Calculating Biological Quantities

CSCI 2897

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2021, Lecture 21

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• HW #5 due @ 11:59 P.M.

Last time on CSCI 2897

diagonal matrix

$$\begin{pmatrix} M_{11} & & 0 \\ & M_{22} & \\ & & \ddots & \\ 0 & & & M_{nn} \end{pmatrix}$$

$D = D^T$ if D diagonal

upper triangular matrix

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & \cdots \\ 0 & M_{22} & & \vdots \\ 0 & 0 & \ddots & \\ \vdots & & 0 \quad 0 & \\ 0 & \cdots & 0 & M_{nn} \end{pmatrix}$$

lower triangular matrix

$$\begin{pmatrix} M_{11} & 0 & 0 & \cdots & 0 \\ M_{21} & M_{22} & 0 & & 0 \\ M_{31} & & \ddots & 0 & \vdots \\ \vdots & & & 0 & \vdots \\ & & & & 0 \\ & \cdots & & M_{nn} & \end{pmatrix}$$

symmetric matrix

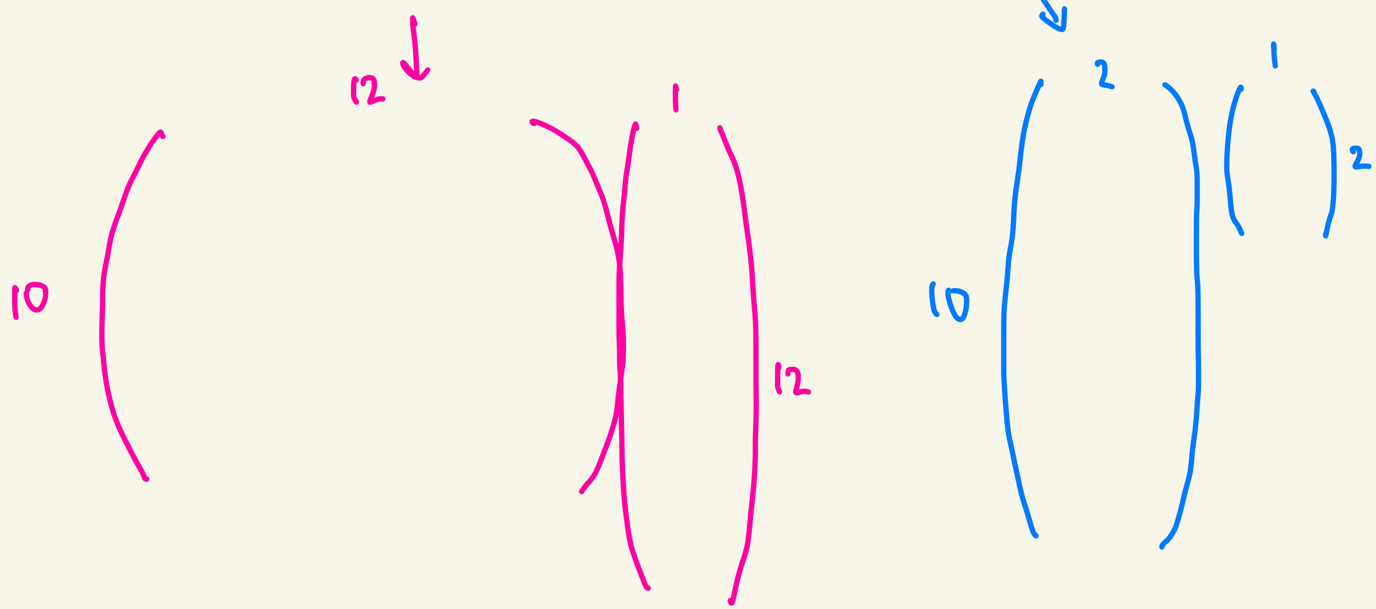
$S = S^T$

Multiplication (side note)

- consider work by hand
- consider computer memory as well!

$$P^{10\times 2} \quad Q^{2\times 12} \quad V^{12\times 1}$$

$$PQV = (PQ)^{10\times 12}\, V^{12\times 1} = P^{10\times 2}(QV)^{2\times 1}$$



Hard Way

Easy Way

Complex Eigenvalues

Unreal! Sometimes we can have eigenvalues which are *complex numbers*.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightarrow \quad \lambda_1, \lambda_2 = \frac{\text{tr}(A) \pm \sqrt{\text{tr}^2(A) - 4\text{det}(A)}}{2}$$

If $\text{tr}^2(A) - 4\text{det}(A) < 0$, then λ_1, λ_2 will be **complex numbers**.

