CHAPTER – 2

QUADRATIC EQUATIONS

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"If a variable occurs in an equation with all positive integer powers and the highest power is two, then it is called a Quadratic Equation (in that variable)."

In other words, a second degree polynomial in x equated to zero will be a quadratic equation. For such an equation to be a quadratic equation, the co-efficient of x^2 should not be zero.

The most general form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$ (and a, b, c are real)

Some examples of quadratic equations are

$x^2 - 5x + 6 = 0$ .... (1)
$x^2 - x - 6 = 0$ .... (2)
$2x^2 + 3x - 2 = 0$ .... (3)
$2x^2 + x - 3 = 0$ .... (4)

Like a first degree equation in x has one value of x satisfying the equation, a quadratic equation in x will have TWO values of x that satisfy the equation. The values of x that satisfy the equation are called the ROOTS of the equation. These roots may be real or imaginary.

For the four quadratic equations given above, the roots are as given below:

Equation (1) : $x = 2$ and $x = 3$
Equation (2) : $x = -2$ and $x = 3$
Equation (3) : $x = 1/2$ and $x = -2$
Equation (4) : $x = 1$ and $x = -3/2$

In general, the roots of a quadratic equation can be found out in two ways.
(i) by factorising the expression on the left-hand side of the quadratic equation
(ii) by using the standard formula

All the expressions may not be easy to factorise whereas applying the formula is simple and straightforward.

Finding the roots by factorisation

If the quadratic equation $ax^2 + bx + c = 0$ can be written in the form $(x - \alpha)(x - \beta) = 0$, then the roots of the equation are α and β.

To find the roots of a quadratic equation, we should first write it in the form of $(x - \alpha)(x - \beta) = 0$, i.e., the left hand side $ax^2 + bx + c$ of the quadratic equation $ax^2 + bx + c = 0$ should be factorised into two factors.

For this purpose, we should go through the following steps. We will understand these steps with the help of the equation $x^2 - 5x + 6 = 0$ which is the first of the four quadratic equations we looked at as examples above.

- First write down b (the co-efficient of x) as the sum of two quantities whose product is equal to ac.

  In this case –5 has to be written as the sum of two quantities whose product is 6. We can write –5 as (–3) + (–2) so that the product of (–3) and (–2) is equal to 6.

- Now rewrite the equation with the 'bx' term split in the above manner.
  In this case, the given equation can be written as $x^2 - 3x - 2x + 6 = 0$

- Take the first two terms and rewrite them together after taking out the common factor between the two of them. Similarly, the third and fourth terms should be rewritten after taking out the common factor between the two of them. In other words, you should ensure that what is left from the first and the second terms (after removing the common factor) is the same as that left from the third and the fourth terms (after removing their common factor).

  In this case, the equation can be rewritten as $x(x - 3) - 2(x - 3) = 0$; Between the first and second terms as well as the third and fourth terms, we are left with $(x - 3)$ as a common factor.

- Rewrite the entire left-hand side to get the form $(x - \alpha)(x - \beta)$.
  In this case, if we take out $(x - 3)$ as the common factor, we can rewrite the given equation as $(x - 3)(x - 2) = 0$

- Now, α and β are the roots of the given quadratic equation.
  $\therefore$ For $x^2 - 5x + 6 = 0$, the roots of the equation are 3 and 2.

For the other three quadratic equations given above as examples, let us see how to factorise the expressions and get the roots.

For equation (2), i.e., $x^2 - x - 6 = 0$, the co-efficient of x which is –1 can be rewritten as (–3) + (+2) so that their product is –6 which is equal to ac (1 multiplied by –6). Then we can rewrite the equation as $(x - 3)(x + 2) = 0$ giving us the roots as 3 and –2.

For equation (3), i.e., $2x^2 + 3x - 2 = 0$, the co-efficient of x which is 3 can be rewritten as (+4) + (–1) so that their product is –4 which is the value of ac (–2 multiplied by 2). Then we can rewrite the equation as $(2x - 1)(x + 2) = 0$ giving the roots as 1/2 and –2.For equation (4), i.e., $2x^2 + x - 3 = 0$, the co-efficient of x which is 1 can be rewritten as (+3) + (–2) so that their product is –6 which is equal to ac (2 multiplied by –3). Then we can rewrite the given equation as$(x - 1)(2x + 3) = 0$ giving us the roots as 1 and –3/2.

Finding the roots by using the formula

If the quadratic equation is $ax^2 + bx + c = 0$, then we can use the standard formula given below to find out the roots of the equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The four quadratic equations we took as examples above can be taken and their roots found out by using the formula. The student is advised to check it out for himself/herself that the roots can be obtained by using this formula also.

SUM AND PRODUCT OF ROOTS OF A QUADRATIC EQUATION

For the quadratic equation $ax^2 + bx + c = 0$, the sum of the roots and the product of the roots are given by the following:

Sum of the roots = $-b/a$
Product of the roots = c/a

These two rules will be very helpful in solving problems on quadratic equations.

NATURE OF THE ROOTS

We mentioned already that the roots of a quadratic equation with real co-efficients can be real or complex. When the roots are real, they can be equal or unequal. All this will depend on the expression $b^2 - 4ac$. $b^2 - 4ac$ determines the nature of the roots of the quadratic equation and it is called the "DISCRIMINANT" of the quadratic equation.

If $b^2 - 4ac > 0$, then the roots of the quadratic equation will be real and distinct.

If $b^2 - 4ac = 0$, the roots are real and equal.

If $b^2 - 4ac < 0$, then the roots of the quadratic equation will be complex conjugates.

Thus we can write down the following about the nature of the roots of a quadratic equation when a, b and c are all rational.

when $b^2 - 4ac < 0$	the roots are complex and unequal
when $b^2 - 4ac = 0$	the roots are rational and equal
when $b^2 - 4ac > 0$ and a perfect square	the roots are rational and unequal
when $b^2 - 4ac > 0$ but not a perfect square	the roots are irrational and unequal

Whenever the roots of the quadratic equation are irrational, (a, b, c being rational) they will be of the form $a + \sqrt{b}$ and $a - \sqrt{b}$, i.e. whenever $a + \sqrt{b}$ is one root of a quadratic equation, then $a - \sqrt{b}$ will be the second root of the quadratic equation and vice versa.

SIGNS OF THE ROOTS

We can comment on the signs of the roots, i.e., whether the roots are positive or negative, based on the signs of the sum of the roots and the product of the roots of the quadratic equation. The following table will make clear the relationship between the signs of the sum and the product of the roots and the signs of the roots themselves.

Sign of product of the roots	Signs of sum of the roots	Sign of the roots
+ ve	+ ve	Both the roots are positive
+ ve	– ve	Both the roots are negative
– ve	+ ve	The numerically larger root is positive and the other root is negative.
– ve	– ve	The numerically larger root is negative and the other root is positive.

CONSTRUCTING A QUADRATIC EQUATION

We can build a quadratic equation in the following three cases:

- when the roots of the quadratic equation are given
- when the sum of the roots and the product of the roots of the quadratic equation are given.
- when the relation between the roots of the equation to be framed and the roots of another equation is given.

If the roots of the quadratic equation are given as α and β, the equation can be written as
$(x - \alpha)(x - \beta) = 0$ i.e., $x^2 - x(\alpha + \beta) + \alpha\beta = 0$
If p is the sum of the roots of the quadratic equation and q is the product of the roots of the quadratic equation, then the equation can be written as $x^2 - px + q = 0$.

CONSTRUCTING A NEW QUADRATIC EQUATION BY CHANGING THE ROOTS OF A GIVEN QUADRATIC EQUATION

If we are given a quadratic equation, we can build a new quadratic equation by changing the roots of this equation in the manner specified to us.

For example, let us take a quadratic equation $ax^2 + bx + c = 0$ and let its roots be α and β respectively. Then we can build new quadratic equations as per the following patterns:

(i) A quadratic equation whose roots are the **reciprocals** of the roots of the given equation $ax^2 + bx + c = 0$, i.e., the roots are $1/\alpha$ and $1/\beta$:

This can be obtained by substituting 1/x in place of x in the given equation giving us $cx^2 + bx + a = 0$, i.e., we get the equation required by interchanging the co-efficient of x^2 and the constant term.

(ii) A quadratic equation whose roots are **k more** than the roots of the equation $ax^2 + bx + c = 0$, i.e., the roots are $(\alpha + k)$ and $(\beta + k)$
This can be obtained by substituting $(x - k)$ in place of x in the given equation.

(iii) A quadratic equation whose roots are **k less** than the roots of the equation $ax^2 + bx + c = 0$, i.e., the roots are $(\alpha - k)$ and $(\beta - k)$

This can be obtained by substituting $(x + k)$ in place of x in the given equation.

(iv) A quadratic equation whose roots are **k times** the roots of the equation $ax^2 + bx + c = 0$, i.e., the roots are $k\alpha$ and $k\beta$.
This can be obtained by substituting x/k in place of x in the given equation.

(v) A quadratic equation whose roots are **1/k times** the roots of the equation $ax^2 + bx + c = 0$, i.e., the roots are α/k and β/k

This can be obtained by substituting kx in place of x in the given equation.

An equation whose degree is 'n' will have n roots.

MAXIMUM OR MINIMUM VALUE OF A QUADRATIC EXPRESSION

An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation. An expression of the form $ax^2 + bx + c$ is called a "quadratic expression".
As x varies from $-\infty$ to $+\infty$, (i.e. when x is real) the quadratic expression $ax^2 + bx + c$

(i) has a minimum value whenever $a > 0$ (i.e. a is positive). The minimum value of the quadratic expression is $(4ac - b^2) / 4a$ and it occurs at $x = -b/2a$.

(ii) has a maximum value whenever $a < 0$ (i.e. a is negative). The maximum value of the quadratic expression is $(4ac - b^2) / 4a$ and it occurs at $x = -b/2a$.

Polynomials and Polynomial Equations:

An expression of the form $a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots + a_1x + a_0$ is called a polynomial. If we denote it as f(x), f(x) = 0 is a polynomial equation.

Remainder Theorem:
When a polynomial p(x) of degree n is divided by x – a (a linear polynomial), there results a quotient polynomial q(x) (of degree (n – 1)) and a remainder (of degree 0) i.e. a constant.

i.e., p(x) = (x – a) q(x) + R
This relation is true for all values of x. In particular, for x = a, we get, p(a) = R. This result is the Remainder Theorem.

