Problem 8.5.19

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***Abstract*—This a document explaining a question on the concept of cyclic quadrilateral.**

Download all python codes from

```
svn co https://github.com/SiddharthPh/
    Summer2020/trunk/geometry/Circle/codes
```

and latex-tikz codes from

```
svn co https://github.com/SiddharthPh/
    Summer2020/trunk/geometry/Circle/figs
```

1 Problem

Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadri- lateral is cyclic.

2 Construction

2.1 Constructing QuadrilateralABCD

1. Constructing quadrilateral $ABCD$:
Solution: The design parameters for constructing the quadrilateral ABCD are given in the Table. 2.1.1.

Input Values	
Parameters	Values
a	9
b	6.324
c	4.472
d	5
e	9.219

TABLE 2.1.1: Parameters for Quadrilateral ABCD

$$BC = a = 4.5, CD = b = 5.5, ADc = 4, \tag{2.1.1.1}$$
$$AB = d = 6, BD = e = 7 \tag{2.1.1.2}$$

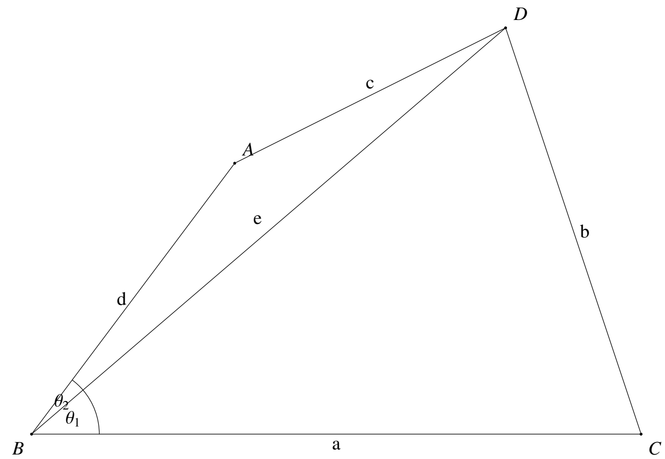

Fig. 2.1.1: QuadrilateralABCD by Latex-Tikz

Solution: The angles θ_1 and θ_2 in Fig. **??** are calculated using the cosine formula as

$$\cos\theta_1 = \frac{a^2+e^2-b^2}{2ae} \tag{2.1.1.3}$$
$$\cos\theta_2 = \frac{d^2+e^2-c^2}{2de} \tag{2.1.1.4}$$

The coordinates are then obtained as

$$\mathbf{A} = d\begin{pmatrix}\cos(\theta_1+\theta_2)\\ \sin(\theta_1+\theta_2)\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}a\\0\end{pmatrix},$$
$$\mathbf{D} = e\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix} \tag{2.1.1.5}$$

2. The values of A,B,C,D are shown in the Table .2.2.5
3. Draw Fig. 2.2.6.
Solution: The following Python code generates Fig. 2.2.6

```
codes/quad1.py
```

and the equivalent latex-tikz code is

```
figs/quad1.tex
```

Input Values	
Parameters	Values
A	$\begin{pmatrix}3\\4\end{pmatrix}$
B	$\begin{pmatrix}0\\0\end{pmatrix}$
C	$\begin{pmatrix}9\\0\end{pmatrix}$
D	$\begin{pmatrix}7\\6\end{pmatrix}$