A **complex number** is a number $c = a + bi$, where a and b are real and $i = \sqrt{-1}$ is "imaginary."

In our formula above, what's the **real part**? And the **imaginary part**?

$$\text{Re} = \frac{\text{tr}(A)}{2} \qquad \text{Im} = i\,\frac{\sqrt{4\,\text{det}(A) - \text{tr}^2(A)}}{2}$$

Complex Eigenvalues

Unreal! Sometimes we can have eigenvalues which are *complex numbers*.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightarrow \quad \lambda_1, \lambda_2 = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}^2(A) - 4\det(A)}}{2}$$

If $\mathrm{tr}^2(A) - 4\det(A) < 0$, then λ_1, λ_2 will be **complex numbers**.

A complex number is a number $c = a + bi$, where a and b are real and $i = \sqrt{-1}$ is "imaginary."

$$a = \frac{\mathrm{tr}(A)}{2}, \quad \text{and} \quad b = \frac{\sqrt{-\mathrm{tr}^2(A) + 4\det(A)}}{2}$$

and therefore $\lambda_1 = a + bi, \quad \lambda_2 = a - bi$

Notice: either both eigenvalues are complex, or both are real.

Euler's Equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\pi} = -1 + 0$$

$$\boxed{e^{i\pi} + 1 = 0}$$

identity

Taylor Series

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \ldots$$

$i^3 = i^2 \cdot i = -1 \cdot i = -i$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + \ldots$$

every other term is imaginary!
(the odd terms)

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \longrightarrow \cos x$$

$$+$$

$$+ i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right) + \ldots \longrightarrow i\sin x$$

Solutions to linear systems

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$e^{ibt} = \cos bt + i\sin bt$

$e^{-ibt} = \cos(-bt) + i\sin(-bt)$

$= \cos bt - i\sin(bt)$

$$\frac{d\vec{n}}{dt} = A\vec{n} \qquad \text{Solution: } \vec{n}(t) = k_1\vec{x_1}e^{\lambda_1 t} + k_2\vec{x_2}e^{\lambda_2 t}$$

So what's going to happen when λ_1 and λ_2 are complex?

$\lambda_1 = a + bi$

$\lambda_2 = a - bi$

$n(t) = k_1 x_1 e^{(a+bi)t} + k_2 x_2 e^{(a-bi)t}$

$= k_1 x_1 e^{at} e^{bit} + k_2 x_2 e^{at} e^{-bit}$

$= e^{at}\left(k_1 x_1 e^{bit} + k_2 x_2 e^{-bit}\right)$

$= e^{at}\left(k_1 x_1 (\cos bt + i\sin bt) + k_2 x_2 (\cos bt - i\sin bt)\right)$

Solutions to linear systems

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\frac{d\vec{n}}{dt} = A\vec{n} \qquad \text{Solution: } \vec{n}(t) = k_1\vec{x}_1 e^{\lambda_1 t} + k_2\vec{x}_2 e^{\lambda_2 t}$$

What are $\vec{x}_1$ and $\vec{x}_2$?

$$(A - \lambda_1 I)\vec{x}_1 = \vec{0}$$

$$\begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(a-\lambda_1)x_1^{(1)} + b\,x_1^{(2)} = 0$$

$$x_1^{(2)} = \frac{(\lambda_1 - a)}{b}x_1^{(1)} \qquad \vec{x}_1 = \begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix}$$

$$\frac{\lambda_1 - a}{b} = \frac{Re(\lambda) + i\,Im(\lambda) - a}{b}$$

$$= \frac{Re(\lambda) - a}{b} + i\,\frac{Im(\lambda)}{b}$$

$$\vec{x}_1 = \begin{pmatrix} 1 \\ \frac{Re(\lambda)-a}{b} \end{pmatrix} + i\begin{pmatrix} 0 \\ \frac{Im(\lambda)}{b} \end{pmatrix}$$

$$\vec{x}_2 = \begin{pmatrix} 1 \\ \frac{Re(\lambda)-a}{b} \end{pmatrix} - i\begin{pmatrix} 0 \\ \frac{Im(\lambda)}{b} \end{pmatrix}$$

$$\boxed{\begin{aligned} \vec{x}_1 &= \vec{y} + i\vec{z} \\ \vec{x}_2 &= \vec{y} - i\vec{z} \end{aligned}}$$

$\vec{x}_1$ and $\vec{x}_2$ are complex conjugates of each other!