Note:
1. If p(a) = 0, we say that 'a' is a zero of the polynomial p(x).
2. If p(x) is a polynomial and 'a' is a zero of p(x), then p(x) = (x – a) q(x).
3. If p(x) is divided by ax + b, then the remainder is given by $p\left(\frac{-b}{a}\right)$.
4. If p(x) is divided by ax – b, then the remainder is given by $p\left(\frac{b}{a}\right)$.
5. The degree of remainder is always less than the degree of divisor.

2.01 If $f(x) = x^2 + 14x + 30$, find the remainder when f(x) is divided by x + 4.

Sol. We know that when f(x) is divided by x + a, the remainder is f(–a),
Given, $f(x) = x^2 + 14x + 30$. The required remainder is $f(-4) = (-4)^2 + 14(-4) + 30 = -10$
$\therefore$ Remainder is –10,

2.02 If $g(x) = 3x^2 + 7x + 15$, find the remainder when g(x) is divided by 3x – 2.

Sol. We know that when f(x) is divided by ax – b, the remainder is f(b/a),
$g(x) = 3x^2 + 7x + 15$

$$\therefore \text{ Rem } \frac{g(x)}{3x-2} = g\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^2 + 7\left(\frac{2}{3}\right) + 15$$

$$= \frac{4}{3} + \frac{14}{3} + 15$$

= 6 + 15 = 21
$\therefore$ Remainder is 21.

2.03 A linear function in x leaves a remainder of 32 when divided by x – 4. If the function is divisible by x + 4, find the linear function.

Sol. Since x + 4 is a factor of the linear function, let the linear function be k(x + 4).
f(x) = k(x + 4)
Given f(4) = 32 $\Rightarrow$ k(4 + 4) = 32 $\Rightarrow$ k = 4
$\therefore$ The linear function = 4(x + 4) = 4x + 16.

Factor theorem:

If R = 0, i.e p(a) = 0, then x – a is a factor of p(x) and conversely, if x – a is a factor of p(x), then p(a) = 0. This immediate consequence of the Remainder Theorem is called the Factor Theorem. This can be restated as follows: The number a is a root of p(x) = 0, if and only if (x – a) is a factor of p(x).

2.04 If $(a-3)x^3 + (a+2)x^2 + x - 2a$ is divisible by x + 2, find the value of a.

Sol. Let $f(x) = (a-3)\,x^3 + (a+2)\,x^2 + x - 2a$
Given x + 2 is a factor of f(x)
$\therefore$ f(–2) = 0
(a – 3) (–8) + (a + 2) (4) – 2 – 2a = 0
$\Rightarrow$ –8a + 24 + 4a + 8 – 2 – 2a = 0
$\Rightarrow$ –6a + 30 = 0 $\Rightarrow$ a = 5.

2.05 If $ax^2 + bx + c$ is divisible by 5x – 3, find the value of 9a + 15b +25c.

Sol. Let $f(x) = ax^2 + bx + c$
Given f(x) is divisible by 5x – 3

$$\therefore f\left(\frac{3}{5}\right) = 0$$

$$a\left(\frac{9}{25}\right) + b\left(\frac{3}{5}\right) + c = 0 \Rightarrow \quad 9a + 15b + 25c = 0.$$

Problems based on factor and remainder theorems:

2.06 If $f(x) = 3x^2 + 9x + 15$, find the remainder when f(x) is divided by 3x + 1.

Sol. We know that when f(x) is divided by ax + b, the remainder is $f\left(\frac{-b}{a}\right)$

$f(x) = 3x^2 + 9x + 15$ and the divisor is $3x + 1$

required remainder is $f\left(\frac{-1}{3}\right)$

$= 3\left(\frac{-1}{3}\right)^2 + 9\left(\frac{-1}{3}\right) + 15 = \frac{1}{3} - 3 + 15 = \frac{37}{3}$

2.07 Q(x) is a quadratic expression and Q(0) = 14. The remainder of Q(x) when divided by x + 1 is 25. The remainder of Q(x) when divided by x – 2 is 10. Find Q(x).

Sol. Let $Q(x) = ax^2 + bx + c$
Given $Q(0) = 14 \Rightarrow c = 14$
$Q(-1) = 25$
$\therefore a - b + c = 25$
$\Rightarrow a - b = 25 - 14 \Rightarrow a - b = 11$ ……. (1)
given $Q(+2) = 10$
$4a + 2b + c = 10 \Rightarrow 4a + 2b = 10 - 14$
$\Rightarrow 4a + 2b = -4$
$\Rightarrow 2a + b = -2$ ……. (2)
Solving (1) and (2) $\rightarrow$ a = 3; b = –8
$\therefore Q(x) = 3x^2 - 8x + 14$.

2.08 $f(x) = ax^3 + 3x^2 - 4x + b$. If f(x) is divisible by $x^2 - 4$, find a + b.

Sol. Given f(x) is divisible by $x^2 - 4$.
$\therefore$ f(x) is also divisible by x – 2 and x + 2.
f(2) = 0; f(–2) = 0
$f(x) = ax^3 + 3x^2 - 4x + b$
$f(2) = 0 \Rightarrow 8a + 12 - 8 + b = 0 \Rightarrow 8a + b = -4$ ….. (1)
$f(-2) = 0 \Rightarrow -8a + 12 + 8 + b = 0 \Rightarrow -8a + b = -20$ …… (2)
Solving (1) and (2) a= 1; b = –12
$\therefore a + b = 1 - 12 = -11$.

2.09 A quadratic expression in x leaves remainders 13 and 8 when divided by x – 2 and x + 3 respectively. If the expression is divisible by x – 1, find the quadratic expression.

Sol. Let the quadratic expression be Q(x)
$= ax^2 + bx + c$
Given Q(1) = 0; and Q(2) = 13, Q(–3) = 8.
$\Rightarrow a + b + c = 0$ …… (1)
$\Rightarrow 4a + 2b + c = 13$ ……. (2)
$\Rightarrow 9a - 3b + c = 8$ …….. (3)
Solving (1), (2) and (3) we will get a = 3, b = 4, c = –7
$\therefore Q(x) = 3x^2 + 4x - 7$.

2.10 If 3x – 2 is a common factor of $12x^2 + (5b + 2)x - (3a + 1)$ and $6ax^2 + (4b + 5)x - 14$, find (a, b).

Sol. Let $f(x) = 12x^2 + (5b + 2)x - (3a + 1)$
$g(x) = 6ax^2 + (4b + 5)x - 14$
given 3x – 2 is a factor of both f(x) and g(x).
$\therefore$ f(2/3) = 0 and g(2/3) = 0

$$12.\frac{4}{9} + (5b+2)\left(\frac{2}{3}\right) - (3a+1) = 0$$

$$\frac{16}{3} + \frac{10b+4}{3} - 3a - 1 = 0$$

10b – 9a = –17 …… (1)

$$g(2/3) = 0 \Rightarrow 6a.\frac{4}{9} + (4b+5)\frac{2}{3} - 14 = 0$$

$$\frac{8a}{3} + \frac{8b}{3} + \frac{10}{3} - 14 = 0$$

8a + 8b = 32
a + b = 4 …….. (2)
solving (1) and (2) a = 3, b = 1., ie (a, b) = (3, 1).

2.11 Find the remainder when x^{888} is divided by $x^2 - 5x + 6$.

Sol. Let $f(x) = x^{888}$
Let f(x) be divided by $x^2 - 5x + 6$.
The remainder is ax + b and Q(x) is the quotient.
By division algorithm
$f(x) = (x^2 - 5x + 6)\,Q(x) + R(x)$
$x^{888} = (x - 3)(x - 2)\,Q(x) + ax + b$
put x = 2
$2^{888} = 2a + b$ …… (1)
Put x = 3
$3^{888} = 3a + b$ …….. (2)
Solving (1) and (2)
$a = 3^{888} - 2^{888}$
$b = 3\,.\,2^{888} - 2.\,3^{888}$
$\therefore$ Required remainder is $(3^{888} - 2^{888})x + 3(2^{888}) - 2(3^{888})$.

2.12 Find the remainder when x^5 is divided by $x^3 - 4x$.

Sol. Let $f(x) = x^5$
Since divisor $x^3 - 4x$ is a cubic expression, the remainder must be a quadratic expression.
If f(x) is divided by $x^3 - 4x$, the remainder is $ax^2 + bx + c$ and quotient is, say q(x)
By division algorithm, $f(x) = (x^3 - 4x)\,q(x) + r(x)$
$x^5 = x(x - 2)(x + 2)\,q(x) + ax^2 + bx + c$.
Set x = 0 we get c = 0
Set x = 2 we get $2^5 = 4a + 2b \Rightarrow 16 = 2a + b$ ……. (1)
Set x = –2
$(-2)^5 = 4a - 2b$
–16 = 2a – b …… (2)
Solving (1) and (2) a = 0, b = 16
$\therefore$ Required remainder is 16x.
Alternately, by dividing x^5 by $x^3 - 4x$ using long division, we will get the same remainder.

Division of a polynomial by a polynomial:

Long division method:

Step 1: First arrange the terms of the dividend and the divisor in the descending order of their degrees.
Step 2: Now the first term of the quotient is obtained by dividing the first term of the dividend by the first term of the divisor.
Step 3: Then multiply all the terms of the divisor by the first term of the quotients and subtract the result from the dividend.
Step 4: Consider the remainder as new dividend and proceed as before.
Step 5: Repeat this process till we obtain a remainder which is either 0 or a polynomial of degree less than that of the divisor.

Example : Divide $2x^3 + 9x^2 + 4x - 15$ by $2x + 5$.

Solution: $2x + 5)\ 2x^3 + 9x^2 + 4x - 15\ (x^2 + 2x - 3$

$$
\begin{array}{r}
2x^3 + 5x^2 \\
(-)\ \ (-) \\
\hline
4x^2 + 4x \\
4x^2 + 10x \\
(-)\ \ (-) \\
\hline
-6x - 15 \\
-6x - 15 \\
(+)\ \ (+) \\
\hline
0 \\
\hline
\end{array}
$$

$\therefore (2x^3 + 9x^2 + 4x - 15) \div (2x + 5) = x^2 + 2x - 3$

Example :

Find the quotient and the remainder when
$x^4 + 4x^3 - 31x^2 - 94x + 120$ is divided by $x^2 + 3x - 4$.

