Lab 3 report

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Question 1

Question 2

We will create a function for sampling from the double exponential (Laplace) distribution with location parameter 0, and scale parameter 1, with the density being

$$g(x) = \frac{1}{2} \cdot \exp\Big(-|x|\Big)$$

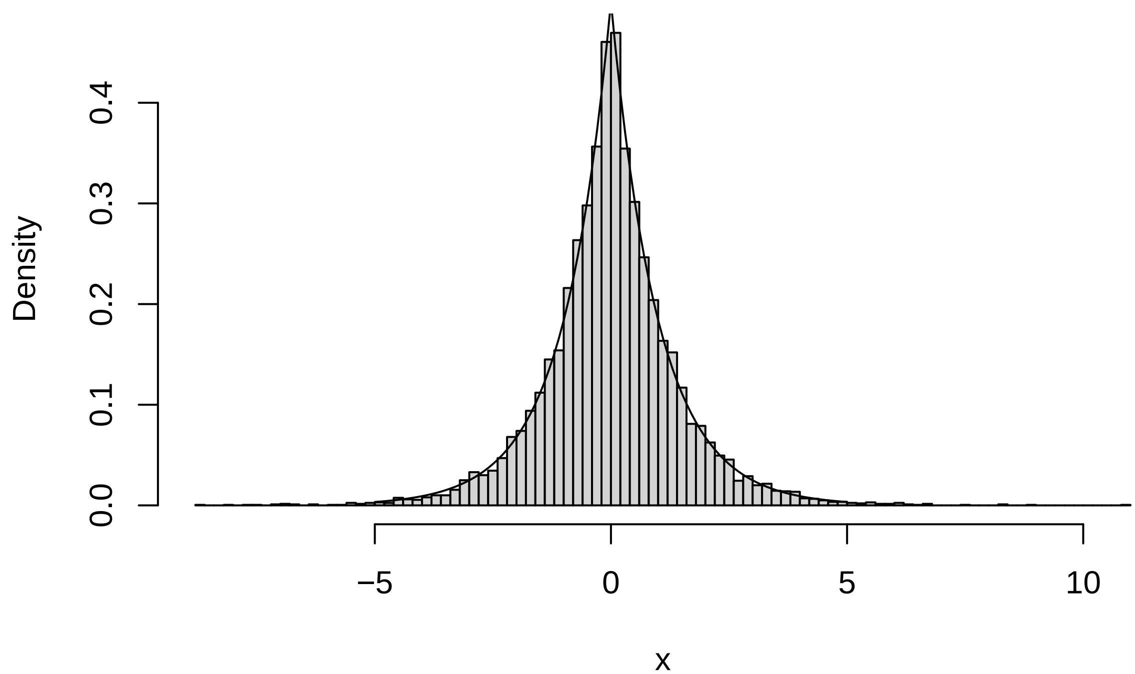
Below is our implementation `rlaplace` which samples from the Laplace(μ, λ) distribution using the *inverse distribution function (IDF) method*. The inverse distribution function $G^-1(x)$ was obtained through our reference for the Laplace distribution. As per the IDF method, `rlaplace` first generates a uniformly random number from $U(0,1)$ for each desired observation. Then the IDF is applied to each such observation, so that the resulting observations are $Laplace(0,1)$ distributed.

We then proceed to generate 10^4 approximately independent observations from `rlaplace`, and plot the result in Figure 2.1, along with the true density curve of $Laplace(0,1)$. The results are quite good, as the distribution of the simulated values closely matches the true density.

```r
# Generate laplace variates using Inverse Distribution Function (IDF)
rlaplace <- function(n=1, location=0, scale=1){
  # Generate n observations from U(0,1)
  u <- runif(n)

  # Calculate (and return) corresponding IDF values
  location - scale*sign(u - 0.5)*log(1-2*abs(u-0.5))
}
```

Fig 2.1. Density of simulated Laplace observations



We will now proceed to use the Rejection Sampling Method (RSM) (see reference) to generate standard normal variates from $N(0,1)$. For the envelope we will use the $Laplace(0,1)$ distribution, scaled appropriately by a constant a. In order to find the optimal a, we construct the following expression and then derive the optimal (maximal) a that satisfies the inequality.

$$h(x) = \frac{g(x)}{f(x)} = \frac{1/2 \cdot \exp(-x)}{1/\sqrt{2\pi} \cdot \exp(-\frac{x^2}{2})} = \sqrt{\frac{\pi}{2}} \cdot \exp\left(\frac{x^2}{2} - x\right) \geq a$$

$$h'(x) = \sqrt{\frac{\pi}{2}} \cdot (x-1) \cdot \exp\left(\frac{x^2}{2} - x\right)$$

