Module-II

VECTOR CALCULUS

Introduction to Vector Calculus in Computer Science & Engineering.

Vector calculus plays an important role in differential geometry and in the study of partial differential equations. Vector calculus originated in the 19th century in connection with the needs of mechanics and physics, when operations on vectors began to be performed directly, without their previous conversion to coordinate form. More advanced studies of the properties of mathematical and physical objects which are invariant with respect to the choice of coordinate systems led to a generalization of vector calculus. It is used extensively in physics and engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow.

Topic Learning Objectives

Upon Completion of this module, student will be able to:

- Understand the existence of vector functions and derivatives of vector.
- The importance of defining vector differential operator ∇ and the operations- Gradient of scalar point functions, Divergence and Curl of vector point functions.
- Understand the fundamentals of the integration of vector point function.
- Solve line, surface and volume integrals.
- Apply Green's Theorem, Stokes' Theorem and Divergence Theorem in solving engineering problems.
- Estimate and apply the concepts of solenoidal and irrotational fields to calculate integrals of vector functions.

Vector Fields

If at each point (x, y, z) there is an associated vector $\vec{v}(x,y,z) = v_1(x,y,z)i + v_2(x,y,z)j + v_3(x,y,z)k$, then $v(x,y,z)$ is a vector function and the field processing such a vector function is called a vector field.

Note: In all the vectors wherever i, j, k are used they have to be treated as unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ along x, y, z directions respectively.

Examples

(i) A magnetic field B in a region of space, $B = B_1 i + B_2 j + B_2 k$

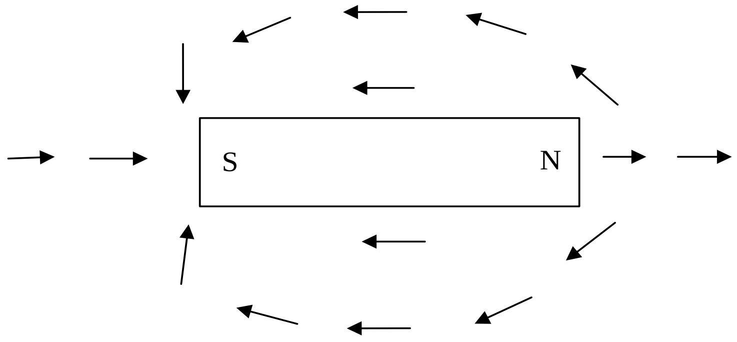

Fig 2. 1 Magnetic Field

(ii) The velocity field of water flowing in a pipe, $v(x, y, z)$.

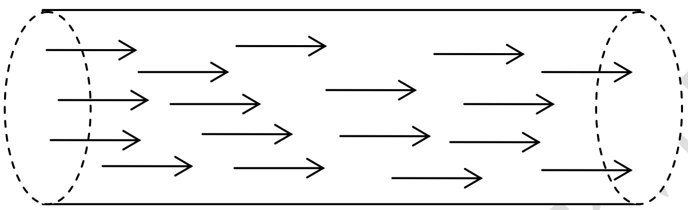

Fig 2.2 Vector Field

Vector function is a function whose domain is set of real numbers and whose range is a set of vectors.

Vector Differentiation

Differentiation of a Vector Function

Let the position vector of a point P (x, y, z) in space be $\vec{r}(t) = x\, i + y\, j + z\, k.$
If *x, y, z* are all functions of *t*, then ***r*** is said to be a vector function of *t*. As the parameter *t* varies the point P traces a curve in space. Therefore $\vec{r}$ (t) = x(t)i + y(t)j + z(t) k is the vector equation of the curve, where *x(t), y(t)*and *z(t)* are real functions of the real variable *t*.

This function can be viewed as describing a space curve. Intuitively it can be regarded as a position vector, expressed as a function of '*t*' that traces out a space curve with increasing values of *t*.

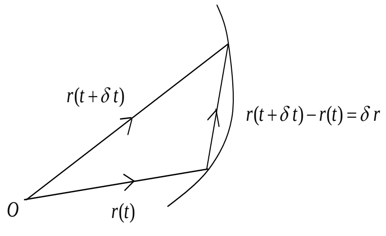

Fig 2.3 Position Vector

If $\vec{r}(t) = x(t)i + y(t)j + z(t)k$ is a vector function of a scalar variable *t* then the derivative of $\vec{r}(t)$ with respect to *t* is

$$\frac{d}{dt}\vec{r}(t) = \lim_{\delta t \to 0}\left[\frac{\vec{r}(t+\delta\, t) - \vec{r}(t)}{\delta\, t}\right]$$
$$= \lim_{\delta t \to 0}\frac{x(t+\delta\, t) - x(t)}{\delta\, t} i + \lim_{\delta t \to 0}\frac{y(t+\delta\, t) - y(t)}{\delta\, t} j + \lim_{\delta t \to 0}\frac{z(t+\delta\, t) - z(t)}{\delta\, t} k$$
$$= \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k$$

- For example, suppose you were driving along a wiggly road with position *r (t)* at time *t*.
- Differentiating *r (t)* should give velocity *v (t)*.
- Differentiating *v (t)* should yield acceleration *a (t)*.
- Differentiating *a (t)* should yield the jerk *j (t)*.

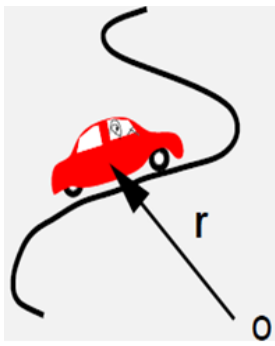

Fig 2.4 Displacement

Velocity and Acceleration

If $\vec{r}\,(t) = x(t)i + y(t)\,j + z(t)\,k$ is the position vector of a particle moving along a smooth curve in space, then $v(t) = \frac{d\vec{r}}{dt}$ is the particle's **velocity vector**, tangent to the curve. At any time *t*, the direction of ***v(t)*** is the **direction of motion**, the magnitude of ***v(t)*** is the particle's **speed**, and the derivative $a(t) = \frac{dv}{dt}$, when it exists, is the particle's **acceleration vector**.

In summary,

- Velocity is the derivative of position vector: $v(t) = \frac{d\vec{r}}{dt}$
- Speed is the magnitude of velocity: $speed = |v(t)|$
- Acceleration is the derivative of velocity: $a(t) = \frac{dv}{dt} = \frac{d^2\vec{r}}{dt^2}$
- Unit Tangent vector $\hat{T} = \frac{v(t)}{|v(t)|}$ is the direction of motion at time *t*.
- Component of velocity along a given vector $\vec{C}$ is $v(t) \cdot \hat{C}$
- Component of acceleration along a given vector $\vec{C}$ is $a(t) \cdot \hat{C}$

Scalar and Vector Point Functions

A physical quantity that can be expressed as a continuous function and which can assume definite values at each point of a region of space is called a point function in that region, and the region containing the point function is called a field.

There are two types of point functions namely **scalar point function** and **vector point function**.

Scalar point function

At each point (x, y, z) of a region R in space if there corresponds a definite scalar $\phi(x, y, z)$, then such a function $\phi(x, y, z)$ is called a scalar point function and the region is called a scalar field.

Examples: Functions representing the temperature, density of a body, gravitational potential etc. are scalar point functions.

Vector point function: At each point (x, y, z) of a region R in space if there corresponds a definite vector $\vec{f}(x, y, z) = f_1(x, y, z)\mathrm{i} + f_2(x, y, z)j + f_3(x, y, z)k$, then such a function $\vec{f}(x, y, z)$ is called a vector point function, the region is called a vector field.

Examples: Functions representing the velocity of moving fluid particle, gravitational force, etc. are vector point functions.

Level surface: The scalar point function $\phi(x, y, z)$ is usually called the potential function and $\phi(x, y, z) = c$ represents the family of surfaces in the scalar field. If at each point on the surface, $\phi(x, y, z) =$ c has the same value then the surface is called the level surface.

Definition: The vector differential operator denoted by ∇ read as **del** or **nabla** is defined by $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$ is called **vector differential operator**. This operator has no meaning on itself but assumes specific meaning depending on how it operates on a scalar or vector point function.

Gradient of a scalar point function: Let $\phi(x, y, z)$ be any scalar point function defined at some point (x, y, z) of a scalar field so that the function is continuously differentiable. Then the vector function $\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$ is called a gradient of scalar function $\phi(x, y, z)$ and it is denoted by $\nabla \phi$ or grad ϕ. Thus $grad\phi = \nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$.

Note:

1. If ϕ is a scalar point function, then $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$ are called components of grad ϕ

2. $|\nabla \phi| = \sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2}$ is called the magnitude of grad ϕ.