Solution: $x^2 + 3x - 4)\ x^4 + 4x^3 - 31x^2 - 94x + 120\ (x^2 + x - 30$

$$
\begin{array}{r}
x^4 + 3x^3 - 4x^2 \\
-\ \ -\ \ + \\
\hline
x^3 - 27x^2 - 94x + 120 \\
x^3 + 3x^2 - 4x \\
-\ \ -\ \ + \\
\hline
-30x^2 - 90x + 120 \\
-30x^2 - 90x + 120 \\
+\ \ +\ \ - \\
\hline
0
\end{array}
$$

$\therefore$ the quotient is $x^2 + x - 30$ and the remainder is '0'

Relations between Roots and Coefficients:

An n^{th} order equation has n roots. Corresponding to every root, there is a factor. If α is a root of $f(x) = 0$, then $x - \alpha$ is a factor of $f(x)$. Sometimes $(x - \alpha)^2$ may also be a factor. In such a case, α is said to be a double root. Similarly equations can have triple roots, quadruple roots and roots of multiplicity m. If m is the greatest value of k, for which $(x - \alpha)^k$ is a factor of f(a), then α is said to be a root of multiplicity m. If all the roots are counted by taking their multiplicity into account, the number of roots is equal to n, the degree of the equation.

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ (not necessarily distinct) are the roots of $f(x) = 0$, then
$f(x) = a_n(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$
$= a_n[x^n - S_1 x^{n-1} + S_2 x^{n-2} \ldots + (-1)^n S_n]$
where S_1 = the sum of the roots
S_2 = the sum of the products of the roots taken 2 at a time
S_3 = the sum of the product of the roots taken 3 at a time and so on.
S_n = the 'sum' of the product of the roots taken n(or all) at a time. Thus, S_n is a single term.
$S_n = \alpha_1 \alpha_2 \ldots \alpha_n$
Let us write down the polynomial f(x) in two forms:

The standard form
$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} \ldots + a_1 x + a_0$
In terms of the roots of the corresponding equation.
$f(x) = a_n [x_n - S_1 x^{n-1} + S_2 x^{n-2} + \ldots + (-1)^{x-1} S_{n-1} x + (-1)^n S_n]$

These polynomials are identically equal, i.e, equal for all values of x. Therefore the corresponding coefficients are equal. The sum of the roots $S_1 = -a_{n-1} / a_n$

The sum of the products of the roots, taken two at a time, $S_2 = a_{n-2} / a_n$

The sum of the products of the roots, taken three at a time, $S_3 = -a_{n-3} / a_n$ and so on.

The 'sum' of the 'products' of the roots taken m ($m \le n$) at a time $S_m = \Sigma \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_m = (-1)^m \frac{a_{n-m}}{a_n}$

$\therefore S_n = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n = (-1)^n \frac{a_0}{a_n}$

For example, consider the polynomial equation
$(x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6 = 0$
(We can see immediately that the roots are 1, 2, 3)
The sum of roots = $(1 + 2 + 3) = -(-6)/1$

The sum of the products of the roots, taken two at a time
$S_2 = 1(2) + 1(3) + 2(3) = 11 = 11/1$
We can drop the word 'sum' and 'products' for the last relation, because there is only one term (only one product). The product $= 1(2)(3) = 6 = -(-6)/1$.

Roots of Equations and Descartes' Rule:

If the coefficients are all real and the complex number z_1, is a root of $f(x) = 0$, then the conjugate of z_1, viz, $\overline{z_1}$ is also a root of $f(x) = 0$. Thus, for equations with real, coefficients, complex roots occur in pairs.
A consequence of this is that any equation of an odd degree must have at least one real root.
The number of roots is related to very simple properties of the equation as illustrated below.
Let α_1 be a positive root, ie $x - \alpha_1$, is a factor.
Let α_2 be another positive root, ie, $x^2 - (\alpha_1 + \alpha_2)x + \alpha_1 \alpha_2$ is a factor.
Let α_3 be another positive root ie $x^3 - (\alpha_1 + \alpha_2 + \alpha_3)x^2 + (\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1) x - \alpha_1 \alpha_2 \alpha_3$ is a factor
We note that every positive root introduces a sign change in the polynomial. For 1 root, there is 1 sign change (the coefficient of x is positive and $-\alpha_1$ is negative)

The second root results in a second sign change [$x^2 - (\alpha_1 + \alpha_2)x + \alpha_1 \alpha_2$ has 2 sign changes] and so on.
But every sign change need not correspond to a real positive root. (For example $x^2 - 2x + 4$ has two sign changes but the corresponding equation $x^2 - 2x + 4 = 0$ has no real roots.)

The number of positive roots of $f(x) = 0$ is at the most equal to the number of sign changes in f(x). It could also be less than that by 2, 4… i.e., if there are k sign changes in f(x), the number of positive roots could be k, k–2, k–4, …
This is called **Descartes' Rule of Signs**. This rule can be extended to negative roots as follows. The number of negative roots of $f(x) = 0$ is at the most equal to the number of sign charges in $f(-x)$.

For example, consider
$f(x) = x^5 – 3x^3 + 6x^2 – 28x + 24$.There are 4 sign changes in f(x)

$\therefore$ The number of positive roots could be 4, 2 or 0,
Consider g(x) = f(–x)
$(–x)^5 – 3(–x)^3 + 6(–x)^2 – 28 (–x) + 24$
$= –x^5 + 3x^3 + 6x^2 + 28x + 24$
There is only one sign change in f(– x). $\therefore$ The number of negative roots of f(x) = 0 is 1. (It can't be –1, –3, …)

The following table shows the various possibilities for the five roots.

Negative	Positive	Complex
1	4	0
1	2	2
1	0	4

We can use more advanced techniques to find the actual roots. But even without that, using only Descartes' Rule, we expect exactly one of the 3 possibilities shown in the table above.

Examples:

2.13. Find the roots of the equation $2x^2 + 13x + 18 = 0$.

Sol: To find the roots of a quadratic equation, the following steps are required. First write the coefficient of x i.e. 13 as the sum (or difference) of two parts such that the product of these two parts is equal to the coefficient of x^2 term and constant term i.e. product of 2 and 18 which is 36.
We see that 13 can be written as the sum of 4 and 9 and the product of these two numbers is 36.
$2x^2 + 13x + 18 = 0 \Rightarrow 2x^2 + 4x + 9x + 18 = 0$
Taking 2x common from the first two terms and taking 9 common from the last two terms, we have:
$\Rightarrow 2x (x + 2) + 9 (x + 2) = 0$
$\Rightarrow (x + 2) (2x + 9) = 0$, $x = –2$ or $-\frac{9}{2}$.

2.14. Find the roots of the equation $x^2 + x – 12 = 0$.

Sol: Given equation is $x^2 + x – 12 = 0$
Applying the procedure described above, we have
$\Rightarrow x^2 + 4x – 3x – 12 = 0$
$\Rightarrow x (x + 4) – 3 (x + 4) = 0$
$\Rightarrow (x – 3) (x + 4) = 0 \Rightarrow x = 3$ or $x = –4$.

2.15. Find the roots of the equation $11x^2 – 37x + 30 = 0$.

Sol: We have to write –37 as the sum of two parts whose product should be equal to $(11) \times (30)$
$(–22) + (–15) = –37$ and $(–22) (–15) = 11 \times 30$
Therefore, $11x^2 – 37x + 30 = 0$
$\Rightarrow 11x^2 – 22x – 15x + 30 = 0$
$\Rightarrow 11x (x – 2) – 15 (x – 2) = 0$
$\Rightarrow (11x – 15) (x – 2) = 0$ $x = \frac{15}{11}$ or 2.

2.16. Discuss the nature of the roots of the equation $8x^2 – 2x – 4 = 0$.

Sol: For the quadratic equation $ax^2 + bx + c = 0$ the nature of the roots is given by the discriminant $b^2 – 4ac$.
Discriminant of $8x^2 – 2x – 4 = 0$ is
$(–2)^2 – 4(8) (–4) = 132$.
Since the discriminant is positive but not a perfect square, the roots of the equation are irrational and unequal.

2.17. Comment on the nature of the roots of $3x^2 – x – 4 = 0$.

Sol: Discriminant of $3x^2 – x – 4 = 0$ is $(–1)^2 – 4 (3) (– 4) = 1 + 48 = 49$. Since the discriminant is positive and a perfect square, the roots of the equation are rational and unequal.

2.18. If the sum of the roots of the equation $Rx^2 + 5x – 24 = 0$ is 5/11, then find the product of the roots of that equation.

Sol: For a quadratic equation $ax^2 + bx + c = 0$, the sum of the roots is (–b/a) and the product of the roots is (c/a).
The sum of the roots of the equation
$Rx^2 + 5x – 24 = 0$ is $\left(\frac{-5}{R}\right)$ which is given as $\frac{5}{11}$
$\therefore R = –11$
In the given equation, product of the roots $= \frac{-24}{R} = \frac{-24}{-11} = +\frac{24}{11}$.

2.19. Find the value of k, so that the roots of $6x^2 – 12x – k = 0$ are reciprocals of each other.

Sol: If the roots of the equation are reciprocals of each other, then the product of the roots should be equal to 1.
$\Rightarrow \frac{-k}{6} = 1$. Therefore $k = –6$.

2.20. If $4 + \sqrt{7}$ is one root of a quadratic equation with rational co-efficients, then find the other root of the equation.

Sol: When the coefficients of a quadratic equation are rational and the roots are irrational, they occur only in pairs like $p \pm \sqrt{q}$ i.e., if $p + \sqrt{q}$ is one root, then the other root of the equation will be $p – \sqrt{q}$. So, in this case, the other root of the equation will be $4 – \sqrt{7}$.

2.21. Find the positive value of k if one root of the equation $x^2 – kx + 243 = 0$ is three times the other root.

Sol: If one root of the equation is α, then the other root will be 3α.
We have $(\alpha) (3\alpha) = 3\alpha^2 = 243$
$\Rightarrow \alpha^2 = 81$
$\Rightarrow \alpha = \pm 9$. Hence $3\alpha = \pm 27$.
Sum of the roots $= -\left(\frac{-k}{1}\right) = k = 4\alpha = \pm 36$.
Since we need the positive value of k, so k = 36.

2.22. Form a quadratic equation whose roots are 4 and 21.

Sol: The sum of the roots = 4 + 21 = 25.
The product of the roots = $4 \times 21 = 84$.
We know that if p is the sum of the roots and q is the product of the roots of a quadratic equation, the equation will be $x^2 - px + q = 0$. Hence the required equation will be $x^2 - 25x + 84 = 0$.

