

Calculating Biological Quantities

CSCI 2897

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Today:

1. Linear models with more than one variable
2. Matrices and vectors

Models with more than one dynamic variable

Let's go back to exponential growth in continuous time.

$$\frac{dn}{dt} = rn$$

$$\int \frac{dn}{n} = \int r dt$$

$$\ln n = rt + c$$

$$n = e^{rt+c}$$

$$n = e^{rt} \cdot e^c$$

$$n = ke^{rt}$$

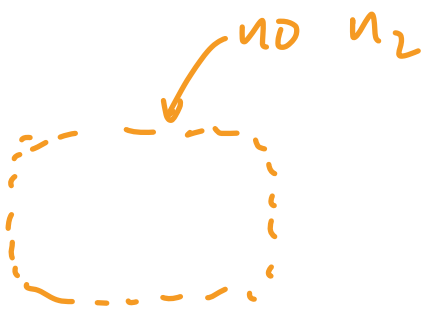
We know by now that this is called exponential growth because

$$n(t) = ke^{rt}$$

where $k = n(0)$ is the initial condition.

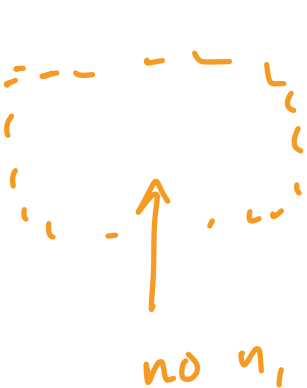
Models with more than one dynamic variable

Now let's imagine that we have two populations, n_1 and n_2

$$\frac{dn_1}{dt} = r_1 n_1$$


$$n_1(t) = k_1 e^{r_1 t}$$

not coupled.

$$\frac{dn_2}{dt} = r_2 n_2$$


$$n_2(t) = k_2 e^{r_2 t}$$

This one is easy too: the populations are totally independent of each other, so we can solve each equation by itself.

Models with more than one dynamic variable

What are the equilibrium solutions for this set of equations?

$$\frac{dn_1}{dt} = r_1 n_1 = 0 \quad \rightarrow n_1 = 0$$

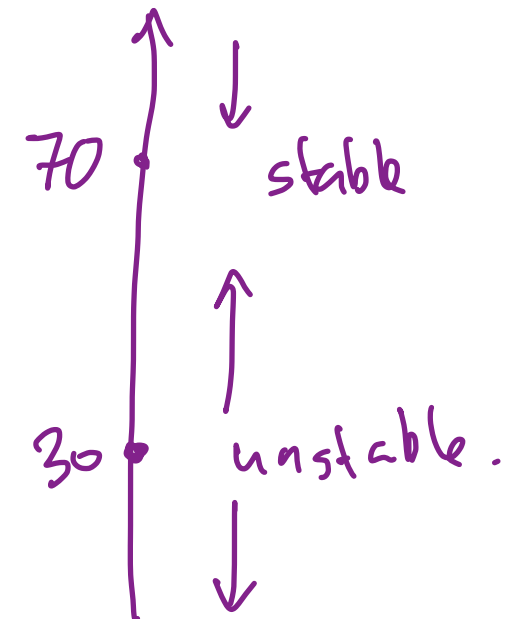
$$\frac{dn_2}{dt} = r_2 n_2 = 0 \quad \rightarrow n_2 = 0$$

equilibrium $(n_1, n_2) = (0, 0)$

What can we say about stability of the equilibrium solution(s)?

use vector fields.

recall exam 8:

