Calculating Biological Quantities

CSCI 2897

- HW 1 posted (.tex source)
- OH M 9-11 BIOT A414, Zoom
  W 4-6

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2021, Lecture 3

Lecture 3 Plan

1. **A little notation & vocabulary**
2. **What does it mean to "solve" a differential equation?**
3. **Checking an analytical solution**
4. **Creating a numerical solution**

Notation

- "Leibniz" Notation: $\dfrac{dy}{dt} + y = 2021$
- Prime Notation: $y' + y = 2021$
- Dot Notation: $\dot{y} + y = 2021$

dot $\rightarrow$ w.r.t. time
physicists, appl. math

- Note: $\dfrac{d^2y}{dt^2} = y'' = \ddot{y}$

Vocab: ODE

- An **ODE** is an ordinary differential equation. ← this class.
- A **PDE** is a partial differential equation.

- ODEs have ordinary derivatives in them. PDEs have partial derivatives in them.
- Note: partial derivatives come up in Calc 3, but tbh they're not that complicated. Ask me in office hours!

- Ordinary derivatives look like $\dfrac{dy}{dx}$ while partial derivates look like $\dfrac{\partial y}{\partial x}$

\partial y

Vocab: Order

- The **order** of a differential equation is the highest derivative.
- Examples:
  - $y' + y = \pi$

    $y' \to$ first order ODE
  - $\dddot{z} - \ddot{z} = z$

    $\dddot{z} \to$ third order ODE
  - $\dfrac{d^2y}{dx^2} + 5\left(\dfrac{dy}{dx}\right)^3$

    $\dfrac{d^2y}{dx^2} \to 2^{nd}$ order

Linearity

- A *n*th order ODE is **linear** if we can write the ODE in this form:

derivatives

functions of t

$$a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_x(t)\frac{dy}{dt} + a_0(t)y = g(t)$$

- Two special cases that come up often are linear first order:

$$a_1(t)y' + a_0(t)y = g(t)$$

linear: $\sin(t)y'' + t^2 y' + y = 0$

- and linear second order:

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = g(t)$$

- A **nonlinear** ODE is simply one which is not linear.

Ex: $y' + y^2 = 1$

$y''' - y^4 = 10$

$t\ddot{y} + t^2\dot{y} + \sin(\dot{y}) = 0$

Practice makes the master!

- **Write down a third order linear ODE.**

Highest Order? | Linear?

$$y''' + y'' + y' + y = 1$$

3 ✓

$$e^t y''' + t y'' + \tan(t) y' + t^3 y = 5$$

3 ✓

$$e^t y''' = 0$$

3 ✓

- **Write down a second order non-linear ODE.**

Highest Order? | Linear?

$$t^3 (y'')^3 = 4$$

2 nope

$$y'' y = 1$$

2 nope

$$t^3 y'' = 4$$

(this is linear)

What does it mean to "solve" an ODE?

- What does it mean to solve $x + 3 = 9$?

Find a value of x, such that, when I plug it into the equation, I get LHS = RHS

$x = 6$
$6+3=9$
$9=9$ ✓
solves

$x=2$
$2+3=9$
$5 \neq 9$
✗
does not solve.

- Suppose that I give you $\sqrt{z} + z^2 - e^{z-4} = 17$. Is $z = 1$ a solution?

$\sqrt{1} + 1^2 - e^{1-4} = 17$

$1 + 1 - e^{-3} = 17$

$2 - e^{-3} = 17$? nope!

$z=1$ is <u>not</u> solution.

- What *is* the solution above? How do we know?

$z=4$

$\sqrt{4} + 4^2 - e^{4-4} \stackrel{?}{=} 17$

$2 + 16 - 1 = 17$ ✓ solves.

ODEs are the same: solving means satisfying

- Example: $\dot{y} = y$. Show that $y = e^t$ is a solution, but that $y = e^{2t}$ is not.

$\frac{dy}{dt} = y$

take a deriv. of proposed solution

$y = e^t$

$\frac{d}{dt}(y) = \frac{d}{dt}(e^t) = e^t$

$e^t = e^t$

yes!

$\frac{dy}{dt} = y$

$\frac{d}{dt}(e^{2t}) = 2e^{2t}$

e^{2t}

$2e^{2t} = e^{2t}$

nope!

ODEs are the same: solving means satisfying

- Example: $\dfrac{dy}{dx} = x\sqrt{y}$. Show that $y = \dfrac{1}{16}x^4$ is a solution.

need one derivative.