Geometrical interpretation of gradient: $grad\ \phi$ is a vector normal to the surface $\phi = constant$ and has a magnitude equal to the rate of change of f along this normal.

Properties of Gradient

1. The differential $d\varphi$ of $\varphi$is given by $d\varphi = \nabla\varphi . dr$ where $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$
2. For any scalar function φ and $\psi$and any scalar $\alpha$and β

$$\nabla(\alpha\, \varphi \pm \beta\, \psi) = \alpha\, \nabla\varphi \pm \beta\, (\nabla\psi)$$

3. For any scalar function φ and ψ

i) $\nabla(\varphi\psi) = \varphi\, \nabla\psi + \psi\, \nabla\varphi$ ii) $\nabla\left(\frac{\varphi}{\psi}\right) = \frac{(\psi\, \nabla\varphi - \varphi\, \nabla\psi)}{\psi^2}$ if $\psi \neq 0$

Unit normal vector:

Since $\nabla\varphi$ is normal vector to surface $\varphi(x,y,z) = c$ then unit vector is denoted by $\hat{n}$ and is defined as, $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\nabla\varphi}{|\nabla\varphi|}$ where $\hat{n} = \nabla\varphi$ = normal vector.

Note: The angle between the normal's to the surfaces is given by $\cos\theta = \hat{n}_1 . \hat{n}_2$.

Directional derivative:

If $\vec{a}$ is any vector incline at an angle θ to the direction of $\nabla\varphi$ where $\varphi$is scalar point function then

$$\nabla\phi.\hat{a} = \left(\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k\right) \cdot \frac{(a_1 i + a_2 j + a_3 k)}{|\vec{a}|} = a_1 \frac{\partial\phi}{\partial x} + a_2 \frac{\partial\phi}{\partial y} + a_3 \frac{\partial\phi}{\partial z}.$$

It represents component of $\nabla\varphi$ in the direction of $\vec{a}$which is known as directional derivative of φ in the direction of $\vec{a}$.

Maximum Directional Derivative:

The direction derivative will be maximum in the direction of $\nabla\varphi$ $(\vec{a} = \nabla\varphi)$ and maximum value of the directional derivative $= \frac{\nabla\varphi.\nabla\varphi}{|\nabla\varphi|} = \frac{|\nabla\varphi|^2}{|\nabla\varphi|} = |\nabla\varphi|$

Maximum directional derivative is also called normal derivative.

$\therefore$ normal derivative $= |\nabla\varphi|$

Problems

1. If $\phi(x,y,z) = xy^2z^3 - x^3y^2z$ then find $\nabla\phi$ and $|\nabla\phi|$ at (1, -1, 1).

Solution: Given $\phi(x,y,z) = xy^2z^3 - x^3y^2z$

$$\frac{\partial\phi}{\partial x} = y^2z^3 - 3x^2y^2z, \frac{\partial\phi}{\partial y} = 2xyz^3 - 2x^3yz, \frac{\partial\phi}{\partial z} = 3xy^2z^3 - x^3y^2$$

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k = \left(y^2z^3 - 3x^2y^2z\right)i + \left(2xyz^3 - 2x^3yz\right)j + \left(3xy^2z^3 - x^3y^2\right)k$$

At (1, -1, 1), $\nabla\phi = -2i + 0j + 2k$

$$|\nabla\phi| = \sqrt{(-2)^2 + 0^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

2. Find the gradient of the following scalar function $f(x,y,z) = x^2y^2 + xy^2 - z^2$ at (3,1,1)

Solution: $\nabla f = x^2y^2 + xy^2 - z^2$

$$grad\ f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = (2xy^2 + y^2)i + (2x^2\mathrm{y} + 2\mathrm{xy})\mathrm{j} + (-2\mathrm{z})\mathrm{k}$$

At (3, 1, 1), $\nabla f = 7i + 24j - 2k$.

3. Find the directional derivative of $\phi(x,y,z) = xy^2 + yz^3$ at (2, -1, 1) in the direction $2i + j + 2k$.

Solution: $\phi(x,y,z) = xy^2 + yz^3$

$$grad\phi = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (y^2)i + (2xy + z^3)j + (3yz^2)k$$

At (2, -1, 1) $\nabla\phi = i - 3j - 3k$

Given $\vec{a} = (2i + j + 2k) \Rightarrow |\vec{a}| = 3$

$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}(2i + j + 2k)$

The directional derivative of $\phi(x, y, z)$ is $\nabla\emptyset.\hat{a} = (i - 3j - 3k).\frac{1}{3}(2i + j + 2k)$

The directional derivative$\nabla\emptyset.\hat{a} = \frac{1}{3}(2 - 3 - 6) = -\frac{7}{3}$.

4. Find the directional derivative of $\phi(x, y, z) = x^4 + y^4 + z^4$ at the point (-1, 2, 3) in the direction towards the point (2, -1, -1).

Solution: Given $\phi(x, y, z) = x^4 + y^4 + z^4$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = 4x^3 i + 4y^3 j + 4z^3 k$$

At (-1, 2, 3) $\nabla\phi = -4i + 32j + 108k$

Let $P = (-1,2,3)$ and $Q = (2,-1,-1)$

$\vec{a} = \overrightarrow{OQ} - \overrightarrow{OP} = (3i - 3j - 4k) \Rightarrow |\vec{a}| = \sqrt{34}$

The directional derivative is $\nabla\emptyset.\hat{a} = (-4i + 32j + 108k).\frac{1}{\sqrt{34}}(3i - 3j - 4k)$

The directional derivative$\nabla\emptyset.\hat{a} = \frac{1}{\sqrt{34}}(-12 - 96 - 432) = -\frac{540}{\sqrt{34}}$.

5. Find the unit normal vector to the surface $xy^3z^2 = 4 \;\; at \;\; (-1, -1, 2)$.

Solution: Let $f = xy^3z^2 - 4$.

$grad\, f = \nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k} = (y^3z^2)i + (3xy^2z^2)\mathrm{j} + (2xy^3\mathrm{z})\mathrm{k}$

At (3, 1, 1), $\nabla f = -4i - 12j + 4k \;\; \Rightarrow |\nabla f| = \sqrt{176}$,

The unit normal vector $\hat{n} = \frac{\nabla f}{|\nabla f|} = -\frac{(\vec{i} + 3\vec{j} - \vec{k})}{\sqrt{11}}$.

6. Find the maximum directional derivative of $\phi(x, y, z) = x^2y + yz^2 - xz^3$ at $(-1,2,1)$.

What is the greatest rate of increase of $\phi(x, y, z) = x^2y + yz^2 - xz^3$ at $(-1,2,1)$.

Solution: Given $\phi(x, y, z) = x^2y + yz^2 - xz^3$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (2xy - z^3)i + (x^2 + z^2)j + (2yz - 3xz^2)k$$

At $(-1,2,1)$ $\nabla\phi = -5i + 2j + 7k$

The maximum directional derivative = $|\nabla\phi| = \sqrt{(-5)^2 + 2^2 + 7^2} = \sqrt{78}$.

7. If the directional derivatives of $f = axy^2 + byz + cz^2x^3$at $(1,2,-1)$has a maximum magnitude of 64 units in the direction parallel to the z-axis. Find the values a, b, c.

Solution: Let $f = axy^2 + byz + cz^2x^3$.

$grad\, f = \nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k} = (ay^2 + 3cx^2z^2)i + (2axy + bz)\mathrm{j} + (by + 2czx^3)\mathrm{k}$

At (1, 2, -1), $\nabla f = (4a + 3c)i + (4a - b)j + (2b - 2c)k$

Maximum directional derivative is along ∇f and in the direction parallel to the z-axis
$\nabla f.k = 64 \Rightarrow 2b - 2c = 64.$
Also, since ∇f is parallel to the z-axis, we must have $4a + 3c = 0, 4a - b = 0.$
Solving the above three equation, we get $a = 6, b = 24, c = -8.$

8. The temperature of the points in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (1,1,1) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move?
Solution: If $T = c$ is the level surface, then ∇T gives the direction of maximum rate of change we have

$$\nabla T = 2xi + 2yj - k$$

At(1,1,2)

$$\nabla T = 2i + 2j - k$$

It should move in the direction of the unit vector normal along ∇T i.e., $\frac{1}{3}(2\vec{i} + 2\vec{j} - \vec{k})$.

