# Continuous Methods for the Quadratic Assignment Problem

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# Linear Assignment Problem

**Problem:** Given $A$ and $B$, $n$x$n$ matrices find a permutation matrix $P \varepsilon \mathcal{P}_n$ to

$$\min \sum_{ij} (PB)_{ij} a_{ij}$$

**Equivalently:** Given $C = AB^T$, find a

doubly stochastic matrix $X \varepsilon \mathcal{D}_n$ to

$$\min \sum_{ij} c_{ij} x_{ij}$$

# Quadratic Assignment Problem
## (An NP-Hard Problem)

**Problem:** Given $A$ and $B$ find a permutation matrix $P \varepsilon \mathcal{P}_n$ to

$$\min \sum_{ij} (PBP^T)_{ij} a_{ij}$$

**Relaxed Problem:** Given $A$ and $B$ find a doubly stochastic matrix $X \varepsilon \mathcal{D}_n$ to

$$\min \sum_{ij} (XBX^T)_{ij} a_{ij}$$

# Continuous Relaxed Problem

**Find $X \varepsilon \mathcal{D}_n$ to**

$$\min f(X) = \sum_{ij} (XBX^T)_{ij} a_{ij}$$

**Continuous indefinite quadratic programming**

- **may have many local maxima**
- **may have interior solution**
- **quadratic cost, linear gradient**

# Frank Wolfe (SLP) Method

**Given (A,B) and $X_0 \varepsilon \mathcal{D}_n$**

**For k=0,1,...**

**1. Find $Q \varepsilon \mathcal{P}_n$ to**

$$\min \sum_{ij} \nabla f(X_k)_{ij} q_{ij}$$

**2. Find $\alpha \varepsilon [0,1]$**

$$\min f((1-\alpha)X_k + \alpha Q)$$

$$X_{k+1} = (1-\alpha)X_k + \alpha Q$$

# Mapping Interior Solution to Vertex

If $X \varepsilon \mathcal{D}_n$ solves the relaxed problem,

find nearest $P \varepsilon \mathcal{P}_n$ by

$$\min \| X - P \|_F \equiv \max \sum_{ij} x_{ij} p_{ij}$$

Solve a linear assignment problem.

# Cost Per Iteration

**Gradient:**

$$\nabla f(X_k) = AX_kB^T + A^TX_kB$$

**Can reduce to two matrix multiplies.**

**Linear Assignment:** $O(n^3)$

**Two Function Evaluations:**

**2 implicit matrix multiplies,**
**can be reduced to $O(n^2)$.**

# Results

## Table 1. # Starts to Best Known Solution

| Name | Best Known Solution | Lower Bound | FW Solution | #starts |
|---|---|---|---|---|
| esc032a | 130 | 35 | 132 | 186 |
| esc032b | 168 | 96 | 168 | 26 |
| esc032c | 642 | 464 | 642 | 2 |
| esc032d | 200 | 106 | 200 | 7 |
| esc032e | 2 | 0 | 2 | 1 |
| esc032f | 2 | 0 | 2 | 1 |
| esc064a | 116 | 47 | 116 | 2 |
| sko42 | 15812 | 13830 | 15818 | 168 |
| wil50 | 48816 | 47098 | 48816 | 328 |
| sko64 | 48498 | 43668 | 48508 | 9 |

# Summary

- Very often found best known solution
- Cubic Runtime
- Good Suboptimal Solutions Efficiently
- Interior point method? e.g., Boggs et. al. under investigation.