CHAPTER – 3

INEQUALITIES AND MODULUS

If 'a' is any real number, then 'a' is either positive or negative or zero. When 'a' is positive, we write a > 0 which is read 'a is greater than zero'. When 'a' is negative, we write a < 0 which is read 'a is less than zero'. If 'a' is zero, we write a = 0 and in this case, 'a' is neither positive nor negative.

Symbols and Notations:

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'>' means 'greater than'
'<' means 'less than'
'≥' means 'greater than or equal to'
'≤' means 'less than or equal to'

For any two non-zero real numbers a and b,
(i) a is said to be greater than b when a – b is positive.
(ii) a is said to be less than b when a – b is negative.

These two statements are written as
(i) a > b when a – b > 0 and
(ii) a < b when a – b < 0.

For example,
3 is greater than 2 because 3 – 2 = 1 and 1 is greater than zero. –3 is less than –2 because –3 – (–2) = –1 and –1 is less than zero.

Certain properties and useful results pertaining to inequalities are given below. A thorough understanding of these properties/results is very essential for being able to solve the problems pertaining to inequalities.

[In the following list of properties and results, numbers like a, b, c, d, etc. are real numbers]

- For any two real numbers a and b, either a > b or a < b or a = b.
- If $a \nless b$, then $a \geq b$ and if $a \ngtr b$, then $a \leq b$.
- If a > b and b > c, then a > c.
- If a < b and b < c, then a < c.
- If a > b, then a ± c > b ± c.
- If a > b and c > 0, then ac > bc.
- If a < b and c > 0, then ac < bc.
- If a > b and c < 0, then ac < bc.
- If a < b and c < 0, then ac > bc.
- If a > b and c > d, then a + c > b + d.
- If a < b and c < d, then a + c < b + d.
- The square of any real number is always greater than or equal to 0.
- The square of any non-zero real number is always greater than 0.
- If a > 0, then –a < 0 and if a > b, then –a < –b.
- If a and b are positive numbers and a > b, then
  (i) 1/a < 1/b,
  (ii) a/c > b/c if c > 0 and
  (iii) a/c < b/c if c < 0.
- For any two positive numbers a and b,
  If a > b, then $a^2 > b^2$.
  If $a^2 > b^2$, then a > b.
  If a > b, then for any positive value of n, $a^n > b^n$.
- Let, A, G and H be the Arithmetic mean, Geometric mean and Harmonic mean respectively of n positive real numbers. Then A ≥ G ≥ H, the equality occurring only when the numbers are all equal.

- If the sum of two positive quantities is given, their product is the greatest when they are equal; and if the product of two positive quantities is given, their sum is the least when they are equal.

- If a > b and c > d, then we cannot say anything conclusively about the relationship between (a – b) and (c – d); depending on the values of a, b, c and d, it is possible to have

(a – b) > (c – d), (a – b) = (c – d) or (a – b) < (c – d)

When two numbers a and b have to be compared, we can use one of the following two methods:

- If both a and b are positive, we can take the ratio a/b and depending on whether a/b is less than, equal to or greater than 1, we can conclude that a is less than, equal to or greater than b.

  In other words, for two positive numbers a and b,
  if a/b < 1 then a < b
  if a/b = 1 then a = b
  if a/b > 1 then a > b

- If one or both of a and b are not positive or we do not know whether they are positive, negative or zero, then we can take the difference of a and b and depending on whether (a – b) is less than, equal to or greater than zero, we can conclude that a is less than, equal to or greater than b.
  In other words, for any two real numbers a and b,
  if a – b < 0, then a < b,
  if a – b = 0, then a = b,
  if a – b > 0, then a > b.

- For any positive number x ≥ 1,

$$2 \leq \left(1+\frac{1}{x}\right)^x < 2.8.$$

The equality in the first part will occur only if x = 1.

- For any positive number, the sum of the number and its reciprocal is always greater than or equal to 2, i.e. $x+\frac{1}{x} \geq 2$ where x > 0. The equality in this relationship will occur only when x = 1.

Absolute Value:

(written as |x| and read as "modulus of x")

For any real number x, the absolute value is defined as follows:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \text{ and} \\ -x, & \text{if } x < 0 \end{cases}$$

Properties of Modulus:

For any two real numbers x and y,

1. $x = 0 \Leftrightarrow |x| = 0$
2. $|x| \geq 0$ and $-|x| \leq 0$
3. $|x + y| \leq |x| + |y|$
4. $||x| - |y|| \leq |x - y|$
5. $-|x| \leq x \leq |x|$
6. $|x \cdot y| = |x|\,|y|$
7. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$; $(y \neq 0)$
8. $|x|^2 = x^2$

In inequalities, the variables generally take a range of values unlike in the case of equations where the variables in general, take one value or a finite set of values.

Interval Notation:

Generally, the solution set or the range of values satisfied by inequalities are not discrete.

So it is important to understand the "interval notation".

(a, b) read as "open interval a, b" means all real numbers between a and b excluding a and b (a < b).
[a, b] read as "closed interval a, b" means all real numbers between a and b including a and b (a < b). [a, b) means all numbers between a and b, with a being included and b excluded (a < b).

The problems on inequalities normally fall into 3 categories:

(a) consisting of first degree expressions in x.
(b) consisting of second degree expressions in x, directly in the problem or consisting single expression which reduces to quadratic expression.
(c) consisting of expressions including "modulus".