Solutions to linear systems

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$x_1 = y + iz$

$x_2 = y - iz$

$$\frac{d\overrightarrow{n}}{dt} = A\overrightarrow{n}$$

Solution: $\overrightarrow{n}(t) = k_1\overrightarrow{x_1}e^{\lambda_1 t} + k_2\overrightarrow{x_2}e^{\lambda_2 t}$

$$= e^{at}\left(k_1 x_1 (\cos bt + i\sin bt) + k_2 x_2 (\cos bt - i\sin bt)\right)$$

$$= e^{at}\left(k_1 (y + iz)(\cos bt + i\sin bt) + k_2 (y - iz)(\cos bt - i\sin bt)\right)$$

$$= e^{at}\left(k_1 \left[\begin{matrix} y\cos bt & + \; i\,y\sin bt \ldots \\ -z\sin bt & + \; iz\cos bt \end{matrix}\right] + k_2 \left[\begin{matrix} y\cos bt & - \; i\,y\sin bt \\ -z\sin bt & - \; iz\cos bt \end{matrix}\right]\right)$$

$$= e^{at}\left((k_1 + k_2)\vec{y}\cos bt + (k_1 - k_2)\,i\,\vec{y}\sin bt - (k_1 + k_2)\vec{z}\sin bt + (k_1 - k_2)\,i\vec{z}\cos bt\right)$$

Solutions to linear systems

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$k_1 + k_2 \equiv C_1$$
$$(k_1 - k_2)i \equiv C_2$$

$$\frac{d\vec{n}}{dt} = A\vec{n} \qquad \text{Solution: } \vec{n}(t) = k_1\vec{x_1}e^{\lambda_1 t} + k_2\vec{x_2}e^{\lambda_2 t}$$

$$= e^{at}\left(\underbrace{(k_1+k_2)}_{C_1}\vec{y}\cos bt + \underbrace{(k_1-k_2)i}_{C_2}\,\vec{y}\sin bt - \underbrace{(k_1+k_2)}_{-C_1}\vec{z}\sin bt + \underbrace{(k_1-k_2)i}_{C_2}\vec{z}\cos bt\right)$$

$$\vec{n}(t) = \underbrace{e^{at}}_{\text{growth (decay)}}\underbrace{\left[C_1\left(\vec{y}\cos bt - \vec{z}\sin bt\right) + C_2\left(\vec{z}\cos bt + \vec{y}\sin bt\right)\right]}_{\text{rotation/oscillation}}$$

Solutions to linear systems

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\frac{d\overrightarrow{n}}{dt} = A\overrightarrow{n}$$

Solution: $\overrightarrow{n}(t) = k_1\overrightarrow{x_1}e^{\lambda_1 t} + k_2\overrightarrow{x_2}e^{\lambda_2 t}$

What do solutions look like if eigenvalues are complex?

