{A-B}}{||A-B||} + \frac{\mathbf{C-B}}{||C-B||}$$

Vector form of angular bisector of $\underline{/A}$ is

$$\mathbf{L3} = \mathbf{A} + u(\mathbf{R3}) \tag{2.2.4.5}$$

$$\mathbf{L3} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + u\begin{pmatrix} 0.294 \\ -0.352 \end{pmatrix} \tag{2.2.4.6}$$

Where **R3** is the d.r of the line **L3** obtained by the formula

$$\mathbf{R3} = \frac{\mathbf{B-C}}{||B-C||} + \frac{\mathbf{D-C}}{||D-C||}$$

Vector form of angular bisector of $\underline{/D}$ is

$$\mathbf{L4} = \mathbf{D} + v(\mathbf{R4}) \tag{2.2.4.7}$$

$$\mathbf{L4} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} + v\begin{pmatrix} -0.578 \\ -1.395 \end{pmatrix} \tag{2.2.4.8}$$

Where **R4** is the d.r of the line **L4** obtained by the formula

$$\mathbf{R4} = \frac{\mathbf{A-D}}{||A-D||} + \frac{\mathbf{C-D}}{||C-D||}$$

Here s,t,u,v are constants used to define a line in vector form, where a unique position vector is obtained for unique values of (s,t,u,v) of the respective line.

5. To find the point of intersection of the angular bisectors, equate the respective line equations. **Solution:** **E** is obtained by equating line equations **L1** and **L2**

$$\begin{pmatrix} 1.6s \\ 0.8s \end{pmatrix} = \begin{pmatrix} 9-1.316t \\ 0.948t \end{pmatrix} \tag{2.2.5.1}$$

$$\begin{pmatrix} 1.6s+1.316t-9 \\ 0.8s-0.948t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.5.2}$$

By solving the two equations we obtain the values of s,t.
By substituting the values in **L1** we obtain **E**

$$\mathbf{E} = \begin{pmatrix} 5.3137 \\ 2.6568 \end{pmatrix}$$

F is obtained by equating line equations **L1** and **L3**

$$\begin{pmatrix} 1.6s \\ 0.8s \end{pmatrix} = \begin{pmatrix} 3+0.294u \\ 4-0.352u \end{pmatrix} \tag{2.2.5.3}$$

$$\begin{pmatrix} 1.6s-0.294u-3 \\ 0.8s+0.352u-4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.5.4}$$

By solving the two equations we obtain the values of s,u.
By substituting the values in **L1** we obtain **F**

$$\mathbf{F} = \begin{pmatrix} 4.472 \\ 2.236 \end{pmatrix}$$

G is obtained by equating line equations **L3** and **L4**

$$\begin{pmatrix} 3+0.294u \\ 4-0.352u \end{pmatrix} = \begin{pmatrix} 7-0.578v \\ 6-1.395v \end{pmatrix} \tag{2.2.5.5}$$

$$\begin{pmatrix} 0.294u+0.578v-4 \\ -0.352u+1.395v-2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.5.6}$$

By solving the two equations we obtain the values of u,v.
By substituting the values in **L3** we obtain **G**

$$\mathbf{G} = \begin{pmatrix} 5.119 \\ 1.460 \end{pmatrix}$$

H is obtained by equating line equations **L2** and **L4**

$$\begin{pmatrix} 9-1.316t \\ 0.948t \end{pmatrix} = \begin{pmatrix} 7-0.578v \\ 6-1.395v \end{pmatrix} \tag{2.2.5.7}$$

$$\begin{pmatrix} -1.316t+0.578v+2 \\ 0.948t+1.395v-6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.5.8}$$

By solving the two equations we obtain the

values of t,v.
By substituting the values in **L2** we obtain **H**.

$$\mathbf{H} = \begin{pmatrix} 5.545 \\ 2.489 \end{pmatrix}$$

The values are listed in Table. 2.2.5

Derived values	
Parameter	Value
$\mathbf{E}$	$\begin{pmatrix} 5.313 \\ 2.656 \end{pmatrix}$
$\mathbf{F}$	$\begin{pmatrix} 4.472 \\ 2.236 \end{pmatrix}$
$\mathbf{G}$	$\begin{pmatrix} 5.119 \\ 1.460 \end{pmatrix}$
$\mathbf{H}$	$\begin{pmatrix} 5.545 \\ 2.489 \end{pmatrix}$