An example of each category is given below.

Examples

3.01. Express all real numbers between 2 and 10 in the interval form where
(A) 2 and 10 are excluded.
(B) 2 and 10 are included.
(C) 2 is included and 10 is excluded.
(D) 2 is excluded and 10 is included.

Sol: (A) (2, 10) i.e., $2 < x < 10$ and x is a real number.
(B) [2, 10] i.e., $2 \leq x \leq 10$ and x is a real number.
(C) [2, 10) i.e., $2 \leq x < 10$ and x is a real number.
(D) (2, 10] i.e., $2 < x \leq 10$ and x is a real number.

3.02. Express the following in the interval notation.
(A) All non-negative real numbers
(B) All non-positive real numbers
(C) All non-zero real numbers
(D) All real numbers

Sol: (A) $[0, \infty)$ i.e., $0 \leq x < \infty$
(B) $(-\infty, 0]$ i.e., $-\infty < x \leq 0$
(C) $(-\infty, 0) \cup (0, \infty)$
(D) $(-\infty,\infty)$

3.03. Express the following in the interval notation.
(A) All real numbers less than –2
(B) All real numbers greater than or equal to 3

Sol: (A) $(-\infty, -2)$, i.e. $-\infty < x < -2$
(B) $[3, \infty)$, i.e. $3 \leq x < \infty$

3.04. If $13x - 19 \leq 4x + 26$, find the range of x.

Sol: $13x - 4x \leq 26 + 19$
$x \leq 5$.
$(-\infty, 5]$ in the interval notation.

3.05. Solve the following inequalities : $5x + 21 < 46$ and $4x + 18 < 54$.

Sol: $5x + 21 < 46 \Rightarrow x < 5 \quad \rightarrow \quad (1)$
$4x + 18 < 54 \Rightarrow x < 9 \quad \rightarrow \quad (2)$
The common inequality satisfying (1) and (2) is x < 5 or $(-\infty, 5)$ in the interval notation.

3.06. Which of the numbers 50^{51} and 51^{50} is greater?

Sol: Let $a = 50^{51}$ and $b = 51^{50}$.

$$\frac{b}{a} = \frac{51^{50}}{50^{51}} = \left(\frac{51}{50}\right)^{50}\left(\frac{1}{50}\right) = \left(1+\frac{1}{50}\right)^{50}\left(\frac{1}{50}\right)$$

$\left(1+\frac{1}{x}\right)^x$ where x > 0 always lies between 2 and 2.8.

$\therefore \frac{b}{a}$ lies between

$\frac{2}{50} = 0.04$ and

$\frac{2.8}{50} = 0.056$

$\therefore \frac{b}{a} < 1$

$\therefore a > b$.

3.07. Solve the simultaneous inequations :
$10x + 7 > 2x + 39$ and $9x - 14 > 11x - 28$

Sol: $10x + 7 > 2x + 39 \Rightarrow x > 4 \quad \rightarrow \quad (1)$
$9x - 14 > 11x - 28 \Rightarrow x < 7 \quad \rightarrow \quad (2)$
From (1) and (2), $4 < x < 7$

3.08. Solve for x if $4x^2 - 21x + 20 > 0$

Sol: $4x^2 - 21x + 20 > 0 \Rightarrow (4x - 5)(x - 4) > 0$
Both factors are positive (i.e. the smaller is positive) or both are negative (i.e. the greater is negative)
i.e., $x > 4$ or $x < \frac{5}{4}$
or it can be expressed in the interval notation as $(4, \infty) \cup \left(-\infty, \frac{5}{4}\right)$

3.09. Solve for x, if $\frac{x^2 + 5x - 24}{2x^2 - 5x - 3} < 0$

Sol: $x^2 + 5x - 24 = (x + 8)(x - 3)$
Similarly $2x^2 - 5x - 3 = (2x + 1)(x - 3)$

Given : $\frac{x^2 + 5x - 24}{2x^2 - 5x - 3} < 0 \Rightarrow \frac{(x+8)(x-3)}{(2x+1)(x-3)} < 0$

$\Rightarrow \frac{x+8}{2x+1} < 0$

$$\frac{(x+8)(2x+1)}{(2x+1)^2} < 0$$

$\Rightarrow (x + 8)\left(x+\frac{1}{2}\right) < 0$

$\therefore -8 < x < -\frac{1}{2}$.

3.10. Solve the inequality $|3x + 6| > -12$.

Sol: The modulus of any number is always non-negative.
$\therefore |3x + 6| \geq 0$.
$\therefore$ The given inequality is always satisfied.
$\therefore -\infty < x < \infty$

3.11. Solve the inequality $|2x + 4| < -6$.

Sol: The modulus of any number is always non negative.
$\therefore |2x + 4| \geq 0$
$\therefore$ The given inequality will never hold.
There is no solution.

3.12. Solve for x: $|2x - 3| = 5$

Sol: $2x - 3 = 5$ or $2x - 3 = -5$
(If $|y| = a$, $y = \pm a$) $\Rightarrow x = 4$ or $x = -1$.

