Calculating Biological Quantities

CSCI 2897

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Last time on CSCI 2897:

row vector $(1, 3, 5)$ 1x3

column vector $\begin{pmatrix} 1 \\ 19 \end{pmatrix}$ 2x1

- one of a vector's dimensions is 1.
- dimensions = rows × columns

A **vector** is a list of elements.
A **matrix** is a table of elements.

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ 2x2 $\quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 2x2 $\quad \begin{pmatrix} 1 & 4 \\ 5 & 2 \\ 3 & 6 \end{pmatrix}$ 3x2

Rule: you can add two matrices or two vectors **only if** they have the same dimensions.

Rule: you can **multiply** a matrix or a vector **by a constant**.

$$\begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 10 \end{pmatrix} \qquad 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

To take the **transpose** of a matrix, think of its columns as column vectors, and then write them as row vectors. The first column becomes the first row.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Recap: multiplying two vectors

Rule: we can multiply a **row vector** by a **column vector** provided that they have the same number of elements.

Formula: Step **across the row vector** and **down the column vector**, multiplying each pair of elements. Then **add the products**.

$$\underset{1\times 3}{\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} a \\ b \\ 10 \end{pmatrix}} = 2\cdot a + 1\cdot b + 10\cdot 0 = 2a + b$$

Dot Product is related:

$$(3, 1, 4) \cdot (2, 0, 5) = 2\cdot 3 + 1\cdot 0 + 4\cdot 5 = 26$$

\cdot

Recap: multiplying a matrix and a vector

Suppose we have a **NxN matrix** and a **Nx1 vector**.

1. Multiply the 1st row of the matrix by the vector.
2. Multiply the 2nd row of the matrix by the vector.
3. Multiply the j^{th} row of the matrix by the vector, etc.
4. Stack the answers in a new vector.

Example:

outer dimensions tell me what shape the resulting matrix will have

$$\overset{3\times 3}{\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 3 & -1 \end{pmatrix}} \overset{3\times 1}{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}} = \overset{3\times 1}{\begin{pmatrix} \boxed{8} \\ \boxed{6} \\ \boxed{2} \end{pmatrix}}$$

$$-1\cdot 3 + 2\cdot 3 + -1\cdot 1$$

$$-3 + 6 - 1 = 6-4 = 2$$

Multiplying two matrices

On the prev. slide, we took the idea of multiplying two vectors and expanded it:
We treated a **matrix on the left** as a set of **stacked row vectors**.

To multiply two matrices, what should we do with the **matrix on the right**?

treat like column vectors!

$$\begin{array}{c} \\ i \end{array}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{2\times 2} \overset{j}{\begin{pmatrix} 0 & 1 \\ 5 & 2 \end{pmatrix}}^{2\times 2} = \begin{pmatrix} \overset{(i,j)}{1\cdot 0 + 2\cdot 5 = 10} & 1\cdot 1 + 2\cdot 2 = 5 \\ 3\cdot 0 + 4\cdot 5 = 20 & 3\cdot 1 + 4\cdot 2 = 11 \end{pmatrix}^{2\times 2} = \begin{pmatrix} 10 & 5 \\ 20 & 11 \end{pmatrix}$$

Keys:

1. inner dimensions must match.
2. resulting matrix has outer dimensions
3. result from mult. row i (left matrix) with column j (right matrix) $\rightarrow$ entry i,j in the product matrix.

Does matrix multiplication *commute*?

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 0+2\cdot 5 & 1\cdot 1+2\cdot 2 \\ 3\cdot 0+4\cdot 5 & 3\cdot 1+4\cdot 2 \end{pmatrix}_{2\times 2} = \begin{pmatrix} 10 & 5 \\ 20 & 11 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0\cdot 1+1\cdot 3 & 0\cdot 2+1\cdot 4 \\ 5\cdot 1+2\cdot 3 & 5\cdot 2+2\cdot 4 \end{pmatrix}_{2\times 2} = \begin{pmatrix} 3 & 4 \\ 11 & 18 \end{pmatrix}$$

In general, we cannot say that $AB = BA$.

In most cases $AB \neq BA$.

Some Exceptions:

① $B=A$ then $AB = A\cdot A = B\cdot B = B\cdot A$

② $A=0$ or $B=0$.