TABLE 2.2.5: Cyclic Quadrilateral EFGH

6. Draw Fig. 2.2.1.
Solution: The following Python code generates Fig. 2.2.6

```
codes/quad.py
```

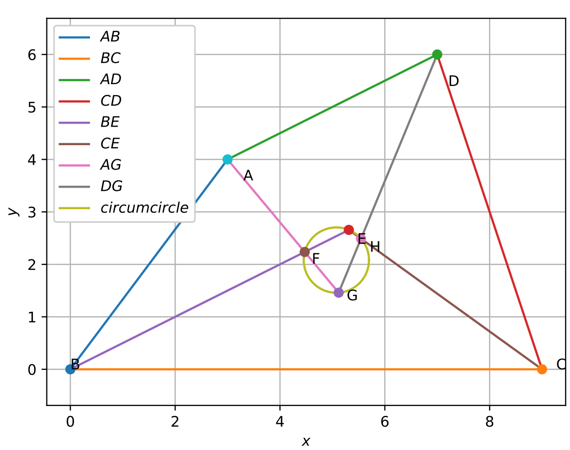

Fig. 2.2.6: Quadrilateral generated using python

and the equivalent latex-tikz code generating Fig. 2.2.1 is

```
figs/quad.tex
```

The above latex code can be compiled as a standalone document as

```
figs/quad_fig.tex
```

3 Solution

3.1. Show that $\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}$ lies on a circle.
Solution: Let $\mathbf{V}$ be a general vector that satisfies the circle equation.
Then, $||V - O|| = r$ will be the equation, where $\mathbf{O}, R$ are Centre of circle, and radius respectively.
Find a point $\mathbf{O}$ that is equidistant from the vertices of $\triangle EFG$ for $e = 1.010, f = 1.212, g = 0.940$.

$$||E - O|| = ||F - C|| = ||G - C|| = R \tag{3.1.1}$$

From (3.1.1),

$$||\mathbf{E} - \mathbf{O}||^2 - ||\mathbf{F} - \mathbf{O}||^2 = 0 \tag{3.1.2}$$

$$\implies (\mathbf{E} - \mathbf{O})^T (\mathbf{E} - \mathbf{O}) - (\mathbf{F} - \mathbf{O})^T (\mathbf{F} - \mathbf{O}) = 0 \tag{3.1.3}$$

which can be simplified as

$$(\mathbf{E} - \mathbf{F})^T \mathbf{O} = \frac{||\mathbf{E}||^2 - ||\mathbf{F}||^2}{2} \tag{3.1.4}$$

Similarly,

$$(\mathbf{F} - \mathbf{G})^T \mathbf{O} = \frac{||\mathbf{F}||^2 - ||\mathbf{G}||^2}{2} \tag{3.1.5}$$

(3.1.4) and (3.1.5), can be combined to form the matrix equation

$$\mathbf{N}^T \mathbf{O} = \mathbf{g} \tag{3.1.6}$$

$$\implies \mathbf{O} = \mathbf{N}^{-T} \mathbf{g} \tag{3.1.7}$$

where

$$\mathbf{N} = \begin{pmatrix} \mathbf{E} - \mathbf{F} & \mathbf{F} - \mathbf{G} \end{pmatrix} \tag{3.1.8}$$

$$\mathbf{g} = \frac{1}{2} \begin{pmatrix} ||\mathbf{E}||^2 - ||\mathbf{F}||^2 \\ ||\mathbf{F}||^2 - ||\mathbf{G}||^2 \end{pmatrix} \tag{3.1.9}$$