3.13. Solve the inequality $|4x - 5| > 3$

Sol: From the definition of modulus,
$|4x - 5| = 4x - 5$ if $4x - 5 \geq 0$ i.e., $x \geq \frac{5}{4}$
$= -(4x - 5)$ if $x < \frac{5}{4}$
$\therefore$ If $x \geq \frac{5}{4}$, $4x - 5 > 3 \Rightarrow x > 2$
and If $x < \frac{5}{4}$, $-4x + 5 > 3 \Rightarrow x < \frac{1}{2}$

3.14. Find the maximum value of $g(x) = 16 - |-x - 6|$; $x \in R$.

Sol: g (x) is maximum when $|-x - 6|$ is minimum.
The minimum value of the modulus of all numbers is 0.
$\therefore$ The maximum value of $g(x) = 16 - 0 = 16$.

Some useful models:

Quite often, when dealing with positive real numbers we come across situations where the sum (or product) of certain variables is given and we are required to maximise (or minimise) the product (or sum) of the same. We illustrate the technique involved with a couple of examples.

Model 1: If $ax + by = k$ where a, b, x, y are all positive, maximise $x^m y^n$ where m and n are positive integers.

3.15. If $2x + 3y = 30$ where $x, y > 0$, find the maximum value of x^2y^3.

Sol: $x^2y^3 = \left(\frac{2x}{2}\right)^2\left(\frac{3y}{3}\right)^3$

sum of all the factors of the above expression
$= 2\left(\frac{2x}{2}\right) + 3\left(\frac{3y}{3}\right) = 2x + 3y = 30$ (a constant)

When the sum of two or more positive quantities is constant their product is maximum when all the quantities are equal.

$\therefore \left(\frac{2x}{2}\right)^2\left(\frac{3y}{3}\right)^3$ is maximum when $\frac{2x}{2} = \frac{3y}{3}$, i.e. x = y.
When x = y, $2x + 3y = 5y = 30$
$\therefore y = 6$.
$\therefore$ Maximum value $= 6^2 . 6^3 = 7776$.

Note: When the expression ax + by is constant, the maximum value of $x^m y^n$ is realized when $\frac{ax}{m} = \frac{by}{n}$.

Model 2: If $x^m y^n = k$ where $x > 0$, $y > 0$ and m and n are positive integers, minimise ax + by where $a > 0$, $b > 0$.

3.16. If x, y are positive and $xy^2 = 216$, find the minimum value of $32x + y$.

Sol: The given condition (C) is $xy^2 = 216$. We need the minimum value of the expression E : $32x + y$. If the product of several factors is constant, we can obtain the minimum value of the sum of all the factors. We should modify C, so that the sum of all the factors is E. After modifying, $8(xy^2) = 8\ (216)$. We should think of the factors on the LHS as 32x, $\frac{y}{2}$ and $\frac{y}{2}$. The sum of these factors is E.
$C \Rightarrow (32x)\left(\frac{y}{2}\right)\left(\frac{y}{2}\right) = 8\ (216)$

The sum of the 3 factors, i.e. $32x + y$ is minimum, when $32x = \frac{y}{2} = 2\ (6) = 12$
Thus minimum value is $24 + 12 = 36$

Note: When the expression $x^m y^n$ is constant, the minimum value of ax + by is realized when $\frac{ax}{m} = \frac{by}{n}$.

Model 3: The greatest value of $(a - x)^m (b + x)^n$, for any real value of x numerically less than a, b and $m,n \in Z^+$, occurs when $\frac{a-x}{m} = \frac{b+x}{n}$ or at $x = \frac{an-bm}{m+n}$.

3.17. Find the maximum value of $(9 - x)^2(-4 + x)^3$.

Sol: m = 2, n = 3 are positive integers. The given expression is maximum when $\frac{9-x}{2} = \frac{x-4}{3}$
$\Rightarrow x = 7$
$\therefore$ The maximum value is $(9 - 7)^2 (-4 + 7)^3$
$= 2^2\ 3^3 = 108$

Model 4: If a and b are two positive numbers, the mean of their m^{th} powers (say M) and the m^{th} power of their mean (say P) are related as follows, depending on the value of M.

(i) If m < 0 or m > 1, then P is less than or equal to M.

i.e., $\left(\frac{a+b}{2}\right)^m \le \left(\frac{a^m+b^m}{2}\right)$ (For example,

$$\left(\frac{a+b}{2}\right)^2 \le \frac{a^2+b^2}{2}\Bigg)$$

The equality holds if a = b.

(ii) If m = 0 or 1, then P = M

i.e., $\left(\frac{a+b}{2}\right)^0 = \frac{a^0+b^0}{2}$ =1 and $\left(\frac{a+b}{2}\right)^1 = \frac{a^1+b^1}{2} =$

$$\frac{a+b}{2}$$

(iii) If 0 < m < 1, then M is less than or equal to P.

i.e., $\frac{a^m+b^m}{2} \le \left(\frac{a+b}{2}\right)^m$

(For example, $\frac{\sqrt{a}+\sqrt{b}}{2} < \sqrt{\frac{a+b}{2}}$) The equality holds iff a = b.

This rule can be extended for 3 or more quantities. If a > 0, b > 0, c > 0, then

(i) If m < 0 or m > 1, $\left(\frac{a+b+c}{3}\right)^m \le \frac{a^m+b^m+c^m}{3}$.

The equality holds iff a = b = c.