9. Find the angle between the normals to the surface $2x^2 + 3y^2 = 5z$ at the points (2, -2, 4) and (-1, -1, 1).
Solution: Let $\phi(x, y, z) = 2x^2 + 3y^2 - 5z$

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\therefore \nabla\phi = 4xi + 6yj - 5k$$

Now at (2, -2, 4), $\nabla\phi_1 = 8i - 12j - 5k \Rightarrow |\nabla\phi_1| = \sqrt{8^2 + (-12)^2 + (-5)^2} = \sqrt{233}$

At (-1, -1, 1), $\nabla\phi_2 = -4i - 6j - 5k \Rightarrow |\nabla\phi_2| = \sqrt{(-4)^2 + (-6)^2 + (-5)^2} = \sqrt{77}$

Unit normal vector to the surface at (2, -2, 4) is $\hat{n}_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{1}{\sqrt{233}}(8i - 12j - 5k)$

Unit normal vector to the surface at (-1, -1, 1) is $\hat{n}_2 = \frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{1}{\sqrt{77}}(-4i - 6j - 5k)$

Angle between the normals is given by $\cos\theta = \hat{n}_1 . \hat{n}_2$

$$\cos\theta = \frac{1}{\sqrt{233}}(8i - 12j - 5k).\frac{1}{\sqrt{77}}(-4i - 6j - 5k) = \frac{1}{\sqrt{17941}}(-32 + 72 + 25) = \frac{65}{\sqrt{17941}}$$

$\theta = \cos^{-1}\left(\frac{65}{\sqrt{17941}}\right)$ is the angle between the normals.

10. Find the angle between the surfaces $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 - z = 4$ at the point (2, -1,2) common to them.
Solution: The angle between the two surfaces at common point is angle between the normals drawn to the surfaces at that point.
Let $\phi_1(x, y, z) = x^2 + y^2 + z^2$, $\nabla\phi_1 = 2xi + 2yj + 2zk$
At (2, -1, 2) $\nabla\phi_1 = 4i - 2j + 4k \Rightarrow |\nabla\phi_1| = \sqrt{4^2 + (-2)^2 + 4^2} = 6$

Now $\hat{n}_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{1}{6}(4i - 2j + 4k)$

Let $\phi_2(x,y,z) = x^2 + y^2 - z$, $\nabla\phi_2 = 2xi + 2yj - k$

At (2, -1, 2) $\nabla\phi_2 = 4i - 2j - k \Rightarrow |\nabla\phi_2| = \sqrt{4^2 + (-2)^2 + (-1)^2} = \sqrt{21}$

Now $\hat{n}_1 = \frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{1}{\sqrt{21}}(4i - 2j - k)$

Angle between the normals is $\cos\theta = \hat{n}_1 . \hat{n}_2$

$$\cos\theta = \frac{1}{6}(4i - 2j + 4k).\frac{1}{\sqrt{21}}.(4i - 2j - k) = \frac{1}{6.\sqrt{21}}(16 + 4 - 4) = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

11. Find whether the surfaces $4x^2 - z^3 = 4$ and $5x^2 - 2yz = 7x$ intersect orthogonally at the point $(1,-1,-2)$.

Solution: Let $\phi_1(x,y,z) = 4x^2 - z^3 - 4$, $\nabla\phi_1 = 8x\, i + 0\, j - 3z^2$ k

At $(1,-1,-2)$, $\nabla\phi_1 = 8i + 0j - 12k \Rightarrow |\nabla\phi_1| = \sqrt{64 + 144} = \sqrt{208}$

$$\hat{n}_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{1}{\sqrt{208}}(8i + 0j - 12k)$$

$\phi_2(x,y,z) = 5x^2 - 7x - 2yz$, $\quad \nabla\phi_2 = (10\,x - 7)i - 2z\,j - 2y\,k$

At (1, -1, -2), $\nabla\phi_2 = 3i + 4j + 2k \Rightarrow |\nabla\phi_2| = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}$

$$\hat{n}_2 = \frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{3i + 4j + 2k}{\sqrt{29}}$$

Angle between two normals is $\cos\theta = \hat{n}_1 . \hat{n}_2$

$$\cos\theta = \frac{1}{\sqrt{208}}(8i + 0j - 12k).\frac{1}{\sqrt{29}}(3i + 4j + 2k) = \frac{1}{\sqrt{6032}}(24 + 0 - 24) = 0$$

$$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

Therefore, the surfaces intersect orthogonally.

12. Find the constants a and b so that the surface $3x^2 - 2y^2 - 3z^2 + 8 = 0$ is orthogonal to the surface $ax^2 + y^2 = bz$ at the point (-1, 2, 1).

Solution: Let $\phi_1(x,y,z) = 3x^2 - 2y^2 - 3z^2 + 8$

$\Rightarrow \nabla\phi_1 = 6xi - 4yj - 6zk$

At (-1, 2, 1) $\nabla\phi_1 = -6i - 8j - 6k \Rightarrow |\nabla\phi_1| = \sqrt{(-6)^2 + (-8)^2 + (-6)^2} = \sqrt{136} = 2\sqrt{34}$

Now $\hat{n}_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{1}{2\sqrt{34}}(-6i - 8j - 6k) = -\frac{1}{\sqrt{34}}(3i + 4j + 3k)$

$\phi_2(x,y,z) = ax^2 + y^2 - bz$, $\nabla\phi_2 = 2axi + 2yj - bk$

At (-1, 2, 1) $\nabla\phi_2 = -2ai + 4j - bk \Rightarrow |\nabla\phi_2| = \sqrt{4a^2 + 16 + b^2}$

$$\hat{n}_2 = \frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{-2ai+4j-bk}{\sqrt{4a^2+b^2+16}}$$

Since the surfaces intersect orthogonally

$$\hat{n}_1 . \hat{n}_2 = 0 \Rightarrow -\frac{1}{\sqrt{34}}(3i+4j+3k).\frac{1}{\sqrt{4a^2+b^2+16}}(-2ai+4j-bk) = 0$$

$$\Rightarrow (3i+4j+3k).(-2ai+4j-bk) = 0 \Rightarrow -6a+16-3b = 0$$

i.e. $6a+3b=16$............. (1)

Also, the point (-1, 2, 1) lies on the surface $ax^2+y^2=bz \Rightarrow a+4=b$

i.e. $a-b=-4$ .... (2)

Solving the equation (1) and (2) $a=\frac{4}{9}$ and $b=\frac{40}{9}$

Exercise:

1. Find the gradient of the following scalar function $f(x,y,z) = x^2y^2 + xy^2 - z^2$ at (3,1,1).
2. Find the directional derivative of $\phi(x,y,z) = xyz - xy^2z^3$ at (1, 2,-1) in the direction of $i-j-3k$.
3. Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point P (1,2,3) in the direction of the line PQ, where Q is the point (5,0,4). In what direction it will be maximum? Find also the magnitude of this maximum
4. Find the unit normal vector to the surface $3x^2+2y^2+4z^2=9$ at (1, -1, 1).
5. Find the direction that a person standing at the origin should move to get warm as quickly as possible given that the temperature field is $x\sin z - y\cos z$.
6. Find the angle between the normals to the surface $z^2-xy=0$ at the points (4, 1, 2) and (3, 3, -3).
7. Find the angle between the normals to the surface $2x^2+3y^2=3z$ at the points (2, -2, 4) and (-1, -1, 1).
8. Find whether the surfaces $4x^2-z^3=4$ and $5x^2-2yz=7x$ intersect orthogonally at the point $(1,-1,-2)$.
9. Find the angle between the normals to the surface $x\log z = y^2-1$ and $x^2y=2-z$ at the point (1, 1, 1) common to them.
10. Find the constants a and b so that the surfaces $x^2+ayz=3x$ and $bx^2y+z^3=(b-8)y$ intersect orthogonally at the point (1, 1,-2).

Answers:

1. $\nabla f_{(3,1,1)} = 7i + 24j - 2k$.
2. $\frac{29}{\sqrt{11}}$
3. $\frac{28}{\sqrt{21}}, 2\sqrt{21}$
4. $\frac{1}{\sqrt{29}}(3i-2j+4k)$
5. y-axis
6. $\cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$
9. $\cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$

7. $\cos^{-1}\left(\dfrac{65}{\sqrt{17941}}\right)$ 10. $a = \frac{5}{2}, b = 1$ 8. Orthogonal

Divergence of a vector function:

Let $\vec{f}(x,y,z) = f_1(x,y,z)i + f_2(x,y,z)j + f_3(x,y,z)k$ be a continuously differentiable vector function, then divergence of a vector point function is denoted by $\nabla \cdot \vec{f}$ or $div\,\vec{f}$ and defined as

$$\nabla \cdot \vec{f} \text{ or } div\,\vec{f} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) \cdot (f_1 i + f_2 j + f_3 k) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Clearly divergence of a vector point function is a scalar point function.