$\lambda_{1,2} = a \pm ib$

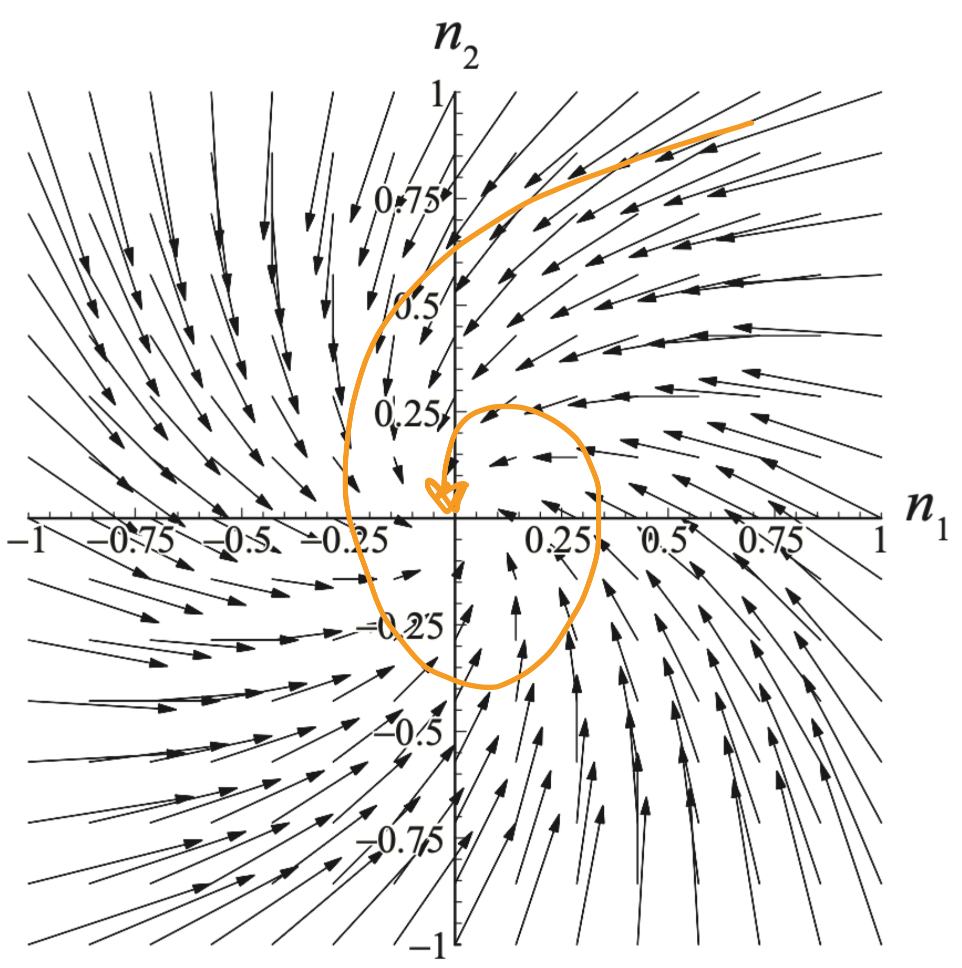
growth/decay

speed of rotation

Solutions to linear systems

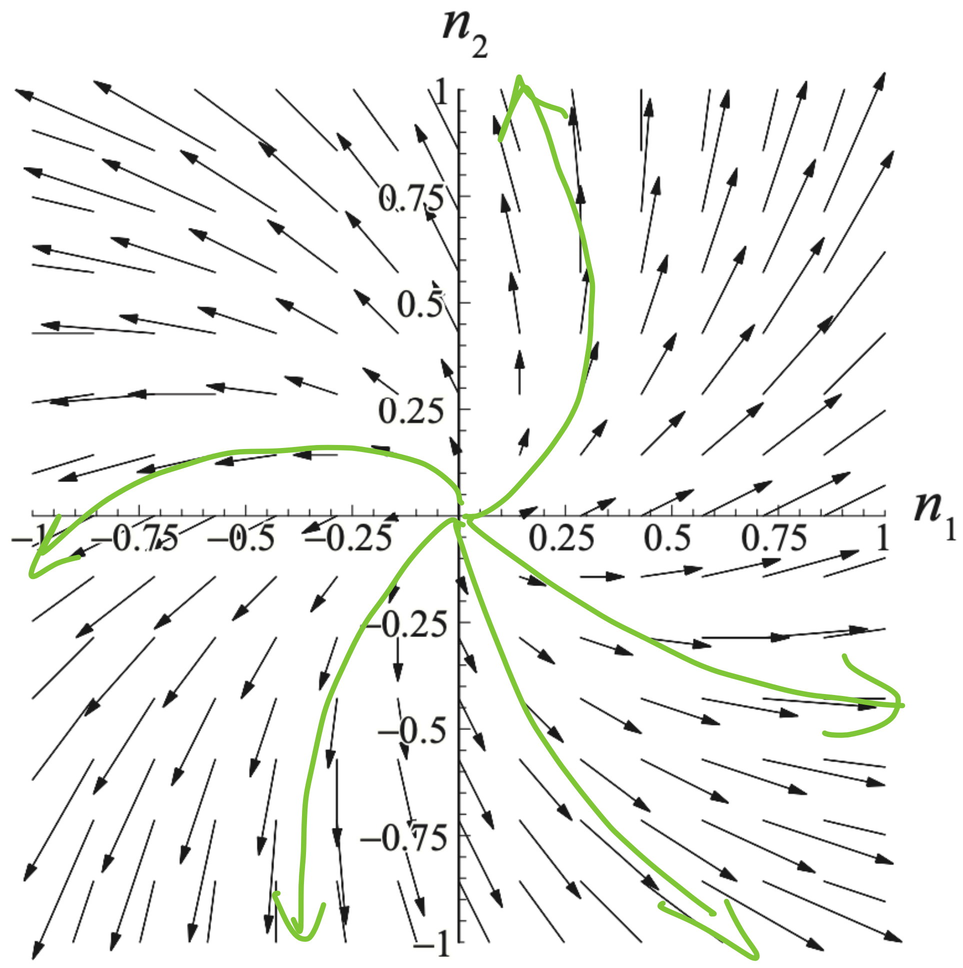
$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\frac{d\vec{n}}{dt} = A\vec{n} \qquad \text{Solution: } \vec{n}(t) = k_1\vec{x_1}e^{\lambda_1 t} + k_2\vec{x_2}e^{\lambda_2 t}$$