$\dfrac{d}{dx}\left(\dfrac{1}{16}x^4\right) = 4\cdot\dfrac{1}{16}x^3$

$= \dfrac{1}{4}x^3$

just x

$\sqrt{\dfrac{1}{16}x^4}$

$= \dfrac{1}{4}x^2$

plug it all in

$\dfrac{1}{4}x^3 = x\,\dfrac{1}{4}x^2$

$\dfrac{1}{4}x^3 = \dfrac{1}{4}x^3$ ✓

1. compute derivs as needed
2. plug in all y, y', y'', etc.
3. simplify to see if LHS=RHS.

ODEs are the same: solving means satisfying

- Ex: $y'' - 2y' + y = 0$. For what values of the constant k is $y = kte^t$ a solution?

kte^t

$\frac{d}{dt}(kte^t) = k(1 \cdot e^t + t \cdot e^t)$

prod. rule

$\frac{d}{dt}(k(e^t + te^t)) = k(e^t + e^t + te^t)$

Plug in:

$k(2e^t + te^t) - 2k(e^t + te^t) + kte^t = 0$

$e^t(2k - 2k) + te^t(k - 2k + k) = 0$

$e^t(0) + te^t(0) = 0$

$0 = 0$

All values of k solve.

"Ansatz"

Educated Guess.

Some ODEs have *families* of solutions

- Definition: a **family of solutions** is a **set of solutions** that all solve an ODE.
- Typically, a family of solutions will have **arbitrary constants**. The number of constants is typically equal to the order of the ODE.
- Ex: $\dot{y} = y$

$y(t) = k e^{t}$ k can be anything

- Ex: $\ddot{y} = -y$

$y(t) = A\cos(t) + B\sin(t)$ 2^{nd} order $\rightarrow$ 2 const.

$y(t) = A\cos(t)$

Exercise: DIY ODEs

1. Write down a solution to an ODE that has not yet been written down. In other words, write down a function.
2. Take a couple derivatives and write those down.
3. Combine them in an equation to create your own ODE.
4. Then swap with someone else, and **verify** (meaning confirm) the solution.

E.C. HW1.

Challenge: DIY recurrence equations?

- On the last slide, we made up our own ODEs and solutions. Can you puzzle out how to do the same kind of thing, but with a **recurrence** equation?
- Recall that a recurrence equation looks like: $n(t+1) =$ some function of $n(t)$

who'd you swap with?