Physical interpretation: If $\vec{f}$ represents a velocity field of a gas or fluid then $div\,\vec{f}$ represents the **rate of expansion per unit volume under the flow of gas or fluid**.

Definition: A vector function $\vec{f}$ is said to be a **Solenoidal** if $div\,\vec{f} = 0$.

Clearly constant vector function is a solenoidal vector function.

Curl of vector function: Let $\vec{f}(x,y,z) = f_1(x,y,z)i + f_2(x,y,z)j + f_3(x,y,z)k$ be a continuously differentiable vector function, then∇ operating vectorially on $\vec{f}$ is denoted by$curl\,\vec{f}$ or $\nabla \times \vec{f}$ is given by

$$\nabla \times \vec{f} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) \times (f_1 i + f_2 j + f_3 k) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\nabla \times \vec{f} = \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right]i + \left[\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right]j + \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]k$$

Clearly curl of a vector function is a vector function.

Physical interpretation: The curl of a vector function represents **rotational motion.**

Definition: A vector function $\vec{f}$ is said to be irrotational vector function if $curl\,\vec{f} = \vec{0}$.

Laplacian of a scalar field

Let $\varphi = \varphi(x,y,z)$ be a given scalar field. Then $\nabla\varphi$is a vector field given by,

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}$$

$\therefore$ divergence of $\nabla\varphi$ is given by

$$div(\nabla\varphi) = \nabla.(\nabla\varphi) = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)$$

$$div(\nabla\varphi) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

The RHS is call Laplacian of $\varphi$and denoted by $\nabla^2\varphi$.

$\therefore$ By definition $\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$ …. (1)

$$\therefore div(\nabla\varphi) = \nabla\nabla\varphi = \nabla^2\varphi$$

Equation (1) can be rewritten as,

$$\nabla^2\varphi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi$$

∇^2 is the differential operator given by, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and is called Laplacian operator.

Problems

1. If $\vec{f} = x^2yi - 2xzj + 2yzk$ then find $div\,\vec{f}$.

Solution: $div\,\vec{f} = \nabla.\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(2yz)$

$$\nabla\cdot\vec{f} = 2xy + 0 + 2y = 2y(x+1).$$

2. If $\vec{f} = 3xyi + x^2zj - y^2e^{2z}k$ then find $\nabla\cdot\vec{f}$ at (1, 2, 0).

Solution: $div\,\vec{f} = \nabla.\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(3xy) + \frac{\partial}{\partial y}(x^2z) + \frac{\partial}{\partial z}(-y^2e^{2z})$

$$= 3y + 0 - 2y^2e^{2z}$$

At (1, 2, 0) $\nabla\cdot\vec{f} = -2.$

3. If $\vec{f} = \frac{xi + yj}{x+y}$ then find $div\,\vec{f}$.

Solution: $\vec{f} = \frac{x}{x+y}i + \frac{y}{x+y}j$

$$div\,\vec{f} = \frac{\partial}{\partial x}\left(\frac{x}{x+y}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x+y}\right) = \frac{y}{(x+y)^2} + \frac{x}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{x+y}$$

4. If $\phi(x,y,z) = 2x^3y^2z^4$ then find $div(grad\phi)$.

Solution: $\phi(x,y,z) = 2x^3y^2z^4$

$$grad\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k \Rightarrow grad\phi = 6x^2y^2z^4i + 4x^3yz^4j + 8x^3y^2z^3k$$

$$div(grad\phi) = \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3) = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^3$$

5. Show that the vector function $\vec{f} = 2xyzi + (xy - y^2z)j + (x^2 - zx)k$ is solenoidal.

Solution: Consider $div\,\vec{f} = \frac{\partial}{\partial x}(2xyz) + \frac{\partial}{\partial y}(xy - y^2z) + \frac{\partial}{\partial z}(x^2 - zx)$

$$div\,\vec{f} = 2yz + x - 2yz - x = 0$$

$\therefore \vec{f}$ is solenoidal vector function.

6. Find the value of a if the vector $\vec{f}=(ax+3y+4z)i+(x-2y+3z)j+(3x+2y-z)k$ has zero divergence.

Solution: If $\vec{f}$ is solenoidal then $div\,\vec{f}=0$

Hence $\frac{\partial}{\partial x}(ax+3y+4z)+\frac{\partial}{\partial y}(x-2y+3z)+\frac{\partial}{\partial z}(3x+2y-z)=0$

$\Rightarrow a-2-1=0\Rightarrow a=3$

7. If $\vec{r}=xi+yj+zk$ and $r=|\vec{r}|$ then show that $\frac{\vec{r}}{r^3}$ is solenoidal.

Solution: Given $r=|\vec{r}|=\sqrt{x^2+y^2+z^2}$, $r^2=x^2+y^2+z^2=\sum x^2$

$$2r\frac{\partial r}{\partial x}=2x \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}$$

$$2r\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y}=\frac{y}{r},\frac{\partial r}{\partial z}=\frac{z}{r}$

Now $\frac{\vec{r}}{r^3}=\frac{1}{r^3}(xi+yj+zk)$

Consider $div\left[\frac{\vec{r}}{r^3}\right]=\sum\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)=\sum\frac{r^3.1-x.3r^2\frac{\partial r}{\partial x}}{(r^3)^2}$

$$div\left(\frac{\vec{r}}{r^3}\right)=\sum\frac{r^3-3.x.r^2.\frac{x}{r}}{r^6}=\sum\frac{r^3-3r,x^2}{r^6}=\sum\frac{r^3}{r^6}-\frac{3r}{r^6}\sum x^2$$

$$div\left(\frac{\vec{r}}{r^3}\right)=\sum\frac{1}{r^3}-\frac{3r}{r^6}.r^2\Rightarrow div\left(\frac{\vec{r}}{r^3}\right)=\frac{3}{r^3}-\frac{3}{r^3}=0$$

$\therefore \frac{\vec{r}}{r^3}$ is solenoidal vector field.

8. If $\vec{r}=xi+yj+zk$ and $r=|\vec{r}|$ then prove that $\nabla^2 f(r)=f''(r)+\frac{2}{r}f'(r)$.

Solution: Given $r=|\vec{r}|=\sqrt{x^2+y^2+z^2}$, $r^2=x^2+y^2+z^2=\sum x^2$

$$2r\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y}=\frac{y}{r},\frac{\partial r}{\partial z}=\frac{z}{r}$

$$\nabla^2 f(r)=\sum\frac{\partial^2}{\partial x^2}(f(r))=\sum\frac{\partial}{\partial x}\left(f'(r)\frac{\partial r}{\partial x}\right)=\sum\frac{\partial}{\partial x}\left(f'(r)\frac{x}{r}\right)=\sum\frac{\partial}{\partial x}\left(\frac{x\,f'(r)}{r}\right)$$

$$\nabla^2 f(r) = \sum \left[\frac{r\left\{ f'(r) + x\, f''(r)\frac{\partial r}{\partial x} \right\} - x\, f'(r)\frac{\partial r}{\partial x}}{r^2} \right] = \sum \left[\frac{r\left\{ f'(r) + x\, f''(r)\frac{x}{r} \right\} - x\, f'(r)\frac{x}{r}}{r^2} \right]$$

$$\nabla^2 f(r) = \sum \frac{f'(r)}{r} + \frac{f''(r)}{r^2}\sum x^2 - \frac{f'(r)}{r^3}\sum x^2 = 3\frac{f'(r)}{r} + \frac{f''(r)}{r^2}.r^2 - \frac{f'(r)}{r^3}.r^2$$

$$\nabla^2 f(r) = \frac{2f'(r)}{r} + f''(r)$$

9. If $\vec{f} = xy^2 i + 2x^2 yzj - 3y^2 zk$ then find $curl\,\vec{f}$.

Solution: $curl\,\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2 yz & -3y^2 z \end{vmatrix} = i(-6yz - 2x^2 y) - j(0-0) + k(4xyz - 2xy)$

10. Show that $\vec{f} = (\sin x + z)i + (\cos y - z)j + (x - y)k$ is irrotational.

Solution: Consider $curl\,\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x + z & \cos y - z & x - y \end{vmatrix} = i(-1+1) - j(1-1) + k(0-0)$

$curl\,\vec{f} = 0 \Rightarrow \vec{f}$ is irrotational.

