

Math Document Template

Pothukuchi Siddhartha

Download all python codes from

svn co <https://github.com/SiddharthPh/Summer2020/trunk/geometry/LinearAlgebra/codes>

and latex-tikz codes from

svn co <https://github.com/SiddharthPh/Summer2020/trunk/geometry/LinearAlgebra/figs>

1 TRIANGLE EXERCISE.

1.0.1 Problem:

1. Draw the graphs of the equations

$$(1 \ -1)\mathbf{x} + 1 = 0 \quad (1.0.1.1)$$

$$(3 \ 2)\mathbf{x} - 12 = 0 \quad (1.0.1.2)$$

Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.

1.1 Solution

1. Let

$$\mathbf{x} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (1.1.1.1)$$

Substituting in (1.0.1.1),

$$(1 \ -1)\begin{pmatrix} a \\ 0 \end{pmatrix} = -1 \quad (1.1.1.2)$$

$$\Rightarrow a = -1 \quad (1.1.1.3)$$

Simiarly, substituting

$$\mathbf{x} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad (1.1.1.4)$$

in (1.0.1.1),

$$b = 1 \quad (1.1.1.5)$$

The intercepts on the x and y-axis from above are

$$\mathbf{A} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.1.1.6)$$

\mathbf{A} is the x-intercept of the line and is the point where it meets x-axis.

Using the above method, the intercepts on x and y-axis for the equation (1.0.1.2) are

$$\mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \quad (1.1.1.7)$$

\mathbf{C} is the x-intercept of the line and is the point where it meets x-axis.

The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 12 \end{pmatrix} \quad (1.1.1.8)$$

The augmented matrix for the above equation is row reduced as follows

$$\left(\begin{array}{cc|c} 1 & -1 & -1 \\ 3 & 2 & 12 \end{array} \right) \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 3 \end{array} \right) \quad (1.1.1.9)$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array} \right) \quad (1.1.1.10)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

2. And the vertices of triangle (1.1.2) formed due to the intersection of lines and x-axis are:

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (1.1.2.1)$$

$$\mathbf{B} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (1.1.2.2)$$

$$\mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (1.1.2.3)$$

Where \mathbf{B} and \mathbf{C} are X-intercepts of line (1.0.1.1) and (1.0.1.2) respectively (from (1.1.1) and (1.1.1)). The equivalent python

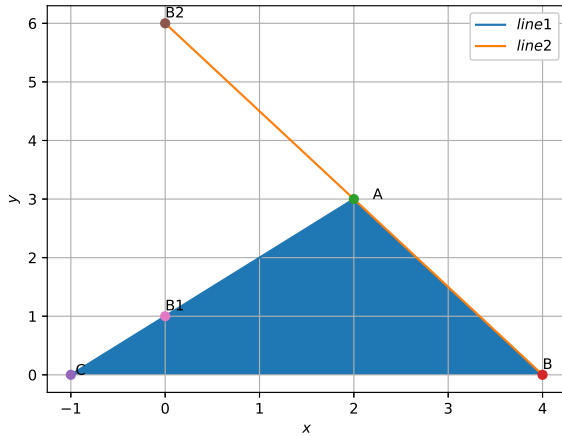


Fig. 1.1.2: Shaded Triangle

code for figure (1.1.2) is

codes/triangle/shaded.py

2 QUADRILATERAL EXERCISE

2.1 Problem

The angles of quadrilateral are in the ratio 3 : 5 : 9 : 13. Find all the angles of the quadrilateral.

2.2 Solution

- Let the measure of angles $\angle A, \angle B, \angle C, \angle D$ of a quadrilateral are $3x, 5x, 9x$ and $13x$ respectively, where x is a real number.