2.23. Form a quadratic equation with rational coefficients, one of whose roots is $5 + \sqrt{6}$.

Sol: If $5 + \sqrt{6}$ is one root, then the other root is $5 - \sqrt{6}$ (because the coefficients are rational).
The sum of the roots = $5 + \sqrt{6} + 5 - \sqrt{6} = 10$.
The product of the roots = $(5 + \sqrt{6})(5 - \sqrt{6})$
$= 25 - 6 = 19$.
Thus the required equation is
$x^2 - 10x + 19 = 0$.

2.24. If the price of each book goes up by ₹5, then Atul can buy 20 books less for ₹1200. Find the original price and the number of books Atul could buy at the original price.

Sol: Let the original price of each book be x.
Then the new price of each book will be x + 5.
The number of books that can be bought at the original price = $\frac{1200}{x}$
The number of books that can be bought at the new price = $\frac{1200}{x+5}$
Given that Atul gets 20 books less at new price
i.e. $\frac{1200}{x} - \frac{1200}{x+5} = 20$
$\Rightarrow \frac{60}{x} - \frac{60}{x+5} = 1 \Rightarrow \frac{60(x+5-x)}{x^2+5x} = 1$
$\Rightarrow 300 = x^2 + 5x \Rightarrow x^2 + 5x - 300 = 0$
$\Rightarrow (x + 20)(x - 15) = 0 \Rightarrow x = -20$ or 15
As the price cannot be negative, the original price is ₹15.

2.25. If α and β are the roots of the equation $x^2 - 3x - 180 = 0$ such that $\alpha < \beta$, then find the values of

(i) $\alpha^2 + \beta^2$ (ii) $\frac{1}{\alpha} + \frac{1}{\beta}$ (iii) $\alpha - \beta$

Sol: From the given equation, we get $\alpha + \beta = 3$ and $\alpha\beta = -180$
(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
$= (3)^2 - 2(-180) = 369$
(ii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{+3}{-180} = \frac{-1}{60}$
(iii) $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$
$\Rightarrow \alpha - \beta = \pm\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$
$= \pm\sqrt{(+3)^2 - 4(-180)} = \pm\sqrt{9 + 720}$
$= \pm\sqrt{729} = \pm 27$; as $\alpha < \beta$, $\alpha - \beta = -27$

2.26. If $\sqrt{x+4} + \sqrt{x+8} = 7$, then find the value of x.

Sol: Given $\sqrt{x+4} + \sqrt{x+8} = 7$
Squaring on both sides, we get
$x + 4 + x + 8 + 2\left(\sqrt{x+4}\sqrt{x+8}\right) = 49$
$\Rightarrow 2x + 12 + 2\sqrt{x^2 + 12x + 32} = 49$
$\Rightarrow 2x - 37 = -2\sqrt{x^2 + 12x + 32}$
Squaring again on both sides, we have
$(2x - 37)^2 = 4(x^2 + 12x + 32)$
$\Rightarrow 4x^2 - 148x + 1369 = 4x^2 + 48x + 128$
$\Rightarrow 1241 = 196x$
$\Rightarrow x = \frac{1241}{196}$

2.27. If $4^{2x+1} + 4^{x+1} = 80$, then find the value of x.

Sol: Given $4^{2x+1} + 4^{x+1} = 80$
$\Rightarrow 4^{2x} \cdot 4 + 4^x \cdot 4 = 80$
$4^{2x} + 4^x = 20$
Substituting $4^x = a$,
we get $a^2 + a = 20$
$\Rightarrow a^2 + a - 20 = 0$
$\Rightarrow (a + 5)(a - 4) = 0$
$\Rightarrow a = -5$ or 4
If $4^x = -5$, there is no possible value for x as no power of 4 gives negative value.
If $4^x = 4$, then x = 1.

2.28. Find the nature of roots of the equation, $f(x) = x^3 + x - 2 = 0$.

Sol: There is only 1 change of sign in f(x).
We know that when f(x) has r changes of sign then f(x) has r, r – 2, r – 4, ….. positive roots.
$\therefore$ f(x) = 0 has one positive root.
Now
$f(-x) \equiv -x^3 - x - 2 = 0$. q = 0
Since there is no change of sign in f(–x), f(x) has no negative roots. The number of complex roots is even.
$\therefore$The equation has one positive root, and two complex roots.
Hence f(x) = 0 has 1 real root and two complex roots.

2.29. How many non-real-roots does the equation $x^4 - 2x^2 + 3x - 2 = 0$ have?

Sol: Let $f(x) = x^4 - 2x^2 + 3x - 2$
f(x) has 3 sign changes
$\therefore$ f(x) has 3 or 1 positive roots.
$f(-x) = x^4 - 2x^2 - 3x - 2$
$\therefore$ f(– x) has one sign change
$\therefore$ f(x) has exactly one negative root.
As the sum of the co-efficient of f(x) is zero, x = 1 is a root f f(x) = 0
$\therefore f(x) = (x - 1)(x^3 + x^2 - x + 2) = (x - 1) f_1(x)$
By trial, $f_1(-2) = 0$
$\therefore f_1(x) = (x + 2)(x^2 - x + 1)$
We can see that $x^2 - x + 1 = 0$ has two non-real

roots.
$\therefore$ f(x) has one positive, one negative and two non-real roots.

2.30. If p – q, p, p + q are the roots of the equation $x^3 - 18x^2 + 99x - 162 = 0$, then find the values of p and q.

Sol: Given p – q, p, p + q are the roots of the equation.
$\therefore$The sum of the roots is (p – q) + p + (p + q) = 18
$\Rightarrow 3p = 18 \Rightarrow p = 6$
and the product of the roots is (p – q) p (p + q) = 162
$p^2 - q^2 = \frac{162}{6} = 27 \Rightarrow 36 - q^2 = 27$
$\Rightarrow q = \pm 3 \therefore p = 6$ and $q = \pm 3$

2.31. Find the range of the expression $\frac{x-2}{x^2+x+3}$ where x is real.

Sol: Let $f(x) = \frac{x-2}{x^2+x+3} = y$
$\Rightarrow x^2y + xy + 3y = x - 2$
i.e, $x^2y + x(y - 1) + 3y + 2 = 0$
f(x) can have any value y, provided the above equation in x has real roots
$\therefore b^2 - 4ac \geq 0$
$\Rightarrow (y - 1)^2 - 4y(3y + 2) \geq 0$
i.e, $11y^2 + 10y - 1 \leq 0$
$(11y - 1)(y + 1) \leq 0 \Rightarrow -1 \leq y \leq \frac{1}{11}$
$\therefore$ The range of y is $\left[-1, \frac{1}{11}\right]$

2.32. Solve the equation $x^4 - 2x^3 - 19x^2 + 8x + 60 = 0$, given that two of the roots are α and $-\alpha$, where $\alpha > 0$.

Sol: Let the roots of the equation be $\alpha, -\alpha, \beta, \gamma$. Let $\beta > \gamma$
The sum of the roots = $\alpha - \alpha + \beta + \gamma = 2$
$\Rightarrow \beta + \gamma = 2$
The sum of the products of the roots taken three at a time =
$-\alpha^2\beta - \alpha^2\gamma + \alpha\beta\gamma - \alpha\beta\gamma = -8$
$\alpha^2(\beta + \gamma) = 8. \therefore \alpha^2 = 4$
The product of the roots = $-\alpha^2\beta\gamma = 60$. $\beta\gamma = -15$
$\beta + \gamma = 2$ and $\beta\gamma = -15$, $\alpha^2 = 4$
$\beta = 5$ and $\gamma = -3$, $\alpha = \pm 2$
$\Rightarrow$ The roots are –2, 2, – 3 and 5.

2.33. Find the sum of the squares of the roots of the equation $x^3 - 4x^2 + x + 6 = 0$.

Sol: Let α, β, γ be the roots of the given equation.
$\alpha + \beta + \gamma = 4$, $\alpha\beta + \beta\gamma + \alpha\gamma = 1$ (and $\alpha\beta\gamma = -6$).
$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$
$= (4)^2 - 2(1) = 14$

2.34. Find the roots of the equation $x^2 + 9x - 10 = 0$.

Sol: $x^2 + 9x - 10 = 0$
$\Rightarrow x^2 + 10x - x - 10 = 0$
$\Rightarrow x(x + 10) - 1(x + 10) = 0$
$\Rightarrow (x + 10)(x - 1) = 0 \therefore x = -10$ or 1

2.35. Find the roots of the equation $4x^2 - 17x + 4 = 0$.

Sol: $4x^2 - 17x + 4 = 0$
$\Rightarrow 4x^2 - 16x - x + 4 = 0$
$\Rightarrow (4x - 1)(x - 4) = 0$
$\Rightarrow x = \frac{1}{4}$ or 4

2.36. Find the nature of the roots of the equation $9x^2 - 3x + 1 = 0$.

Sol: Discriminant = $(-3)^2 - 4(9)(1) = -27$
($\because$ discriminant = $b^2 - 4ac$)
Since the discriminant is negative, the roots are complex conjugates.

2.37. Find the nature of the roots of the equation $5x^2 - x - 4 = 0$.

Sol: Discriminant = $(-1)^2 - 4(5)(-4) = 81$. As the coefficients are rational and the discriminant is positive and a perfect square, the roots are rational and unequal.

2.38. If the sum of the roots of the equation $kx^2 - 52x + 24 = 0$ is $\frac{13}{6}$, find the product of its roots.

Sol: $\frac{13}{6} = \frac{52}{k}$ ($\because$ sum of the roots of $ax^2 + bx + c = 0$ is $-\frac{b}{a}$)
k = 24
The product of its roots = $\frac{24}{k} = 1$
($\because$ product of the roots of $ax^2 + bx + c = 0$ is $\frac{c}{a}$).

2.39. If the roots of the equation $6x^2 - 7x + b = 0$ are reciprocals of each other, find b.

Sol: Let the roots be α and $\frac{1}{\alpha}$
The product of the roots = $\frac{b}{6}$
$= (\alpha)\left(\frac{1}{\alpha}\right) = 1$
$\therefore b = 6$

2.40. The roots of a quadratic equation are a and –a. The product of its roots is –9. Form the equation in variable x.

Sol: The sum of the roots = 0 and the product of the roots = –9
The quadratic equation whose roots are α and β is $x^2 - x(\alpha + \beta) + \alpha\beta = 0$ i.e
$x^2 - x$ (sum of the roots) + product = 0
Hence the quadratic equation is $x^2 - 9 = 0$

2.41. The roots of the equation $x^2 - 12x + k = 0$ are in the ratio 1 : 2. Find k.

Sol: Let the roots be m and 2m.
$m + 2m = -(-12) = 12$
$\Rightarrow m = 4$
k = the product (m) (2m) = 32

2.42. A quadratic equation has rational coefficients. One of its roots is $2+\sqrt{2}$. Find its other root.

Sol: If a quadratic equation has rational coefficients, and one of its roots is $p+\sqrt{q}$ (where p, q are rational), the other root is $p - \sqrt{q}$.
$\therefore$The other root of the given equation is $2-\sqrt{2}$.