(ii) if m = 0 or 1, $\left(\frac{a+b+c}{3}\right)^m = \frac{a^m+b^m+c^m}{3}$

(iii) if 0 < m < 1, $\frac{a^m+b^m+c^m}{3} \le \left(\frac{a+b+c}{3}\right)^m$.

The equality holds iff a= b = c.

3.18. Find the minimum value of $\frac{(3+x)(6+x)}{2+x}$.

Sol: Let E = $\frac{(3+x)(6+x)}{2+x} = \frac{(2+x+1)(2+x+4)}{2+x}$

$= \frac{(2+x)^2+5(2+x)+4}{2+x}$

$= (2+x)+5+\left(\frac{4}{2+x}\right)$ = F + 5.

The minimum value of F is obtained as follows.

Since x > –2 ⇒ 2 + x > 0

$\therefore$ A.M$\left(2+x, \frac{4}{2+x}\right) \ge$ G.M$\left(2+x, \frac{4}{2+x}\right)$ = 2

$\therefore$ $2+x+\frac{4}{2+x} \ge 4$ (The equality holds if and only if $2+x = \frac{4}{2+x}$. As x + 2 > 0, this means 2 + x = 2)

$\therefore E_{min} =$ Min $\frac{(3+x)(6+x)}{2+x} = 9$

The generalisation of this is given below

Model 5: The minimum value of $\frac{(x+a)(x+b)}{x+c}$, where a > c, b > c and x + c > 0 is given by $a - c + b - c + 2\sqrt{(a-c)(b-c)}$ and the corresponding value of x is $\sqrt{(a-c)(b-c)} - c$.

Concept Review Questions

Directions for questions 1 to 30: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. **(a)** The set of all real numbers lying between 3 and 4 is represented as
(A) [3, 4) (B) (3, 4] (C) (3, 4) (D) [3, 4]

(b) The set of all real numbers lying between 5 and 6 including 5 is represented as
(A) (5, 6) (B) [5, 6) (C) [5, 6] (D) (5, 6]

(c) The set of all real numbers from 4 to 6 is represented as
(A) [4, 6] (B) (4, 6) (C) [4, 6) (D) (4, 6]

2. When $x < 0$, then $|x| =$
(A) 0 (B) x (C) ±x (D) –x

3. If $a > b$ and $c > 0$, then which of the following statements is / are true?
(A) $a + c > b + c$ (B) $a – c > b – c$
(C) $ac > bc$ (D) All the above

4. If $a < b$ and $c < 0$ then which of the following is true?
(A) $ac < bc$ (B) $\frac{a}{c} < \frac{b}{c}$
(C) $ac > bc$ (D) None of these

5. When $x > 0$, the minimum value of $x + \frac{1}{x}$ is

6. Which of the following intervals is such that, any number in that interval can be expressed as $\left(1+\frac{1}{x}\right)^x$, for some positive real number x?
(A) [1.5, 3.5] (B) [2, 3)
(C) (1, 2) (D) [1, 3)

7. If p and q are two real numbers, then which of the following statements is always true?
(A) $\frac{p}{q} < 1 \Rightarrow p < q$
(B) $p > 0$, $q > 0$ and $\frac{p}{q} > 1 \Rightarrow p > q$
(C) $\frac{p}{q} > 1 \Rightarrow p > q$
(D) All the above

8. All real numbers less than or equal to 5 are included in
(A) $(–\infty, 5]$ (B) $(–\infty, 5)$
(C) $(–\infty, –3) \cup (5, \infty)$ (D) $(5, \infty)$

9. If $– 3 \le x \le 8$, then x belongs to
(A) (–3, 8) (B) [–3, 8)
(C) [–3, 8] (D) (–3, 8)

10. If $3x – 7 \le 5$, then the maximum of x is

11. If $7x – 4 \ge 31$, then the minimum value of x is

12. If $8 – 12x \ge –16$ then the maximum value of x is

13. Solve: $–7x + 5 \ge 5x – 19$.
(A) $x \ge 2$ (B) $x \ge –2$
(C) $x \le 2$ (D) $x \le –2$

14. The solution set of the inequality $–9x – 5 < 7x + 27$ is
(A) $(2, \infty)$ (B) $(–\infty, –2)$
(C) $(–2, \infty)$ (D) $(–\infty, 2)$

15. If $–2x \ge 8$ then
(A) $x \ge 8$ (B) $x \ge 4$
(C) $x \le 8$ (D) $x \le –4$

16. **(a)** If $2x + 7 \le 9x$ then
(A) $x \le 1$ (B) $x < 1$ (C) $x \ge 1$ (D) $x > 1$

(b) If $4x + 34 > 7x + 31$, then
(A) $x > 1$ (B) $x < 1$
(C) $x > – 1$ (D) $x < – 1$

(c) If $5x – 17 \ge 2x – 15$, then
(A) $x \ge \frac{-2}{3}$ (B) $x \le \frac{-2}{3}$
(C) $x \ge \frac{2}{3}$ (D) $x \le \frac{2}{3}$

17. **(a)** If $5x + 3 > 7x – 9$, then
(A) $x < –6$ (B) $x > 6$
(C) $x < 6$ (D) $x > –6$

