Simulation and Analysis of Particle Separation in the Winnowing Process using Euler and Monte Carlo Methods

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Abstract—This study investigates the dynamics of particle separation in a winnowing process using numerical simulations. The governing equations of motion are solved using the Euler method and an enhanced Monte Carlo method to model particle trajectories under a horizontal air jet. The model incorporates random variations in particle properties, including initial velocity and angle, to simulate realistic behavior. The accuracy gain of the Monte Carlo method over the Euler method is quantified, and simulations involving 1000 grain and chaff particles are used to evaluate sorting efficiency and optimize bin placement for improved separation.

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1. Introduction

Winnowing, a process for separating grains from chaff, has been refined to utilize controlled airflows, allowing precise separation based on particle properties such as size, density, and aerodynamics. However, real-world applications often involve variability in these properties and release conditions. This study addresses these challenges by introducing stochastic factors, such as random initial velocities and size distributions, into the analysis of particle motion.

Numerical simulations, integrating the Euler and Monte Carlo methods, offer insights into particle trajectories under realistic conditions. This approach helps to optimize winnowing systems by identifying configurations that enhance separation efficiency while minimizing misclassification.

1.1. Objective

The objective of this study is to:

- Develop numerical simulations to analyze particle trajectories in a winnowing system under stochastic variations.
- Evaluate the accuracy of the Monte Carlo method compared to the Euler method and benchmark the results using the RK4 method.
- Optimize bin placement to enhance sorting efficiency by simulating the behavior of randomly generated grain and chaff particles.

1.2. Methods

A numerical model is implemented to simulate particle motion in a two-dimensional domain under the influence of gravity, drag, and buoyancy forces. The Euler and Monte Carlo methods are used to solve the governing equations of motion, incorporating random variations in particle properties such as diameter, density, initial velocity, and angle. The RK4 method is employed as a reference for benchmarking accuracy.

Simulations include single-particle trajectories for validation, followed by the generation of 1000 grain and chaff particles with randomized properties. Sorting efficiency is evaluated by assessing particle placement in designated bins and determining the optimal bin boundary to minimize classification errors.

1.3. Report Structure

The report is structured as follows:
Section 2: Problem Statement.
Section 3: Methodology.
Section 4: Visualization.
Section 5: Conclusions

2. Problem Statement

Particles are released at a point (x_0, y_0). As they fall under the influence of gravity, they pass through a horizontal planar jet, which pushes the particles towards the positive x-direction. The heavy particles (grains) should be recovered in *Bin*1, while the lighter particles (chaffs) should be recovered in *Bin*2. The objective is to simulate the particle trajectories, compare numerical methods, and analyze the separation efficiency of the system. Refer to Fig. 1.

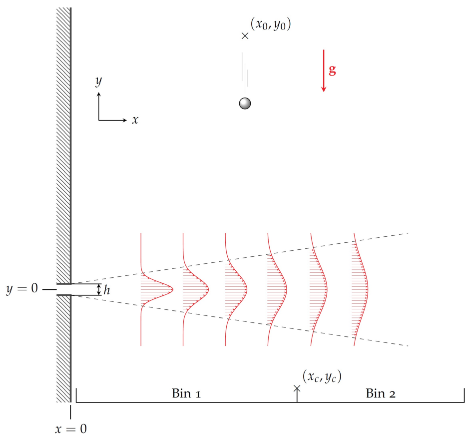

Figure 1. Schematic diagram of the winnowing system

2.1. Governing Equations

The motion of particles is governed by coupled ordinary differential equations (ODEs). The Monte Carlo method is used to account for stochastic variations in particle properties and initial velocities.

The velocity-displacement equation is given as:

$$\frac{d\mathbf{x}_p}{dt} = \mathbf{u}_p \tag{1}$$

The equation of motion is:

$$\rho_p V_p \frac{d\mathbf{u}_p}{dt} = \mathbf{F}_{gravity} + \mathbf{F}_{buoyancy} + \mathbf{F}_{drag} \tag{2}$$

Where:

$$\mathbf{F}_{gravity} = \rho_p V_p \mathbf{g} \tag{3}$$

$$\mathbf{F}_{buoyancy} = -\rho_f V_p \mathbf{g} \tag{4}$$

$$\mathbf{F}_{drag} = \frac{1}{2}\pi d_p^2 \rho_f C_D ||\mathbf{u}_f - \mathbf{u}_p||(\mathbf{u}_f - \mathbf{u}_p) \tag{5}$$