2.43. I can buy a certain number of books for ₹1050. If the price of the book increases by ₹15, the number of books I can buy for the same amount decreases by 9. Find the original price and the number of books I could buy at that price.

Sol: Let the original price be ₹x per book
The number of books I could buy at that price
$= \frac{1050}{x}, \frac{1050}{x+15}$
$= \frac{1050}{x} - 9$
$\frac{1050}{x} - \frac{1050}{x+15} = 9$
$(1050)\,15 = 9\,(x + 15)x$
$x^2 + 15x - 1750 = 0$
$x = \frac{-15 \pm \sqrt{15^2 - 4(1)(-1750)}}{2}$ = 35 or – 50
as x > 0, x = 35

2.44. P and Q are the roots of the equation $x^2 - 22x + 120 = 0$.
Find the value of
(i) $P^2 + Q^2$

(ii) $\frac{1}{P} + \frac{1}{Q}$

(iii) difference of P and Q

Sol: P + Q = 22 and PQ = 120
(i) $P^2 + Q^2 = (P + Q)^2 - 2PQ$
$= 22^2 - 2\,(120) = 244$

(ii) $\frac{1}{P} + \frac{1}{Q} = \frac{Q+P}{PQ} = \frac{22}{120} = \frac{11}{60}$

(iii) (Difference of P and Q)^2
$= (P + Q)^2 - 4PQ = 22^2 - 4\,(120) = 4$
$\therefore$ Difference of P and Q = 2

2.45. Find the remainder when $f(x) = x^2 + 6x + 8$ is divided by $2x + 1$.

Sol: Given polynomial is $f(x) = x^2 + 6x + 8$ and divisor is $2x + 1$.
$\therefore$ Remainder = $f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 + 6\left(-\frac{1}{2}\right) + 8$
$= \frac{1}{4} - 3 + 8 = 5\frac{1}{4}$

2.46. If $4^{x+2} + 4^{2x+1} = 1280$, find x.

Sol: $4^{x+2} + 4^{2x+1} = 1280$
$(4^x)(4^2) + (4^x)^2\,(4) = 1280$
$4\,(4^x)^2 + 16\,(4^x) = 1280$
Dividing both sides by 4
$(4^x)^2 + 4\,(4^x) = 320$
let $4^x = y$
$\Rightarrow y^2 + 4y - 32 = 0$
$\Rightarrow (y + 20)(y - 16) = 0$
$\Rightarrow y = -20$ or 16
$\therefore 4^x = 16$ or -20
as $4^x > 0$, $4^x = 16 \Rightarrow x = 2$

2.47. The minimum value of $2x^2 + bx + c$ is known to be $\frac{15}{2}$ and occurs at $x = -\frac{5}{2}$. Find the values of b and c.

Sol: The minimum value of $ax^2 + bx + c$ (where a > 0) occurs at $x = -b/2a$ and it equals $\frac{4ac-b^2}{4a}$.
$\Rightarrow \frac{-b}{2(2)} = -\frac{5}{2}$
b = 10

minimum value = $\frac{4(2)c - 10^2}{4(2)} = \frac{15}{2}$
$\Rightarrow 8c - 100 = 60$

$\Rightarrow c = \frac{100+60}{8} = 20$
$\therefore$ The values of b and c are 10 and 20 respectively.

2.48. Find the number of positive and negative roots of the equation
$x^3 - ax + b = 0$ where a > 0 and b > 0.

Sol: Let $f(x) = x^3 - ax + b$
In f(x) there are two changes of sign (+ – +)
$\therefore$ By Descartes' rule there exist at most two positive roots. The number of positive roots could be 2 or 0.
$f(-x) = (-x)^3 - a\,(-x) + b$
$f(-x) = -x^3 + ax + b$
There is only one sign change in f(–x). Therefore there exists at most one negative root. As the number of negative roots cannot be negative, it is exactly equal to 1. Therefore, f (x) = 0 has 2 or 0 positive roots and exactly 1 negative root.

2.49. If –1 and 2 are two of the roots of the equation $x^4 - 3x^3 + 2x^2 + 2x - 4 = 0$, then find the other two roots.

Sol: Given equation is $x^4 - 3x^3 + 2x^2 + 2x - 4 = 0$ and –1, 2 are the roots of it.

We can find the Quotient as follows

$$
\begin{array}{r|l}
 & x^3 - 4x^2 + 6x - 4 \\
\hline
x + 1 & x^4 - 3x^3 + 2x^2 + 2x - 4 \\
 & x^4 + x^3 \\
\hline
 & -4x^3 + 2x^2 \\
 & -4x^3 - 4x^2 \\
\hline
 & 6x^2 + 2x \\
 & 6x^2 + 6x \\
\hline
 & -4x - 4 \\
 & -4x - 4 \\
\hline
 & 0
\end{array}
$$

$$
\begin{array}{r|l}
 & x^2 - 2x + 2 \\
\hline
x - 2 & x^3 - 4x^2 + 6x - 4 \\
 & x^3 - 2x^2 \\
\hline
 & -2x^2 + 6x \\
 & -2x^2 + 4x \\
\hline
 & 2x - 4 \\
 & 2x - 4 \\
\hline
 & 0
\end{array}
$$

$\therefore$ The quotient when the given expression is divided by (x + 1) (x – 2) is $x^2 - 2x + 2$ which is a quadratic equation whose discriminant is negative.

The roots of $x^2 - 2x + 2 = 0$ are $\frac{2+\sqrt{4-8}}{2}$ i.e. $1 \pm i$.

The division shown above can be performed in very compact way, using Horner's Method of Synthetic Division.

x = –1	1	–3	2	2	–4
	0	–1	4	–6	4
	1	–4	6	–4	0
x = 2	0	2	–4	–4	0
	1	–2	2	0	

2.50. If one root of the cubic equation $ax^3 + bx^2 + cx + d = 0$ is the negative of another, what is the condition that the coefficients satisfy?

Sol: Let the roots be $\alpha, -\alpha, \beta$

$\therefore \alpha - \alpha + \beta = \frac{-b}{a}$, ie $\frac{-b}{a}$ is a root of the equation

$$\therefore\ a\left(\frac{-b^3}{a^3}\right)+b\left(\frac{b^2}{a^2}\right)+c\left(\frac{-b}{a}\right)+d = 0$$

$\Rightarrow ad = bc$

2.51. If α, β, γ are the roots of $x^3 + 10x^2 + x + 7 = 0$, find the value of $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

Sol: Let $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = E$

$\therefore\ \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma} = E$

$\alpha\beta + \beta\gamma + \gamma\alpha = 1$

And $\alpha\beta\gamma = -7$ $\therefore E = -1/7$

2.52. Solve the following equation
$F(x) = x^4 - 5x^3 + 10x^2 - 20x + 24 = 0$

Sol: By trial
$F(1) = 1 - 5 + 10 - 20 + 24 = 10 \neq 0$
$F(-1) = 1 + 5 + 10 + 20 + 24 \neq 0$
$F(2) = 16 - 40 + 40 - 40 + 24 = 0$
$\therefore$ One of the roots is 2 and x – 2 is a factor of f(x).
dividing f(x) by x–2, we get,
$f(x) = (x-2)(x^3 - 3x^2 + 4x - 12)$
$= (x-2)g(x)$ say
By trial g(3) [no need to check for g(1) or g(–1)]
$g(3) = 27 - 27 + 12 - 12 = 0$
Dividing by x – 3, we get
$F(x) = (x-2)(x-3)(x^2+4)$
$\therefore$ The roots are 2, 3, $\pm 2i$

2.53. For the cubic equation $ax^3 + bx^2 + bx + a = 0$, find the roots.

Sol: Let α be a root ie
$a\alpha^3 + b\alpha^2 + b\alpha + a = 0 \rightarrow$ (1)
consider $x = 1/\alpha$

$$\frac{a}{\alpha^3}+\frac{b}{\alpha^2}+\frac{b}{\alpha}+a$$

$$= \frac{1}{\alpha^3}\left(a + b\alpha + b\alpha^2 + a\alpha^3\right) = 0 \text{ (from 1)}$$

$\therefore 1/\alpha$ is also a root.
All together there are 3 roots. Also, if α is a root $1/\alpha$ is also a root. $\therefore$ one of the roots has to be its own reciprocal (1, or –1)
If it is 1, $a + b = 0 \rightarrow$ (A)
If it is -1, no additional condition has to be satisfied. Consider A. The equation would be
$ax^3 - ax^2 - ax + a = 0$
$\Rightarrow (x^2 - 1)(x - 1) = 0$
The roots are 1, –1, 1.
Consider the other possibility.
One of the roots is –1. The three roots are α, $1/\alpha$, –1.
The equation would be $(x^2 + px + 1)(x + 1) = 0$.
Or $x^3 + (p + 1)x^2 + (p + 1)x + 1 = 0 \rightarrow$ (2)

The roots of (2) are –1, $\frac{-p \pm \sqrt{p^2-4}}{2}$

$\therefore$The roots of (1) are –1, $\frac{\left(\frac{-b}{a}+1\right) \pm \sqrt{\left(\frac{-b}{a}+1\right)^2 - 4}}{2}$

($\because p + 1 = \frac{b}{a}$)

Concept Review Questions

Directions for questions 1 to 25: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. Factorise: $12x^2 + 23x + 5$
(A) $(3x + 1)(2x + 5)$ (B) $(4x + 1)(3x + 5)$
(C) $(6x + 1)(x + 5)$ (D) $(2x + 1)(6x + 5)$

2. (a) Find the roots of the quadratic equation $x^2 - x - 20 = 0$.
(A) –5, 4 (B) –5, –4 (C) 5, –4 (D) 5, 4

(b) What are the roots of the quadratic equation $2x^2 - 5x - 3 = 0$?
(A) –1/2, –3 (B) 1/2, –3
(C) 1/2, 3 (D) –1/2, 3

3. Find a quadratic equation whose roots are 3 and 4.
(A) $x^2 + 7x + 12 = 0$ (B) $x^2 - 7x + 12 = 0$
(C) $x^2 - 7x - 12 = 0$ (D) $x^2 + 7x - 12 = 0$