$\lambda = -2 \pm i$

inward spiral.



$\lambda = 2 \pm i$

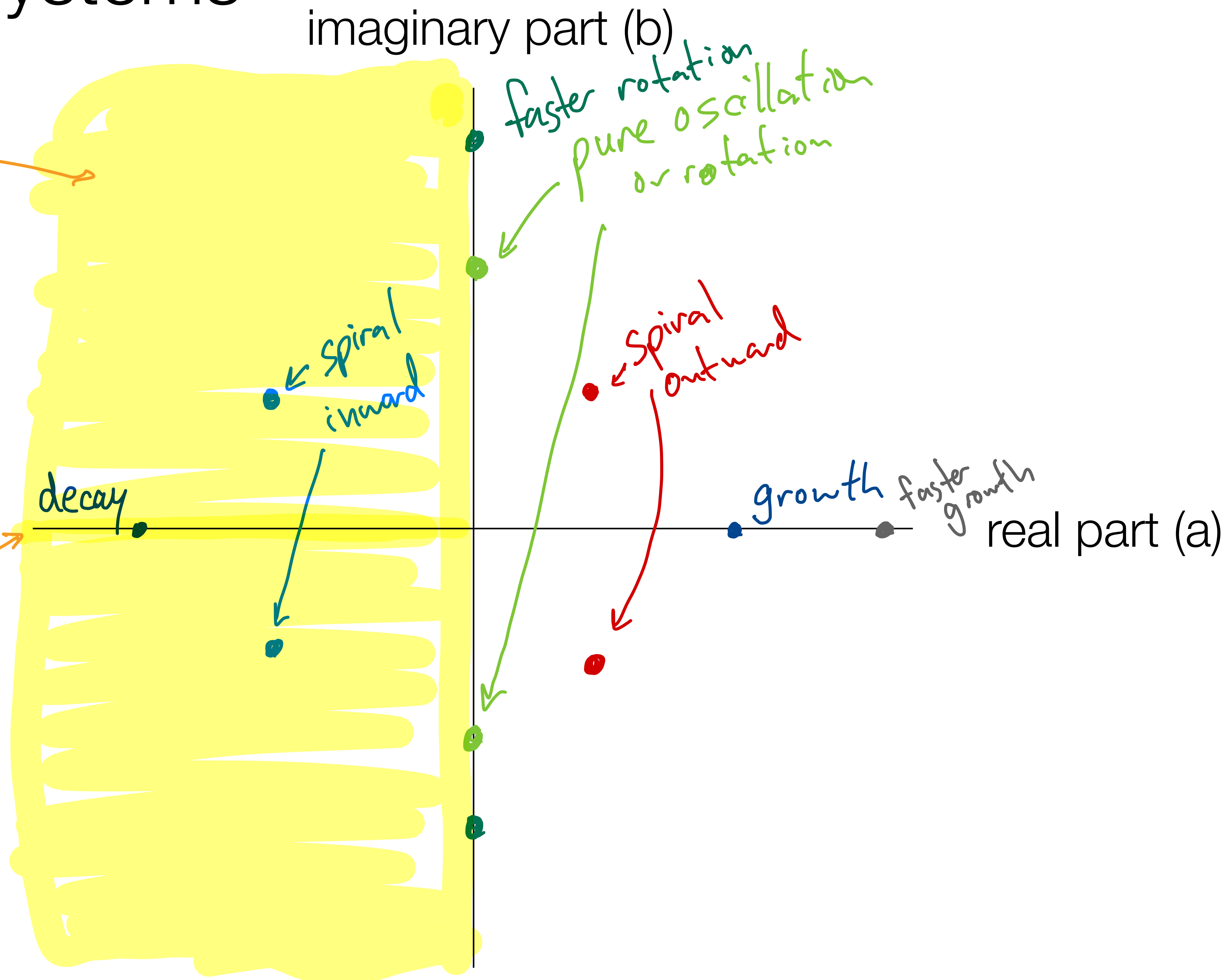
Outward spiral

Solutions to linear systems

$\lambda = a \pm bi$

Complex:

Real parts of eigenvalues must be negative to have stability.



Real: both eigs must be neg. to have stability.

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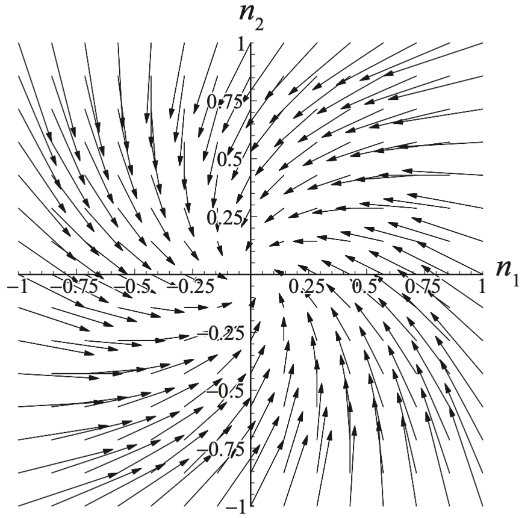
The eigenvalues of a **diagonal** or **triangular (upper or lower)** matrix are easy to get: they are just the values on the diagonal of the matrix!

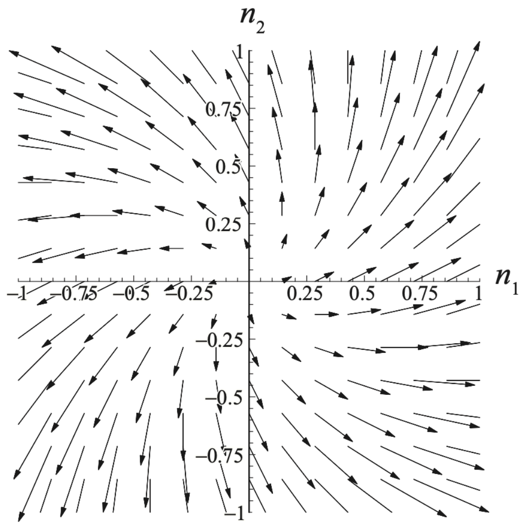
Stability of equilibria (real eigenvalues):

- If all eigenvalues are negative, the system is stable.
- If one or more eigenvalues are positive, the system is unstable.



Stability of equilibria (complex eigenvalues):

- If the real part of all eigenvalues is negative, the system is stable.
- The complex part of the eigenvalues tells us about rotation.



The **complex conjugate** of a complex number $a + bi$ is $a - bi$.
If all the entries of a matrix are real, then the eigenvalues are real or come in conjugate pairs — no lon~~g~~e complex eigenvalues.

Class structured populations

The study of population *age* structure or *size* structure is known as **demography**.

three

There are ~~four~~ kinds of questions we can ask which commonly come up:

1. What is the **long-term growth rate** of a population? conservation.

2. What is the **long-term class structure** of a population?
(age, life stage, body size)

3. Which **classes contribute most** to the long-term growth rate of a population.

conservation