Problem 8.5.19

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***Abstract*—This a document explaining a question on the concept of cyclic quadrilateral.**

Download all python codes from

```
svn co https://github.com/SiddharthPh/
    Summer2020/trunk/geometry/Circle/codes
```

and latex-tikz codes from

```
svn co https://github.com/SiddharthPh/
    Summer2020/trunk/geometry/Circle/figs
```

1 Problem

Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadri- lateral is cyclic.

2 Construction

2.1 Constructing QuadrilateralABCD

1. Constructing quadrilateral $ABCD$:
Solution: The design parameters for constructing the quadrilateral ABCD are given in the Table. 2.1.1.

Input Values	
Parameters	Values
a	9
b	6.324
c	4.472
d	5
e	9.219

TABLE 2.1.1: Parameters for Quadrilateral ABCD

$$BC = a = 4.5, CD = b = 5.5, AD = c = 4, \tag{2.1.1.1}$$

$$AB = d = 6, BD = e = 7 \tag{2.1.1.2}$$

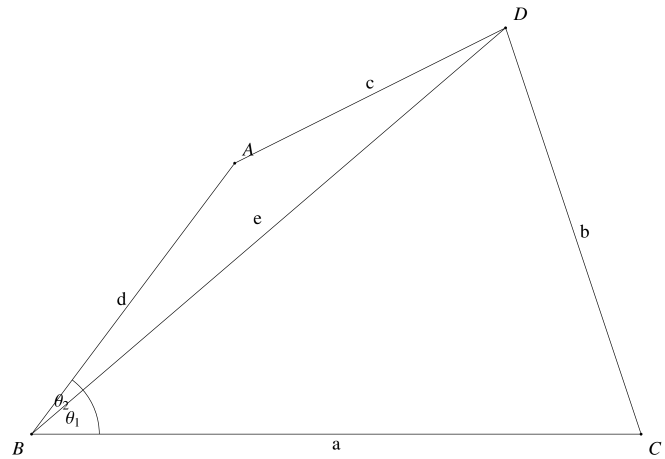

Fig. 2.1.1: QuadrilateralABCD by Latex-Tikz

Solution: The angles θ_1 and θ_2 in Fig. **??** are calculated using the cosine formula as

$$\cos\theta_1 = \frac{a^2 + e^2 - b^2}{2ae} \tag{2.1.1.3}$$

$$\cos\theta_2 = \frac{d^2 + e^2 - c^2}{2de} \tag{2.1.1.4}$$

The coordinates are then obtained as

$$\mathbf{A} = d\begin{pmatrix}\cos(\theta_1 + \theta_2)\\ \sin(\theta_1 + \theta_2)\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}a\\0\end{pmatrix},$$

$$\mathbf{D} = e\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix} \tag{2.1.1.5}$$

2. The values of A,B,C,D are shown in the Table .2.3.4
3. Draw Fig. 2.3.5.
Solution: The following Python code generates Fig. 2.3.5

```
codes/quad1.py
```

and the equivalent latex-tikz code is

```
figs/quad1.tex
```

2.2 Proof for Angular Bisector.

1. Finding the equation of angular bisectors if $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are given, where $\mathbf{BD}$ is the angular

Input Values	
Parameters	Values
A	$\begin{pmatrix}3\\4\end{pmatrix}$
B	$\begin{pmatrix}0\\0\end{pmatrix}$
C	$\begin{pmatrix}9\\0\end{pmatrix}$
D	$\begin{pmatrix}7\\6\end{pmatrix}$

TABLE 2.1.2: Vertices A,B,C,D

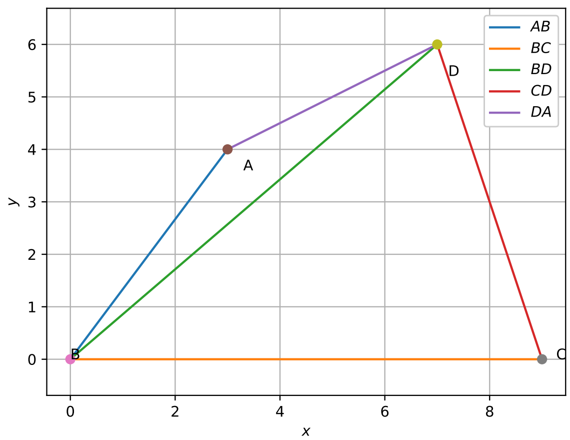

Fig. 2.1.3: QuadrilateralABCD generated using python

bisector.
Solution: To find the equation of the bisector **BD**.
From the figure 2.2.1, let **B** be origin, and