11. If $\vec{f} = x^2 i - 2xzj + 2yzk$ then find $curl\left(curl\,\vec{f}\right)$.

Solution: $curl\,\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -2xz & 2yz \end{vmatrix} = i(2z + 2x) - j(0-0) + k(-2z)$

$$curl\left(curl\,\vec{f}\right) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x+2z) & 0 & -2z \end{vmatrix} = i(0-0) - j(0-2) + k(0-0) = 2j$$

12. For what value of a the vector field $\vec{f} = (axy - z^3)i + (a-2)x^2 j + (1-a)xz^2 k$ is irrotational.

Solution: If $\vec{f}$ is irrotational, then $curl\,\vec{f} = 0$

Hence $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0$

$$\Rightarrow [0-0]i - [(1-a)z^2 - (0-3z^2)]j + [2x(a-2) - ax]k = 0$$
$$\Rightarrow 0i - [4-a]z^2 j + [a-4]xk = 0 \Rightarrow a = 4$$

13. Find the constants a, b, c so that $\vec{f} = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)k$ is irrotational and find the function ϕ such that $\vec{f} = grad\ \emptyset$.

Solution: If $\vec{f}$ is irrotational, then $curl\ \vec{f} = 0$

Hence $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$

$$\Rightarrow [c+1]i - [4-a]j + [b-2]k = 0$$
$$\Rightarrow c+1 = 0, 4-a = 0, b-2 = 0 \Rightarrow a = 4, b = 2, c = -1$$

We have to find the function ϕ such that $\vec{f} = grad\ \emptyset$

$$(x+2y+4z)i + (2x-3y-z)j + (4x-y+2z)k = \frac{\partial \emptyset}{\partial x}\mathbf{i} + \frac{\partial \emptyset}{\partial y}\mathbf{j} + \frac{\partial \emptyset}{\partial z}\mathbf{k}$$

$$\frac{\partial \emptyset}{\partial x} = x + 2y + 4z \Rightarrow \emptyset = \frac{x^2}{2} + 2xy + 4xz + f(y,z)$$
$$\frac{\partial \emptyset}{\partial y} = 2x - 3y - z \Rightarrow \emptyset = 2xy - \frac{3y^2}{2} + 4xz + g(x,z)$$
$$\frac{\partial \emptyset}{\partial z} = 4x - y + 2z \Rightarrow \emptyset = 4xz - yz + z^2 + h(x,y)$$

We get, $\emptyset = 2xy + 4xz - yz + z^2 + \frac{x^2}{2} - \frac{3y^2}{2}$.

14. Show that $\vec{f} = (\sin y + z)i + (x\cos y - z)j + (x-y)k$ is irrotational. Find the function ϕ such that $\vec{f} = grad\ \emptyset$.

Solution: Given that $\vec{f} = (\sin y + z)i + (x\cos y - z)j + (x-y)k$

Consider $curl\ \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & xcoy - z & x-y \end{vmatrix}$

$$curl\ \vec{f} = i[-1+1] - j[1-1] + k[\cos y - \cos y] = i(0) + j(0) + k(0) = 0$$

Therefore $\vec{f}$ is irrotational.

Find the function ϕ such that $\vec{f} = grad\phi$

$$(\sin y + z)i + (x\cos y - z)j + (x-y)k = \frac{\partial \phi}{\partial x}i + \frac{\partial \phi}{\partial y}j + \frac{\partial \phi}{\partial z}k$$
$$\Rightarrow \frac{\partial \phi}{\partial x} = \sin y + z, \frac{\partial \phi}{\partial y} = x\cos y - z, \frac{\partial \phi}{\partial z} = x - y$$
$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$
$$d\phi = (\sin y + z)dx + (x\cos y - z)dy + (x-y)dz$$

Regrouping the terms

$d\phi = \sin y\, dx + x\cos y\, dy + z\, dx + x\, dz - z\, dy - y\, dz$

$d\phi = d(x\sin y) + d\,(x\,z) - d(yz) \Rightarrow d\emptyset = d(x\sin y + x\,z - yz)$

$\phi = x\sin y + x\,z - yz + c.$

15. If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ then show that $r^n\vec{r}$ is irrotational for all values of n and solenoidal for $n = -3$.

Solution: Given $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$, $r^2 = x^2+y^2+z^2 = \sum x^2$

$2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$r^n\vec{r} = r^n(xi+yj+zk) = r^n xi + r^n yj + r^n zk$

$$curl\,\vec{f} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix} = \sum\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right]i$$

$$curl\,\vec{f} = \sum[nr^{n-1}\frac{\partial r}{\partial y}.z - nr^{n-1}\frac{\partial r}{\partial z}\,y] = \sum\left[n\,r^{n-1}\,\frac{y}{r}\,z - nr^{n-1}\,\frac{z}{r}\,y\right]i$$

$$curl\,\vec{f} = \sum\left[n\,r^{n-2}\,yz - n\,r^{n-2}\,yz\right] = \sum 0\,i = \vec{0}$$

$\therefore r^n\vec{r}$ is irrotational for all values of n.

Consider $div\left[r^n\vec{r}\right] = \sum\dfrac{\partial}{\partial x}(r^n x) = \sum\left[r^n.1 + x.nr^{n-1}.\dfrac{\partial r}{\partial x}\right] = \sum\left[r^n + nr^{n-1}.x.\dfrac{x}{r}\right]$

$$div\left[r^n\vec{r}\right] = \sum[r^n + n.r^{n-2}.x^2] = \sum r^n + n.r^{n-2}\sum x^2 = 3r^n + n.r^{n-2}.r^2$$

$$div\left[r^n\vec{r}\right] = 3r^n + n.r^n = (3+n)r^n$$

$r^n\vec{r}$ is solenoidal implies $div\left[r^n\vec{r}\right] = 0 \Rightarrow (n+3)r^n = 0 \Rightarrow n+3 = 0,\ \therefore n = -3$.

Exercise:

1. Find $divF$ and $curl\vec{F}$ if $\vec{F} = xy^2\vec{i} - 2yz\vec{j} + xyz\vec{k}$ at the point (1,-1,2).
2. If $\vec{F} = x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}$ find $curl(curl\vec{F})$ and $div(curl\vec{F})$.
3. Evaluate i) ∇r^n ii) $\nabla\cdot[\nabla r^n]$
4. Show that the vector field $\vec{f} = (x+3y)i + (y-3z)k + (x-2z)k$ is solenoidal.
5. Determine the constant a such that the vector field

   $\vec{f} = (x+3y)i + (y-2z)j + (x-az)k$ is solenoidal.
6. If $\vec{f} = x(y+z)i + y(z+x)j + z(x+y)k$ then find $curl\,\vec{f}$
7. Find the values of "a" for which the vector field $\vec{F} = xy^2\vec{i} - yz^2\vec{j} + (a-5)x^2z\vec{k}$ at (1,1,1) is a source or sink field. a>5, a<5 .

8. Show that $\vec{f}=(6xy+z^3)i+(3x^2-z)j+(3xz^2-y)k$ is irrotational.

9. If $\vec{f}=(2x+3y+az)i+(bx+2y+3z)j+(2x+cy+3z)k$ is irrotational vector Field, then find the constants a,b,c.

10. If $\vec{r}=xi+yj+zk$ and $r=|\vec{r}|$ then find $\nabla^2(r^n)$ and hence show that $\nabla^2\left(\frac{1}{r}\right)=0$

11. Show that $\vec{f}=(\sin y+z\cos x)i+(x\cos y+\sin z)j+(y\cos z+\sin x)k$ is irrotational and find the function ϕ such that $\vec{f}=\nabla\phi$

12. Show that $\nabla^2(\log r)=\frac{1}{r^2}$

Answers:

1. -4, 2(j+k)
2. $2(z\vec{i}+x\vec{j}+y\vec{k}), 0$
3. $nr^{n-2}\vec{r}, n(n+1)r^{n-2}$
5. $a=2$
6. $(z-y)i+(z-x)j+(y-x)k$
9. $a=2, b=3, c=3$
10. $\frac{-2\vec{r}}{r^3}$
11. $\phi=x\sin y+y\sin z+z\sin x+c$

The concept of orthogonal curvilinear coordinates

The cartesian orthogonal coordinate system is very intuitive and easy to handle. Once an origin has been fixed in space and three orthogonal scaled axis are anchored to this origin, any point in space is uniquely determined by three real numbers, its cartesian coordinates. A curvilinear coordinate system can be defined starting from the orthogonal cartesian one. If x, y, z are the cartesian coordinates, the curvilinear ones, u, v, w, can be expressed as smooth functions of x, y, z, according to: $u = u(x, y, z)$, $v = v(x, y, z)$ and $w = w(x, y, z)$. These functions can be inverted to give x, y, z-dependency on u, v, w: $x = x(u, v, w)$, $y = y(u, v, w)$ and $z = z(u, v, w)$. There are infinitely many curvilinear systems that can be defined using above equations. We are mostly interested in the so-called orthogonal curvilinear coordinate systems, defined as follows. Any point in space is determined by the intersection of three warped planes: u = const ; v = const ; w = const. We could call these surfaces as coordinate surfaces. Three curves, called coordinate curves, are formed by the intersection of pairs of these surfaces. Accordingly, three straight lines can be calculated as tangent lines to each coordinate curve at the space point.