4. Find the roots of the quadratic equation $x^2 - 12x + 13 = 0$.
(A) 1, 13 (B) –1, –13
(C) $6 + \sqrt{23}$, $6 - \sqrt{23}$ (D) None of these

5. (a) If the sum of the roots and the product of the roots of a quadratic equation are 13 and 30 respectively, find its roots.
(A) 10, 3 (B) –10, –3 (C) 10, –3 (D) –10, 3

(b) Find the sum and the product of the roots of the equation $\sqrt{5}\,x^2 + 25x + 2\sqrt{5} = 0$.
(A) $-5\sqrt{5}$, 2 (B) $5\sqrt{5}$, 2
(C) $25\sqrt{5}$, 2 (D) $-25\sqrt{5}$, 2

6. (a) The sum of the squares of two consecutive positive integers added to their product is equal to 331. Find the two integers.
(A) 9, 10 (B) 10, 11 (C) 11, 12 (D) 12, 13

(b) The sum of squares of three consecutive positive integers is 869. Find the numbers.
(A) 14, 15, 16 (B) 15, 16, 17
(C) 16, 17, 18 (D) 17, 18, 19

7. An integer exceeds its reciprocal by $\frac{143}{12}$, the integer is $\boxed{}$

8. (a) The roots of the quadratic equation $2x^2 - 7x + 2 = 0$ are
(A) rational and unequal
(B) real and equal
(C) imaginary
(D) irrational

(b) Find the nature of the roots of the quadratic equation $2x^2 + 6x - 5 = 0$.
(A) Complex conjugates
(B) Real and equal
(C) Conjugate surds
(D) Unequal and rational

(c) If the square of the sum of the roots of a quadratic equation is equal to 4 times the product of its roots, the roots are
(A) complex conjugates.
(B) equal.
(C) conjugate surds.
(D) unequal and rational.

9. The value of the discriminant of the equation $3x^2 + 7x + 2 = 0$ is $\boxed{}$

10. How many roots (both real and complex) does $(x^n - a)^2 = 0$ have?
(A) 2 (B) n + 1 (C) 2n (D) n

11. Find the signs of the roots of the equation $x^2 + x - 420 = 0$.
(A) Both are positive.
(B) Both are negative.
(C) The roots are of opposite signs with the numerically larger root being positive.
(D) The roots are of opposite signs with the numerically larger root being negative.

12. (a) Construct a quadratic equation whose roots are 2 more than the roots of the equation $x^2 + 9x + 10 = 0$.
(A) $x^2 + 5x - 4 = 0$ (B) $x^2 + 13x + 32 = 0$
(C) $x^2 - 5x - 4 = 0$ (D) $x^2 - 13x + 32 = 0$

(b) Construct a quadratic equation whose roots are reciprocals of the roots of the equation $2x^2 + 8x + 5 = 0$.
(A) $5x^2 + 8x + 2 = 0$
(B) $8x^2 + 5x + 2 = 0$
(C) $2x^2 + 5x + 8 = 0$
(D) $8x^2 + 2x + 5 = 0$

(c) Construct a quadratic equation whose roots are one third of the roots of $x^2 + 6x + 10 = 0$.
(A) $x^2 + 18x + 90 = 0$
(B) $x^2 + 16x + 80 = 0$
(C) $9x^2 + 18x + 10 = 0$
(D) $x^2 + 17x + 90 = 0$

13. The sum of the roots of a quadratic equation is 33 and the product of its roots is 90, the sum of the squares of its roots is $\boxed{}$

14. A quadratic equation in x has its roots as reciprocals of each other. The coefficient of x is twice the coefficient of x^2, the sum of the squares of its roots is $\boxed{}$

15. A quadratic equation in x has the sum of its roots as 19 and the product of its roots as 90, the difference of its roots is $\boxed{}$

16. If one root of the quadratic equation $4x^2 - 8x + k = 0$, is three times the other root, the value of k is $\boxed{}$

17. The roots of the quadratic equation $(m - k + \ell)x^2 - 2mx + (m - \ell + k) = 0$ are

(A) $1, \frac{\ell + m - k}{k + m - \ell}$ (B) $1, \frac{2m}{\ell + m - k}$

(C) $1, \frac{k + m - \ell}{\ell + m - k}$ (D) $1, \frac{2k}{k - m + \ell}$

18. The square of the sum of the roots of a quadratic equation E is 8 times the product of its roots. The value of the square of the sum of the roots divided by the product of the roots of the equation whose roots are reciprocals of those of E is $\boxed{}$

19. The common root of $x^2 + 10x + 24 = 0$ and $x^2 + 14x + 48 = 0$ is $\boxed{}$

20. **(a)** The expression $\frac{4ac - b^2}{4a}$ represents the maximum/minimum value of the quadratic expression $ax^2 + bx + c$. Which of the following is true?

(A) It represents the maximum value when $a > 0$.
(B) It represents the minimum value when $a < 0$.
(C) Both (A) and (B)
(D) Neither (A) nor (B)

(b) The quadratic expression $ax^2 + bx + c$ has its maximum/minimum value at x = ______.

(A) $-\frac{b}{2a}$ (B) $\frac{b}{2a}$

(C) $\frac{-2b}{a}$ (D) $\frac{2b}{a}$

(c) Find the maximum value of the quadratic expression $-3x^2 + 4x + 5$.

(A) 19/3 (B) 31/12 (C) 3/19 (D) – 19/3

21. **(a)** If α, β, γ and δ are the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then $\alpha + \beta + \gamma + \delta$ is

(A) – b (B) $\frac{-b}{a}$ (C) – e (D) $\frac{-e}{a}$

(b) In the above question the value of $\alpha\beta\gamma\delta$ =

(A) –e (B) e (C) $\frac{-e}{a}$ (D) $\frac{e}{a}$

22. The lowest possible degree of an equation, with real coefficients two of whose roots are $\sqrt{3}$ and 3 + 2i is

$\boxed{}$

23. Find the degree of the equation $(x^3 - 3)^2 - 6x^5 = 0$

(A) 5 (B) 6
(C) 9 (D) None of these

24. The number of sign changes in $16x^4 - 64x^3 + 56x^2 + 16x - 15$ is $\boxed{}$

25. If an equation is such that for every root α, $1/\alpha$ is also a root, the equation is called a reciprocal equation. Which of the following is a reciprocal equation?

(A) $x^4 - 20x^3 + 33x^2 - 20x + 1 = 0$
(B) $3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0$
(C) $x^5 - 5x^3 + 5x^2 - 1 = 0$
(D) All the above

Exercise – 2(a)

Directions for questions 1 to 40: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. Solve for x.

(i) $x^4 - 35x^2 + 196 = 0$

(A) $\pm\sqrt{7}, \pm 2\sqrt{7}$ (B) $\pm\sqrt{7}, \pm 7$

(C) $\pm 2\sqrt{7}, \pm 7$ (D) $\pm 7, \pm 14$

(ii) $2\{3^{2(1+x)}\} - 4(3^{2+x}) + 10 = 0$

(A) $-1, \log_3\left(\frac{5}{3}\right)$ (B) $-1, \log_3 2$

(C) $-1, \frac{5}{3}$ (D) $-1, \log_3\left(\frac{3}{5}\right)$

2. The value of k for which $(k + 12)x^2 + (k + 12)x - 2 = 0$ has equal roots is ☐

3. In a school, 5/2 times the square root of the total number of children play football. One fourth of the total number of children play tennis. The remaining 28 children play basketball, the total number of children in the school is ☐

4. If $\left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + \frac{23}{4} = 0$, the value of $\left(x + \frac{1}{x}\right)$ is ☐

5. If k is a natural number and $(k^2 - 3k + 2)(k^2 - 7k + 12) = 120$, then k = ☐

6. If the product of the roots of the equation $x^2 - (R + 7)x + 2(2R - 2) = 0$ is three times the sum of the roots, R = ☐

7. Both A and B were trying to solve a quadratic equation. A copied the coefficient of x wrongly and got the roots of the equation as 12 and 6. B copied the constant term wrongly and got the roots as 1 and 26. Find the roots of the correct equation.

(A) 6, 16 (B) –6, –16

(C) 24, 3 (D) –3, –24

8. If $\sqrt{5x-4} - \sqrt{2x+1} = 1$, then x = ______.

(A) 4 (B) 5 (C) 6 (D) 8

9. If the roots of the equation $(x - k_1)(x - k_2) + 1 = 0$, k_1 and k_2 are integers, then which of the following must be true?

(A) k_1, k_2 are two consecutive integers.

(B) $k_2 - k_1 = 2$

(C) $k_1 - k_2 = 2$

(D) Either (B) or (C)

10. The area of a playground is 153 m^2. If the length of the playground is decreased by 4 m and the breadth is increased by 4 m, the playground becomes a square. The side of the square (in meters) is ☐

11. If the roots of the equation $3x^2 + 17x + 6 = 0$ are in the ratio of p : q, compute $\frac{\sqrt{p}}{\sqrt{q}} + \frac{\sqrt{q}}{\sqrt{p}}$.

(A) $\frac{17\sqrt{2}}{6}$ (B) $\frac{17}{6\sqrt{2}}$

(C) $-\frac{17\sqrt{2}}{6}$ (D) $-\frac{17}{6\sqrt{2}}$

12. If x + y = 4, find the maximum / minimum possible value of $x^2 + y^2$.

(A) Minimum, 8 (B) Maximum, 8

(C) Maximum, 16 (D) Minimum, 16

13. If the roots of the equation $ax^2 + bx + c = 0$ are α and β, find the equation whose roots are α^2 and β^2.

(A) $a^2x^2 + (b^2 - 2ca)x + c^2 = 0$

(B) $a^2x^2 - b^2x - 2cax + c^2 = 0$

(C) $a^2x^2 - b^2x + 2cax + c^2 = 0$

(D) $a^2x^2 + b^2 + 2ca\,x + c^2 = 0$

14. The sides of a right-angled triangle are such that the sum of the lengths of the longest and that of the shortest side is twice the length of the remaining side, the longest side of the triangle (in cm) if the longer of the sides containing the right angle is 9 cm more than half the hypotenuse is ☐

15. The roots of the equation $ax^2 + bx + c = 0$ are k less than those of the equation $px^2 + qx + r = 0$. Find the equation whose roots are k more than those of $px^2 + qx + r = 0$.

(A) $ax^2 + bx + c = 0$

(B) $a(x - 2k)^2 + b(x - 2k) + c = 0$

(C) $a(x + 2k)^2 + b(x + 2k) + c = 0$

(D) $a(x - k)^2 + b(x - k) + c = 0$

16. If α and β are the roots of the equation $x^2 - 11x + 24 = 0$, find the value of $1/\alpha - 1/\beta$ given that it is positive.