$$\mathbf{P_1} = \frac{\mathbf{C}}{\|\mathbf{C}\|} \tag{2.2.1.1}$$

$$\mathbf{P_2} = \frac{\mathbf{A}}{\|\mathbf{A}\|} \tag{2.2.1.2}$$

Then $\mathbf{P_1P_2} \perp \mathbf{BD}$

$$\mathbf{D}^T(\mathbf{P_1} - \mathbf{P_2}) = 0 \tag{2.2.1.3}$$

However, $\|\mathbf{P_1}\| = \|\mathbf{P_2}\|$
$\implies (\|\mathbf{P_1}\|)^2 = (\|\mathbf{P_2}\|)^2$
$\implies (\mathbf{P_1} - \mathbf{P_2})^T(\mathbf{P_1} + \mathbf{P_2}) = 0$

$$\implies \mathbf{P_1} + \mathbf{P_2} \perp \mathbf{P_1} - \mathbf{P_2} \tag{2.2.1.4}$$

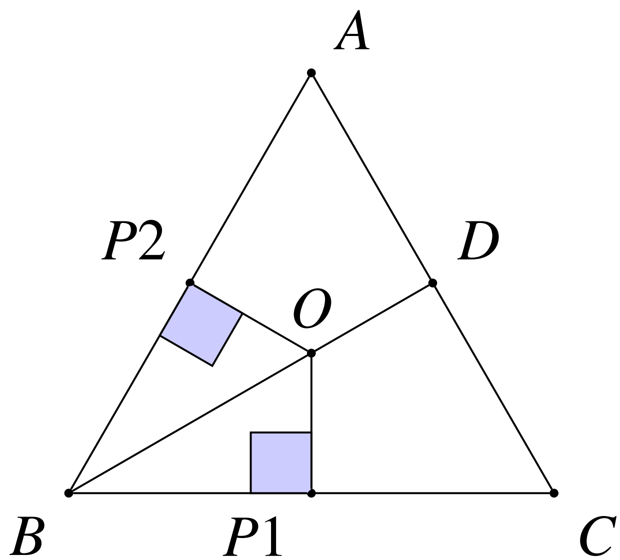

Fig. 2.2.1: Triangle with angular bisectors.

From 2.2.1.3 and 2.2.1.4

$$\mathbf{D} = k(\mathbf{P_1} + \mathbf{P_2}) \tag{2.2.1.5}$$

Then equation of **BD** is:

$$\mathbf{BD} : \mathbf{x} = \lambda(\mathbf{P_1} + \mathbf{P_2}) \tag{2.2.1.6}$$

$$\mathbf{x} = \lambda\left(\frac{\mathbf{A}}{\|\mathbf{A}\|} + \frac{\mathbf{C}}{\|\mathbf{C}\|}\right) \tag{2.2.1.7}$$

If **B** is not at origin then,

$$\mathbf{BD} : \mathbf{x} = \mathbf{B} + \lambda\left(\frac{\mathbf{A} - \mathbf{B}}{\|\mathbf{A} - \mathbf{B}\|} + \frac{\mathbf{C} - \mathbf{B}}{\|\mathbf{C} - \mathbf{B}\|}\right) \tag{2.2.1.8}$$

2.3 Cyclic QuadrilateralEFGH using angular bisectors.

1. The Figure of the quadriletral as obtained in the question looks like Fig. 2.3.1. with angles $\angle A, \angle C$ and $\angle B$ and$\angle D$ and sides a, b and c and d.
2. The design parameters for construction are:
Solution: See Table. 2.3.2.
3. Find the angular bisectors of each angle in Fig. 2.3.1
Solution: From the given information, the line equation of acute angular bisector of $\angle B$ in vector form is

$$\mathbf{L_1} \implies \begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 0 \tag{2.3.3.1}$$