Algebra -vs- Linear Algebra

Associative Law

$$(AB)C = A(BC)$$

sequence of multiplication doesn't matter, as long as the left-to-right order is preserved.

$$2 \cdot 3 \cdot 4 = 2 \cdot (12) = (6) \cdot 4 = 24$$

Left Distributive Law

$$(A+B)C = AC + BC$$

$$(4+1) \cdot 3 = 3 \cdot 4 + 3 \cdot 1 = 12 + 3 = 15$$

Right Distributive Law

$$A(B+C) = AB + AC$$

$$2(1+5) = 2 \cdot 1 + 2 \cdot 5 = 12.$$

Commutative Law for Scalars

$$\underset{\text{scalar}}{k}(\underbrace{AB}_{\text{matrices}}) = (kA)B = A(kB) = (AB)k$$

scalars have a special passport that allows them to commute.

Tricks of the Transpose

We already learned one cute transpose trick: $\left(A^T\right)^T = A$

Here's another one: $(A+B)^T = A^T + B^T$
In other words, the transpose of the sum = the sum of the transposes.

Here's the *tricky* one: $(AB)^T = B^T A^T$ ★
Rule: To transpose a product, you can only "distribute" the transpose if you reverse the order of the product!

$$(ABC)^T = C^T B^T A^T$$

$$\left(A(BC)\right)^T \overset{★}{=} (BC)^T A^T \overset{★}{=} C^T B^T A^T$$

$$(ABCDE)^T$$

$$\left((ABC)(DE)\right)^T \qquad \left(A(BCDE)\right)^T$$

$$(DE)^T (ABC)^T \qquad (BCDE)^T A^T$$

$$E^T D^T C^T B^T A^T \qquad \vdots$$

$$E^T D^T C^T B^T A^T$$

The Zero Matrix & the Identity Matrix

In **algebra**, **zero** is the number that doesn't do anything in addition: $5 + 0 = 5$

In **algebra**, **one** is the number that doesn't do anything in multiplication: $9 \times 1 = 9$

In **linear algebra**, the **zero matrix** is doesn't do anything in addition:

$$\begin{pmatrix} 5 & 9 \\ 19 & 29 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 5+0 & 9+0 \\ 19+0 & 29+0 \end{pmatrix} = \begin{pmatrix} 5 & 9 \\ 19 & 29 \end{pmatrix}$$

In **linear algebra**, the **identity matrix** doesn't do anything in multiplication:

<u>want:</u> $AI = A$

$IA = A$

The Identity Matrix - what does it look like?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} \overset{\text{want}}{=} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$aI_{11} + bI_{21} \overset{\text{want}}{=} a$

$aI_{12} + bI_{22} = b$

$cI_{11} + dI_{21} = c$

$cI_{12} + dI_{22} = d$

$$\begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$I_{11}a + I_{12}c = a$

$I_{11}b + I_{12}d = b$

$I_{21}a + I_{22}c = c$

$I_{21}b + I_{22}d = d$

$I_{11} = 1 \quad I_{12} = 0$

$I_{21} = 0 \quad I_{22} = 1$

$\xrightarrow{e} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 2x2

- Ones on the diagonal
- zeroes elsewhere.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ 3x3

$$\begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \\ & & & & & 1 \end{pmatrix}$$ nxn