(A) 5/24 (B) 7/24 (C) 1/24 (D) 1/8

17. If one root of the equation $x^2 - 10x + 16 = 0$ is half of one of the roots of $x^2 - 4Rx + 8 = 0$. The value of R such that both the equations have integral roots is ☐

18. How many equations of the form $x^2 + 4x + p = 0$ exist such that the equation has real roots and p is a positive integer?

(A) 2 (B) 3 (C) 4 (D) 5

19. Two software professionals Ranjan and Raman had 108 floppies between them. They sold them at different prices, but each received the same sum. If Raman had sold his at Ranjan's price, he would have received ₹722 and if Ranjan had sold his at Raman's price, he would have received ₹578. The number of floppies Ranjan had is ☐

20. Find positive integral value(s) of p such that the equation $2x^2 + 8x + p = 0$ has rational roots.
(A) 8 (B) 4
(C) 6 (D) (A) or (C)

21. The coefficients of the equation $ax^2 + bx + c = 0$ satisfy the condition $64ac = 15b^2$. Find the condition satisfied by the coefficients of the equation whose roots are the reciprocals of the roots of the equation, $ax^2 + bx + c = 0$.
(A) $15ac = 64b^2$ (B) $64ac = 15b^2$
(C) $15bc = 64a^2$ (D) $15ab = 64c^2$

22. If α and β are the roots of the equation $ax^2 + bx + c = 0$ where $c^3 + abc + a^3 = 0$, which of the following is true?
(A) $\alpha\beta^2 = 1$ or $\alpha^2\beta = 1$ (B) $\alpha\beta^3 = 1$ or $\alpha^3\beta = 1$
(C) $\alpha = \beta^2$ or $\alpha^2 = \beta$ (D) $\alpha = \beta^3$ or $\alpha^3 = \beta$

23. Two equations have a common root which is positive. The other roots of the equations satisfy $x^2 - 9x + 18 = 0$. The product of the sums of the roots of the two equations is 40. The common root is $\boxed{}$

24. Both the roots of each of the equations $x^2 + px + q = 0$ and $x^2 + qx + p = 0$ are real. Which of the following is the condition or combination of conditions for the two equations to have exactly one common root?
(I) $1 + p + q = 0$ (II) $p = q$ (III) $p \neq -\frac{1}{2}$
(A) I only (B) II only (C) III only (D) I and III

25. The sum and product of the roots of a quadratic equation E are a and b respectively. Find the equation whose roots are the product of first root of E and the square of the second root of E, and the product of the second root of E and the square of the first root of E.
(A) $x^2 - abx + b^3 = 0$ (B) $x^2 + abx + b^3 = 0$
(C) $x^2 + abx - b^3 = 0$ (D) $x^2 - abx - b^3 = 0$

26. The roots [the values of x (and not |x|)] of the equation $|x|^2 + 6|x| - 55 = 0$ are α and β. One of the roots of $py^2 + qy + r = 0$ is $\alpha\beta$ times the other root. Which of the following can be concluded?
(A) $25q^2 = -576pr$ (B) $25pr = -576q^2$
(C) $25q^2 = 576pr$ (D) $25pr = 576q^2$

27. Find the respective values of p and q such that for the equation $x^2 - px + q = 0$, the sum of the roots is 36 and the product of the roots is 12.
(A) 36, –12 (B) 36, 12
(C) –36, –12 (D) –36, 12

28. If a, b and $c \in Q$ and $3 + \sqrt{7}$ is a root of $ax^2 + bx + c = 0$, which of the following sets contain possible values of *a, b* and *c*?
(A) {1, –6, 16} (B) {1, –9, 2}
(C) {1, –9, 16} (D) {1, –6, 2}

29. The roots of $x^3 + px^2 + qx + r = 0$ are consecutive positive integers. Which of the following can never be the value of q?
(A) 47 (B) 11 (C) 107 (D) 27

30. α, β, γ are the roots of the equation $x^3 - 15x^2 + 71x - 105 = 0$. If α, β, γ are in arithmetic progression, then the difference between the least and the greatest of the roots is $\boxed{}$

31. If the equations $x^3 - 4x^2 + x + 6 = 0$ and $x^3 - 3x^2 - 4x + k = 0$ have a common root, which of the following could be a value of k?
(A) –6 (B) –3 (C) 12 (D) 6

32. If one root of the equation $x^3 - 11x^2 + 37x - 35 = 0$ is $3 - \sqrt{2}$, then find the other two roots.
(A) $5, 3 - \sqrt{2}$ (B) $-5, 3 + \sqrt{2}$
(C) $5, 3 + \sqrt{2}$ (D) $-5, 3 - \sqrt{2}$

33. F(x) is a polynomial in x of degree 7. The coefficient of x^7 in F(x) is 1. F(1) = 5, F(2) = 7, F(3) = 9, F(4) = 11, F(5) = 13, F(6) = 15, F(7) = 17. Find F(8).
(A) 21 (B) 4943 (C) 5059 (D) 5071

34. If the equation $x^5 + 15x^4 + 85x^3 + 225x^2 + 274x + a - 119 = 0$ has exactly 5 negative roots, then the value of a can be
(A) 100 (B) 85 (C) 120 (D) 90

35. If the equation $x^3 - 6x^2 - ax - 6 = 0$ has all positive roots, then the value of 'a' could be
(A) 6 (B) 11 (C) 5 (D) –11

36. Both $x^2 + 16x - q = 0$ and $x^2 - 11qx + 25 = 0$ have real roots. The number of positive integral values of q is _____ .
(A) 39 (B) 70 (C) 60 (D) 64

37. The number of non - real roots of the equation $x^3 - 2x^2 - 2x - 3 = 0$ is $\boxed{}$

38. If $x + \frac{1}{x} = \sqrt{2}$, find the value of $x^{80} + x^{76} + x^{72} + x^{68} + x^{64} + 4$.
(A) 6 (B) 4 (C) 9 (D) 5

39. What is the remainder when $x^{2030} - x^3 + x + 1$ is divided by $x - 1$?
(A) 1 (B) 0 (C) 2 (D) – 2

40. What is the remainder when $x^{87} + x^{69} + x^{51} + x^{33} + x^{15}$ is divided by $x^3 - x$?
(A) 5x (B) – 5x (C) 5x + 4 (D) 4 – 5x

Exercise – 2(b)

Directions for questions 1 to 45: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. Find the value of k if k is a positive integer and $(k + 1)(k + 2)(k + 3)(k + 4) = 360$. $\boxed{}$

2. Find the equation whose roots are thrice the roots of the equation $2x^2 – 15x + 18 = 0$.
(A) $x^2 + 45x + 324 = 0$ (B) $2x^2 – 45x + 81 = 0$
(C) $x^2 + 45x – 324 = 0$ (D) $2x^2 – 45x + 162 = 0$

3. Which of the following options represent(s) a condition for the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ to have exactly one common root, given that the roots of both the equations are real?
(A) $a – b = 1$ (B) $b – a = 1$
(C) $1 + a + b = 0$ (D) Either (A) or (B)

4. A person bought a certain number of oranges for ₹70. If the price of each orange was ₹2 less, he would have bought 4 more oranges for the same amount, the number of oranges he bought originally is $\boxed{}$

5. The equation $x^2 – 2x – 8 = 0$ will have
(A) the numerically larger root as positive.
(B) the numerically larger root as negative.
(C) both roots as negative.
(D) both the roots as positive.

6. The value of p in the equation $x^2 + qx + p = 0$, where one of the roots of the equation is $(2 + \sqrt{3})$ and q and p are integers is $\boxed{}$

7. If a positive number is increased by three and then squared, the result is 23 more than the original number. The original number is $\boxed{}$

8. If k is a perfect square, the roots of the equation $4kx^2 + 4\sqrt{k}\, x – k = 0$ are
(A) always rational.
(B) rational for only some of the values of k.
(C) always irrational.
(D) always complex.

9. Find the value of R, so that one of the roots of $x^2 + 6Rx + 64 = 0$ is the square of the other root.
(A) –10/3 (B) 8/3 (C) 5/3 (D) 7/3

10. Find the values of k for which the roots of $x^2 + x(14 – k) – 14k + 1 = 0$ are equal integers.
(A) –11, –13 (B) –12, –16
(C) –13, –15 (D) –11, –12

11. If the roots of $2x^2 + (4m + 1)x + 2(2m – 1) = 0$ are reciprocals of each other, m = $\boxed{}$

12. If the roots of $2^m x^2 + 8x + 64^m = 0$ are real and equal, find m.
(A) 2/3 (B) 1/2 (C) 7/4 (D) 4/7

13. If a and b are positive numbers, what is the nature of the roots of the equation $(a + b)x^2 + 2\,abx + \frac{(a+b)^3}{16} = 0$?
(A) Real and distinct.
(B) Real and equal.
(C) Non-real and distinct.
(D) Either (B) or (C)

14. If the value of p in the equation $x^2 + 2(p + 1)x + 2p = 0$, is real, the roots of the equation are
(A) rational and unequal.
(B) irrational and unequal.
(C) real and unequal.
(D) real and equal.