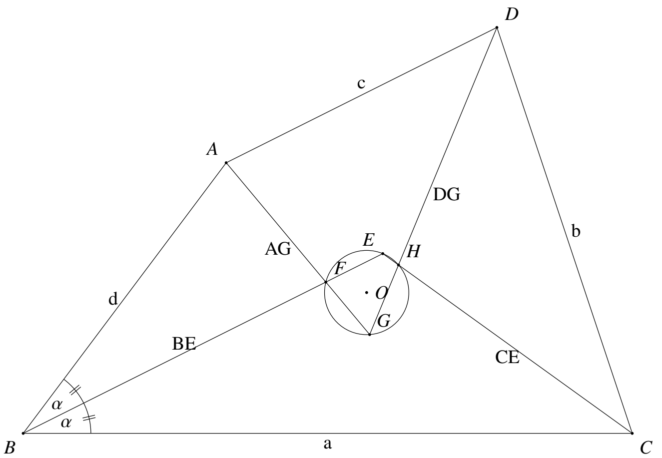

Fig. 2.3.1: Quadrilateraal by Latex-Tikz

Input Values	
Parameters	Values
A	$\begin{pmatrix}3\\4\end{pmatrix}$
B	$\begin{pmatrix}0\\0\end{pmatrix}$
C	$\begin{pmatrix}9\\0\end{pmatrix}$
D	$\begin{pmatrix}7\\6\end{pmatrix}$

TABLE 2.3.2: Quadrilateral ABCD

Where the direction ratio of the line $\mathbf{L_1}$ are obtained by equation (2.2.1.7)) Vector form of angular bisector of $\angle C$ is

$$\mathbf{L_2} \implies \begin{pmatrix}0.72 & 1\end{pmatrix}\mathbf{x} = 6.48 \tag{2.3.3.2}$$

Vector form of angular bisector of $\angle A$ is

$$\mathbf{L_3} \implies \begin{pmatrix}1.20 & 1\end{pmatrix}\mathbf{x} = 7.6 \tag{2.3.3.3}$$

Vector form of angular bisector of $\angle D$ is

$$\mathbf{L_4} \implies \begin{pmatrix}-2.4 & 1\end{pmatrix}\mathbf{x} = -10.9 \tag{2.3.3.4}$$

4. To find the point of intersection of the angular bisectors? **Solution:** **E** is obtained by using line equations $\mathbf{L_1}$ and $\mathbf{L_2}$ as matrix equations.

$$\begin{pmatrix}1 & -1\\0.72 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}0\\6.48\end{pmatrix} \tag{2.3.4.1}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}1 & -2 & 0\\0.72 & 1 & 6.48\end{pmatrix} \xleftrightarrow{R_2\leftarrow\frac{R_2-0.72R_1}{2.44}} \begin{pmatrix}1 & -2 & 0\\0 & 1 & 2.65\end{pmatrix} \tag{2.3.4.2}$$

$$\xleftrightarrow{R_1\leftarrow R_1+2R_2} \begin{pmatrix}1 & 0 & 5.3\\0 & 1 & 2.65\end{pmatrix} \tag{2.3.4.3}$$

$$\implies \mathbf{x} = \begin{pmatrix}5.31\\2.65\end{pmatrix} \tag{2.3.4.4}$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix}1 & -2 & 0\\0.72 & 1 & 6.48\end{pmatrix} \tag{2.3.4.5}$$

results in a matrix with 2 nonzero rows, its rank is 2. Similarly, the rank of the matrix

$$\begin{pmatrix}1 & -2\\0.72 & 1\end{pmatrix} \tag{2.3.4.6}$$

is 2, from 2.3.4.3.

$$\therefore \mathbf{E} = \begin{pmatrix}5.3137\\2.6568\end{pmatrix}$$

F is obtained by using line equations $\mathbf{L_1}$ and $\mathbf{L_3}$

$$\mathbf{F} = \begin{pmatrix}4.472\\2.236\end{pmatrix} \tag{2.3.4.7}$$

G is obtained by equating line equations $\mathbf{L_3}$ and $\mathbf{L_4}$

$$\mathbf{G} = \begin{pmatrix}5.119\\1.460\end{pmatrix} \tag{2.3.4.8}$$

H is obtained by equating line equations $\mathbf{L_2}$ and $\mathbf{L_4}$

$$\mathbf{H} = \begin{pmatrix}5.545\\2.489\end{pmatrix} \tag{2.3.4.9}$$

The values are listed in Table. 2.3.4

5. Draw Fig. 2.3.1.
Solution: The following Python code generates Fig. 2.3.5

```
codes/quad.py
```

and the equivalent latex-tikz code generating Fig. 2.3.1 is

```
figs/quad.tex
```