(b) If $4x + 3 \ge 3x – 12$, then
(A) $x \le 5$ (B) $x \le –15$
(C) $x \ge 5$ (D) $x \ge –15$

18. If $5x – 8 < 2x + 9$ and $4x + 7 > 7x – 8$, then the range of x is
(A) $(5, \infty)$ (B) $(–\infty, 5)$
(C) $\left(5, {}^{17}\!/_{3}\right)$ (D) $\left(-\infty, {}^{17}\!/_{3}\right)$

19. If $\frac{1}{3x+1} \ge \frac{1}{3}$ and $x > 0$, then which of the following holds?
(A) $3x + 1 \ge 3$
(B) $1 < 3x + 1 \le 3$
(C) $–(3x + 1) \ge 3$
(D) $3x + 1 \le 1/3$

20. Solve : $2x – 3 \ge 7$ and $5x – 7 < 3$.
(A) ϕ (B) $[5, \infty)$
(C) $(–\infty, 2)$ (D) [2, 5]

21. Which of the following is always true?
(A) $x > y \Rightarrow x^2 > y^2$.
(B) $x > y \Rightarrow x^2 < y^2$.
(C) If $x > 0$, $y > 0$ and $x > y$, then $x^2 > y^2$.
(D) None of the above

22. (a) The solution set of the inequality $|2x + 1| < 0$ is

(A) $\left(-\infty, -\frac{1}{2}\right)$ (B) $\left(\frac{1}{2}, \infty\right)$

(C) $\left(-\frac{1}{2}, \infty\right)$ (D) ϕ

(b) The solution set of the inequality $|2x - 3| \geq 0$ is

(A) ϕ (B) $\left(\frac{3}{2}, \infty\right)$

(C) R (D) None of these

23. If $x^2 - 9x - 36$ is negative, then find the range of x.

(A) $(-3, 12)$ (B) $[-3, 12]$

(C) $(-12, 3)$ (D) $[-12, 3]$

24. Solve $|x^2 - 16| = 0$.

(A) $\{0, 4\}$ (B) $[-4, 4]$

(C) $(-4, 4)$ (D) $\{-4, 4\}$

25. (a) Solve for x: $5x^2 - 3x - 2 \geq 0$.

(A) $\left[\frac{-2}{5}, 1\right]$ (B) $R - \left(\frac{-2}{5}, 1\right)$

(C) $[1, \infty)$ (D) $R - (0, 1)$

(b) The number of integral values of x, that do not satisfy the inequation $\frac{x+5}{x-2} \geq 0$ is $\boxed{}$

(c) Solve for x: $4x^2 - 7x - 30 < 0$

(A) $\left(-2, \frac{15}{4}\right)$ (B) $\left[-2, \frac{15}{4}\right]$

(C) $(-\infty, -2)$ (D) $\left[-2, \frac{15}{4}\right)$

26. The solution set of the inequality $|x - 5| < 9$ is

(A) $(0, 14)$ (B) $(-4, 14)$

(C) $(-4, 0)$ (D) $(9, 14)$

27. If $|5x - 7| = 12$, then the positive value of x is

$\boxed{}$

28. If $(x + 5)(x + 9)(x + 3)^2 < 0$, then the solution set for the inequality is

(A) $(-9, -3)$

(B) $(-9, -5)$

(C) $(-3, \infty)$

(D) $(-9, \infty)$

29. Which of the following is true?

(A) $|x + y| \leq |x| + |y|$

(B) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, $y \neq 0$

(C) $|x - y| \geq ||x| - |y||$

(D) All the above

30. If $6x + 8 > 7x - 9$ and $4x - 7 < 6x - 3$, then the range of the values of x is

(A) $(-17, 2)$ (B) $(2, 17)$

(C) $(-2, 17)$ (D) $(-\infty, 17)$

Exercise – 3(a)

Directions for questions 1 to 30: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. Find the range of the values of x satisfying the inequalities $3x + 4 \geq -5$ and $8x - 13 \leq 19$.
(A) [–4, 3] (B) [–3, 4]
(C) $(-\infty, -3] \cup [4, \infty]$ (D) (–3, 4)

2. Find the range of the values of x satisfying $8 - 3x \leq 5$ and $4x + 5 \leq -7$.
(A) [–3, 1] (B) $(-\infty, -3] \cup [1, \infty)$
(C) (–3, 1) (D) ϕ

3. What are the values of x that satisfy the inequality $-x^2 + x + 90 > 0$?
(A) $(-\infty, -9) \cup (10, \infty)$ (B) (–10, 9)
(C) (–9, 10) (D) $(-\infty, -10) \cup (9, \infty)$

4. What are the values of x that satisfy the inequality $4x^2 + x - 5 > 0$?
(A) (–5/4, 1)
(B) (–1, 5/4)
(C) $(-\infty, -1) \cup (5/4, \infty)$
(D) $(-\infty, -5/4) \cup (1, \infty)$

5. What are the values of x that satisfy the inequality $\frac{2x-3}{x+4} < -3$, $(x \neq -4)$?
(A) $(-\infty, -4) \cup (-9/5, \infty)$
(B) $(-\infty, -9/5) \cup (4, \infty)$
(C) (–4, –9/5)
(D) (–9/5, 4)