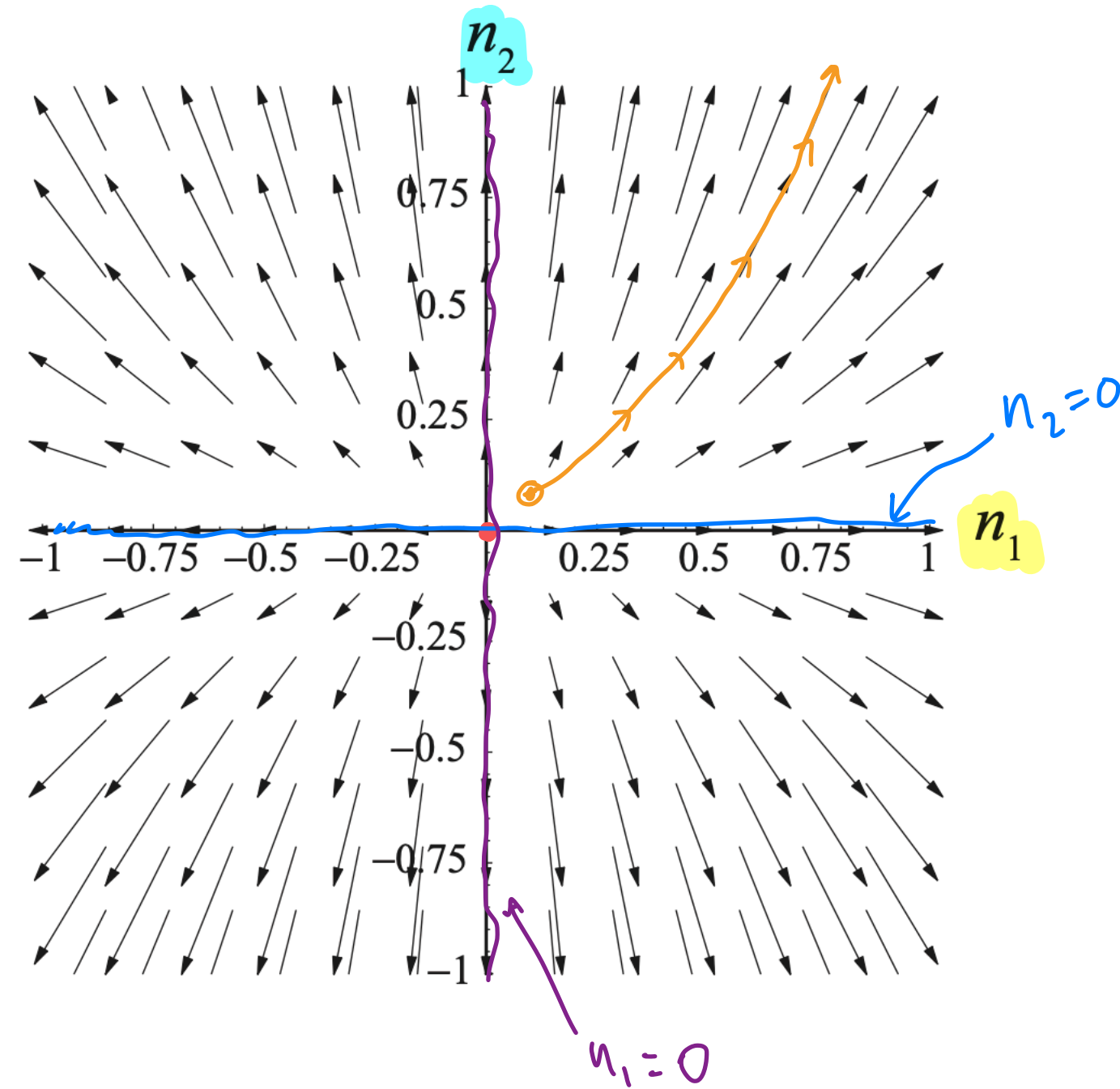


• equilibrium

Phase portrait
Vector Field

$$\frac{dn_1}{dt} = r_1 n_1$$

$$\frac{dn_2}{dt} = r_2 n_2$$



$$r_1 > 0$$