Using angle sum property, the sum of interior angles of a quadrilateral is 360 degree.

$$3x + 5x + 9x + 13x = 360^\circ \quad (2.2.1.1)$$

$$30x = 360^\circ \quad (2.2.1.2)$$

$$x = 12^\circ \quad (2.2.1.3)$$

From the above calculations,

$$\angle A = 3x = 3(12) = 36^\circ \quad (2.2.1.4)$$

$$\angle B = 5x = 5(12) = 60^\circ \quad (2.2.1.5)$$

$$\angle C = 9x = 9(12) = 108^\circ \quad (2.2.1.6)$$

$$\angle D = 13x = 13(12) = 156^\circ \quad (2.2.1.7)$$

3 LINE EXERCISES

3.1 Point and Vector Exercise

3.1.1 Problem:

- Find the distance between the following pairs of points

a)

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (3.1.1.1.1)$$

b)

$$\begin{pmatrix} -5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (3.1.1.1.2)$$

c)

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} -1 \\ b \end{pmatrix} \quad (3.1.1.1.3)$$

3.1.2 Solution:

- The distance between two vectors is given by

$$\|\mathbf{A} - \mathbf{B}\| \quad (3.1.2.1.1)$$

- The distance between $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$$\text{is } \left\| \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\| = 2.828 \text{ (From (3.1.2.1.1))}$$

- The distance between $\mathbf{A} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$ and

$$\mathbf{B} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \text{ is}$$

$$\left\| \begin{pmatrix} -5 \\ 7 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\| = 5.656 \text{ (From (3.1.2.1.1))}$$

- The distance between $\mathbf{A} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 \\ b \end{pmatrix}$

$$\text{is } \left\| \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} -1 \\ b \end{pmatrix} \right\| = a + 1 \text{ (From (3.1.2.1.1))}$$

3.2 Point on a line

3.2.1 Problem: Find the coordinates of the point which divides the join of

$$\begin{pmatrix} -1 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad (1.1)$$

in the ratio 2 : 3.

3.2.2 Solution:

- $\mathbf{A} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

Then \mathbf{C} that divides \mathbf{A}, \mathbf{B} in the ratio $k : 1$ is

$$\mathbf{C} = \frac{k\mathbf{A} + \mathbf{B}}{k + 1} \quad (3.2.2.1.1)$$

For the given problem $k=2 : 3$

Using the equation 3.2.2.1.1, the desired point is

$$C = \frac{\frac{2}{3} \begin{pmatrix} -1 \\ 7 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix}}{\frac{2}{3} + 1} \quad (3.2.2.1.2)$$

$$\therefore C = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (3.2.2.1.3)$$

The following code plots the figure ??

codes/line/section.py

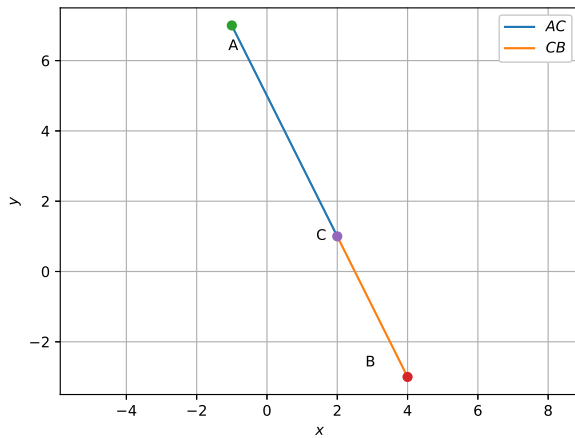


Fig. 3.2.2.1

3.3 Lines and Planes

3.3.1 Problem:

1. Verify whether the following are zeroes of the polynomial, indicated against them.

- $p(x) = 3x + 1, x = \frac{1}{3}$
- $p(x) = 5x - \pi, x = \frac{4}{5}$
- $p(x) = 5lx + m, x = -\frac{m}{l}$
- $p(x) = 2x + 1, x = \frac{1}{2}$

3.3.2 Solution:

1. Let

$$y = 3x + 1 \implies \begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = -1 \quad (3.3.2.1.1)$$

Thus,

$$y = 0 \quad (3.3.2.1.2)$$

$$\implies 3x + 1 = 0 \quad (3.3.2.1.3)$$

$$\text{or, } x = -\frac{1}{3} \quad (3.3.2.1.4)$$

Hence $x = \frac{1}{3}$ is not a zero. This is verified in Fig. 3.3.2.1.