TABLE 2.1.2: Vertices A,B,C,D

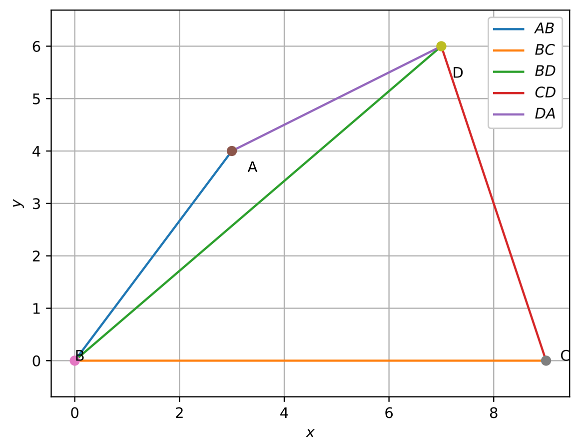

Fig. 2.1.3: QuadrilateralABCD generated using python

2.2 Cyclic QuadrilateralEFGH using angular bisectors.

1. The Figure of the quadriletral as obtained in the question looks like Fig. 2.2.1. with angles $\angle A, \angle C$ and $\angle B$ and$\angle D$ and sides a, b and c and d.
2. The design parameters for construction are:
   Solution: See Table. 2.2.2.
3. **Proof**: Finding angular bisector using unit vectors.
   Solution: : Let the angle between AB and BC be θ and between **R** and **BC** be α.

$$\mathbf{R} = \frac{\mathbf{A}-\mathbf{B}}{\|A-B\|} + \frac{\mathbf{C}-\mathbf{B}}{\|C-B\|} \tag{2.2.3.1}$$

$$\mathbf{R}\cdot\mathbf{BC} = \|R\|\,\|BC\|\cos\theta \tag{2.2.3.2}$$

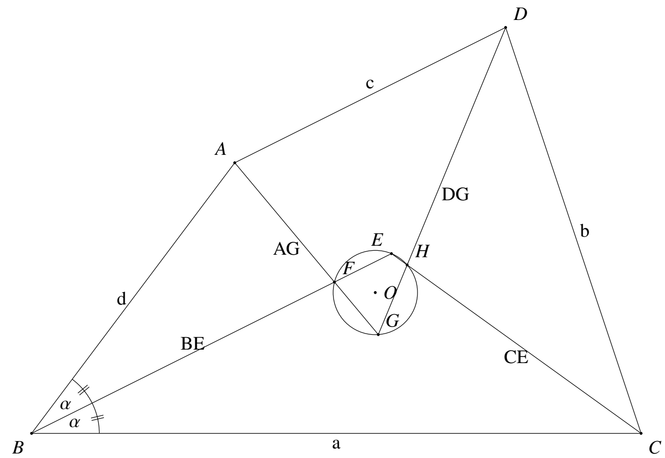

Fig. 2.2.1: Quadrilateraal by Latex-Tikz

Input Values	
Parameters	Values
A	$\begin{pmatrix}3\\4\end{pmatrix}$
B	$\begin{pmatrix}0\\0\end{pmatrix}$
C	$\begin{pmatrix}9\\0\end{pmatrix}$
D	$\begin{pmatrix}7\\6\end{pmatrix}$

TABLE 2.2.2: Quadrilateral ABCD

The resulting equation after simplifying is,

$$\cos\theta + 1 = \sqrt{2+2\cos\theta}\cos\alpha \tag{2.2.3.3}$$

By squaring on both sides

$$(\cos\theta+1)^2 = 2+2\cos\theta(\cos\alpha)^2 \tag{2.2.3.4}$$

$$\cos\theta = 2\cos^2\alpha - 1 \tag{2.2.3.5}$$

The above equation is the formula of $\cos 2\theta$ $\therefore \alpha = \frac{\theta}{2}$

4. Find the angular bisectors of each angle in Fig. 2.2.1
   Solution: From the given information, the line equation of acute angular bisector of $\angle B$ in vector form is

$$\mathbf{L1} = \mathbf{B} + s(\mathbf{R1}) \tag{2.2.4.1}$$

$$\mathbf{L1} = \begin{pmatrix}0\\0\end{pmatrix} + s\begin{pmatrix}1.6\\0.8\end{pmatrix} \tag{2.2.4.2}$$

Where **R1**(from .(2.2.3)) is the direction ration of the line **L1** obtained by the formula

$$\mathbf{R1} = \frac{\mathbf{A} - \mathbf{B}}{\|A - B\|} + \frac{\mathbf{C} - \mathbf{B}}{\|C - B\|}$$

Vector form of angular bisector of $\underline{/C}$ is

$$\mathbf{L2} = \mathbf{C} + t(\mathbf{R2}) \tag{2.2.4.3}$$

$$\mathbf{L2} = \begin{pmatrix} 9 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1.316 \\ 0.948 \end{pmatrix} \tag{2.2.4.4}$$