$$r_2 > 0$$

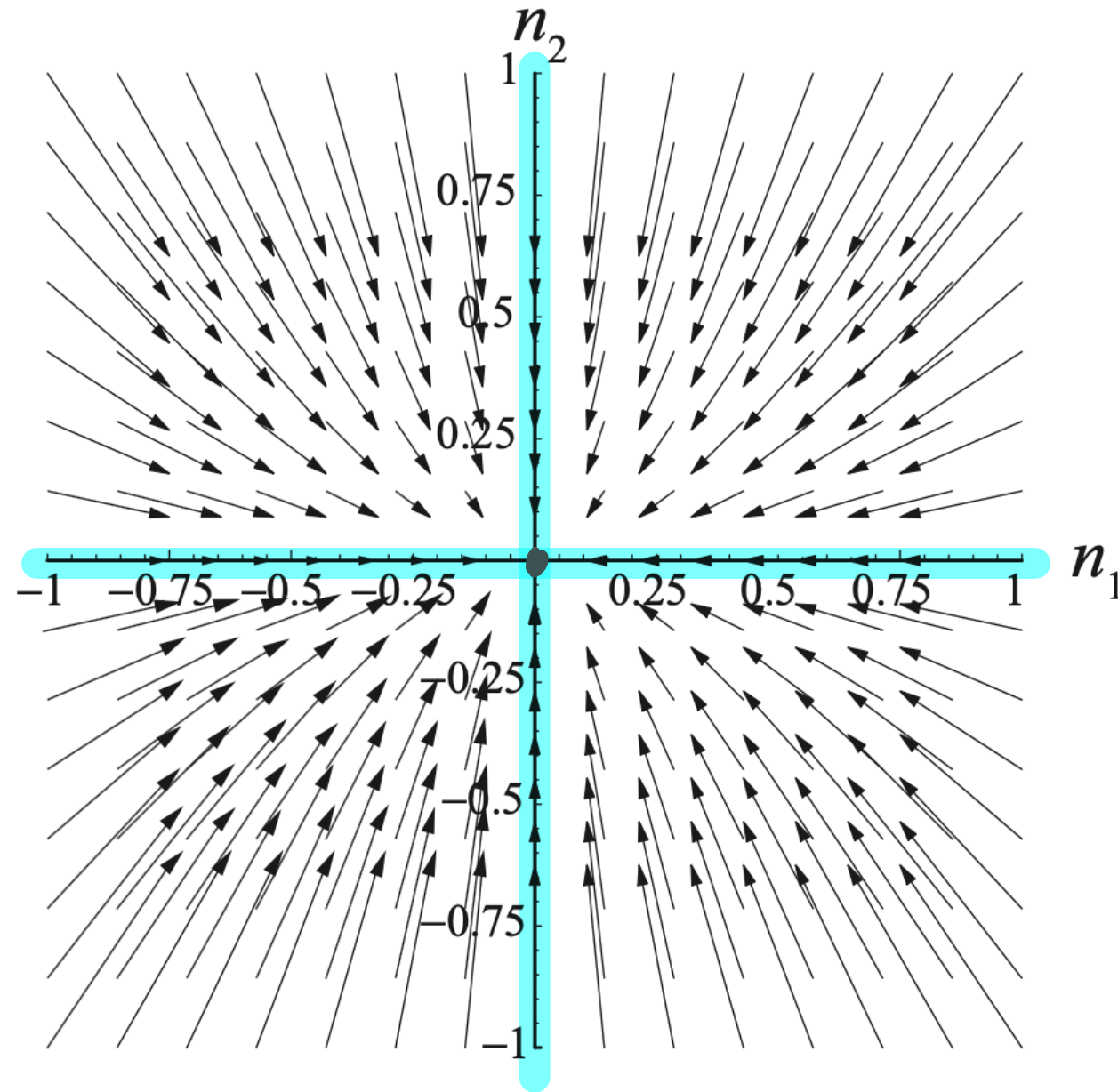
$$r_2 > r_1$$

Unstable

$$\frac{dn_1}{dt} = r_1 n_1$$

$$\frac{dn_2}{dt} = r_2 n_2$$

stable