6. What are the values of x that satisfy the inequality $\frac{3x^2 + 7x - 6}{x^2 - 9x + 8} < 0$, $(x \in R\ x \neq 1, 8)$?
(A) $(-3, 2/3) \cup (1, 8)$
(B) $[-\infty, -3) \cup (2/3, 1) \cup (8, \infty)$
(C) [1, 8]
(D) $(-3, 2/3) \cup (8, \infty)$

7. Which of the following is/are true?
(A) $30^{31} < 31^{30}$ (B) $71^{69} > 70^{70}$
(C) $(155)^{29} < (150)^{30}$ (D) Both (B) and (C)

8. The minimum value of $6 + |4 - 7x|$ is ☐

9. For what value of x is $-|x - 3| + \frac{21}{2}$ maximum?
☐

10. The maximum value of $3 - |2x - 1|$ is ☐

11. Find the range of all values of x if $|2x + 3| \geq 7$.
(A) (–5, 2) (B) [–5, 2]
(C) (–5, –2) (D) $(-\infty, -5] \cup [2, \infty)$

12. Find the range of all values of x if $|3x + 5| < 5x - 11$.
(A) $(8, \infty)$
(B) $(-\infty, -5/3) \cup (8, \infty)$
(C) (–5/3, 8)
(D) $(-5/3, \infty)$

13. If x, y, z are positive numbers, then which of the following holds?
(A) $(x + y)(y + z)(z + x) \geq 8xyz$
(B) $(xy + yz + zx)^2 \geq 3xyz(x + y + z)$
(C) $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \geq \frac{1}{xy} + \frac{1}{yz} + \frac{1}{xz}$
(D) All the above

14. If ac = bd = 2, then the minimum value of $a^2 + b^2 + c^2 + d^2$ is ☐

15. If x, y > 0 and x + y = 3 then
(A) $xy \leq 0.72$ (B) $xy \leq 1.8$
(C) $xy \leq 2.25$ (D) $xy \leq 1.25$

16. The number of integral solutions of the inequality $|x - 5| + |x - 1| < 2$ is ☐

17. Find the number of integers satisfying the inequality $(3x^2 - 7x - 6)(x^2 - 5x + 4) < 0$.
(A) 1 (B) 2 (C) 0 (D) 3

18. The difference between the largest and the smallest integer that satisfies the inequality $\frac{1}{|2x-7|} > \frac{2}{9}$; $x \neq \frac{7}{2}$ is ☐

19. Find the range of values of x that satisfies $|x - 16| > x^2 - 7x + 24$.
(A) (0, 2) (B) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (C) (1, 3) (D) (2, 4)

20. For which of the following range of values of x is $x^2 + x$ less than $x^3 + 1$?
(A) $(-\infty, -1)$ (B) $(1, \infty)$
(C) $(-1, 1) \cup (1, \infty)$ (D) [–1, 1]

21. If $a^2 b^3 = (540)(35)^2$, then the minimum value of $5a + 7b$ is
(A) 200 (B) 70 (C) 175 (D) 105

22. If a + b + c = 24, then the maximum value of $a^2 b^3 c$ is
(A) $2^3 3^{10}$ (B) $6^3 2^{10}$ (C) $6^3 2^8$ (D) $2^{14} 3^3$

23. The minimum value of
$$\frac{1}{a^3 b^3 c^3}\left[(a^3 + b^3)^2 c^3 + (b^3 + c^3)^2 a^3 + (a^3 + c^3)^2 b^3\right]$$
where a, b, c are positive numbers is ______.
(A) 10 (B) 12 (C) 15 (D) 18

24. The solution set of the inequality $x^2 - 14x + 56 < 0$ is
(A) R – (3, 7) (B) R – {3, 7}
(C) $\varnothing$ (D) R

25. If $|5x + 3| > 14$ and $x > 0$, then x > ☐

26. The number of distinct solutions of the equation $x^2 - 15 | x | + 56 = 0$ is $\boxed{}$

27. The number of distinct solutions of the equation $\left|2x - |5x-3|\right| = 18$ is $\boxed{}$

28. If x, y, z are positive, then the value of $\frac{(4x^2+x+4)(5y^2+y+5)(7z^2+z+7)}{xyz}$ can be

(A) 400 (B) 500 (C) 1000 (D) 1500

29. The range of a for which $ax^2 - 6x + 9 > 0$ for all real values of x is

(A) $(-\infty, 1)$ (B) $(1, \infty)$
(C) $(-1, 1)$ (D) $(-\infty\ -1)$

30. The range of x for which $2x^2 - 5x - 8 \le |2x^2 + x|$ is

(A) $\left[-\frac{4}{3}, \infty\right)$ (B) $\left(-\frac{4}{3}, -1\right)$
(C) $[-1, \infty)$ (D) $[-1, 2]$

Exercise – 3(b)

Directions for questions 1 to 40: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. The number of integral values of x, for which the inequation $\frac{x-5}{x+7} > 4$ is satisfied, is $\boxed{}$

2. If x, y, z are positive numbers and xyz = 216, which of the following is not a possible value of xy + yz + zx?

(A) 326 (B) 433 (C) 291 (D) 96

3. If $1 \le x \le 3$ and $2 \le y \le 5$, then the minimum value of $\frac{x+y}{y}$ is $\boxed{}$

4. The solution set for the inequation $3x + 2 < |2x + 5| < 8x + 9$ is

(A) $\left(-\frac{2}{3}, 3\right)$ (B) $\left(\frac{2}{3}, 3\right)$
(C) $\left[-\frac{2}{3}, 3\right]$ (D) $\left(\frac{-2}{3}, 4\right)$