In an orthogonal curved system these three tangents will be orthogonal for all points in space (see Figure 1).

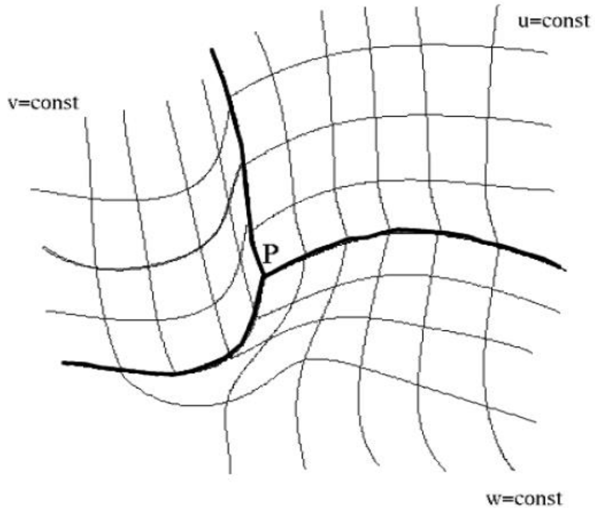

Fig 2.5. General orthogonal cure coordinate system.

Basic vector and scale factor

Consider the u_1-curve through a given point and let $\vec{r}$ be the position vector of P then the partial derivative $\dfrac{\partial \vec{r}}{\partial u}$ gives tangential vector u_1-curve at P

The unit tangential vector $\hat{e}_1$ to the u_1-curve is

$$\hat{e}_1 = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u_1}, \text{ where } h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|$$

Similarly $$\hat{e}_2 = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial u_1}, \text{ where } h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|$$

$$\hat{e}_3 = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial u_3}, \text{ where } h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right|$$

Where h_1, h_2, h_3 are called scale factors.

1. **Cartesian coordinates** (x, y, z)

The Cartesian coordinates of a point (x,y,z) are determined by following straight paths starting from the origin: first along the x-axis, then parallel to the y-axis, then parallel to the z-axis, as in Figure 1. In *curvilinear coordinate systems*, these paths can be curved. The two types of curvilinear coordinates which we will consider are cylindrical and spherical coordinates. Instead of referencing a point in terms of sides of a rectangular parallelepiped, as with Cartesian coordinates, we will think of the point as lying on a cylinder or sphere. Cylindrical coordinates are often used when there is symmetry around the z-axis; spherical coordinates are useful when there is symmetry about the origin.

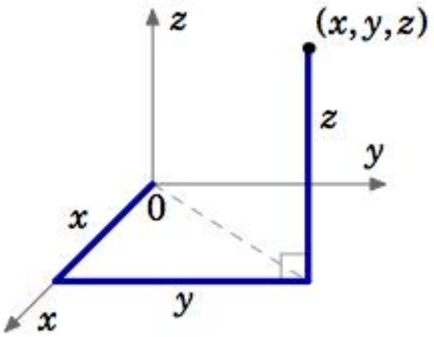

Fig.2.6 Cartesian coordinates system

Let P=(x,y,z) be a point in Cartesian coordinates in R3, and let P_0=(x,y,0) be the projection of P upon the xy-plane. Treating (x,y) as a point in R2, let (r,θ) be its polar coordinates (see Figure 1.7.2). Let ρρ be the length of the line segment from the origin to P, and let φ be the angle between that line segment and the positive z-axis (see Figure 2). φ is called the *zenith angle*. Then the **cylindrical coordinates** (r,θ,z) and the **spherical coordinates** (r,θ,ϕ) of P(x,y,z) are defined as follows:

2.cylindrical coordinates (r,θ,z)

$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$

$$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = z \qquad z = z$$

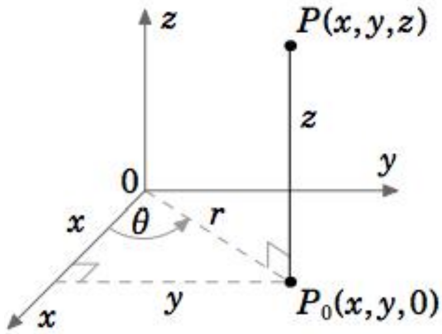

Figure 2.7 Cylindrical coordinates

Scale factors

Then the be the position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j} + z\hat{k}$$

$$\frac{\partial\vec{r}}{\partial r} = \cos\theta\hat{i} + \sin\theta\hat{j} \Rightarrow h_1 = \left|\frac{\partial\vec{r}}{\partial r}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\hat{i} + r\cos\theta\hat{j} \Rightarrow h_2 = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = r\sqrt{\cos^2\theta + \sin^2\theta} = r$$

$$\frac{\partial \vec{r}}{\partial z} = \hat{k} \Rightarrow h_3 = \left|\frac{\partial \vec{r}}{\partial z}\right| = 1$$

Then the unit vector

$$\hat{e}_1 = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial r} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

$$\hat{e}_2 = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

$$\hat{e}_3 = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial z} = \hat{k}\,.$$

2. Spherical coordinates (r, θ, ϕ)

$$x = r\sin\theta\cos\phi \qquad r = \sqrt{x^2 + y^2 + z^2}$$

$$y = r\sin\theta\sin\phi \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = r\cos\theta$$

$$\phi = \cos^{-1}\left(\frac{z}{x^2 + y^2 + z^2}\right)$$

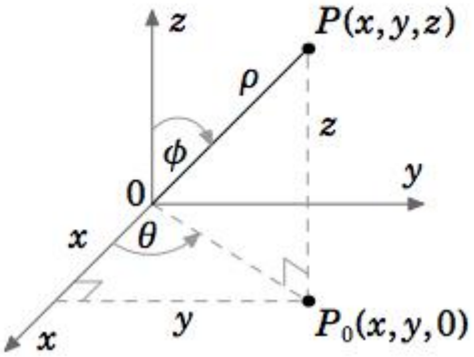

Figure 2.8 Spherical coordinates

Scale factors

Then the be the position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{r} = r\sin\theta\cos\phi\hat{i} + r\sin\theta\sin\phi\hat{j} + r\cos\theta\hat{k}$$

$$\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k} \Rightarrow h_1 = \left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} = 1$$

$$\frac{\partial \vec{r}}{\partial \theta} = r\cos\theta\cos\phi\hat{i} + r\cos\theta\sin\phi\hat{j} - r\sin\theta\hat{k} \Rightarrow h_2 = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = r\sqrt{\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta} = r$$

$$\frac{\partial \vec{r}}{\partial \phi} = -r\sin\theta\sin\phi\hat{i} + r\sin\theta\cos\phi\hat{j} \Rightarrow h_3 = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = r\sin\theta\sqrt{\sin^2\phi + \cos^2\phi} = r\sin\theta$$

Then the unit vector

$$\hat{e}_1 = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

$$\hat{e}_2 = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}$$

$$\hat{e}_3 = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial \phi} = \sin\phi\hat{i} + \cos\phi\hat{j} .$$

Problems

1. Show that cylindrical coordinate system is orthogonal.

Solution: Let (r,θ,z) cylindrical coordinates**,** we have

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Then the position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j} + z\hat{k}$$

$$\frac{\partial \vec{r}}{\partial r} = \cos\theta\hat{i} + \sin\theta\hat{j} \Rightarrow h_1 = \left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\hat{i} + r\cos\theta\hat{j} \Rightarrow h_2 = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = r\sqrt{\cos^2\theta + \sin^2\theta} = r$$

$$\frac{\partial \vec{r}}{\partial z} = \hat{k} \Rightarrow h_3 = \left|\frac{\partial \vec{r}}{\partial z}\right| = 1$$

Then the unit vector

$$\hat{e}_1 = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial r} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

$$\hat{e}_2 = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

$$\hat{e}_3 = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial z} = \hat{k} .$$

$$\hat{e}_1.\hat{e}_2 = \left(\cos\theta\hat{i} + \sin\theta\hat{j}\right).\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right) = 0$$

$$\hat{e}_1.\hat{e}_3 = \left(\cos\theta\hat{i} + \sin\theta\hat{j}\right).\left(\hat{k}\right) = 0.$$

$$\hat{e}_3.\hat{e}_2 = \left(\hat{k}\right).\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right) = 0.$$

Therefore cylindrical coordinate system is orthogonal.