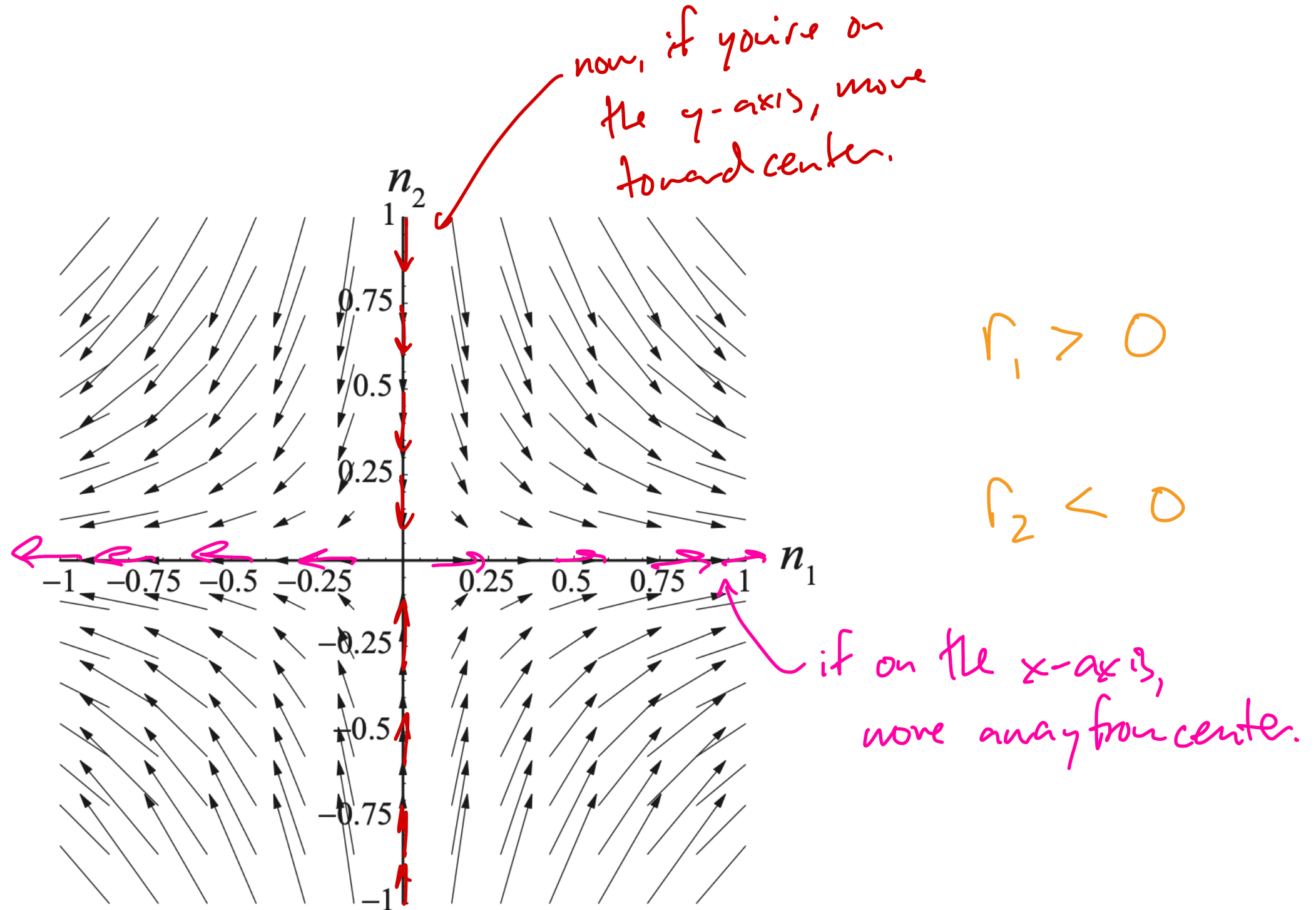
$$r_1 < 0$$

$$r_2 < 0$$

note: even though
we changed the
signs of r_1, r_2 ,
the "directions" given
by the axes are special.