5. If 2x + 3y = 10, where x and y are positive numbers then the maximum value of $x^3 y^2$ is $\boxed{}$

6. The solution set of the inequality $\frac{x^2-7x+10}{x^2+6x-40} < 1$ is

(A) $\left(\frac{70}{13}, \frac{50}{3}\right)$ (B) $\left(-10, \frac{50}{13}\right) \cup (4, \infty)$
(C) $(4, \infty)$ (D) $R - (-4, 0)$

7. If $|b| \ge 5$ and $x = |a|\, b$, which of the following is true?

(A) $a - xb > 0$ (B) $a + xb < 0$
(C) $a + xb > 0$ (D) $a - xb \le 0$

8. The number of solutions of the equation $\left|x - |x-2|\right| = 6$ is $\boxed{}$

9. If a, b and c are positive numbers and $a^3 + b^3 + c^3 = 27$, then the maximum value of abc is $\boxed{}$

10. If $x \le 4$, $y \ge -2$, which of the following is necessarily true?

(A) $xy \ge -8$ (B) $x + y > 2$
(C) $x - y \le 6$ (D) (A) and (B)

11. Let $a_1, a_2, - - - a_{3n}$ be 3n positive numbers such that their product is 1. The minimum value of $a_1 + a_2 + a_3 + - - - - + a_{3n}$ is

(A) n (B) $3^{1/n}$ (C) 3^n (D) 3n

12. If a, b, c, d are positive numbers, the minimum value of the product $(a + b + c + d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$ is $\boxed{}$

13. If a, b, c, d and e are positive numbers such that abcde = 32, then the minimum value of (1 + a) (1 + b) (1 + c) (1 + d) (1 + e) is $\boxed{}$

14. If a, b, c are positive and if a + b + c = 12, then the maximum value of (b + c) (c + a) (a + b) is

(A) 676 (B) 408 (C) 256 (D) 512

15. If x, y and z are positive numbers, then the minimum value of $\frac{x^2y+y^2z+z^2x+xy^2+yz^2+zx^2}{xyz}$ is $\boxed{}$

16. If $2 \le x \le 5$ and $-7 \le y \le -3$, then the minimum value of which of the following is the least?

(A) $x^2 y$ (B) xy^2
(C) $3xy$ (D) $-2xy^2$

17. The range of values of y that satisfy the inequation $y^{2/3}-3y^{1/3}-10 \le 0$ are

(A) $-2 \le y \le 5$ (B) $8 \le y \le 125$
(C) $-8 \le y \le 125$ (D) $-9 \le y \le 5$

18. The number of integral values of x that satisfy the inequation $|x-3| + |x-4| \le 7$ is $\boxed{}$

19. The solution set of the system of inequations $x^2 - 5x + 6 > 0$ and $x^2 - 3x + 2 > 0$ is

(A) $(-\infty, 2)$ (B) $(-\infty, 1) \cup (3, \infty)$
(C) $(1, 3)$ (D) $(1, \infty)$

20. The number of integral values of x, for which $\frac{x-3}{x+2} < 0$ is $\boxed{}$

21. The solution set of the system of inequations $3x + 17 < 5x - 19$ and $4x + 15 > 9x + 21$.

(A) $(18, \infty)$ (B) R
(C) $\varnothing$ (D) $\left(-\infty, \frac{-6}{5}\right)$

22. For which of the following values of x does the expression $|x+3|+|x-5|$ + 7 have its least value?

(A) 4 (B) –6 (C) 7 (D) –5

23. The maximum value of the expression $10-|3x+5|$ is $\boxed{}$

24. For how many integral values of x, is the expression $9 - 4x - 5x^2$ non-negative?

(A) 7 (B) 6 (C) 5 (D) 3

25. If x, y are positive and xy = 27, then the minimum value of 3x + 4y is $\boxed{}$

26. At what value of x is |2x – 7| – 8 minimum?

(A) –8 (B) $\frac{7}{2}$ (C) $-\frac{7}{2}$ (D) 15

27. Which of the following is/are true?

(A) $(15!)^2 < 15^{15}$ (B) $(12)^{12} < (12!)^2$
(C) $(18!)^2 < 18^{18}$ (D) both (B) and (C)

28. Find the range of values of x for which |x + 4| < 3x – 7.

(A) (11/2, ∞) (B) (3/4, ∞)
(C) (–∞, –3/4) (D) (3/4, 11/2)

29. If x, y and z are positive numbers and x + y + z equals k, then xy + yz + zx will be _____.

(A) at most $\frac{k^2}{3}$ (B) at least $\frac{k^2}{3}$
(C) less than $\frac{k^2}{3}$ (D) more than $\frac{k^2}{3}$

30. If 3x – 7 ≤ 6x + 8 and 2x – 5 ≥ 7x + 10, then the range of the values of x that satisfies the inequalities is

(A) [–5, – 3] (B) (–∞, – 3)
(C) (– ∞, –5] (D) [–5, ∞)