15. The roots of the equation $6x^4 – 6x^3 – 24x^2 – 6x + 6 = 0$ are
(A) 1 and $\frac{6 \pm \sqrt{10}}{2}$ (B) –1 and $\frac{3 \pm \sqrt{5}}{2}$
(C) ±3 and $\frac{-6 \pm \sqrt{10}}{2}$ (D) ±4 and $\frac{6 - \sqrt{10}}{2}$

16. If the roots of the equation $ax^2 + bx + c = 0$ are α and β, find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} - 2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + 2\alpha\beta$.
(A) $(b^2 + 2ac)/ac$ (B) $(b^2 – 2ac)/ac$
(C) $(b^2 + 4ac)/ac$ (D) None of these

17. If the price of a book goes down by ₹20 per dozen, a person can purchase 50 dozen books more for ₹30,000. The original price (in ₹) of each book is $\boxed{}$

18. If 31 is split up into two parts such that the sum of the squares of the two parts is 481, the difference between the two parts is $\boxed{}$

19. Solve for x
(i) $x^4 – 42x^2 + 216 = 0$
(A) $\pm\sqrt{6}$, ±6 (B) $\pm 2\sqrt{6}$, ±6
(C) $\pm 3\sqrt{6}$, ±6 (D) $\pm 4\sqrt{6}$, ±6
(ii) $16(3^{2x+1}) – 32(3^x) + 4 = 0$
(A) $–\log_3 2, –\log_3 6$ (B) $–\log_3 4, –\log_3 6$
(C) $–1, –\log_3 12$ (D) $–\log_3 6, –\log_3 8$

20. In a class, eight students play basketball. The remaining students, who represent 7 times the square root of the strength of the class, play football. The strength of the class is _______.
(A) 36 (B) 16 (C) 64 (D) 100

21. If $x^2 + \frac{1}{x^2} - 2\left(x - \frac{1}{x}\right) - \frac{5}{4} = 0$, which of the following can be the value of $x - \frac{1}{x}$?
(A) $\frac{7}{2}$ (B) $\frac{1}{2}$ (C) $-\frac{1}{2}$ (D) $-\frac{3}{2}$

22. Solve for x: $\sqrt{2x+3}+\sqrt{4x+13}=8$
(A) 2 (B) –3 (C) 3 (D) 11

23. Find the equation whose roots are twice the roots of the equation $3x^2 - 7x + 4 = 0$.
(A) $3x^2 - 14x + 8 = 0$ (B) $3x^2 + 14x + 16 = 0$
(C) $3x^2 + 14x - 16 = 0$ (D) $3x^2 - 14x + 16 = 0$

24. The area of a playground is 247 sq m. If its length is decreased by 2 m and its breadth is increased by 4 m, it becomes a square. The side of the square (in m) is $\boxed{}$

25. The length of a rectangle is 1 cm more than its breadth. If its diagonal is 29 cm, the measure of its breadth (in cm) is $\boxed{}$

26. In a right-angled triangle, the larger of the sides containing the right angle is 8 cm longer than the smaller of the sides. The sum of the lengths of the sides containing the right angle is 16 cm more than the length of the other side. The length (in cm) of the smallest side is $\boxed{}$

27. The roots of the equation $4x^2 + 14x + 3 = 0$ are a and b. Find $\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}$.
(A) $-\frac{7\sqrt{3}}{6}$ (B) $-\frac{7\sqrt{3}}{3}$. (C) $\frac{7\sqrt{3}}{3}$ (D) $\frac{7\sqrt{3}}{6}$

28. The roots of the equation $x^2 - 3x - 108 = 0$ are α and β, where $|\alpha| > |\beta|$. Which of the following holds true?
(A) $\alpha - \beta = 3$ (B) $\alpha - \beta = -3$
(C) $\alpha - \beta = -21$ (D) $\alpha - \beta = 21$

29. P and Q were trying to solve a quadratic equation. P copied the coefficient of x wrong and obtained 12 and 9 as the roots. Q copied the constant term wrong and obtained 8 and 16 as the roots. Find the roots of the equation.
(A) 18, –6 (B) –18, 6 (C) 18, 6 (D) –18, –6

30. A is any single-digit prime number and B is any natural number. The number of equations of the form $x^2 - 4\sqrt{A}\,x + 3B = 0$ having both real roots is $\boxed{}$

31. The roots of the equation $ax^2 + bx + c = 0$ are d less than those of the equation $ez^2 + fz + g = 0$. The roots of the equation $py^2 + qy + r = 0$ are d more than those of the equation $ez^2 + fz + g = 0$. Which of the following is true?
(A) $y = x - d$ (B) $y = x + d$
(C) $y = x - 2d$ (D) $y = x + 2d$

32. If $3-\sqrt{-5}$ is one root of the equation $3x^3 - 23x^2 + 72x - 70 = 0$, then the real root of the equation is
(A) $\frac{3}{5}$ (B) $\frac{5}{3}$ (C) $\frac{-3}{5}$ (D) $\frac{-5}{3}$

33. If the roots of the quadratic equation $x^2 - px + q = 0$ are two successive multiples of 5, then the value of $p^2 - 4q$ is $\boxed{}$

34. If a > 0, b > 0, the number of real roots of the equation $2x^7 - ax^5 - 3x^4 - bx^2 + 7 = 0$ can be
(A) 2 (B) 3 (C) 4 (D) 0

35. If a > 0, b > 0, the number of complex roots of the equation $x^7 - ax^4 + bx^3 - 8 = 0$ can be
(A) 2 (B) 3 (C) 4 (D) 5

36. If the roots of the equation $x^3 - 6x^2 + 3x + k = 0$ are in arithmetic progression, the value of k is

$\boxed{}$

37. Find two positive integers which differ by seven, such that the square of the larger integer exceeds the product of the two integers by 84.
(A) 7, 12 (B) 5, 12
(C) 8, 13 (D) 4, 11

38. Find the respective values of α and k if the roots of the quadratic equation $27x^2 - 87x + k = 0$, are α and 8/3.
(A) 5/9, 40 (B) 4/9, 32
(C) 7/9, 72 (D) 2/9, 16

39. The sum of the squares of three consecutive even positive integers is 440. Find the numbers.
(A) 11, 12, 13 (B) 8, 10, 12
(C) 4, 6, 8 (D) 10, 12, 14

40. Find the quadratic equation whose roots are the reciprocals of the roots of $3x^2 - 8x + 4 = 0$.
(A) $-4x^2 + 8x + 3 = 0$ (B) $-8x^2 + 4x + 3 = 0$
(C) $4x^2 - 8x + 3 = 0$ (D) $8x^2 + 3x - 4 = 0$

41. The expression $E = ax^2 + bx + c$ is positive at x = – 1 and negative at x = 1. Which of the following can be concluded?
(A) c < 0 (B) a < 0
(C) b < 0 (D) either (A) or (B)

42. The polynomial equation of the lowest degree with real coefficients, whose roots are $2 - i\sqrt{5}$ and 3, could be
(A) $x^3 - 7x^2 + 21x - 27 = 0$
(B) $x^3 + 7x^2 + 21x + 27 = 0$
(C) $x^3 - 7x^2 - 21x - 27 = 0$
(D) $x^3 + 7x^2 - 21x + 27 = 0$

43. If f(x) = 0 is a tenth degree equation and it has 3 negative roots, then which of the following can be the number of sign changes in f(–x)?
(A) 1 (B) 2 (C) 5 (D) 6

44. The expression f(x) is cubic in x, in which the coefficient of x^3 is 1. If f(1) = 5, f(2) = 8, f(3) = 11, then f(4) = $\boxed{}$

45. If $2x^2 + 5x - 3$ is a factor of $6x^5 + 11x^4 - px^3 - 33x^2 + qx + 6$, then (q, p) is ______.
(A) (33, 11) (B) (–33, 11)
(C) (11, 33) (D) (–11, –33)

Directions for questions 46 to 55: Each question is followed by two statements, I and II. Answer each question using the following instructions:

Choose (A) if the question can be answered by using the statement I alone but not by using statement II alone

(or)

if the question can be answered by using the statement II alone but not by using statement I alone.

Choose (B) if the question can be answered by using either of the statements alone.

Choose (C) if the question can be answered by using both the statements together but not by either of the statements alone.

Choose (D) if the question cannot be answered on the basis of the two statements.

46. Is $x = 2$?
- I. x is a number such that $x^2 - 4x + 3 = 0$.
- II. x is a number such that $x^2 - x + 2 = 0$.

47. What is the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$?
- I. The product of the roots is the same as the sum of the roots.
- II. Sum of the roots of the quadratic equation is $cx^2 + ax - b = 0$ is 2.

48. If $ax^2 + bx + c = 0$, does x have a real value?
- I. $0 < a < c < 1$.
- II. $b > 3$.

49. Are the roots of the quadratic equation real?
- I. The square of the sum of its roots is at most equal to 4 times the product of its roots.
- II. The square of the sum of its roots is at least equal to 4 times the product of its roots.

50. In $Ax^2 + Bx + C$, one of A, B and C is negative. Which of them is negative?
- I. $Ax^2 + Bx + C$ is positive when $x = 1$ and negative when $x = -1$.
- II. C is not negative.

51. A and B are natural numbers. The equation $Ax^2 + Bx + 4A = 0$ has real roots, neither of which exceeds 10. Find A.
- I. $B < 5$
- II. $A \leq 2$

52. P and Q are natural numbers. Find $\frac{P}{Q}$.
- I. $Px^2 + Qx + Q = 0$ has real roots.
- II. $Qx^2 + 4Px + P = 0$ has real roots.

53. Q is a quadratic expression in x. Find its maximum value.
- I. The discriminant of Q = 0 is –12.
- II. The coefficient of x^2 in Q is –1.

54. Are the roots of $x^2 + ax + b = 0$ equal given that they are integers?
- I. $a + b = 0$.
- II. The roots are reciprocal to each other.

55. The roots of $(a^2 + b^2)x^2 + 2(ab + bc)x + b^2 + c^2 = 0$, where a, b, c are real numbers, are p and q. Find $\frac{b^2}{ac}$.
- I. p and q are real.
- II. $b = -2$, $a = c = 2$

Key

Concept Review Questions

1. B	(b) C	12. (a) A	18. 8	23. B
2. (a) C	7. 12	(b) A	19. –6	24. 3
(b) D	8. (a) D	(c) C	20. (a) D	25. D
3. B	(b) C	13. 909	(b) A	
4. C	(c) B	14. 2	(c) A	
5. (a) A	9. 25	15. 1	21. (a) B	
(b) A	10. C	16. 3	(b) D	
6. (a) B	11. D	17. C	22. 3	

Exercise – 2(a)

1. (i) A	7. C	14. 30	21. B	28. D	35. D
(ii) A	8. A	15. B	22. A	29. D	36. D
2. –20	9. D	16. A	23. 2	30. 4	37. 2
3. 64	10. 13	17. 1.5	24. D	31. C	38. D
4. 2.5	11. A	18. C	25. A	32. C	39. C
5. 6	12. A	19. 51	26. A	33. C	40. A
6. 25	13. C	20. D	27. B	34. C	

Exercise – 2(b)

1. 2	9. A	17. 10	24. 17	32. B	40. C	48. C
2. D	10. B	18. 1	25. 20	33. 25	41. C	49. A
3. C	11. 1	19. (i) A	26. 24	34. B	42. A	50. C
4. 10	12. D	(ii) A	27. C	35. C	43. C	51. A
5. A	13. D	20. C	28. D	36. 10	44. 20	52. C
6. 1	14. C	21. B	29. C	37. B	45. C	53. C
7. 2	15. B	22. C	30. 21	38. A	46. B	54. B
8. C	16. D	23. D	31. D	39. D	47. C	55. B