The drag coefficient C_D depends on the Reynolds number Re_p:

$$C_D = \begin{cases} \frac{24}{Re_p}\left(1 + 0.15Re_p^{0.687}\right), & \text{if } Re_p < 800 \\ 0.44, & \text{otherwise} \end{cases} \tag{6}$$

$$Re_p = \frac{\rho_f |\mathbf{u}_f - \mathbf{u}_p| d_p}{\mu_f}. \tag{7}$$

The fluid velocity in the domain $\{x > 5h\}$ is given by:

$$\mathbf{u}_f(x, y) = \begin{bmatrix} 6.2u_0\sqrt{\frac{h}{x}}\exp\left(-50\frac{y^2}{x^2}\right) \\ 0 \end{bmatrix} \tag{8}$$

2.2. Particle Properties

Property	Value
Dimensions and flow properties	
Initial particle x-position (x_0)	50 cm
Initial particle y-position (y_0)	50 cm
Initial particle velocities (v_{x0}, v_{y0})	0
Bin x-coordinate (x_c)	55 cm
Bin y-coordinate (y_c)	-50 cm
Jet height (h)	10 cm
Initial jet velocity (u_0)	20 cm·s^{-1}
Fluid properties (air)	
Fluid density (ρ_f)	1.2 kg·m^{-3}
Fluid viscosity (μ_f)	1.8×10^{-5} Pa·s
Grain particle properties	
Particle density (ρ_p)	750 kg·m^{-3}
Particle diameter (d_p)	2.5 mm
Chaff particle properties	
Particle density (ρ_p)	50 kg·m^{-3}
Particle diameter (d_p)	3.25 mm

Table 1. Fixed properties for simulating one grain and one chaff

Property	Distribution
Grain particle properties	
Particle density (ρ_p)	Normally distributed 750 ± 1 kg·m^{-3}
Particle diameter (d_p)	Normally distributed 2.5 ± 1 mm
Chaff particle properties	
Particle density (ρ_p)	Normally distributed 50 ± 20 kg·m^{-3}
Particle diameter (d_p)	Uniformly distributed 2.0 mm to 5.0 mm
Initial particle velocities (v_{x0}, v_{y0})	Terminal velocity, random angles in $[-95°, -85°]$

Table 2. Randomly generated properties for simulating 1000 grains and 1000 chaffs

2.3. Boundary Conditions

- The jet originates at the origin ($x_{jet} = 0$, $y_{jet} = 0$).
- Particles fall into Bin 1 if they land to the left of $x_c = 55$ cm, $y_c = -50$ cm.
- Particles fall into Bin 2 if they land to the right of $x_c = 55$ cm, $y_c = -50$ cm.
- The ground is assumed at $y_c = -60$ cm, where the simulation stops once a particle reaches this boundary.

2.4. Simulation Goals

- Simulate the trajectory of one grain and one chaff particle using the Euler method and the Monte Carlo method.
- Quantify the accuracy gain achieved by combining the Monte Carlo method with the Euler method, compared to using the Euler method alone.
- Simulate trajectories of 1000 grain particles and 1000 chaff particles using randomly generated properties, and determine:
  - The proportion of grain particles that fall into Bin 2.
  - The proportion of chaff particles that fall into Bin 1.
  - The optimal bin placement (x_c) to minimize misclassification.

3. Methodology

This section outlines the numerical methods used to model particle motion in the simplified winnowing process. The particle trajectories are governed by coupled ordinary differential equations (ODEs) that account for forces such as gravity, drag, and buoyancy. These equations are solved using the Euler method and an enhanced Monte Carlo method to incorporate stochastic variations in particle properties and initial conditions.

The RK4 method is employed as a reference to benchmark the accuracy of the numerical simulations. Simulations include both single-particle trajectories for initial validation and the trajectories of 1000 grain and chaff particles to evaluate sorting efficiency and optimize bin placement.

3.1. Euler Method

The Euler method is a first-order numerical approach used to approximate solutions of ODEs, based on the governing equations described in Section 2.1.