Derived values	
Parameter	Value
$\mathbf{E}$	$\begin{pmatrix}5.313\\2.656\end{pmatrix}$
$\mathbf{F}$	$\begin{pmatrix}4.472\\2.236\end{pmatrix}$
$\mathbf{G}$	$\begin{pmatrix}5.119\\1.460\end{pmatrix}$
$\mathbf{H}$	$\begin{pmatrix}5.545\\2.489\end{pmatrix}$

TABLE 2.3.4: Cyclic Quadrilateral EFGH

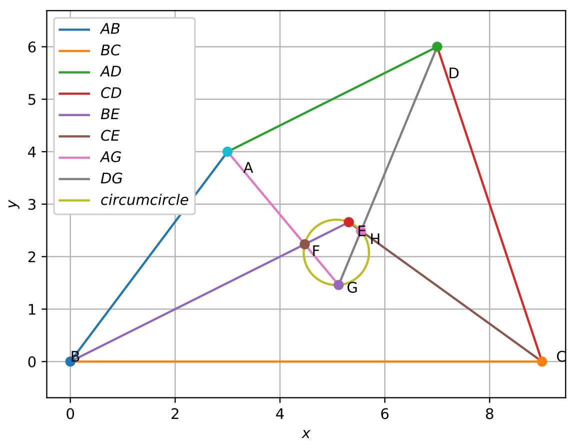

Fig. 2.3.5: Quadrilateral generated using python

The above latex code can be compiled as a standalone document as

```
figs/quad_fig.tex
```

3 Solution

3.1. Show that $\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}$ lies on a circle.
Solution: Let $\mathbf{V}$ be a general vector that satisfies the circle equation.
Then, $\|V - O\| = r$ will be the equation, where $\mathbf{O}, R$ are Centre of circle, and radius respectively.
Find a point $\mathbf{O}$ that is equidistant from the vertices of $\triangle EFG$ for $e = 1.010, f = 1.212, g = 0.940$.

$$\|E - O\| = \|F - C\| = \|G - C\| = R \tag{3.1.1}$$

From (3.1.1),

$$\|\mathbf{E} - \mathbf{O}\|^2 - \|\mathbf{F} - \mathbf{O}\|^2 = 0 \tag{3.1.2}$$

$$\implies (\mathbf{E} - \mathbf{O})^T (\mathbf{E} - \mathbf{O}) - (\mathbf{F} - \mathbf{O})^T (\mathbf{F} - \mathbf{O}) = 0 \tag{3.1.3}$$

which can be simplified as

$$(\mathbf{E} - \mathbf{F})^T \mathbf{O} = \frac{\|\mathbf{E}\|^2 - \|\mathbf{F}\|^2}{2} \tag{3.1.4}$$

Similarly,

$$(\mathbf{F} - \mathbf{G})^T \mathbf{O} = \frac{\|\mathbf{F}\|^2 - \|\mathbf{G}\|^2}{2} \tag{3.1.5}$$

(3.1.4) and (3.1.5), can be combined to form the matrix equation

$$\mathbf{N}^T \mathbf{O} = \mathbf{g} \tag{3.1.6}$$

$$\implies \mathbf{O} = \mathbf{N}^{-T} \mathbf{g} \tag{3.1.7}$$

where

$$\mathbf{N} = \begin{pmatrix}\mathbf{E} - \mathbf{F} & \mathbf{F} - \mathbf{G}\end{pmatrix} \tag{3.1.8}$$

$$\mathbf{g} = \frac{1}{2}\begin{pmatrix}\|\mathbf{E}\|^2 - \|\mathbf{F}\|^2\\ \|\mathbf{F}\|^2 - \|\mathbf{G}\|^2\end{pmatrix} \tag{3.1.9}$$