$\mathbf{O}$ can be computed using the python code below

```
codes/quad.py
```

and the equivalent latex-tikz code to draw Fig. 3.1 is

```
figs/quad.tex
```

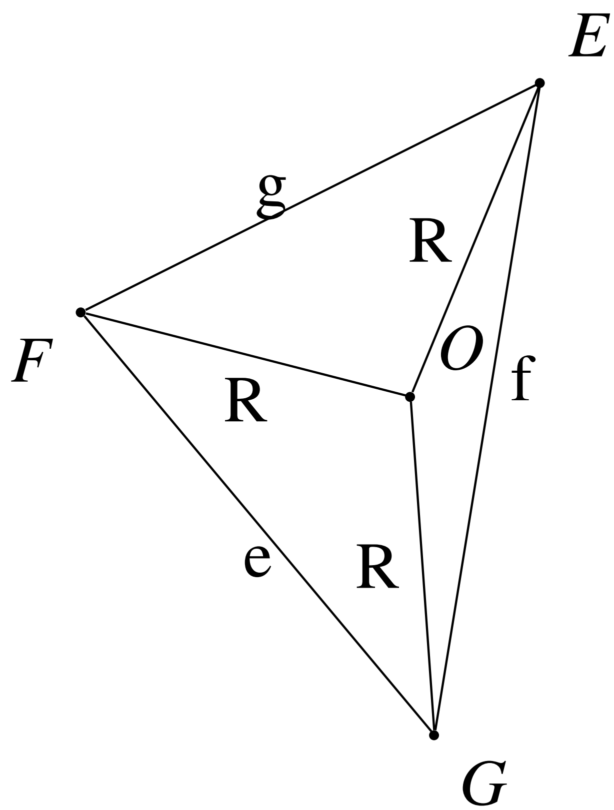

Fig. 3.1: Circumcentre O of $\triangle EFG$

3.2. $\therefore \mathbf{O} = \begin{pmatrix} 5.075 \\ 2.081 \end{pmatrix}$

3.3. In $\triangle OFG$, $OF = OG = R$. Such a triangle is known as an *isoceles triangle*.

3.4. Show that $\angle OFG = \angle OGF$. In an isoceles triangle, opposite sides and corresponding opposite angles are equal.
Solution: Using the sine formula ,

$$\frac{\sin \angle OFG}{R} = \frac{\sin \angle OGF}{R} \tag{3.4.1}$$

$$\implies \sin \angle OFG = \sin \angle OGF \tag{3.4.2}$$

3.5. Show that $\angle FOG = 2\angle E$.
Solution: In Fig. 3.1,

$$E = \theta_2 + \theta_3 \tag{3.5.1}$$

$$F = \theta_1 + \theta_2 \tag{3.5.2}$$

$$G = \theta_3 + \theta_1 \tag{3.5.3}$$

$$\implies 2\left(\theta_1 + \theta_2 + \theta_3\right) = E + G + H = 180^\circ \tag{3.5.4}$$

$$\implies \theta_1 + \theta_2 + \theta_3 = 90^\circ \tag{3.5.5}$$

From (3.5.1) and (3.5.5),

$$E = 90^\circ - \theta_1 \tag{3.5.6}$$

Also, in $\triangle OFG$, all angles add up to 180°. Hence,

$$\angle FOG + 2\theta_1 = 180^\circ \tag{3.5.7}$$

$$\implies \angle FOG = 180^\circ - 2\theta_1 = 2\left(90^\circ - \theta_1\right) = 2\angle E \tag{3.5.8}$$

upon substituting from (3.5.6).

3.6. Let $\mathbf{L}$ be the mid point of FG. Show that $OL \perp FG$.
Solution: From (3.1.4),

$$\left(\mathbf{F} - \mathbf{G}\right)^T \mathbf{O} = \frac{\|\mathbf{F}\|^2 - \|\mathbf{G}\|^2}{2} \tag{3.6.1}$$

$$\implies \left(\mathbf{F} - \mathbf{G}\right)^T \mathbf{O} = \frac{1}{2}\left(\mathbf{F} - \mathbf{G}\right)^T\left(\mathbf{F} + \mathbf{G}\right) \tag{3.6.2}$$

$$\implies \left(\mathbf{F} - \mathbf{G}\right)^T \left(\mathbf{O} - \frac{\mathbf{F} + \mathbf{G}}{2}\right) = 0 \tag{3.6.3}$$

$$\text{or, } \left(\mathbf{F} - \mathbf{G}\right)^T \left(\mathbf{O} - \mathbf{L}\right) = 0 \tag{3.6.4}$$

$\because \mathbf{L} = \frac{\mathbf{F}+\mathbf{G}}{2}$ is the mid point of FG. From (**??**) we then conclude that $OL \perp FG$.