2. Express the vector $\vec{F} = z\hat{i} - 2x\hat{j} + y\hat{k}$ in cylindrical coordinates.

Solution: We have

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Then vector $\vec{F} = z\hat{i} - 2r\cos\theta\hat{j} + r\sin\theta\hat{k}$ (1)

The position vector $\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j} + z\hat{k}$
the unit vectors are

$$\hat{e}_1 = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial r} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

$$\hat{e}_2 = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

$$\hat{e}_3 = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial z} = \hat{k}.$$

Let the expression in cylindrical coordinates be $\vec{F} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$ (2)

$$f_1 = \vec{F}.e_1 = \left(z\hat{i} - 2r\cos\theta\hat{j} + r\sin\theta\hat{k}\right).\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right) = z\cos\theta - 2r\cos\theta\sin\theta$$

$$f_2 = \vec{F}.e_2 = \left(z\hat{i} - 2r\cos\theta\hat{j} + r\sin\theta\hat{k}\right).\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right) = -z\sin\theta - 2r\cos^2\theta.$$

$$f_3 = \vec{F}.e_3 = \left(z\hat{i} - 2r\cos\theta\hat{j} + r\sin\theta\hat{k}\right).\left(\hat{k}\right) = r\sin\theta$$

$$\vec{F} = (z\cos\theta - 2r\cos\theta\sin\theta)\hat{e}_1 - \left(z\sin\theta + 2r\cos^2\theta\right)\hat{e}_2 + (r\sin\theta)\hat{e}_3.$$

3. Show that spherical coordinate system is orthogonal.

Solution: Let (r,θ,ϕ) spherical coordinates**,** we have

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

Then the be the position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{r} = r\sin\theta\cos\phi\hat{i} + r\sin\theta\sin\phi\hat{j} + r\cos\theta\hat{k}$$

$$\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k} \Rightarrow h_1 = \left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} = 1$$

$$\frac{\partial \vec{r}}{\partial \theta} = r\cos\theta\cos\phi\hat{i} + r\cos\theta\sin\phi\hat{j} - r\sin\theta\hat{k} \Rightarrow h_2 = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = r\sqrt{\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta} = r$$

$$\frac{\partial \vec{r}}{\partial \phi} = -r\sin\theta\sin\phi\hat{i} + r\sin\theta\cos\phi\hat{j} \Rightarrow h_3 = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = r\sin\theta\sqrt{\sin^2\phi + \cos^2\phi} = r\sin\theta$$

Then the unit vector

$$\hat{e}_1 = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

$$\hat{e}_2 = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}$$

$$\hat{e}_3 = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial \phi} = \sin\phi\hat{i} + \cos\phi\hat{j}.$$

$$\hat{e}_1.\hat{e}_2=\left(\sin\theta\cos\phi\hat{i}+\sin\theta\sin\phi\hat{j}+\cos\theta\hat{k}\right).\left(\cos\theta\cos\phi\hat{i}+\cos\theta\sin\phi\hat{j}-\sin\theta\hat{k}\right)=0$$

$$\hat{e}_1.\hat{e}_3=\left(\sin\theta\cos\phi\hat{i}+\sin\theta\sin\phi\hat{j}+\cos\theta\hat{k}\right).\left(\sin\phi\hat{i}+\cos\phi\hat{j}\right)=0.$$

$$\hat{e}_3.\hat{e}_2=\left(\sin\phi\hat{i}+\cos\phi\hat{j}\right).\left(\cos\theta\cos\phi\hat{i}+\cos\theta\sin\phi\hat{j}-\sin\theta\hat{k}\right)=0.$$

Therefore spherical coordinate system is orthogonal.

4. Express the vector $\vec{F}=2y\hat{i}-z\hat{j}+3x\hat{k}$ in spherical coordinates.

Solution: We have

$$x=r\sin\theta\cos\phi \qquad y=r\sin\theta\sin\phi \qquad z=r\cos\theta$$

Then vector $\vec{F}=2r\sin\theta\sin\phi\hat{i}-r\cos\theta\hat{j}+3r\sin\theta\cos\phi\ \hat{k}$ (1)

The position vector $\vec{r}=r\sin\theta\cos\phi\hat{i}+r\sin\theta\sin\phi\hat{j}+r\cos\theta\hat{k}$

the unit vectors are

$$\hat{e}_1=\frac{1}{h_1}\frac{\partial\vec{r}}{\partial r}=\sin\theta\cos\phi\hat{i}+\sin\theta\sin\phi\hat{j}+\cos\theta\hat{k}$$

$$\hat{e}_2=\frac{1}{h_2}\frac{\partial\vec{r}}{\partial\theta}=\cos\theta\cos\phi\hat{i}+\cos\theta\sin\phi\hat{j}-\sin\theta\hat{k}$$

$$\hat{e}_3=\frac{1}{h_3}\frac{\partial\vec{r}}{\partial\phi}=\sin\phi\hat{i}+\cos\phi\hat{j}\ .$$

Let the expression in spherical coordinates be $\vec{F}=f_1\hat{e}_1+f_2\hat{e}_2+f_3\hat{e}_3$ (2)

$$f_1=\vec{F}.e_1=\left(2r\sin\theta\sin\phi\hat{i}-r\cos\theta\hat{j}+3r\sin\theta\cos\phi\ \hat{k}\right).\left(\sin\theta\cos\phi\hat{i}+\sin\theta\sin\phi\hat{j}+\cos\theta\hat{k}\right)$$

$$f_1=2r\sin^2\theta\sin\phi\cos\phi-r\cos\theta\sin\theta\sin\phi+3r\sin\theta\cos\theta\cos\phi.$$

$$f_2=\vec{F}.e_2=\left(2r\sin\theta\sin\phi\hat{i}-r\cos\theta\hat{j}+3r\sin\theta\cos\phi\ \hat{k}\right).\left(\cos\theta\cos\phi\hat{i}+\cos\theta\sin\phi\hat{j}-\sin\theta\hat{k}\right)$$

$$f_2=2r\sin\theta\cos\theta\sin\phi\cos\phi-r\cos^2\theta\sin\phi-3r\sin^2\theta\cos\phi.$$

$$f_3=\vec{F}.e_3=\left(2r\sin\theta\sin\phi\hat{i}-r\cos\theta\hat{j}+3r\sin\theta\cos\phi\ \hat{k}\right).\left(\sin\phi\hat{i}+\cos\phi\hat{j}\right)$$

$$f_3=-\left(2r\sin^2\phi\sin\theta+r\cos\theta\cos^2\phi\right)$$

Therefore the expression in spherical coordinates be $\vec{F}=f_1\hat{e}_1+f_2\hat{e}_2+f_3\hat{e}_3$

$$\vec{F}=\left(2r\sin^2\theta\sin\phi\cos\phi-r\cos\theta\sin\theta\sin\phi+3r\sin\theta\cos\theta\cos\phi\right)\hat{e}_1+$$

$$\left(2r\sin\theta\cos\theta\sin\phi\cos\phi-r\cos^2\theta\sin\phi-3r\sin^2\theta\cos\phi\right)\hat{e}_2-\left(2r\sin^2\phi\sin\theta+r\cos\theta\cos^2\phi\right)\hat{e}_3.$$

Important Formulae

Dot Product	$Cos\theta = A_x B_x + A_y B_y + A_z B_z$ with $a_x . a_x = a_y . a_y = a_z . a_z = 1$ *and* $a_x . a_y = a_x . a_y = a_y . a_z = 0.$
Cross Product	$\vec{A} \times \vec{B} = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$ with $a_x \times a_x = a_y \times a_y = a_z \times a_z = 0$ *and* $a_x \times a_y = a_z, a_y \times a_z = a_x, a_y \times a_y = -a_z, a_z \times a_y = -a_x.$
Differential length	$dl = dx\, a_x + dy\, a_y + dz\, a_z$ (Cartesian coordinates) $dl = d\rho\, a_\rho + \rho d\phi\, a_\phi + dz\, a_z$ (Cylindrical coordinates) $dl = dr\, a_r + r d\theta\, a_\theta + r \sin\theta d\phi\, a_\phi$ (Spherical coordinates)
Differential surface area	$\left.\begin{matrix} dS_x = dy\,dz \\ dS_y = dx\,dz \\ dS_z = dx\,dy \end{matrix}\right\}$ (Cartesian coordinates) $\left.\begin{matrix} dS_\rho = \rho d\phi\, dz \\ dS_\phi = d\rho\, dz \\ dS_z = \rho d\rho\, d\phi \end{matrix}\right\}$ (Cylindrical coordinates) $\left.\begin{matrix} dS_r = r^2 \sin\theta d\phi\, d\theta \\ dS_\theta = r \sin\theta dr\, d\phi \\ dS_\phi = r dr\, d\theta \end{matrix}\right\}$ (Spherical coordinates)
Differential volume	$dV = dxdydz$ (Cartesian coordinates) $dV = \rho\, d\rho\, d\phi dz$ (Cylindrical coordinates) $dV = r^2 \sin\theta\, dr\, d\theta d\phi$ (Spherical coordinates)
Differential operator	$\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial z} a_y + \frac{\partial}{\partial z} a_z$
Gradient of a Scalar	$\nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial z} a_y + \frac{\partial V}{\partial z} a_z$ (Cartesian coordinates) $\nabla V = \frac{\partial V}{\partial \rho} a_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} a_\phi + \frac{\partial V}{\partial z} a_z$ (Cylindrical coordinates) $\nabla V = \frac{\partial V}{\partial r} a_r + \frac{1}{r \sin\theta} \frac{\partial V}{\partial \theta} a_\theta + \frac{1}{r} \frac{\partial V}{\partial \phi} a_\phi$ (Spherical coordinates)