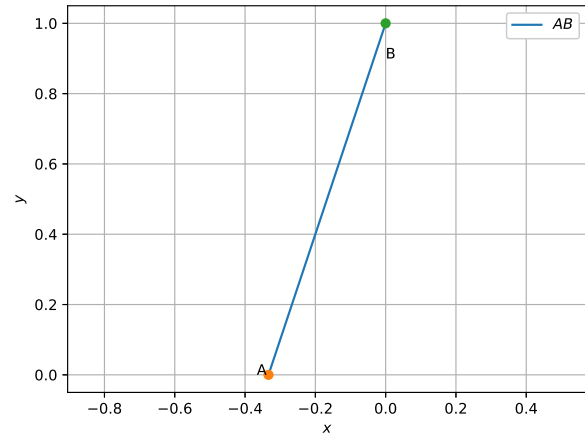


Fig. 3.3.2.1

2. Let

$$y = 5x - \pi \implies \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \pi \quad (3.3.2.2.1)$$

Thus,

$$y = 0 \quad (3.3.2.2.2)$$

$$\implies 5x - \pi = 0 \quad (3.3.2.2.3)$$

$$\text{or, } x = \frac{\pi}{5} \quad (3.3.2.2.4)$$

Hence $x = \frac{4}{5}$ is not a zero. This is verified in Fig. 3.3.2.2.

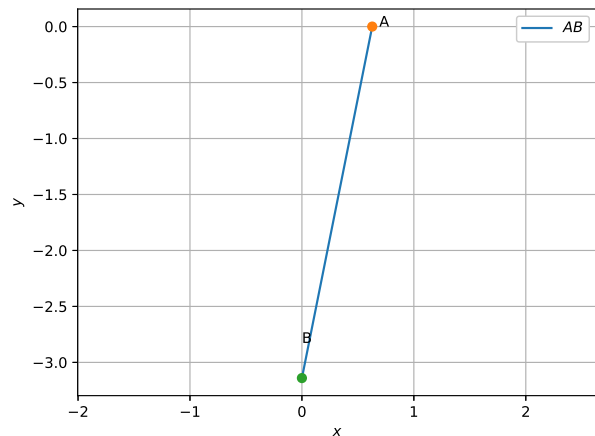


Fig. 3.3.2.2

3. Let

$$y = 5lx + m \Rightarrow (5l \ -1)\mathbf{x} = -m \quad (3.3.2.3.1)$$

Thus,

$$y = 0 \quad (3.3.2.3.2)$$

$$\Rightarrow 5lx + m = 0 \quad (3.3.2.3.3)$$

$$\text{or, } x = -\frac{m}{5l} \quad (3.3.2.3.4)$$

Hence $\mathbf{x} = -\frac{m}{l}$ is not a zero. This is verified in Fig. 3.3.2.3.

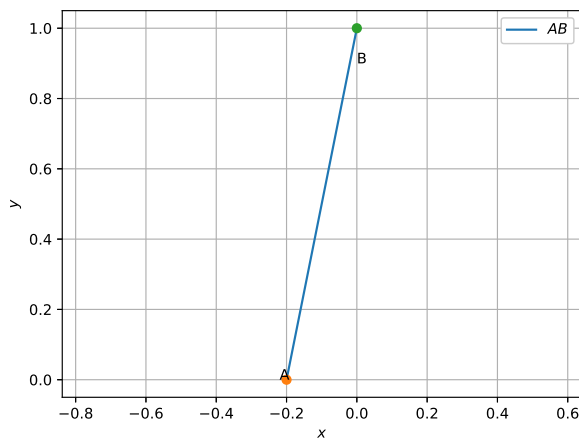


Fig. 3.3.2.3

4. Let

$$y = 2x + 1 \Rightarrow (2 \ -1)\mathbf{x} = -1 \quad (3.3.2.4.1)$$

Thus,

$$y = 0 \quad (3.3.2.4.2)$$

$$\Rightarrow 2x + 1 = 0 \quad (3.3.2.4.3)$$

$$\text{or, } x = -\frac{1}{2} \quad (3.3.2.4.4)$$

Hence $\mathbf{x} = \frac{1}{2}$ is not a zero. This is verified in Fig. 3.3.2.4.

