# Control Systems

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## Contents



*Abstract*—**This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

## 1 Stability

### 1.1 Second order System

1.1. State the general model of a state space system specifying the dimensions of the matrices and vectors.
**Solution:** The model is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1.1.1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{1.1.2}$$

1.2. Find the transfer function $\mathbf{H}(s)$ for the general system.
**Solution:** Taking Laplace transform on both sides we have the following equations

$$s\mathbf{I}X(s) - x(0) = \mathbf{A}X(s) + \mathbf{B}U(s) \tag{1.2.1}$$

$$(s\mathbf{I} - \mathbf{A})X(s) = \mathbf{B}U(s) + x(0) \tag{1.2.2}$$

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) + (s\mathbf{I} - \mathbf{A})^{-1}x(0) \tag{1.2.3}$$

and

$$Y(s) = \mathbf{C}X(s) + D\mathbf{I}U(s) \tag{1.2.4}$$

Substituting from (1.2.3) in the above,

$$Y(s) = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\mathbf{I})U(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0) \tag{1.2.5}$$

1.3. Find $H(s)$ for a SISO (single input single output) system.
**Solution:**

$$H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + DI \tag{1.3.1}$$

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1.4. Given

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{1.4.1}$$

$$D = 0 \tag{1.4.2}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.4.3}$$

find $\mathbf{A}$ and $\mathbf{C}$ such that the state-space realization is in *controllable canonical form.*
**Solution:**

$$\because \frac{Y(s)}{U(s)} = \frac{Y(s)}{V(s)} \times \frac{V(s)}{U(s)}, \tag{1.4.4}$$

letting

$$\frac{Y(s)}{V(s)} = 1, \tag{1.4.5}$$

results in

$$\frac{U(s)}{V(s)} = s^3 + 3s^2 + 2s + 1 \tag{1.4.6}$$

giving

$$U(s) = s^3V(s) + 3s^2V(s) + 2sV(s) + V(s) \tag{1.4.7}$$

so equation 0.1.13 can be written as

$$\begin{pmatrix} sV(s) \\ s^2V(s) \\ s^3V(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{1.4.8}$$

So

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \tag{1.4.9}$$

$$Y = X_1(s) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix} \tag{1.4.10}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{1.4.11}$$

1.5. Obtain $\mathbf{A}$ and $\mathbf{C}$ so that the state-space realization in in *observable canonical form.*
**Solution:**
Given that

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{1.5.1}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{1.5.2}$$

$$Y(s) \times (s^3 + 3s^2 + 2s + 1) = U(s) \tag{1.5.3}$$

$$s^3Y(s) + 3s^2Y(s) + 2sY(s) + Y(s) = U(s) \tag{1.5.4}$$
$$s^3Y(s) = U(s) - 3s^2Y(s) - 2sY(s) - Y(s) \tag{1.5.5}$$
$$Y(s) = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{1.5.6}$$

let $Y = aU + X_1$
by comparing with equation 1.5.6 we get a=0 and

$$Y = X_1 \tag{1.5.7}$$

inverse laplace transform of above equation is

$$y = x_1 \tag{1.5.8}$$

so from above equation 1.5.6 and 1.5.7

$$X_1 = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{1.5.9}$$
$$sX_1 = -3Y(s) - 2s^{-1}Y(s) + s^{-2}(U(s) - Y(s)) \tag{1.5.10}$$

inverse laplace transform of above equation

$$\dot{x_1} = -3y + x_2 \tag{1.5.11}$$

where

$$X_2 = -2s^{-1}Y(s) + s^{-2}(U(s) - Y(s)) \tag{1.5.12}$$
$$sX_2 = -2Y(s) + s^{-1}(U(s) - Y(s)) \tag{1.5.13}$$

inverse laplace transform of above equation

$$\dot{x_2} = -2y + x_3 \tag{1.5.14}$$

where

$$X_3 = s^{-1}(U(s) - Y(s)) \tag{1.5.15}$$
$$sX_3 = U(s) - Y(s) \tag{1.5.16}$$

inverse laplace transform of above equation

$$\dot{x_3} = u - y \tag{1.5.17}$$

so we get four equations which are

$$y = x_1 \tag{1.5.18}$$
$$\dot{x_1} = -3y + x_2 \tag{1.5.19}$$
$$\dot{x_2} = -2y + x_3 \tag{1.5.20}$$
$$\dot{x_3} = u - y \tag{1.5.21}$$

sub $y = x_1$ in 1.5.19,1.5.20,1.5.21 we get

$$y = x_1 \tag{1.5.22}$$
$$\dot{x_1} = -3x_1 + x_2 \tag{1.5.23}$$
$$\dot{x_2} = -2x_1 + x_3 \tag{1.5.24}$$
$$\dot{x_3} = u - x_1 \tag{1.5.25}$$

so above equations can be written as

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{1.5.26}$$

So

$$\mathbf{A} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \tag{1.5.27}$$

$$y = x_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{1.5.28}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{1.5.29}$$

1.6. Find the eigenvaues of $\mathbf{A}$ and the roots of the characteristic equation of $H(s)$ using a python code.
**Solution:**
as we know tthat the characteristic equation is det(sI-A)

$$\mathbf{sI} - \mathbf{A} = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{pmatrix} \tag{1.6.1}$$

therfore

$$det(sI - A) = s(s^2 + 3s + 2) + 1(1) = s^3 + 3s^2 + 2s + 1 \tag{1.6.2}$$

so from equation 1.6.2 we can see that charcteristic equation is equal to the denominator of the transefer function
by evaluating the charcteristic euation in python
we get -2.32+0j,-0.33+0.56j,-0.33-0.56j as poles
which lie on negative half of s-plane
therfore system is stable

## 2 Routh Hurwitz Criterion

## 3 Compensators

## 4 Nyquist Plot