Divergence of a Vector	$\nabla.\vec{F}=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}$ (Cartesian coordinates) $\nabla.\vec{F}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho)+\frac{1}{\rho}\frac{\partial}{\partial\phi}(F_\phi)+\frac{\partial}{\partial z}(F_z)$ (Cylindrical coordinates) $\nabla.\vec{F}=\frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta F_\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$ (Spherical coordinates)
Curl of a Vector	$\nabla\times\vec{F}=\begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$ (Cartesian coordinates) $\nabla\times\vec{F}=\frac{1}{\rho}\begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial Z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$ (Cylindrical coordinates) $\nabla\times\vec{F}=\frac{1}{r^2\sin\theta}\begin{vmatrix} a_r & ra_\theta & r\sin\theta a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$ (Spherical coordinates)
Laplacian of a scalar field	$\nabla^2V=\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial y^2}+\frac{\partial^2V}{\partial z^2}$ (Cartesian coordinates) $\nabla^2V=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2V}{\partial\phi^2}+\frac{\partial^2V}{\partial z^2}$ (Cylindrical coordinates) $\nabla^2V=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2V}{\partial\phi^2}$ (Spherical coordinates)

Self-Study: Vector integration and Vector line integral
Vector Integration

Line Integral: Any integral which is to be evaluated along a curve is called line integral.
If $\vec{F}(x,y,z)$ is a vector point function and C is any curve then $\int_C \vec{F}.d\vec{r}$ is called the vector line integral. (Tangential line integral or line integral)
Note:

1. C is a called path of integration.

2. If $\vec{F} = f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k}$ then $\int_C \vec{F}.d\vec{r} = \int_C f_1dx + f_2dy + f_3dz$.
3. When C is a simple closed curve, line integral is denoted by $\oint_C \vec{F}.d\vec{r}$ (means the line integral of $\vec{F}$ taken once around C in the anticlock wise direction).
4. If $\vec{F}$ represents force acting on a particle then the line integral $\int_C \vec{F}.d\vec{r}$ represents work done by a force $\overrightarrow{F}$.
5. If $\vec{F}$ represents the velocity of a fluid then $\int_C \vec{F}.d\vec{r}$ represents circulation of $\vec{F}$ around C.
6. Condition for $\vec{F}$ to be conservative is $\nabla \times \vec{F} = 0$.
7. If $curl\,\vec{F} = 0$ then $\int_C \vec{F}.d\vec{r}$ is independent of path.
8. If circulation is "0" then $\int \vec{F}.d\vec{r}$ is irrotational.

Problems

1. If $\vec{F} = (5xy - 6x^2)\hat{\imath} + (2y - 4x)\hat{\jmath}$. Evaluate $\int_C \vec{F}.d\vec{r}$ along $y = x^3$ in XY −plane from $(1,1)$ to $(2,8)$.

Solution: Given

$\vec{F} = (5xy - 6x^2)\hat{\imath} - (2y - 4x)\hat{\jmath}$

$\vec{r} = x\,\hat{\imath} + y\hat{\jmath} \Rightarrow \overrightarrow{dr} = dx\,\hat{\imath} + dy\hat{\jmath}$

$y = x^3 \Rightarrow dy = 3x^2dx$ and $x: 1$ to 2

Consider

$\vec{F}.d\vec{r} = (5x\,x^3 - 6x^2)dx + (2x^3 - 4x)3x^2dx$

$$\int_C \vec{F}.d\vec{r} = \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3)dx = [x^5 - 2x^3 + x^6 - 3x^4]_1^2 = 35.$$

2. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{\imath} + (2xz - y)\hat{\jmath} + z\hat{k}$ along

a) The straight line $(0,0,0)$ to $(2,1,3)$.
b) The curve $x = 2t^2, y = t, z = 4t^2 - t$ from $t = 0$ to $t = 1$.
c) The curve defined by $x^2 = 4y,\ 3x^3 = 8z$ from $x = 0$ to 2.

Solution: Work done $= \int_C \vec{F}.d\vec{r} = \int_C 3x^2dx + (2xz - y)dy + zdz$. ---------- (i)

a) C is a straight line joining $(0,0,0)$ and $(2,1,3)$.

The equation of the line is given by $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$

We have $x = 2t \Rightarrow dx = 2dt,\quad y = t \Rightarrow dy = dt,\quad z = 3t \Rightarrow dz = 3dt$

and $t = 0$ to 1 $[\because t = y,\ y = 0\ to\ 1]$

then equation (i)

$$\Rightarrow \int_C \vec{F}.d\vec{r} = \int_0^1 \{(3(2t)^2)(2dt) + (2(2t)(3t) - t)dt + (3t)3dt\}$$

$$= \int_0^1 (36t^2 + 8t)\,dt = \left[36\frac{t^3}{3} + 8\frac{t^2}{2}\right]\Big|_0^1 = 16.$$

b) Given curve $x = 2t^2 \Rightarrow dx = 4\,tdt$, $y = t \Rightarrow dy = dt$, $z = 4t^2 - t \Rightarrow dz = (8t-1)dt$
then (i) becomes

$$\int_C \vec{F}.d\vec{r} = \int_0^1 \{(3(2t^2)^2)(4tdt) + (2(2t^2)(4t^2-t) - t)dt + (4t^2-t)(8t-1)dt\}$$

$$= \int_0^1 (48t^5 + 16t^4 + 28t^3 - 12t^2)\,dt = \frac{71}{5}$$

c) Given curve $x^2 = 4y \Rightarrow y = \frac{x^2}{4} \Rightarrow dy = \frac{x}{2}dx$, $3x^3 = 8z \Rightarrow z = \frac{3x^3}{8} \Rightarrow dz = \frac{9}{8}x^2dx$ and x: 0 to 2 then (i) becomes

$$\int_C \vec{F}.d\vec{r} = \int_0^2 \left\{3x^2dx + \left(2x\left(\frac{3}{8}x^3\right) - \frac{x^2}{4}\right)\frac{x}{2}dx + \frac{3}{8}x^3\frac{9}{8}x^2\,dx\right\}$$

$$= \int_0^2 (3x^2 + \frac{3}{8}x^5 - \frac{x^3}{8} + \frac{27}{64}x^5)\,dx = 16.$$

Exercise:

1. Find the value of $\int_c \vec{F} \cdot d\vec{r}$,where $\vec{F} = y^2\vec{\imath} + 2xy\vec{\jmath}$ and

i) C is the straight-line path from (0,0) to (1,2)

ii) C is the parabola $y = 2x^2$from (0,0) to (1,2)   Ans: 4

Also show that $\oint_c \vec{F}dr = 0$,where C is the closed curve obatined by the above straight line and parabola and explain why?.

2 Find the work done by the force $\vec{F} = (2x+3)\vec{\imath} + xz\vec{\jmath} + (yz-x)\vec{k}$when it moves a particle from the point (0,0,0) to the piont (2,1,1) along the curve $x = 2t^2, y = t$and $z = t^3$. Ans: $\frac{288}{35}$

Video Links:

1. Vector differentiation
https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/position-vector-functions/v/differential-of-a-vector-valued-function

2. Gradient
https://www.youtube.com/watch?v=fZ231k3zsAA
https://www.youtube.com/watch?v=GkB4vW16QHI

3. Directional derivative
https://www.youtube.com/watch?v=Dcnj1bYEZlY

4. Applications of Gradient, Divergence and curl
https://www.youtube.com/watch?v=qOcFJKQPZfo
https://www.youtube.com/watch?v=vvzTEbp9lrc

5. Line integral
https://www.youtube.com/watch?v=7FUNdFN6ZKI

6. Surface integral
https://www.youtube.com/watch?v=I1dfwKPV75A

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