3.4 Motion in a Plane

3.4.1 Problem:

- Rain is falling vertically with a speed of 35 ms^{-1} after sometime with a speed of 12 ms^{-1} . Winds starts blowing in east to west direction. In which direction should a boy waiting at a bus stop hold his umbrella ?

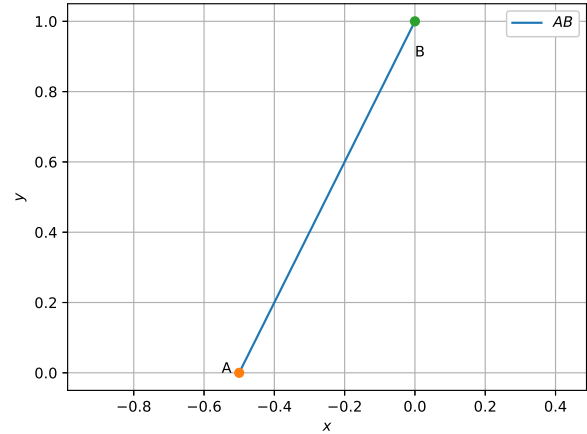


Fig. 3.3.2.4

3.4.2 Solution:

- See Fig. 3.4.2.1. From the given information, the rain velocity is

$$\mathbf{u} = \begin{pmatrix} 0 \\ 35 \end{pmatrix} \quad (3.4.2.1.1)$$

and the wind velocity is

$$\mathbf{v} = -\begin{pmatrix} 12 \\ 0 \end{pmatrix} \quad (3.4.2.1.2)$$

The resulting rain velocity is

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} -12 \\ 35 \end{pmatrix} \quad (3.4.2.1.3)$$

The desired angle is

$$-\tan^{-1} \frac{\mathbf{u} + \mathbf{v}}{35} = \tan^{-1} \frac{12}{35} \quad (3.4.2.1.4)$$

$$\approx 20.05^\circ \quad (3.4.2.1.5)$$

3.5 Matrix Exercise

3.5.1 Problem:

- In the matrix $A = \begin{pmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{pmatrix}$, write

- The order of the matrix
- The number of elements
- Write the elements $a_{31}, a_{21}, a_{33}, a_{24}, a_{23}$.

3.5.2 Solution:

- a) The order of matrix for above problem is 3×4 .

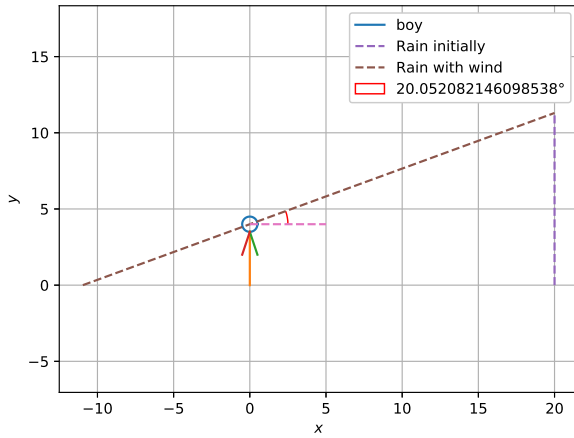


Fig. 3.4.2.1

b) The number of elements=12

c) The elements are

$$a_{31} = \sqrt{3} \quad (3.5.2.1.1)$$

$$a_{21} = 35 \quad (3.5.2.1.2)$$

$$a_{33} = -5 \quad (3.5.2.1.3)$$

$$a_{24} = 12 \quad (3.5.2.1.4)$$

$$a_{23} = \frac{5}{2} \quad (3.5.2.1.5)$$

The python implementation for the above example is given in

codes/line/matrix.py

3.6 Determinants

$$3.6.1 \text{ Problem: } \begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$$

