Problem 8.5.19

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***Abstract*—This a document explaining a question on the concept of cyclic quadrilateral.**

Download all python codes from

```
svn co https://github.com/SiddharthPh/
    Summer2020/trunk/geometry/Circle/codes
```

and latex-tikz codes from

```
svn co https://github.com/SiddharthPh/
    Summer2020/trunk/geometry/Circle/figs
```

1 Problem

Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadri- lateral is cyclic.

2 Construction

2.1 Constructing QuadrilateralABCD

1. Constructing quadrilateral $ABCD$:
   Solution: The design parameters for constructing the quadrilateral ABCD are given in the Table. 2.1.1.

Input Values	
Parameters	Values
a	9
b	6.324
c	4.472
d	5
e	9.219

TABLE 2.1.1: Parameters for Quadrilateral ABCD

$$BC = a = 4.5, CD = b = 5.5, ADc = 4, \tag{2.1.1.1}$$

$$AB = d = 6, BD = e = 7 \tag{2.1.1.2}$$

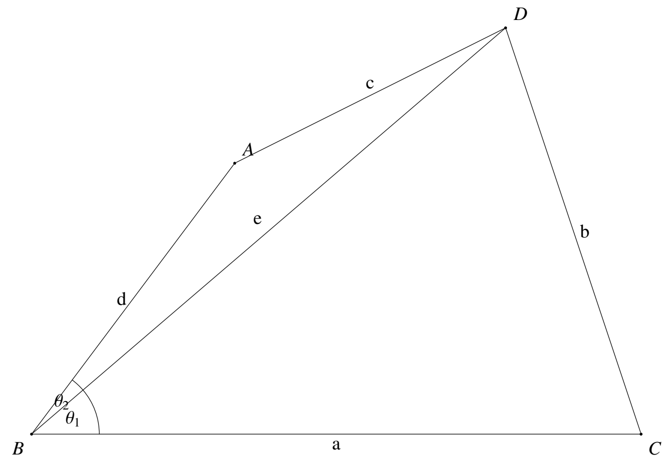

Fig. 2.1.1: QuadrilateralABCD by Latex-Tikz

Solution: The angles θ_1 and θ_2 in Fig. **??** are calculated using the cosine formula as

$$\cos\theta_1 = \frac{a^2 + e^2 - b^2}{2ae} \tag{2.1.1.3}$$

$$\cos\theta_2 = \frac{d^2 + e^2 - c^2}{2de} \tag{2.1.1.4}$$

The coordinates are then obtained as

$$\mathbf{A} = d\begin{pmatrix}\cos(\theta_1 + \theta_2)\\ \sin(\theta_1 + \theta_2)\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}a\\0\end{pmatrix},$$

$$\mathbf{D} = e\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix} \tag{2.1.1.5}$$

2. The values of A,B,C,D are shown in the Table .2.2.4
3. Draw Fig. 2.2.5.
   Solution: The following Python code generates Fig. 2.2.5

```
codes/quad1.py
```

and the equivalent latex-tikz code is

```
figs/quad1.tex
```

Input Values	
Parameters	Values
A	$\begin{pmatrix}3\\4\end{pmatrix}$
B	$\begin{pmatrix}0\\0\end{pmatrix}$
C	$\begin{pmatrix}9\\0\end{pmatrix}$
D	$\begin{pmatrix}7\\6\end{pmatrix}$

TABLE 2.1.2: Vertices A,B,C,D

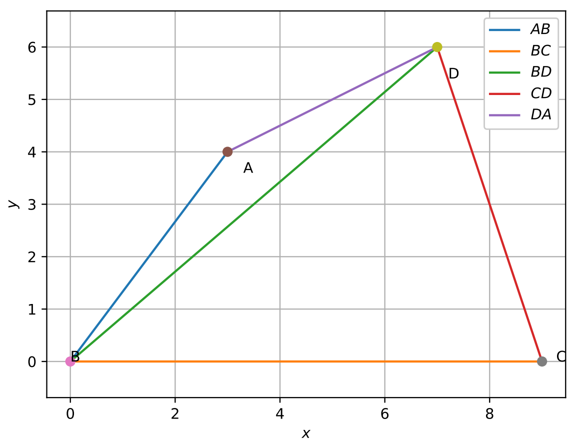

Fig. 2.1.3: QuadrilateralABCD generated using python

2.2 Cyclic QuadrilateralEFGH using angular bisectors.

1. The Figure of the quadriletral as obtained in the question looks like Fig. 2.2.1. with angles $\angle A$, $\angle C$ and $\angle B$ and $\angle D$ and sides a, b and c and d.
2. The design parameters for construction are:
Solution: See Table. 2.2.2.
3. Find the angular bisectors of each angle in Fig. 2.2.1
Solution: From the given information, the line equation of acute angular bisector of $\angle B$ in vector form is

$$\mathbf{L1} = \mathbf{B} + s(\mathbf{R1}) \tag{2.2.3.1}$$

$$\mathbf{L1} = \begin{pmatrix}0\\0\end{pmatrix} + s\begin{pmatrix}1.6\\0.8\end{pmatrix} \tag{2.2.3.2}$$

