

# On the LAP Relaxations of QAP

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In this note we discuss the linear assignment problem (LAP) method as applied to a class of the quadratic assignment problems (QAPs) that arise in classification of 2 by 2 stochastic un-labeled block model. In particular, we show that the LAP is an optimal solution of a relaxed version of the QAP. More generally, the LAP approach solves an assignment problem that corresponds to a linearization of the quadratic function in for symmetric instances of the QAP.

Recall, the QAP is defined by the following, where we seek to find a permutation matrix  $\hat{Q}$  such that

$$\hat{Q}_{\text{QAP}} = \operatorname{argmin}_{Q \in \mathcal{Q}} \|QAQ^T - B\|_F^2 \quad (1)$$

The classification problem studied is a class of homogeneous “kidney-egg” graphs with  $n$  nodes. We are given graphs generated from one of two classes of stochastic block models. Our model problem has the following parameters:  $n = 10$ , so  $\eta^y \in (0, 1)^{n \times n}$ . We further assume a stochastic block model, with two blocks, for both classes. For class 0,  $\eta_{ij}^0 = q_0 = 0.25$  for  $i, j \in \{1, 2, 3\}$ , and for class 1,  $\eta_{ij}^1 = q_1 = 0.75$  for  $i, j \in \{1, 2, 3\}$ . In both classes,  $\eta_{ij} = p = 0.5$  for  $i, j \in \{4, \dots, 10\}$ . For training data we are given a sample matrix for each class ( $B_0$  and  $B_1$ ); the parameters for the block model are known to classifier. In each trail of the simulation we are given a matrix  $A$  and then use an approximate algorithm to find a approximate solutions to the 2 QAPs with  $B = B_i, i = 0, 1$ .

The LAP approximation for this class problem is given by the following equation:

$$\hat{Q}_i^{\text{LAP}} = \operatorname{argmin}_{Q \in \mathcal{Q}} \|AQ^T - B_i\|_F^2, \quad i = 0, 1. \quad (2)$$

As the LAP approximation applies just column operations to the  $A$ , it has power to distinguish between the two classes due to the fact that both classes have non-uniform column distributions. The entries in the columns of are distributed as a mixture of the binomial distribution for the kidney and the egg in both classes. (This panel of the the matrices we might call a “scambled-egg,” as the egg “signal” is mixed into portion of the “kidney” edges.) The ability of the LAP to distinguish between class 0 and 1 relies on the fact that the column distributions of the scambled-egg portions differ from the kidney columns in both classes are non-zero with probability 0.5. In our model problem for class 0, the entries in columns 1,2, and 3 are non-zero with probability  $\frac{17}{40} = \frac{3(0.25)+7(0.5)}{10}$  while the remaining columns’ entries are non-zero with probability  $\frac{1}{2}$ . Similarly, for class 1, entries in columns 1,2, and 3 are non-zero with probabilitiy  $\frac{23}{40} = \frac{3(0.75)+7(0.5)}{10}$ . As long as the stochastic block models for at least one of the classes has a “scambled-egg” subset of columns that differ from the remaining subset of columns, the LAP approximation of this class of QAPs will have power to distinguish between the two classes.

Secondly, the LAP approximation to the QAP solves the same linear assignment algorithm used in the first iteration of the Frank Wolfe algorithm (FW) when started at the identity, when  $A$  and  $B$  are symmetric. In particular, recall that function and its gradient employed by FW are given by

$$f(P) = \sum_{i=1}^m \sum_{j=1}^n (A \circ PBP^t)_{ij}$$

$$\nabla_P f(P) = APB^t + A^tPB.$$

The Frank-Wolfe method (FW) is an iterative method to approximately solve a quadratic programming problem. FW takes a starting point  $X^{(0)} \in \mathcal{D}$ , the set of doubly stochastic matrices, and then proceeds by solving the LAP

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derived from a first order Taylor series expansion about  $X^{(0)}$  as given in Step 1:

$$W^{(k)} = \operatorname{argmin}_{W^{(k)}} \left\{ \sum_{i,j=1}^m \left( \nabla_X f(X^{(k)}) \circ W^{(k)} \right)_{ij} \mid W^{(k)} \in \mathcal{Q} \right\}$$

The solution of this LAP defines the direction to search for the next iterate. An exact line search is used to find the next iterate which is also a doubly stochastic matrix. The procedure is iterated until a desired accuracy is obtained or convergence. Our variation of FW outputs the nearest permutation matrix to the double stochastic matrix on the final iteration.

We note that the LAP approximation to the QAP solves equation 2 as follows:

$$\hat{Q}_{LAP} = \operatorname{argmin}_Q \left\{ \sum_{i,j=1}^m (AB^\top \circ Q)_{ij} \mid Q \in \mathcal{Q} \right\}$$

In the case where  $A$  and  $B$  are symmetric we note that

$$\nabla_X f(I) = 2AB^\top,$$

so in this case the LAP solves the assignment problem that would be tackled by FW on the first iteration given the starting point of the identity matrix.

We note that in our experiments with FW the starting point in the classification experiments was  $\frac{1}{n}J$ , the “flat” doubly stochastic matrix.