3.6.2 Solution:

1. The determinant of a matrix 2x2 matrix is given by:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad (3.6.2.1.1)$$

$$\text{Det} = a_{11}a_{22} - a_{12}a_{21} \quad (3.6.2.1.2)$$

$\therefore \text{Det} = 18.$

3.7 Linear inequation

3.7.1 Problem: Solve $x \geq 3, y \geq 2$ graphically.

3.7.2 Solution:

1. Solve the following system of linear inequalities graphically.

$$\begin{aligned} x &\geq 3 \\ y &\geq 2 \end{aligned} \quad (3.7.2.1.1)$$

Let $u_1 \geq 0, u_2 \geq 0$. This may be expressed as

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \geq \mathbf{0} \quad (3.7.2.1.2)$$

(3.7.2.1.1) can then be expressed as

$$\begin{aligned} x &\geq 3 \\ y &\geq 2 \end{aligned} \quad (3.7.2.1.3)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} \geq \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (3.7.2.1.4)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \mathbf{u} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (3.7.2.1.5)$$

$$\text{or, } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \mathbf{u} \quad (3.7.2.1.6)$$

resulting in

$$\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathbf{u} \quad (3.7.2.1.7)$$

$$\text{or, } \mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{u} \quad (3.7.2.1.8)$$

after obtaining the inverse. Fig. 3.7.2.1 generated using the following python code shows the region satisfying (3.7.2.1.1)

codes/line/line_eq.py

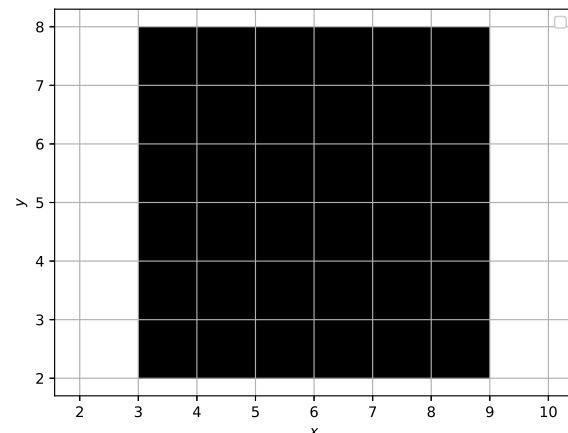


Fig. 3.7.2.1

4 CIRCLE EXAMPLE

4.1 Problem

Find the equation of a circle with centre $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and radius 4.

4.2 Exercise

- The input values for the question are given in the table (5.2.1) The **O** is the centre of

Input Values	
O(centre)	$\begin{pmatrix} -3 \\ 2 \end{pmatrix}$
r(radius)	4

TABLE 4.2.1: Input Values

circle= $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$, let r be the radius of the circle=4.
Let \mathbf{x} satisfy the circle equation

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + (6 \quad -4) \mathbf{x} \quad (4.2.1.1)$$

The python code for the figure (5.2.1) is

codes/circle/circle1.py

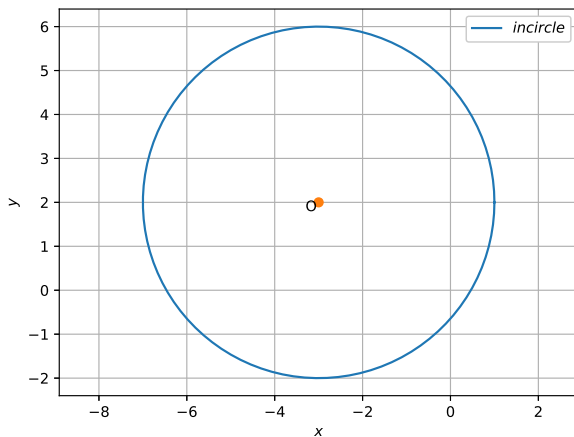


Fig. 4.2.1: Circle using python

5 CIRCLE EXERCISE

5.1 Problem

Find the coordinates of point **A**, where AB is the diameter of circle whose centre is $(2, -3)$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

5.2 Solution

- The input values for the question are given in the table (5.2.1) The **A** is at the end of

Input values	
Parameters	Values
O	$\begin{pmatrix} 2 \\ -3 \end{pmatrix}$
A	$\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

TABLE 5.2.1: Input Values

diameter, so the centre(**O**) is the midpoint of **AB**.