$$\frac{dn_1}{dt} = r_1 n_1$$

$$\frac{dn_2}{dt} = r_2 n_2$$

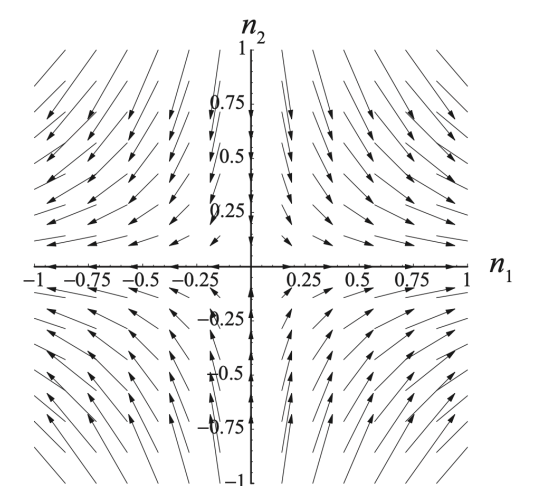
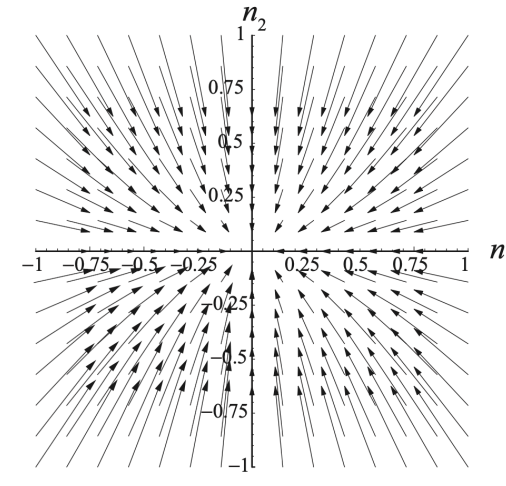
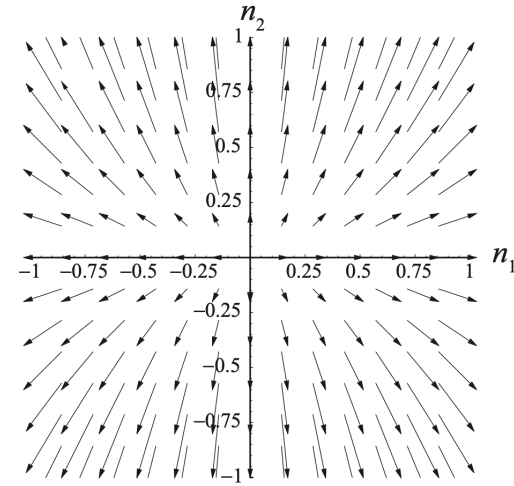


Unstable. \rightarrow must be stable in all directions to be classified stable.

Characteristic directions

$$\frac{dn_1}{dt} = r_1 n_1$$

$$\frac{dn_2}{dt} = r_2 n_2$$



Note: for these equations, if you're on either axis, you never leave.

These directions are therefore special:

$(c,0)$ the horizontal axis

$(0,c)$ the vertical axis

for any arbitrary value of c .