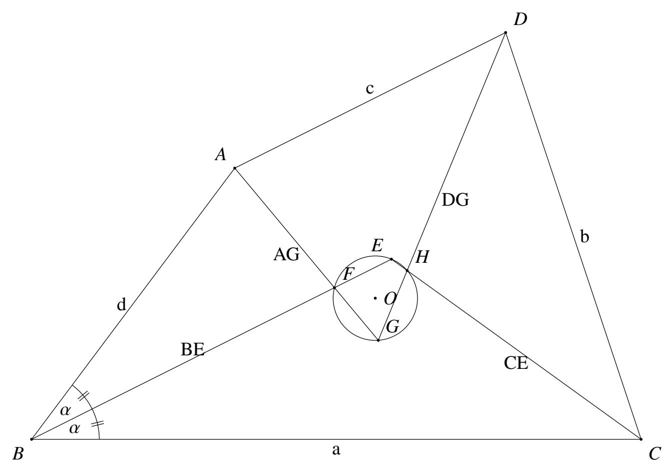

Fig. 2.2.1: Quadrilateraal by Latex-Tikz

Input Values	
Parameters	Values
A	$\begin{pmatrix}3\\4\end{pmatrix}$
B	$\begin{pmatrix}0\\0\end{pmatrix}$
C	$\begin{pmatrix}9\\0\end{pmatrix}$
D	$\begin{pmatrix}7\\6\end{pmatrix}$

TABLE 2.2.2: Quadrilateral ABCD

Where **R1** is the direction ration of the line **L1** obtained by the formula

$$\mathbf{R1} = \frac{\mathbf{A}-\mathbf{B}}{||A-B||} + \frac{\mathbf{C}-\mathbf{B}}{||C-B||}$$

Vector form of angular bisector of $\angle C$ is

$$\mathbf{L2} = \mathbf{C} + t(\mathbf{R2}) \tag{2.2.3.3}$$

$$\mathbf{L2} = \begin{pmatrix}9\\0\end{pmatrix} + t\begin{pmatrix}-1.316\\0.948\end{pmatrix} \tag{2.2.3.4}$$

Where **R2** is the d.r of the line **L2** obtained by the formula

$$\mathbf{R2} = \frac{\mathbf{A}-\mathbf{B}}{||A-B||} + \frac{\mathbf{C}-\mathbf{B}}{||C-B||}$$

Vector form of angular bisector of $\underline{/A}$ is

$$\mathbf{L3} = \mathbf{A} + u(\mathbf{R3}) \tag{2.2.3.5}$$

$$\mathbf{L3} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + u \begin{pmatrix} 0.294 \\ -0.352 \end{pmatrix} \tag{2.2.3.6}$$

Where **R3** is the d.r of the line **L3** obtained by the formula

$$\mathbf{R3} = \frac{\mathbf{B} - \mathbf{C}}{\|B - C\|} + \frac{\mathbf{D} - \mathbf{C}}{\|D - C\|}$$

Vector form of angular bisector of $\underline{/D}$ is

$$\mathbf{L4} = \mathbf{D} + v(\mathbf{R4}) \tag{2.2.3.7}$$

$$\mathbf{L4} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} + v \begin{pmatrix} -0.578 \\ -1.395 \end{pmatrix} \tag{2.2.3.8}$$

Where **R4** is the d.r of the line **L4** obtained by the formula

$$\mathbf{R4} = \frac{\mathbf{A} - \mathbf{D}}{\|A - D\|} + \frac{\mathbf{C} - \mathbf{D}}{\|C - D\|}$$

Here s,t,u,v are constants used to define a line in vector form, where a unique position vector is obtained for unique values of (s,t,u,v) of the respective line.

4. To find the point of intersection of the angular bisectors, equate the respective line equations.
Solution: **E** is obtained by equating line equations **L1** and **L2**

$$\begin{pmatrix} 1.6s \\ 0.8s \end{pmatrix} = \begin{pmatrix} 9 - 1.316t \\ 0.948t \end{pmatrix} \tag{2.2.4.1}$$

$$\begin{pmatrix} 1.6s + 1.316t - 9 \\ 0.8s - 0.948t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.4.2}$$

By solving the two equations we obtain the values of s,t.
By substituting the values in **L1** we obtain **E**

$$\mathbf{E} = \begin{pmatrix} 5.3137 \\ 2.6568 \end{pmatrix}$$

F is obtained by equating line equations **L1** and **L3**

$$\begin{pmatrix} 1.6s \\ 0.8s \end{pmatrix} = \begin{pmatrix} 3 + 0.294u \\ 4 - 0.352u \end{pmatrix} \tag{2.2.4.3}$$

$$\begin{pmatrix} 1.6s - 0.294u - 3 \\ 0.8s + 0.352u - 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.4.4}$$