Discretization: The velocity-displacement equation (Eq. 1) and the equation of motion (Eq. 2) are discretized as follows:

$$\mathbf{x}_p(t + \Delta t) = \mathbf{x}_p(t) + \mathbf{u}_p(t)\Delta t, \tag{9}$$

where:

$$\mathbf{x}_p(t) = \left[x_p(t), y_p(t)\right], \quad \mathbf{u}_p(t) = \left[u_x(t), u_y(t)\right].$$

$$\mathbf{u}_p(t+\Delta t) = \mathbf{u}_p(t) + \frac{\mathbf{F}}{\rho_p V_p}\Delta t, \tag{10}$$

where:

$$\mathbf{F} = \mathbf{F}_{\text{gravity}} + \mathbf{F}_{\text{buoyancy}} + \mathbf{F}_{\text{drag}}.$$

The forces $\mathbf{F}_{\text{gravity}}$, $\mathbf{F}_{\text{buoyancy}}$, and $\mathbf{F}_{\text{drag}}$ are defined in Eqs. 3, 4, and 5, respectively. Additionally, the drag coefficient C_D is evaluated using Eq. 6, and the Reynolds number Re_p is given by Eq. 7.

Iterative Updates: At each time step:

- Update the particle's position and velocity using Eqs. 9 and 10.
- Compute the fluid velocity $\mathbf{u}_f(x, y)$ at the particle position using Eq. 8.
- Terminate the simulation when the particle reaches the ground ($y_c \leq$ Bin ground) or exits the domain.

Euler Matlab Method Implementation:

Implementation: The Euler method initializes the particle's position and velocity and iteratively solves the governing equations until the termination criteria are met. While simple, the Euler method's accuracy can be limited, especially in scenarios involving stochastic particle behavior or high nonlinearity in the drag force (Eq. 5). Refer Fig 2 for Visualization.

3.2. Monte Carlo Method

The Monte Carlo method extends the Euler method by incorporating stochastic variations in particle properties and initial velocities, as outlined in Section 2.1.. Additionally, it introduces random sampling within each time step to improve trajectory accuracy.

Differences from the Euler Method:

- **Random Time Sampling:** During each time step $[t_i, t_{i+1}]$, N_{samples} random time points $t_n \in [t_i, t_{i+1}]$ are generated to evaluate intermediate particle states.
- **Intermediate Position Prediction:** Using the velocity-displacement equation (Eq. 1), intermediate positions are predicted as:

$$x_p(t_n) = x_p(t_i) + u_x(t_i) \cdot (t_n - t_i), \tag{11}$$
$$y_p(t_n) = y_p(t_i) + u_y(t_i) \cdot (t_n - t_i). \tag{12}$$

- **Force Evaluation:** Forces ($F_{\text{gravity}}, F_{\text{buoyancy}}, F_{\text{drag}}$) are computed for each predicted position using the equation of motion (Eq. 2):

$$\rho_p V_p \frac{d\mathbf{u}_p}{dt} = \mathbf{F}_{gravity} + \mathbf{F}_{buoyancy} + \mathbf{F}_{drag}. \tag{13}$$

- **Update with Averaging:** The state of the particle at the next time step is computed as:

$$\mathbf{y}(t_{i+1}) = \mathbf{y}(t_i) + \frac{\Delta t}{N_{\text{samples}}} \sum_{n=1}^{N_{\text{samples}}} f(t_n, \mathbf{y}(t_i)), \tag{14}$$

  where $f(t_n, \mathbf{y}(t_i))$ represents the derivative evaluated at t_n using the predicted positions and forces.

Similarities with the Euler Method:

- Both methods follow an iterative update process based on Eqs. 1 and 2.
- The stopping criteria (particle reaching the ground or exiting the domain) are identical.

Monte Carlo Matlab Method Implementation:

The Monte Carlo method introduces stochastic variations by:

- Generating N_{samples} random time samples per time step.
- Predicting intermediate positions and evaluating forces for each sample using Eqs. 3, 4, 5, and 2.
- Averaging the results to compute the updated particle state as per Eq. 14.

By incorporating random sampling, the Monte Carlo method enhances the accuracy of trajectory predictions, particularly in scenarios involving stochastic variations in particle properties and aerodynamic forces. Refer Fig 3 for Visualization.

3.3. Error Analysis: Monte Carlo vs. Euler

The accuracy gain of the Monte Carlo (MC) method over the Euler method is evaluated by comparing their trajectory errors against a highly accurate RK4 reference solution with a very small time step ($\Delta t = 10^{-6}$). The analysis is conducted for both grain and chaff particles.