$$\boxed{AI = IA = A}$$

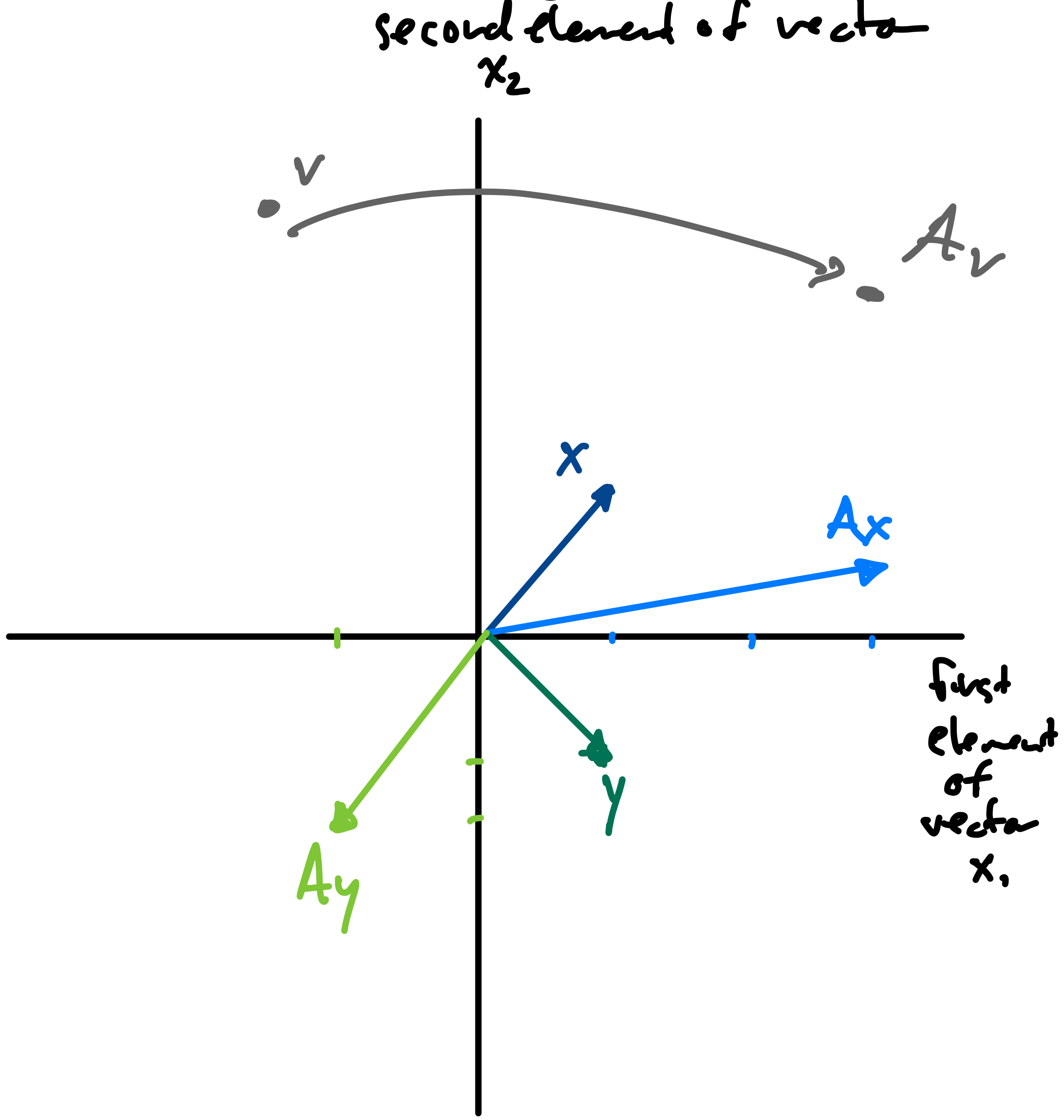
Matrices as Machines "Operators"

A **matrix** is a machine that does stuff to vectors. Take a vector, multiply, get a new vector.

$$A = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \overset{Ax}{Ax = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 1 \\ -\frac{1}{2}\cdot 1 + 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{1}{2} \end{pmatrix}}$$

$$y = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad Ay = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot -1 \\ -\frac{1}{2}\cdot 1 + 1\cdot -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3/2 \end{pmatrix}$$



Trace

The **trace** of a matrix is the sum of the diagonal elements. The trace is a scalar.

$$A = \begin{pmatrix} 1 & 3 \\ 5 & 12 \end{pmatrix} \quad tr(A) = 1+12 = 13 \qquad tr(I^{n\times n}) = tr\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = n$$

Practice: Suppose that the row i column j element of a matrix A is given by A_{ij}. How can we express the trace using summation notation?

$$A = \begin{pmatrix} A_{11} & & & & \\ & A_{22} & & & \\ & & A_{33} & & \\ & & & \ddots & \\ & & & & A_{nn} \end{pmatrix}$$

$$\boxed{tr(A) = \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i=j}}^{n} A_{ij} = \sum_{i=1}^{n} A_{ii}}$$

Note: the trace has no intuitive meaning, but it turns out to be rather convenient later.

Determinant (2x2 matrix)

The **determinant** of a matrix is also a scalar. It has a rather peculiar formula:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Practice:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$$

$$A = \begin{pmatrix} 1 & 2 \\ 0.5 & 1 \end{pmatrix} \quad 1 \cdot 1 - 2 \cdot \frac{1}{2} = 1 - 1 = 0$$

Notes

① det(A) may be negative!

② det(A) may be zero!

③ $\det(A) = \det(A^T)$

Note: the determinant of a matrix is the same as the determinant of its transpose.

Matrices as Machines II

$$A = \begin{pmatrix} 1 & 2 \\ 0.5 & 1 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$