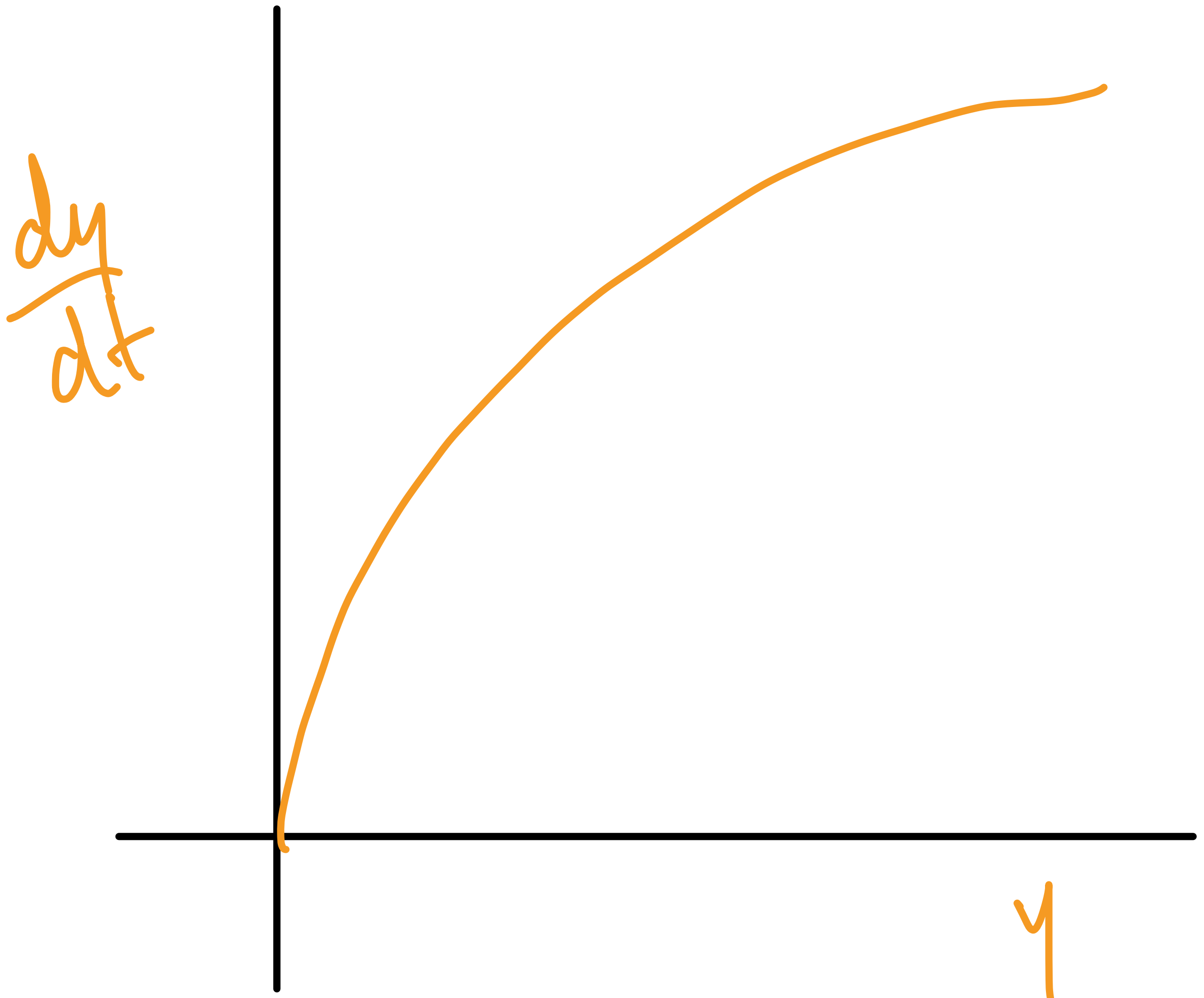
E.C. HW!

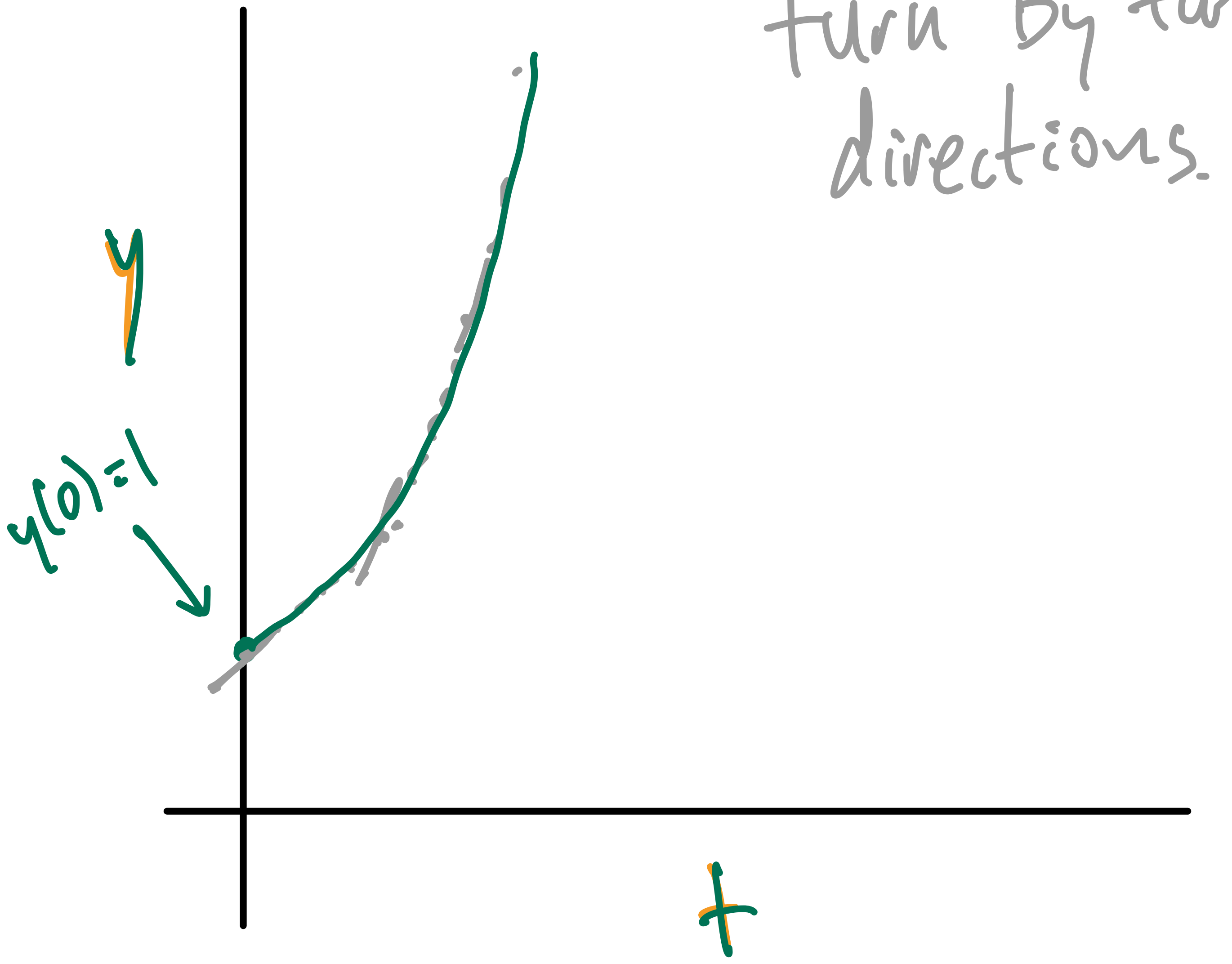
Numerical Solutions to *initial value problems*