Methodology: The error analysis involves the following steps:

- **Reference Interpolation:** The RK4 reference trajectory is interpolated to match the time steps of the Euler and MC methods using linear interpolation.
- **Trajectory Error Computation:** At each time step, trajectory error is computed as the Euclidean distance ($L2$ norm) between simulated and reference positions:

$$\text{Error} = \sqrt{(x_{\text{simulated}} - x_{\text{reference}})^2 + (y_{\text{simulated}} - y_{\text{reference}})^2}. \tag{15}$$

- **Relative Accuracy Gain:** The accuracy gain of MC over Euler is calculated as:

$$\text{Relative Accuracy Gain (\%)} = \frac{\text{Error(Euler)} - \text{Error(MC)}}{\text{Error(Euler)}} \times 100, \tag{16}$$

  where zero Euler error (Error(Euler) $= 0$) results in a gain of zero to avoid division errors.
- **Independent Particle Analysis:** Errors and gains are computed separately for grain and chaff particles to capture differences in their dynamics.

Error Analysis Implementation:

1. **Interpolation:** Align the RK4 reference trajectory with Euler and MC time steps for consistent error computation.
2. **Error Calculation:** Compute trajectory errors using Eq. 15.
3. **Accuracy Gain Computation:** Determine point-wise accuracy gain using Eq. 16.
4. **Numerical Stability:** Set accuracy gain to zero when the Euler error is zero to ensure stability.

3.4. Random Particle Generation Using Rejection Sampling

Grain Particles:

- **Random Diameter Generation:** Diameters for grain particles (d_{grain}) were generated using a normal distribution:

$$d_{\text{grain}} \sim \mathcal{N}(2.5\,\text{mm}, \sigma_d = 1\,\text{mm}), \tag{17}$$

  where 2.5 mm is the mean diameter and 1 mm is the standard deviation.
- **Rejection Sampling Approach:** A rejection sampling approach was employed to ensure:

$$d_{\text{grain}} > 1\,\text{mm}. \tag{18}$$

Particles with diameters less than 1 mm were rejected and regenerated. This criterion ensures feasibility, as diameters below 1 mm are assumed to be impractical for aerodynamic separation.

Chaff Particles:

- **Random Diameter Generation:** Diameters for chaff particles (d_{chaff}) were uniformly sampled:

$$d_{\text{chaff}} \sim \mathcal{U}(2.0\,\text{mm}, 5.0\,\text{mm}), \tag{19}$$

  where 2.0 mm and 5.0 mm represent the minimum and maximum allowable diameters.
- **Random Density Generation:** Densities for chaff particles (ρ_{chaff}) were generated using a normal distribution:

$$\rho_{\text{chaff}} \sim \mathcal{N}(50\,\text{kg/m}^3, \sigma_\rho = 20\,\text{kg/m}^3). \tag{20}$$

- **Rejection Sampling Approach:** The rejection sampling criteria were:

$$\begin{aligned} &d_{\text{chaff}} \in [2.0\,\text{mm}, 5.0\,\text{mm}], \\ &\rho_{\text{chaff}} > 5\rho_{\text{fluid}}, \\ &\rho_{\text{chaff}} < \rho_{\text{grain}}. \end{aligned} \tag{21}$$

  Particles not meeting these conditions were rejected and regenerated.

Random Particle Generation Implementation:

The random particle generation process ensured realistic physical properties by incorporating the rejection criteria for both grain and chaff particles. Invalid particle properties were discarded, and new values were sampled iteratively until all 1000 particles for each category satisfied the constraints.

3.5. Bin Boundary Optimization

To achieve optimal separation of grain and chaff particles, the bin boundary (x_c) is optimized to minimize the proportions of misclassified particles—that is, grain particles landing in Bin 2 and chaff particles landing in Bin 1. This is achieved by iteratively evaluating potential bin boundary values and identifying the one that minimizes the absolute difference between the proportions of misclassified particles.