It is clear that $h(x)$ grows very large as x approaches $\pm\infty$, and thus the lone extrema at $x = 1$ is a global minima. This implies that the optimal a is

$$a = h(1) = \sqrt{\frac{\pi}{2}} \cdot \exp\left(\frac{1^2}{2} - 1\right) \approx 0.76(\text{rounded down})$$

Our envelope is thus

$$e(x) = g(x)/a.$$

We can now proceed to simulate 2000 random numbers from $N(0,1)$ using the rejection sampling method. Below we implement this in `rnorm_rejection`. We also measure and output the average rejection rate when the function is run.

```r
# Our derived optimal a
a <- sqrt(pi/2) * exp(-1/2)

# Generate standard normal variates using rejection sampling
rnorm_rejection <- function(n=1){ #mean=0, sd=1
  output_vector <- c()

  # Keep track of total number of attempts
  n_tries <- 0

  # Run the algorithm once for every n desired values
  for (i in 1:n){
    # Repeat the sampling procedure until a value is accepted
    repeat{
      n_tries <- n_tries + 1
      u <- runif(1, 0, 1)
      Y <- rlaplace(1, 0, 1)
      ratio <- dnorm(Y) / (dlaplace(Y, 0, 1) / a)
      if(u <= ratio){output_vector <- c(output_vector, Y);break}
    }
  }
  cat("The average rejection rate:", (n_tries-n)/n_tries, "\n")
  return(output_vector)
}
```

In Figure 2.2 below, we plot 2000 standard normal variates sampled using `rnorm_rejection`, along with the true standard normal density (instead of values sampled from `rnorm`). The simulated distribution clearly resembles the standard normal, but it is not very smooth, which is desired.

```
## The average rejection rate: 0.2269037
```

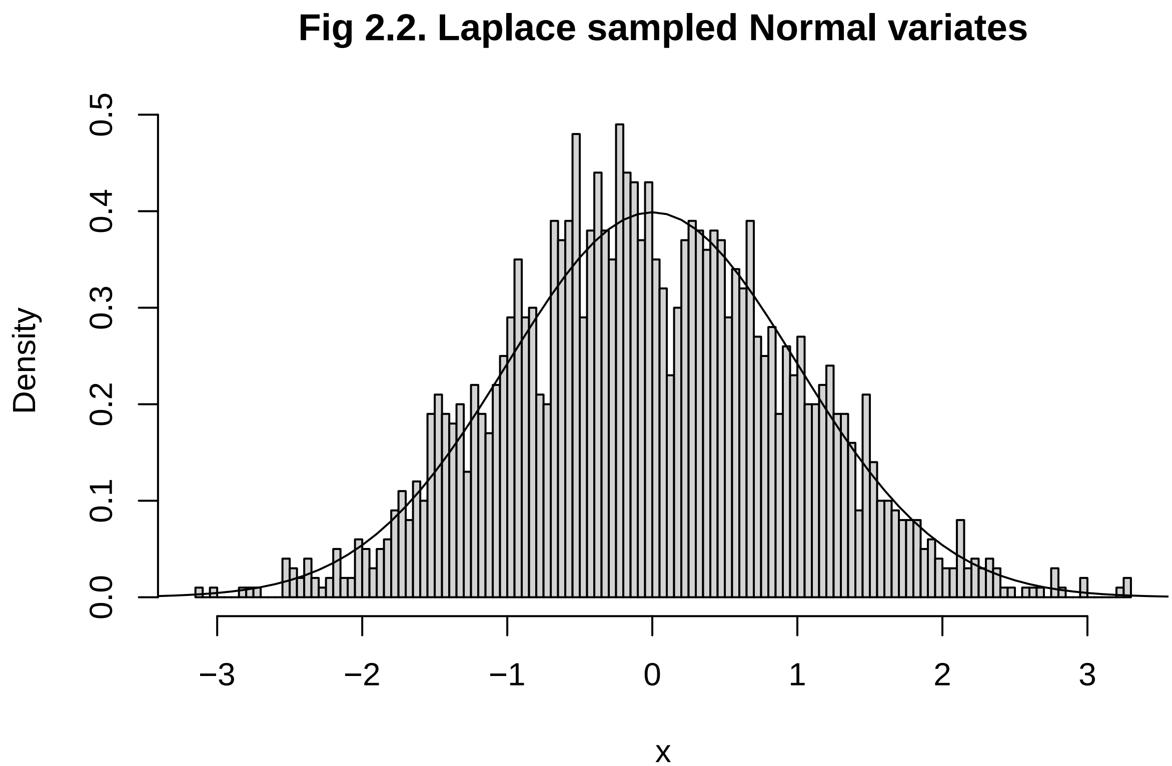


Fig 2.2. Laplace sampled Normal variates
Density
0.0
0.1
0.2
0.3
0.4
0.5
−3
−2
−1
0
1
2
3
x