31. Given $g(x) = |2-x|^x$. Which of the following is true about g(x)?

(A) If 0< a < b < 1, then g(a) < g(b)
(B) If 1 < a < b< 2, then g(a) > g(b)
(C) If 2 < a < b < 3, then g(a) > g(b)
(D) If 1 < a < b < 3, then g(a) < g(b)

32. Which of the following is/are true?

(I) If a < b ⇒ ac < bc
(II) If a > b ⇒ a – c > b – c
(A) Only I
(B) Only II
(C) Both I and II
(D) Neither I nor II

33. The number of distinct solutions of the equation |3x – |5x + 7|| = 10 is $\boxed{}$

34. If –3 < x < 5, –7 < y < 12 and z = 3x – 4y, then which of the following is the range of z?

(A) –57 < z < 43 (B) 19 < z < –33
(C) 19 < z < 43 (D) –57 < z < – 33

35. A = {x : x |x – 5| = 6, x ∈ N} A =

(A) {2, 3, 6} (B) {1, 2, 3}
(C) {1, 2, 3, 6} (D) {1, 3, 6}

36. If p, q, r are positive and

$x = \frac{(2p^2+p+2)(3q^2+q+3)(r^2+r+1)}{15pqr}$, then x cannot be

(A) 8 (B) 10 (C) 7 (D) 6

37. The range of values of k for which $-x^2 + 3kx + 5k + 1 < 0$ is

(A) (∞, –2) ∪ (–2/9, ∞)
(B) $\left[-\frac{9}{2}, -2\right]$
(C) (–2, –2/9)
(D) (–∞, ∞)

38. The range of x for which $|x^2 + x - 2| \le x^2 - x$ is

(A) [–1, 1]
(B) (–∞, –1]
(C) [1, ∞)
(D) No such value of x exists

39. Given $g(x) = |3-x|^x$. Which of the following is true about g(x)?

(A) If a < b < 0, then g(a) > g(b)
(B) If 0 < a < b < 2, then g(a) < g(b)
(C) If 2 < a < b< 4, then |g(a) – g(b)| < 1
(D) If 3 < a < b, then g(a) > g(b)

40. Max $[\min(x+6, x-3), \min(x+5, x-7)]$ =

(A) x + 6 (B) x + 5
(C) x – 3 (D) x – 7

Directions for questions 41 to 50: Each question is followed by two statements, I and II. Answer each question using the following instructions:

Choose (A) if the question can be answered using one of the statements alone, but cannot be answered using the other statement alone.
Choose (B) if the question can be answered using either statement alone.
Choose (C) if the question can be answered using I and II together but not using I or II alone
Choose (D) if the question cannot be answered even using I and II together.

41. Is x < y?

I. $x^2 > y^3$
II. $y^2 > x^3$

42. Is x > 1?

I. $x^2 > x^3$
II. $x^3 > x$

43. What is the maximum value of $|x+6z|$?

I. $|x-2y| \le 5$.
II. $|y+3z| \le 9$.

44. Is $|3x+5| < 5x + 1$?

I. (x + 5) (x + 10) (x – 6) > 0.
II. $8x^2 - 10x - 12 > 0$.

45. What is the value of $\left|\frac{x+5}{x+7}\right|$?

I. $x^2 + 7x - 18 = 0$.
II. $3x^2 + 46x + 171 = 0$.

46. Is $|x+2y-2z| > 9$?

I. $|x+4y| < 3$.
II. $|y+z| > 6$.

47. Is $x < 0$?

I. $x > x^2$
II. $x^2 > x^4$

48. If $x > 0$, is $x < 1$?

I. $x > \sqrt{x}$
II. $x > \sqrt[3]{x}$

49. If $x > 0$, is $x^2 > 1$?

I. $x^4 > 1$
II. $\sqrt{x} > \sqrt[4]{x}$

50. Is $13 - |x+4| + |x+1| > 14$?

I. x is any integer.
II. $-100 \le x \le -4$.

Key

Concept Review Questions

1. (a) C	7. B	15. D	20. A	(c) A
(b) B	8. A	16. (a) C	21. C	26. B
(c) A	9. C	(b) B	22. (a) D	27. 3.8
2. D	10. 4	(c) C	(b) C	28. B
3. D	11. 5	17. (a) C	23. A	29. D
4. C	12. 2	(b) D	24. D	30. C
5. 2	13. C	18. B	25. (a) B	
6. C	14. C	19. B	(b) 7	

Exercise – 3(a)

1. B	6. A	11. D	16. 0	21. C	26. 4
2. D	7. C	12. A	17. A	22. D	27. 2
3. C	8. 6	13. D	18. 3	23. B	28. D
4. D	9. 3	14. 8	19. D	24. C	29. B
5. C	10. 3	15. C	20. C	25. 11/5	30. A

Exercise – 3(b)

1. 3	11. D	21. C	31. B	41. D
2. D	12. 16	22. A	32. B	42. A
3. 1.2	13. 243	23. 10	33. 2	43. C
4. A	14. D	24. D	34. A	44. D
5. 48	15. 6	25. 36	35. A	45. A
6. B	16. D	26. B	36. D	46. C
7. D	17. C	27. B	37. C	47. A
8. 1	18. 8	28. A	38. B	48. B
9. 9	19. B	29. A	39. C	49. B
10. C	20. 4	30. A	40. C	50. A