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B}}{2} \quad (5.2.1.1)$$

$$\mathbf{A} = 2\mathbf{O} - \mathbf{B} \quad (5.2.1.2)$$

$$\therefore \mathbf{A} = \begin{pmatrix} 3 \\ -10 \end{pmatrix} \quad (5.2.1.3)$$

The python code for the figure (5.2.1) is

codes/circle/circle.py

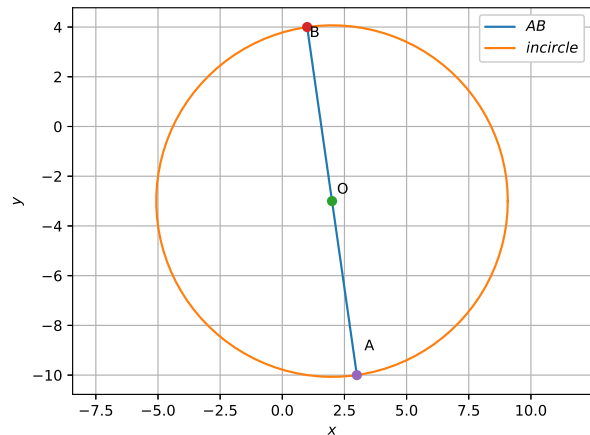


Fig. 5.2.1: Circle using python

6 CONICS EXERCISE

6.1 Problem

- Verify whether the following are zeroes of the polynomial, indicated against them.

a) $p(x) = x^2 - 1, x = 1, -1$

b) $p(x) = (x + 1)(x - 2), x = -1, 2$

c) $p(x) = x^2, x = 0.$

d) $p(x) = 3x^2 - 1, x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}.$

6.2 Solution

1. **Proof** For a general polynomial equation of degree 2,

$$p(x, y) \Rightarrow Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The vector form is

$$\mathbf{x}^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \quad (6.2.0.1.1)$$

2. For eq: $y = x^2 - 1$

Vector form is given by

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} - 1 = 0 \quad (6.2.0.2.1)$$

(From the equation 6.2.0.1.1.)

Thus,

$$y = 0 \quad (6.2.0.2.2)$$

$$\Rightarrow x^2 - 1 = 0 \quad (6.2.0.2.3)$$

$$x = +1, -1 \quad (6.2.0.2.4)$$

Hence +1, -1 are zeros, which can be verified from the figure 6.2.0.2 The python code for the figure 6.2.0.2 is

codes/conics/parab1.py

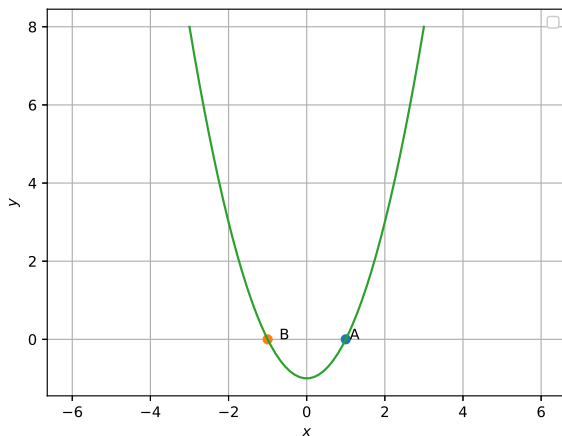


Fig. 6.2.0.2: Parabola 1

3. For eq: $y = (x + 1)(x - 2)$

Equation can be represented as

$$y = x^2 - x - 2$$

Vector form is given by

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} - 2 = 0 \quad (6.2.0.3.1)$$

(From the equation 6.2.0.1.1.)