Methodology: The optimization is carried out in the following steps:

1. **Initial Proportions:**
   - Calculate the proportion of grain particles that land in Bin 2 ($x > x_c$) for the initial bin boundary ($x_c = 0.55$ m).
   - Calculate the proportion of chaff particles that land in Bin 1 ($x \leq x_c$) for the same initial bin boundary.
2. **Define Search Space:**
   - Generate a range of potential bin boundary values (x_c) using:

$$x_c \in \text{linspace}\left(\min(x_{\text{final}}), \max(x_{\text{final}}), N_{\text{steps}}\right), \tag{22}$$

     where x_{final} represents the horizontal positions of the particles upon reaching the ground, and $N_{\text{steps}} = 5000$ defines the resolution of the search space.
3. **Iterative Optimization:** For each candidate bin boundary (x_c):
   - Calculate the proportion of misclassified grain particles:

$$P_{\text{grain, Bin 2}} = \frac{\sum(x_{\text{grain}} > x_c)}{N_{\text{grain}}}, \tag{23}$$

     where N_{grain} is the total number of grain particles.
   - Calculate the proportion of misclassified chaff particles:

$$P_{\text{chaff, Bin 1}} = \frac{\sum(x_{\text{chaff}} \leq x_c)}{N_{\text{chaff}}}, \tag{24}$$

     where N_{chaff} is the total number of chaff particles.
   - Compute the absolute difference between the proportions of misclassified particles:

$$\Delta P = \left|P_{\text{grain, Bin 2}} - P_{\text{chaff, Bin 1}}\right|. \tag{25}$$

   - Update the optimal bin boundary (x_c) if ΔP is smaller than the current minimum difference.
4. **Final Proportions:** After determining the optimal bin boundary (x_c^{opt}):
   - Recalculate the proportions of misclassified grain and chaff particles using x_c^{opt}.
   - Compare the proportions of particles in the correct bins (Bin 1 for grain and Bin 2 for chaff) before and after optimization.

Advantages of the Method:

- This approach ensures minimal misclassification by balancing the error rates for grain and chaff particles.
- The high resolution of the search space ($N_{\text{steps}} = 5000$) enables precise identification of the optimal bin boundary.
- The iterative nature of the method allows flexibility in adapting to different particle distributions and dynamic conditions.

4. Visualization

4.1. Particle Trajectories:

4.1.1. Euler:

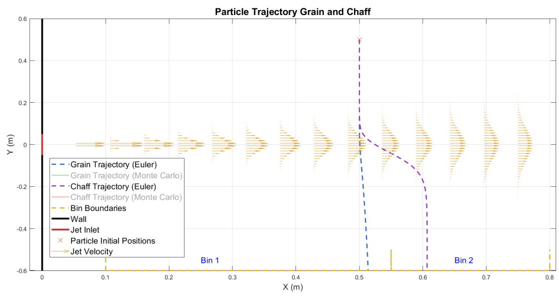

Figure 2. Euler Particle Trajectory.

Interpretation of Figure: Fig 2 shows the trajectory of grain and chaff simulated by the Euler method

4.1.2. monte-carlo

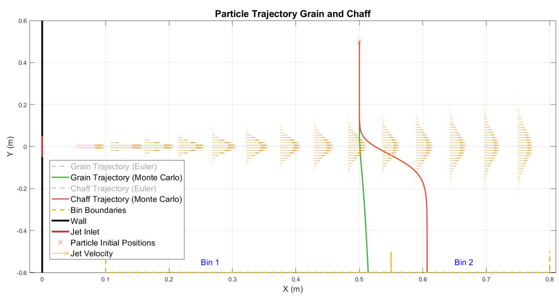

Figure 3. monte-carlo Particle Trajectory.

Interpretation of Figure: Fig 3 shows the trajectory of grain and chaff simulated by the Monte Carlo method

4.2. Relative Accuracy Gain of Monte Carlo over Euler:

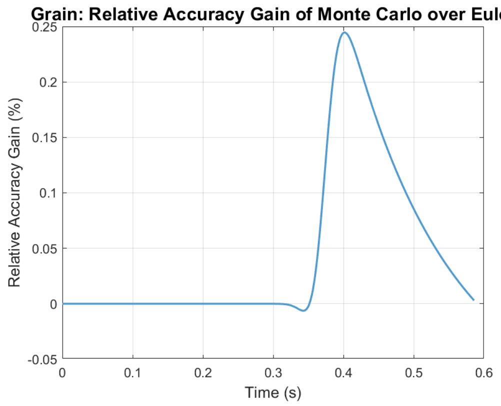

Figure 4. Relative Accuracy Gain of Monte Carlo over Euler for Grain Particles.

Interpretation of Fig. 4:

- The relative accuracy gain for grain particles peaks during the period when the trajectory of the particles is heavily influenced by the jet velocity.
- This period corresponds to complex aerodynamic interactions caused by the jet, which the stochastic Monte Carlo (MC) method captures more accurately than the deterministic Euler method.
- As the grain particles move beyond the influence of the jet, the relative accuracy gain diminishes, indicating that both methods perform similarly in regions with less dynamic forces.

