Calculating Biological Quantities

CSCI 2897

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2021, Lecture 8

Last time on CSCI 2897..

1. Equilibrium solutions

no change

$\Delta n = 0$

$n(t+1) = n(t)$

$\frac{dn}{dt} = 0$

Recipe:

① set $n(t+1) = n(t)$

① set $\frac{dn}{dt} = 0$

② solve for n (other variable, etc)

③ interpret findings.

2. Lotka-Volterra Model of Competition

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t) + \alpha_{12}\, n_2(t)}{K_1}\right)$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t) + \alpha_{21}\, n_1(t)}{K_2}\right)$$

Observations:

① two equations: one for n_1 the other for n_2

→ tracking two versions of similar things.

② $n_1 = f(n_1, n_2)$
$n_2 = g(n_1, n_2)$ coupled via α_{12}, α_{21}

③ very similar to logistic growth.

④ very similar to each other structurally.

Lecture 8 Plan

1. **Consumer-Resource Models**
2. **Reverse engineering an equation — equations to interpretation.**

Consumer-Resource Models

So far we've been thinking about resources as constant.

- light striking a patch of land
- nutrients in a river flowing past a location

resources are not depleted in our previous models.

But in many situations, the resources *get depleted* as they are consumed.

- Bears eat salmon—and decrease the salmon population as a consequence!

We can account for these phenomena using a *consumer-resource* model.

one variable   another variable

$\Rightarrow$ two equations with coupling.

Consumer-Resource Models: General Structure

Here is the general form of a consumer-resource model. What do you see?

only on n_1 → n_1's growth

depends on n_1, n_2 but negative! n_2 is consuming or killing n_1!

res.

$$\frac{dn_1}{dt} = f(n_1) - g(n_1, n_2)$$

how many n_2's do you get every time you eat on n_1

n_2's dynamics (without n_1) are negative. NEEDS n_1 to survive! Apex predator?

cons.

$$\frac{dn_2}{dt} = \epsilon g(n_1, n_2) - h(n_2)$$

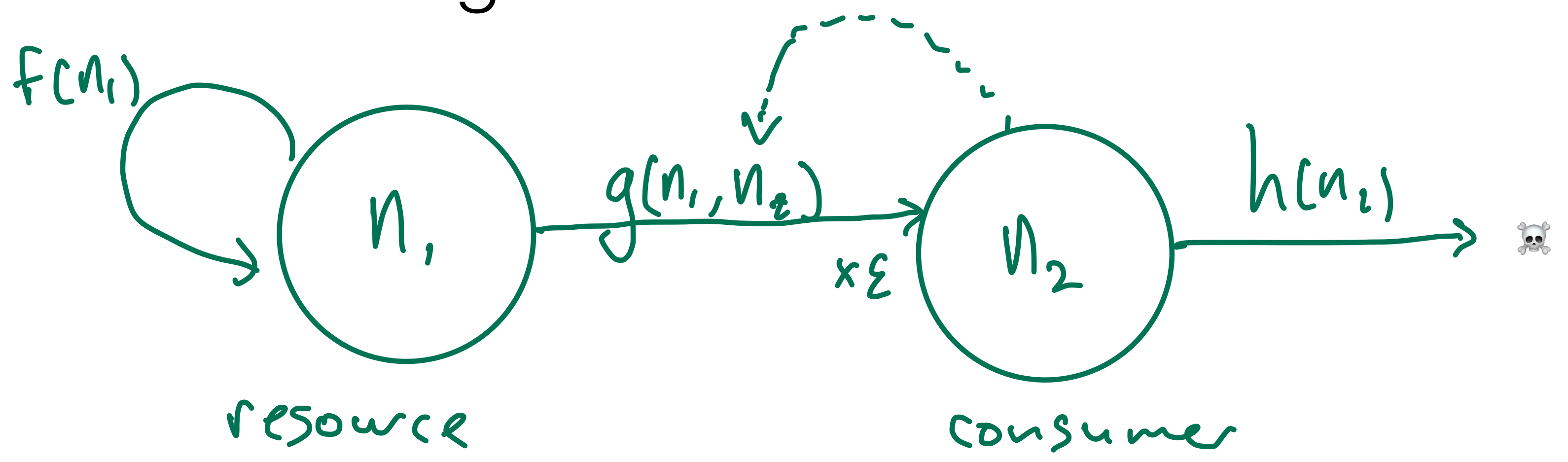
n_1's death at the hands of n_2 leads to ↑ in n_2.
=> n_2 eats n_1, and grows.

- $f(n_1)$ just a function of n_1
- $h(n_2)$ just a function of n_2
- ⊕ in front of f, ⊖ in front of h. => f is about growth ↑n_1
  h is about decline ↓n_2

$g(n_1, n_2)$ is a function of n_1 and n_2

- shows up in both ⟶ interaction/coupling btwn n_1, n_2.
  - ϵ shows up in n_2 equation → exchange rate of resource into consumer
  - -1 coeff in n_1 equation

Construct a flow diagram:



Consumer-Resource Models: General Structure

$$\frac{dn_1}{dt} = f(n_1) - g(n_1, n_2)$$

$$\frac{dn_2}{dt} = \epsilon g(n_1, n_2) - h(n_2)$$

$f(n_1)$: rate of change of the resource via means other than consumption $(n_2 = 0)$.