Models with more than one dynamic variable - Part 2

Imagine that our 2 populations correspond to 2 strains of bacteria. Suppose that

- a is the rate at which **strain 1** produces **strain 1** daughter cells
- b is the rate at which **strain 2** produces **strain 1** daughter cells by mutation
- c is the rate at which **strain 1** produces **strain 2** daughter cells by mutation
- d is the rate at which **strain 2** produces **strain 2** daughter cells

$$\frac{dn_1}{dt} =$$

$$\frac{dn_2}{dt} =$$

Models with more than one dynamic variable - Part 2

We are going to rewrite this in a miraculous way

$$\frac{dn_1}{dt} = an_1 + bn_2$$

$$\frac{dn_2}{dt} = cn_1 + dn_2$$

Vectors and Matrices

A **vector** is a list of elements.

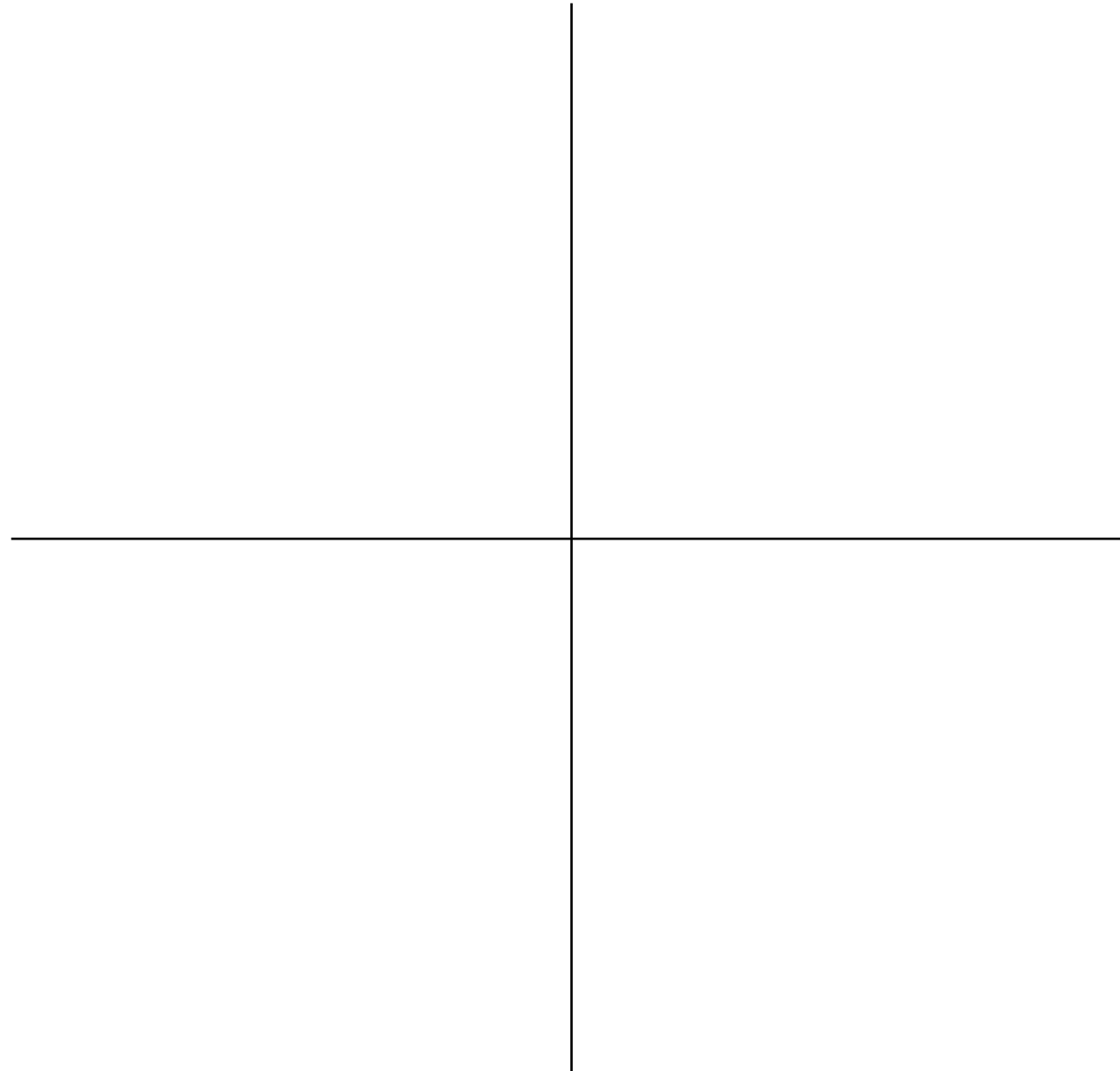
A **matrix** is a table of elements.