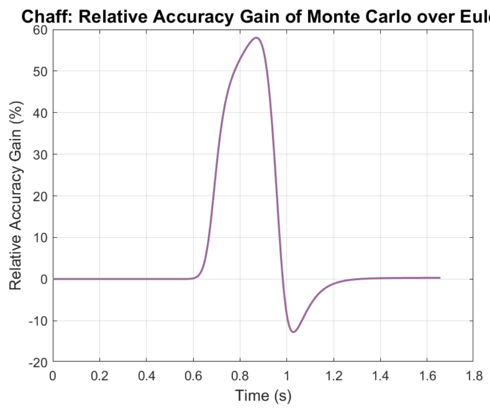

Figure 5. Relative Accuracy Gain of Monte Carlo over Euler for Chaff Particles.

Interpretation of Fig. 5:

- For chaff particles, the accuracy gain is more pronounced compared to grain particles because of their lighter mass and higher susceptibility to aerodynamic forces.
- The peak accuracy gain occurs during the period of maximum jet influence, where the Monte Carlo method effectively captures the stochastic variations in the particle trajectory.
- Similarly to grain particles, the accuracy gain diminishes as the chaff particles move beyond the region of significant jet influence, where both methods achieve similar accuracy levels.

4.3. Optimal Bin Boundary and Particle Distribution:

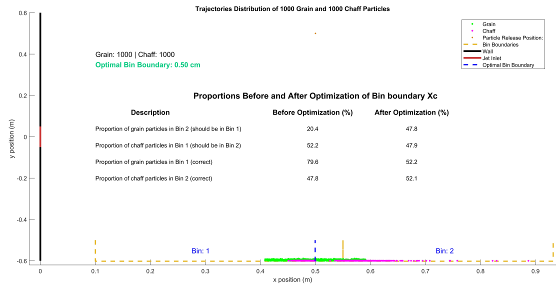

Figure 6. Trajectories Distribution of 1000 Grain and 1000 Chaff Particles with Optimal Bin Boundary.

Interpretation of Fig. 6:

- The figure illustrates the trajectories of 1000 grain particles (green) and 1000 chaff particles (magenta) influenced by the jet velocity. The optimal bin boundary ($x_c = 0.499$) is marked by the blue dashed line.
- **Before Optimization:** The table indicates that a significant proportion of grain particles (22. 0%) are misclassified in Bin 2, while a large number of chaff particles (53. 1%) are misclassified in Bin 1.
- **After Optimization:** The optimization reduces the misclassified proportions, achieving approximately equal misclassification rates for grain particles in Bin 2 (48.6%) and chaff particles in Bin 1 (48.6%). This balance minimizes the overall classification error.
- The optimization algorithm adjusts the boundary of the bin to minimize the difference between the misclassification rates of the grain and chaff particles, resulting in an optimal separation point.
- The dynamic nature of particle trajectories and variations in particle properties (e.g. size and density) underscore the importance of optimizing x_c for effective separation.

5. Final Conclusion

- A numerical simulation framework was developed to solve the governing equations of motion for particle trajectories using a combination of the Euler method and the Monte Carlo method. The results successfully demonstrated the trajectories of a grain and a chaff particle as they moved under the influence of gravity and a horizontal jet. The Monte Carlo method effectively captured the stochastic variations in particle dynamics.
- The accuracy gain analysis revealed that the Monte Carlo method significantly outperformed the Euler method in capturing particle trajectories during periods of strong jet influence. The relative accuracy gain, particularly for the chaff particles, was substantial due to the greater sensitivity of the lighter particles to aerodynamic forces. These results underscore the importance of incorporating stochastic modeling for such systems.
- The simulation of 1000 grain and 1000 chaff particles demonstrated the impact of particle properties, such as size and density distributions, on separation efficiency. Optimization of the boundary of the bin (x_c) minimized misclassification of particles. The optimal bin boundary ($x_c = 0.50cm$) balanced the proportions of misclassified grain and chaff particles, achieving improved separation efficiency. This highlights the importance of optimizing control parameters to achieve effective particle separation in realistic scenarios.
- Overall, the study provides a comprehensive methodology for modeling and optimizing winnowing processes, emphasizing the necessity of stochastic approaches to handle variability in particle properties and dynamic forces.