Where **R2** is the d.r of the line **L2** obtained by the formula

$$\mathbf{R2} = \frac{\mathbf{A} - \mathbf{B}}{\|A - B\|} + \frac{\mathbf{C} - \mathbf{B}}{\|C - B\|}$$

Vector form of angular bisector of $\underline{/A}$ is

$$\mathbf{L3} = \mathbf{A} + u(\mathbf{R3}) \tag{2.2.4.5}$$

$$\mathbf{L3} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + u \begin{pmatrix} 0.294 \\ -0.352 \end{pmatrix} \tag{2.2.4.6}$$

Where **R3** is the d.r of the line **L3** obtained by the formula

$$\mathbf{R3} = \frac{\mathbf{B} - \mathbf{C}}{\|B - C\|} + \frac{\mathbf{D} - \mathbf{C}}{\|D - C\|}$$

Vector form of angular bisector of $\underline{/D}$ is

$$\mathbf{L4} = \mathbf{D} + v(\mathbf{R4}) \tag{2.2.4.7}$$

$$\mathbf{L4} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} + v \begin{pmatrix} -0.578 \\ -1.395 \end{pmatrix} \tag{2.2.4.8}$$

Where **R4** is the d.r of the line **L4** obtained by the formula

$$\mathbf{R4} = \frac{\mathbf{A} - \mathbf{D}}{\|A - D\|} + \frac{\mathbf{C} - \mathbf{D}}{\|C - D\|}$$

Here s,t,u,v are constants used to define a line in vector form, where a unique position vector is obtained for unique values of (s,t,u,v) of the respective line.

5. To find the point of intersection of the angular bisectors, equate the respective line equations. **Solution:** **E** is obtained by equating line equations **L1** and **L2**

$$\begin{pmatrix} 1.6s \\ 0.8s \end{pmatrix} = \begin{pmatrix} 9 - 1.316t \\ 0.948t \end{pmatrix} \tag{2.2.5.1}$$

$$\begin{pmatrix} 1.6s + 1.316t - 9 \\ 0.8s - 0.948t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.5.2}$$

By solving the two equations we obtain the values of s,t.
By substituting the values in **L1** we obtain **E**

$$\mathbf{E} = \begin{pmatrix} 5.3137 \\ 2.6568 \end{pmatrix}$$

F is obtained by equating line equations **L1** and **L3**

$$\begin{pmatrix} 1.6s \\ 0.8s \end{pmatrix} = \begin{pmatrix} 3 + 0.294u \\ 4 - 0.352u \end{pmatrix} \tag{2.2.5.3}$$

$$\begin{pmatrix} 1.6s - 0.294u - 3 \\ 0.8s + 0.352u - 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.5.4}$$

By solving the two equations we obtain the values of s,u.
By substituting the values in **L1** we obtain **F**

$$\mathbf{F} = \begin{pmatrix} 4.472 \\ 2.236 \end{pmatrix}$$

G is obtained by equating line equations **L3** and **L4**

$$\begin{pmatrix} 3 + 0.294u \\ 4 - 0.352u \end{pmatrix} = \begin{pmatrix} 7 - 0.578v \\ 6 - 1.395v \end{pmatrix} \tag{2.2.5.5}$$

$$\begin{pmatrix} 0.294u + 0.578v - 4 \\ -0.352u + 1.395v - 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.5.6}$$

By solving the two equations we obtain the values of u,v.
By substituting the values in **L3** we obtain **G**

$$\mathbf{G} = \begin{pmatrix} 5.119 \\ 1.460 \end{pmatrix}$$

H is obtained by equating line equations **L2** and **L4**

$$\begin{pmatrix} 9 - 1.316t \\ 0.948t \end{pmatrix} = \begin{pmatrix} 7 - 0.578v \\ 6 - 1.395v \end{pmatrix} \tag{2.2.5.7}$$

$$\begin{pmatrix} -1.316t + 0.578v + 2 \\ 0.948t + 1.395v - 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.5.8}$$

