Solution

Hi! For those struggling with the CompStat Lab 6 Question 2 (the infamous EM Algorithm), I can provide some help. After a lot of struggling and then some helpful advice from Bayu and Héctor, I arrived at the following structure/solution which when implemented did in fact converge, to $\hat{\lambda} \approx 1$. I have a lot to do at the moment, so I am only to a *limited* extent available to answer questions you might have.

Also, please *do not* outright copy this explanation or any of its contents, it is completely possible that I have made mistakes in the equations and formulas, and specifically *I bear no responsibility for any mistakes or issues you may make when implementing this solution, and if you read this then I assume that you respect that.* Please be cautious, because I have in fact derived most of these formulas by myself with pen and paper.

Let *un*-censored observations be noted by x_i for $i = 1, \cdots, n$. Let the censored observations be noted by y_i where $i = 1, \cdots, m$. Let the *true* underlying fail times be noted by z_i where $i = 1, \cdots, m$. In other words: "Mom can I have z_i? No we have z_i at home. z_i at home: y_i". We now consider the independent random variables $Z_i|Y = y_i$ (in accordance with lab instructions) as coming from Truncated exponential distributions, e.g.

$$Z|Y = y_i \sim \text{TruncExp}(\lambda, 0, y_i)$$

with density

$$g(z|y) = \begin{Bmatrix} \frac{\lambda \exp(-\lambda z)}{1-\exp(-\lambda y)} & 0 \leq z \leq y \\ 0 & \text{otherwise} \end{Bmatrix}.$$

The Likelihood of our observations comes out to

$$L(\lambda; X, Z|Y) = \left(\prod_{i=1}^{n} f(x_i) \right) \cdot \left(\prod_{i=1}^{m} g(Z_i|y_i) \right).$$

Then we can formulate the Q-function in the following way, where the star in the expectations E_* indicate that we take the expectation with respect to the fact that $Z|Y = y_i \sim \text{TruncExp}(\lambda, 0, y_i)$.

$$Q(\lambda, \lambda^{(t)}) = E_*[\log L(\lambda; X, Z|Y)] = n \cdot \ln \lambda - \lambda n \bar{x} + m \ln \lambda - \lambda \cdot \sum_{i=1}^{m} E_*[Z|Y = y_i] - \sum_{i=1}^{m} \ln \Big(1 - \exp(-\lambda y_i) \Big).$$

The expectation in the expression above can formulated in the following way

$$E_*[Z|Y = y_i] = \frac{1 - \exp(-\lambda^{(t)} y_i) - y_i \lambda^{(t)} \exp(-\lambda^{(t)} y_i)}{\lambda^{(t)} \Big(1 - \exp(-\lambda^{(t)} y_i) \Big)} = \frac{1}{\lambda^{(t)}} - y_i \Big(\exp(\lambda^{(t)} y_i) - 1 \Big)^{-1}$$

Please note the difference between λ and $\lambda^{(t)}$. The former is present outside the expectation, and the latter is only present within the expectation of $Z|Y = y_i$.

All right, that is it for me, good luck understanding and implementing!

$$Q(\lambda, \lambda^{(t)}) = E_* \Big[\log L(\lambda; X, Z|Y) \Big] = E_* \Big[\log \Big(\prod_{i=1}^{n} f(x_i) \Big) \Big] + E_* \Big[\log \Big(\prod_{i=1}^{m} g(Z_i|y_i) \Big) \Big] =$$

$$= E\Big[\log\Big(\prod_{i=1}^{n} f(x_i)\Big)\Big] + E_*\Big[\log\Big(\prod_{i=1}^{m} g(Z_i|y_i)\Big)\Big]$$

$$Q(\lambda, \lambda^{(t)}) = n \cdot \ln\lambda - n + m\ln\lambda - \lambda \cdot \sum_{i=1}^{m} E_*[Z|Y = y_i] - \sum_{i=m}^{m} \ln\Big(1 - \exp(-\lambda y_i)\Big)$$

$$Q(\lambda, \lambda^{(t)}) \approx \ln(\lambda^2) - \frac{\lambda}{\lambda^{(t)}} + \frac{\lambda}{e^{\lambda^{(t)}}} - \sum \ln(1 - e^{-\lambda})$$