By solving the two equations we obtain the values of s,u.
By substituting the values in **L1** we obtain **F**

$$\mathbf{F} = \begin{pmatrix} 4.472 \\ 2.236 \end{pmatrix}$$

G is obtained by equating line equations **L3** and **L4**

$$\begin{pmatrix} 3 + 0.294u \\ 4 - 0.352u \end{pmatrix} = \begin{pmatrix} 7 - 0.578v \\ 6 - 1.395v \end{pmatrix} \tag{2.2.4.5}$$

$$\begin{pmatrix} 0.294u + 0.578v - 4 \\ -0.352u + 1.395v - 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.4.6}$$

By solving the two equations we obtain the values of u,v.
By substituting the values in **L3** we obtain **G**

$$\mathbf{G} = \begin{pmatrix} 5.119 \\ 1.460 \end{pmatrix}$$

H is obtained by equating line equations **L2** and **L4**

$$\begin{pmatrix} 9 - 1.316t \\ 0.948t \end{pmatrix} = \begin{pmatrix} 7 - 0.578v \\ 6 - 1.395v \end{pmatrix} \tag{2.2.4.7}$$

$$\begin{pmatrix} -1.316t + 0.578v + 2 \\ 0.948t + 1.395v - 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.4.8}$$

By solving the two equations we obtain the values of t,v.
By substituting the values in **L2** we obtain **H**.

$$\mathbf{H} = \begin{pmatrix} 5.545 \\ 2.489 \end{pmatrix}$$

The values are listed in Table. 2.2.4

5. Draw Fig. 2.2.1.
Solution: The following Python code generates Fig. 2.2.5

```
codes/quad.py
```

and the equivalent latex-tikz code generating Fig. 2.2.1 is

```
figs/quad.tex
```

Derived values	
Parameter	Value
E	$\begin{pmatrix} 5.313 \\ 2.656 \end{pmatrix}$
F	$\begin{pmatrix} 4.472 \\ 2.236 \end{pmatrix}$
G	$\begin{pmatrix} 5.119 \\ 1.460 \end{pmatrix}$
H	$\begin{pmatrix} 5.545 \\ 2.489 \end{pmatrix}$

TABLE 2.2.4: Cyclic Quadrilateral EFGH

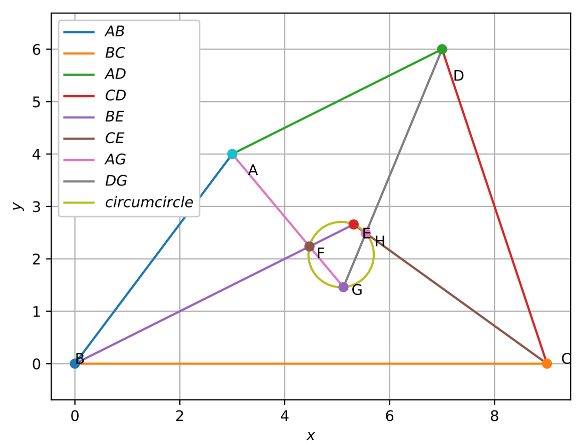

Fig. 2.2.5: Quadrilateral generated using python

The above latex code can be compiled as a standalone document as

figs/quad_fig.tex

3 Solution

3.1. To Prove EFGH is a cyclic quadrilateral, we have to show that the sum of one pair of opposite side is $180°$
In $\triangle AEB$

a) $\angle ABE + \angle BAE + \angle AEB = 180°$
(Angle Sum Property of a triangle)

b) $\angle AEB = 180° - \angle ABE - \angle BAE$

c) $\angle AEB = 180° - \frac{1}{2}(\angle B + \angle A)$
(Since BF bisects $\angle B$ and AH bisects $\angle A$)

3.2. Now, lines AH and BF intersect, so

a) $\angle FEH = \angle AEB$(Vertically Opposite angles)

b) $\therefore \angle FEH = 180° - (\frac{\angle B}{2} + \frac{\angle A}{2})$(From 3.1c)

3.3. Similarly we can prove that
$\angle FGH = 180° - (\frac{\angle C}{2} + \frac{\angle D}{2})$

3.4. Adding (3.2b) and (3.3)

a) $\angle FEH + \angle FGH = 180° - (\frac{\angle A}{2} + \frac{\angle D}{2} + 180° - (\frac{\angle C}{2} + \frac{\angle B}{2})$

b) $\angle FEH + \angle FGH = 360° - \frac{1}{2}(\angle A + \angle B + \angle C + \angle D)$
(Since ABCD is a quadrilateral, Sum of angles in it = $360°$)

c) $\angle FEH + \angle FGH = 360° - \frac{1}{2}(360°)$

d) $\angle FEH + \angle FGH = 180°$

3.5. Thus in EFGH, since the sum of one pair of opposite angles is $180°$.
$\therefore$ EFGH is a cyclic quadrilateral.