$\mathbf{O}$ can be computed using the python code below

```
codes/quad.py
```

and the equivalent latex-tikz code to draw Fig. 3.1 is

```
figs/quad.tex
```

3.2. $\therefore \mathbf{O} = \begin{pmatrix}5.075\\2.081\end{pmatrix}$

3.3. In $\triangle OFG$, $OF = OG = R$. Such a triangle is known as an *isoceles triangle*.

3.4. Show that $\angle OFG = \angle OGF$. In an isoceles triangle, opposite sides and corresponding opposite angles are equal.
Solution: Using the sine formula ,

$$\frac{\sin \angle OFG}{R} = \frac{\sin \angle OGF}{R} \tag{3.4.1}$$

$$\implies \sin \angle OFG = \sin \angle OGF \tag{3.4.2}$$

3.5. Show that $\angle FOG = 2\angle E$.

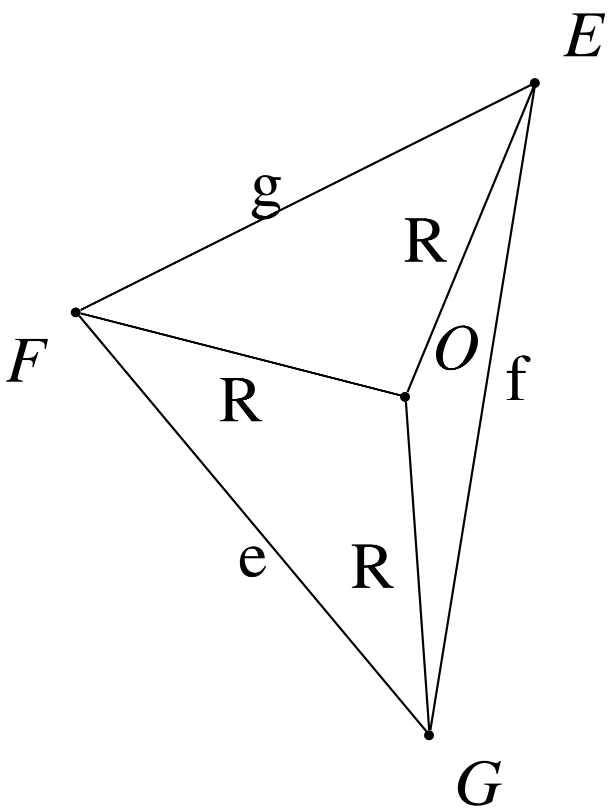

Fig. 3.1: Circumcentre O of $\triangle EFG$

Solution: In Fig. 3.1,

$$E = \theta_2 + \theta_3 \tag{3.5.1}$$
$$F = \theta_1 + \theta_2 \tag{3.5.2}$$
$$G = \theta_3 + \theta_1 \tag{3.5.3}$$
$$\implies 2(\theta_1 + \theta_2 + \theta_3) = E + G + H = 180° \tag{3.5.4}$$
$$\implies \theta_1 + \theta_2 + \theta_3 = 90° \tag{3.5.5}$$

From (3.5.1) and (3.5.5),

$$E = 90° - \theta_1 \tag{3.5.6}$$

Also, in $\triangle OFG$, all angles add up to 180°. Hence,

$$\angle FOG + 2\theta_1 = 180° \tag{3.5.7}$$
$$\implies \angle FOG = 180° - 2\theta_1 = 2(90° - \theta_1) = 2\angle E \tag{3.5.8}$$

upon substituting from (3.5.6).

3.6. Let $\mathbf{L}$ be the mid point of FG. Show that $OL \perp FG$.

Solution: From (3.1.4),

$$(\mathbf{F} - \mathbf{G})^T \mathbf{O} = \frac{\|\mathbf{F}\|^2 - \|\mathbf{G}\|^2}{2} \tag{3.6.1}$$
$$\implies (\mathbf{F} - \mathbf{G})^T \mathbf{O} = \frac{1}{2}(\mathbf{F} - \mathbf{G})^T(\mathbf{F} + \mathbf{G}) \tag{3.6.2}$$
$$\implies (\mathbf{F} - \mathbf{G})^T \left(\mathbf{O} - \frac{\mathbf{F} + \mathbf{G}}{2}\right) = 0 \tag{3.6.3}$$
$$\text{or, } (\mathbf{F} - \mathbf{G})^T (\mathbf{O} - \mathbf{L}) = 0 \tag{3.6.4}$$

$\because \mathbf{L} = \frac{\mathbf{F}+\mathbf{G}}{2}$ is the mid point of FG. From (**??**) we then conclude that $OL \perp FG$.