By solving the two equations we obtain the

values of t,v.
By substituting the values in **L2** we obtain **H**.

$$\mathbf{H} = \begin{pmatrix} 5.545 \\ 2.489 \end{pmatrix}$$

The values are listed in Table. 2.2.5

Derived values	
Parameter	Value
E	$\begin{pmatrix} 5.313 \\ 2.656 \end{pmatrix}$
F	$\begin{pmatrix} 4.472 \\ 2.236 \end{pmatrix}$
G	$\begin{pmatrix} 5.119 \\ 1.460 \end{pmatrix}$
H	$\begin{pmatrix} 5.545 \\ 2.489 \end{pmatrix}$

TABLE 2.2.5: Cyclic Quadrilateral EFGH

6. Draw Fig. 2.2.1.
Solution: The following Python code generates Fig. 2.2.6

```
codes/quad.py
```

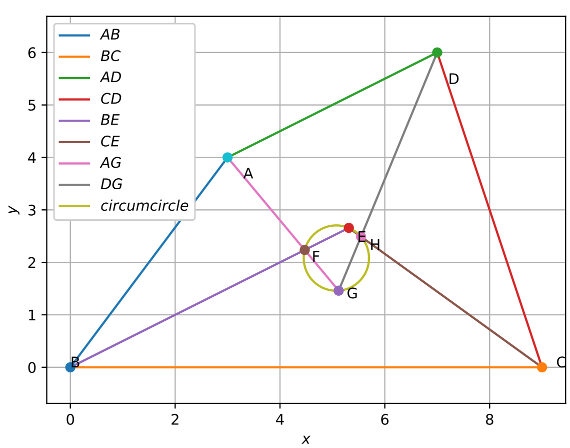

Fig. 2.2.6: Quadrilateral generated using python

and the equivalent latex-tikz code generating Fig. 2.2.1 is

```
figs/quad.tex
```

The above latex code can be compiled as a standalone document as

```
figs/quad_fig.tex
```

3 Solution

3.1. Show that $\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}$ lies on a circle.
Solution: Let **V** be a general vector that satisfies the circle equation.
Then, $\|V - C\| = r$ will be the equation,
where $\mathbf{C}, r$ are Centre of circle, and radius respectively.
Let us assume that the vectors $\mathbf{E}, \mathbf{F}, \mathbf{G}$ lie on the circle with the centre $\mathbf{C} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\|E - C\| = r \tag{3.1.1}$$
$$\|F - C\| = r \tag{3.1.2}$$
$$\|G - C\| = r \tag{3.1.3}$$

By equating .(3.1.1)=.(3.1.2) and equating .(3.1.2)=.(3.1.3), we can find the value of **C**.

$$\left\|\begin{pmatrix} 5.313 - x \\ 2.657 - y \end{pmatrix}\right\| = \left\|\begin{pmatrix} 4.471 - x \\ 2.236 - y \end{pmatrix}\right\| \tag{3.1.4}$$
$$\left\|\begin{pmatrix} 4.471 - x \\ 2.236 - y \end{pmatrix}\right\| = \left\|\begin{pmatrix} 5.119 - x \\ 1.460 - y \end{pmatrix}\right\| \tag{3.1.5}$$

By solving the above two equations we get the value of **C**
$\therefore \mathbf{C} = \begin{pmatrix} 5.075 \\ 2.081 \end{pmatrix}$
and by substituting the value of **C** in the equation .(3.1.1), we get the value of r. $\therefore r = 0.622$.
To prove that Quadrilateral EFGH is a cyclic, then H should aslo lie on the circle.
H should satisfy the general circle equation, $\|V - C\| = r$.
$\|H - C\| = r$

$$\left\|\begin{pmatrix} 5.545 - 5.075 \\ 2.489 - 2.081 \end{pmatrix}\right\| = \left\|\begin{pmatrix} 0.470 \\ 0.408 \end{pmatrix}\right\| = 0.622 \tag{3.1.6}$$

As **H** satisfies the general circle equation.
$\therefore$ Quadrilateral EFGH is a cyclic quadrilateral.