tell you where to start

- Remember this? Can we write down a recipe for *approximately* solving this?
- Ex 4: (A) Sketch the derivative vs ~~time~~ y. (B) Sketch the variable vs time.

$$\frac{dy(t)}{dt} = \sqrt{y}, \quad y(0) = 1$$





turn by turn directions.

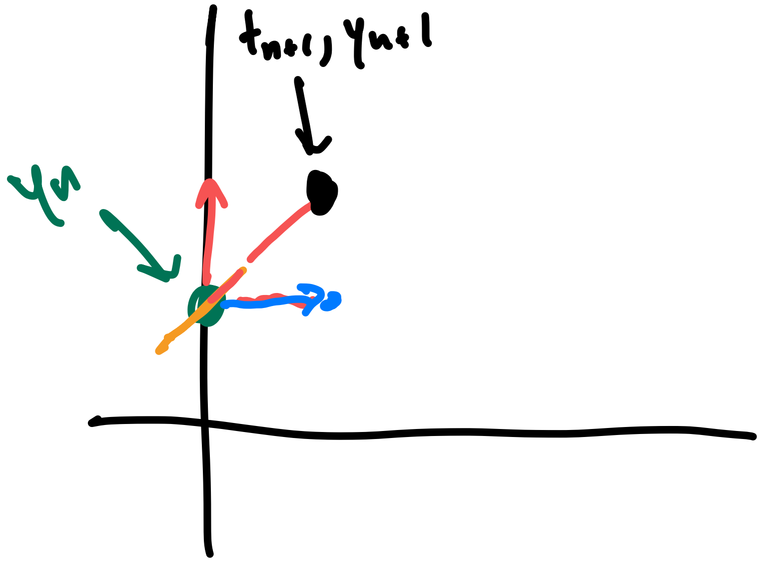
Numerical Solutions to *initial value problems*

- Goal of numerical solution: generate a set of points $\left(t_n, y(t_n)\right)$ that approximate the analytical solution.
- Why might we want to do this?
  - analytical solution too difficult.
  - analytical solution impossible.
  - simulate
- There are many ways to *numerically solve differential equations*, but here is one, referred to as **Euler's Method.**

To solve $y' = f(t, y)$, with $y(t_0) = y_0$ use the formulas

$$y_{n+1} = y_n + \Delta t \cdot f(t_n, y_n) \qquad \frac{\text{rise}}{\text{run}} \cdot \text{run}$$

$$t_{n+1} = t_n + \Delta t$$



Notebook time!

To solve $y' = f(t, y)$, with $y(t_0) = y_0$ use the formulas

$$y_{n+1} = y_n + \Delta t \cdot f(t_n, y_n)$$

$$t_{n+1} = t_n + \Delta t$$

Example: $y' = 2ty, \quad y(1) = 1$

Analytical solution: $y(\cancel{x}\,t) = e^{t^2-1}$