3.7. Perpendicular bisectors of a triangle meet at the circumcentre.

3.8. In the isosceles $\triangle OFG$, if $FL = LG$, $OL \perp FG$.

3.9. Show that

$$\frac{e}{\sin E} = \frac{f}{\sin F} = \frac{g}{\sin G} = 2R. \tag{3.9.1}$$

Solution: In $\triangle OFG$, using the cosine formula,

$$\cos 2A = \frac{R^2 + R^2 - e^2}{2R^2} = 1 - \frac{e^2}{2R^2} \tag{3.9.2}$$

Using the sine formula,

$$\frac{\sin 2E}{e} = \frac{\sin \theta_1}{R} = \frac{\sin\left(90^\circ - E\right)}{R} \tag{3.9.3}$$

$$\implies \sin 2E = \frac{a \cos E}{R} \tag{3.9.4}$$

from (3.5.6) and Baudhanya theorem.

$$\cos^2 2E + \sin^2 2E = 1 \tag{3.9.5}$$

$$\implies \left(1 - \frac{e^2}{2R^2}\right)^2 + \left(\frac{e \cos E}{R}\right)^2 = 1 \tag{3.9.6}$$

upon substituting from (3.9.2) and (3.9.4). Letting

$$x = \left(\frac{e}{R}\right)^2, \tag{3.9.7}$$

in the previous equation yields

$$\left(1 - \frac{x}{2}\right)^2 + x\cos^2 E = 1 \tag{3.9.8}$$

$$\implies 1 - \frac{x^2}{4} - x + x\cos^2 E = 1 \tag{3.9.9}$$

$$\implies x\left(1 - \cos^2 E\right) - \frac{x^2}{4} = 0 \tag{3.9.10}$$

$$x\sin^2 E - \frac{x^2}{4} = 0 \tag{3.9.11}$$

$$\implies x\left(\sin^2 E - \frac{x}{4}\right) = 0 \tag{3.9.12}$$

$$\text{or, } \frac{x}{4} - \sin^2 E = 0 \tag{3.9.13}$$

$\because x \neq 0$. Thus, substituting from (3.9.7),

$$x = \left(\frac{e}{R}\right)^2 = 4\sin^2 E \tag{3.9.14}$$

$$\implies \frac{e}{R} = 2\sin E, \tag{3.9.15}$$

$$\text{or,} \quad \frac{e}{\sin E} = 2R \tag{3.9.16}$$

3.10. Show that

$$\cos 2E = 1 - 2\sin^2 E = 2\cos^2 E - 1 \tag{3.10.1}$$

$$= \cos^2 E - \sin^2 E \quad \text{and} \tag{3.10.2}$$

$$\sin 2E = 2\sin E\cos E \tag{3.10.3}$$

3.11. Find R.
Solution:

$$ar(\triangle EFG) = \frac{1}{2}fg\sin E = \frac{efg}{4R} \tag{3.11.1}$$

$$\implies R = \frac{efg}{4s\sqrt{(s-e)(s-f)(s-g)}} \tag{3.11.2}$$

upon substituting from (3.9.1) and using Hero's formula.

3.12. Show that

$$ar(\triangle OFG) = \frac{1}{2}R^2\sin 2E \tag{3.12.1}$$

3.13. Find the circumradius of $\triangle EFG$ for $e = 1.010, f = 1.212, g = 0.940$.
Solution: The following python code calculates the circumradius

```
codes/quad.py
```

$\therefore$ $\mathbf{R} = 0.622$

3.14. To prove QuadrilateralEFGH to be cyclic **H** should satisfy the general circle equation, $\|V - C\| = R$.
$\|H - C\| = R$

$$\left\|\begin{pmatrix}5.545 - 5.075\\2.489 - 2.081\end{pmatrix}\right\| = \left\|\begin{pmatrix}0.470\\0.408\end{pmatrix}\right\| = 0.622 \tag{3.14.1}$$

As **H** satisfies the general circle equation.
$\therefore$ Quadrilateral EFGH is a cyclic quadrilateral.