3.7. Perpendicular bisectors of a triangle meet at the circumcentre.

3.8. In the isosceles $\triangle OFG$, if $FL = LG$, $OL \perp FG$.

3.9. Show that

$$\frac{e}{\sin E} = \frac{f}{\sin F} = \frac{g}{\sin G} = 2R. \tag{3.9.1}$$

Solution: In $\triangle OFG$, using the cosine formula,

$$\cos 2A = \frac{R^2 + R^2 - e^2}{2R^2} = 1 - \frac{e^2}{2R^2} \tag{3.9.2}$$

Using the sine formula,

$$\frac{\sin 2E}{e} = \frac{\sin \theta_1}{R} = \frac{\sin(90° - E)}{R} \tag{3.9.3}$$
$$\implies \sin 2E = \frac{a \cos E}{R} \tag{3.9.4}$$

from (3.5.6) and Baudhanya theorem.

$$\cos^2 2E + \sin^2 2E = 1 \tag{3.9.5}$$
$$\implies \left(1 - \frac{e^2}{2R^2}\right)^2 + \left(\frac{e \cos E}{R}\right)^2 = 1 \tag{3.9.6}$$

upon substituting from (3.9.2) and (3.9.4). Letting

$$x = \left(\frac{e}{R}\right)^2, \tag{3.9.7}$$

in the previous equation yields

$$\left(1-\frac{x}{2}\right)^2 + x\cos^2 E = 1 \tag{3.9.8}$$

$$\implies 1-\frac{x^2}{4} - x + x\cos^2 E = 1 \tag{3.9.9}$$

$$\implies x\left(1-\cos^2 E\right) - \frac{x^2}{4} = 0 \tag{3.9.10}$$

$$x\sin^2 E - \frac{x^2}{4} = 0 \tag{3.9.11}$$

$$\implies x\left(\sin^2 E - \frac{x}{4}\right) = 0 \tag{3.9.12}$$

$$\text{or, } \frac{x}{4} - \sin^2 E = 0 \tag{3.9.13}$$

$\because x \neq 0$. Thus, substituting from (3.9.7),

$$x = \left(\frac{e}{R}\right)^2 = 4\sin^2 E \tag{3.9.14}$$

$$\implies \frac{e}{R} = 2\sin E, \tag{3.9.15}$$

$$\text{or,} \quad \frac{e}{\sin E} = 2R \tag{3.9.16}$$

3.10. Show that

$$\cos 2E = 1 - 2\sin^2 E = 2\cos^2 E - 1 \tag{3.10.1}$$

$$= \cos^2 E - \sin^2 E \quad \text{and} \tag{3.10.2}$$

$$\sin 2E = 2\sin E\cos E \tag{3.10.3}$$

3.11. Find R.
Solution:

$$ar\,(\triangle EFG) = \frac{1}{2} fg\sin E = \frac{efg}{4R} \tag{3.11.1}$$

$$\implies R = \frac{efg}{4s\sqrt{(s-e)(s-f)(s-g)}} \tag{3.11.2}$$

upon substituting from (3.9.1) and using Hero's formula.

3.12. Show that

$$ar\,(\triangle OFG) = \frac{1}{2} R^2 \sin 2E \tag{3.12.1}$$

3.13. Find the circumradius of $\triangle EFG$ for $e = 1.010, f = 1.212, g = 0.940$.
Solution: The following python code calculates the circumradius

```
codes/quad.py
```

$\therefore$ $\mathbf{R} = 0.622$

3.14. To prove QuadrilateralEFGH to be cyclic **H** should satisfy the general circle equation, $\|V - C\| = R$.

$\|H - C\| = R$

$$\left\|\begin{pmatrix}5.545 - 5.075\\ 2.489 - 2.081\end{pmatrix}\right\| = \left\|\begin{pmatrix}0.470\\ 0.408\end{pmatrix}\right\| = 0.622 \tag{3.14.1}$$

As **H** satisfies the general circle equation.
$\therefore$ Quadrilateral EFGH is a cyclic quadrilateral.