Thus,

$$y = 0 \quad (6.2.0.3.2)$$

$$\Rightarrow (x + 1)(x - 2) = 0 \quad (6.2.0.3.3)$$

$$x = -1, +2 \quad (6.2.0.3.4)$$

Hence -1, +2 are zeros, which can be verified from the figure 6.2.0.3 The python code for the figure 6.2.0.3 is

codes/conics/parab2.py

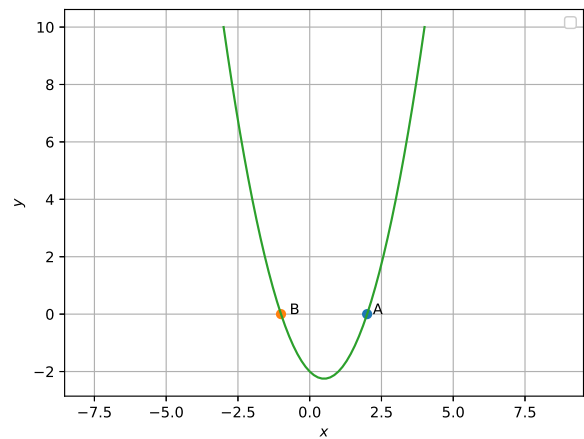


Fig. 6.2.0.3: Parabola 2

4. For eq: $y = x^2$

Vector form is given by

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} = 0 \quad (6.2.0.4.1)$$

(From the equation 6.2.0.1.1.)

Thus,

$$y = 0 \quad (6.2.0.4.2)$$

$$\Rightarrow x^2 = 0 \quad (6.2.0.4.3)$$

$$x = 0 \quad (6.2.0.4.4)$$

Hence 0 is the zero, which can be verified from the figure 6.2.0.4 The python code for the figure 6.2.0.4 is

codes/conics/parab3.py

5. For eq: $y = 3x^2 - 1$

Vector form is given by

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} - 1 = 0 \quad (6.2.0.5.1)$$

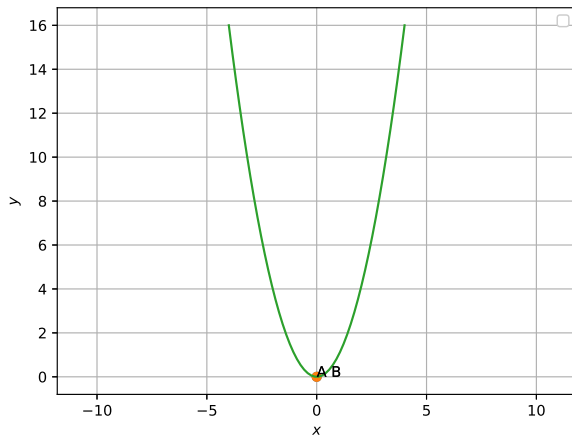


Fig. 6.2.0.4: Parabola 3

(From the equation 6.2.0.1.1.)

Thus,

$$y = 0 \tag{6.2.0.5.2}$$

$$\implies 3x^2 - 1 = 0 \tag{6.2.0.5.3}$$

$$x = +\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \tag{6.2.0.5.4}$$

Hence $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$ are the zeros, which can be verified from the figure 6.2.0.5 The python code for the figure 6.2.0.5 is

codes/conics/parab4.py

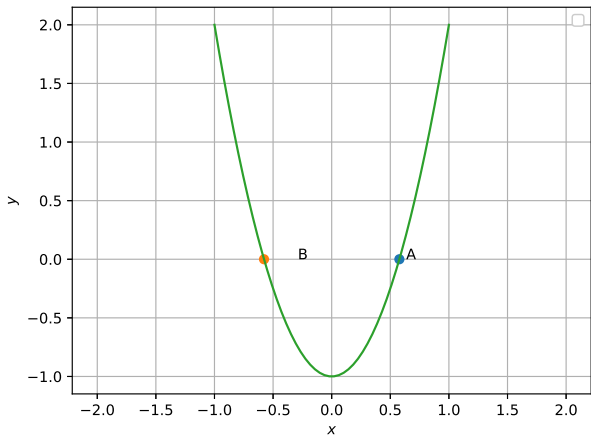


Fig. 6.2.0.5: Parabola 4