NB: the plural of “matrix” is “matrices.”

Vectors in the x-y plane

Remember those characteristic directions from before, $(0,c)$ and $(c,0)$? Those, too, are vectors!

It turns out that points in the x-y plane are also vectors. Why? Because a **vector** is a list of elements.



NB: Because of their use in modeling, we draw vectors as **arrows**, which point in a particular direction, and have a particular magnitude.

Vectors in the x-y-z plane

We can also draw vectors in 3D.

Row vectors and Column vectors

In general, the **dimension** of a vector is:

(# of rows) \times (# of columns)

Because a vector has either 1 row or 1 column, we get two definitions for free:

1. a vector that's a row is called a **row vector**
2. a vector that's a column is called a **column vector**.

Examples:

To denote a vector, we sometimes write a little arrow \overrightarrow{v}

Matrix and Vector addition

Rule: you can add two matrices or two vectors **only if** they have the same dimensions.

Examples:

Matrix and Vector scalar multiplication

Rule: you can **multiply** a matrix or a vectors **by a constant**.

Examples:

Matrix and Vector subtraction

Because we can

1. multiply a matrix or vector by -1 , and
 2. add it to another matrix or vector,
- this means that we we can do subtraction too*!

*Reason: $b - a = b + (-1)a$

Vector-vector multiplication

Rule: we can multiply a **row vector** by a **column vector** provided that they have the same number of elements.

Formula: Step **across the row vector** and **down the column vector**, multiplying each pair of elements. Then **add the products**.

NB: This kind of vector-vector multiplication **produces a scalar**.

Vector-vector multiplication

This kind of vector-vector multiplication **produces a scalar**.

Question: how can we **multiply two column vectors**?

The Transpose

To take the **transpose** of a **row-vector**, flip it & write it as a column vector.

To take the **transpose** of a **column-vector**, flip it & write it as a row vector.

Question: what happens to the *dimensions* of a vector when we take its transpose?

The Transpose - Part 2

To take the **transpose** of a matrix, think of its columns as column vectors, and then write them as row vectors. The first column becomes the first row.

Question: what happens to the *dimensions* of a matrix when we take its transpose?

Matrix-vector multiplication

Suppose we have a **2x2 matrix** and a **2x1 vector**.

We can define matrix-vector multiplication as follows:

1. Multiply the 1st row of the matrix by the vector.
2. Multiply the 2nd row of the matrix by the vector.
3. Stack the answers in a new vector.

Example:

Matrix-vector multiplication

Suppose we have a **3x3 matrix** and a **3x1 vector**.

1. Multiply the 1st row of the matrix by the vector.
2. Multiply the 2nd row of the matrix by the vector.
3. Multiply the 3rd row of the matrix by the vector.
4. Stack the answers in a new vector.

Example:

Matrix-vector multiplication

Suppose we have a **NxN matrix** M and a **Nx1 vector** \vec{x} .

Let's write a formula for the i th element of the resulting vector, $\vec{v} = M\vec{x}$

Rule: To multiply a matrix and a vector, *what must be true of their dimensions?*

Models with more than one dynamic variable - Part 2

We are going to rewrite this in a miraculous way

$$\frac{dn_1}{dt} = an_1 + bn_2$$

$$\frac{dn_2}{dt} = cn_1 + dn_2$$

$$\frac{d\vec{n}}{dt} = M\vec{n}, \text{ where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$