$g(n_1, n_2)$: rate of consumption of the resource by the consumer.

ϵ : the conversion factor by which resource units $\rightarrow$ consumer units.

$h(n_2)$: rate at which the number of consumers changes without resources $(n_1 = 0)$.

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TABLE 3.3
Consumer-resource models. Examples of functions that can be used in the consumer-resource model (3.16), where n_1 refers to the level of resources (e.g., number of prey) and n_2 refers to the level of consumers (e.g., number of predators).

Function	Description
$f(n_1) = \theta$	Inflow of resources at a constant rate
$f(n_1) = -\psi$	Outflow of resources at a constant rate
$f(n_1) = r\, n_1$	Constant per capita growth of resource species
$f(n_1) = rn_1\left(1 - \frac{n_1}{K}\right)$	Per capita growth of resource species declines linearly with resource level (logistic)
$f(n_1) = r\, n_1\, e^{-\alpha n_1}$	Per capita growth of resource species declines exponentially with resource level
$g(n_1, n_2) = a\, c\, n_1\, n_2$	Linear (type I) rate of resource consumption
$g(n_1,n_2) = \frac{a\, c\, n_1}{b + n_1} n_2$	Saturating (type II) rate of resource consumption
$g(n_1,n_2) = \frac{a\, c\, n_1^k}{b + n_1^k} n_2$	Generalized (type III) rate of resource consumption
$h(n_2) = \delta\, n_2$	Constant per capita death rate of consumer
$h(n_2) = (\delta\, n_2)\, n_2$	Per capita death rate of consumer increases linearly with consumer population size

Otto & Day, Table 3.3

Consumer-Resource Models: General Structure

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Consumer-Resource Models: Const. Immigration

const growth rate. not proportional to n_1 → just a const.

$$\frac{dn_1}{dt} = \theta - a\, c\, n_1\, n_2$$

. law of mass action

. "well mixed" contacts

$$\frac{dn_2}{dt} = \epsilon\, a\, c\, n_1\, n_2 - \delta\, n_2$$

→ rates double if you double resource

half half

double/half double/half consume

exchange rate for how much n_2 you get if you actually consume an n_1.

rate of contact between n_1 and n_2.

probability that an n_1-to-n_2 contact results in the consumer consuming the resource.

Consumer-Resource Models: Const. Immigration

$$\frac{dn_1}{dt} = \theta - a\ c\ n_1\ n_2$$

nutrient

const

$$\frac{dn_2}{dt} = \epsilon\ a\ c\ n_1\ n_2 - \delta\ n_2$$

algae

Example: a nutrient flows into a lake at a constant rate.
A population of algae uses that nutrient to grow.

Consumer-Resource Models: Const. Immigration

$$\frac{dn_1}{dt} = \cancel{\theta}^{\,\rightarrow 0} - a\, \underset{\uparrow\, 3c}{c}\, n_1\, n_2$$

$$\frac{dn_2}{dt} = \epsilon\, a\, \overset{3c\,\downarrow}{c}\, n_1\, n_2 - \delta\, n_2$$

beavers dam the inflow stream!

$\theta = 0$

berries + bees disappear, so bears show up to the river 3x as often, eating more salmon

Example: a nutrient flows into a lake at a constant rate. A population of algae uses that nutrient to grow.

Extension: How could we explore a situation where the nutrient no longer flows into the lake? What other scenarios might we explore?

Consumer-Resource Models: Const. Immigration

$$\frac{dn_1}{dt} = \theta - a\ c\ n_1\ n_2 \quad = 0$$

$$\frac{dn_2}{dt} = \epsilon\ a\ c\ n_1\ n_2 - \delta\ n_2 = 0$$

What is an equilibrium solution to these equations?

neither is changing — set $\frac{dn_1}{dt} = 0$ AND $\frac{dn_2}{dt} = 0$

$$\theta - acn_1n_2 = 0$$
$$acn_1n_2 = \theta$$

$$\varepsilon acn_1n_2 - \delta n_2 = 0$$
$$acn_1n_2 = \frac{\delta n_2}{\varepsilon}$$

$$\Rightarrow \frac{\delta n_2}{\varepsilon} = \theta \rightarrow \boxed{n_2 = \frac{\varepsilon\theta}{\delta}}$$

sub in

$$acn_1\left(\frac{\varepsilon\theta}{\delta}\right) = \theta$$

$$n_1 = \frac{\cancel{\theta}}{ac}\ \frac{\delta}{\varepsilon\cancel{\theta}}$$

$$\boxed{n_1 = \frac{\delta}{ac\varepsilon}}$$

Consumer-Resource Models: General Structure

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Consumer-Resource Models: Predator-Prey

only term that changed.

$$\frac{dn_1}{dt} = r\, n_1 - a\, c\, n_1\, \cancel{n_2}^{\,0}$$

$$\frac{dn_2}{dt} = \epsilon\, a\, c\, n_1\, n_2 - \delta\, n_2$$

What happens to n_1 if $\underbrace{\text{there is no } n_2}_{\Rightarrow\, n_2 = 0}$?

$$\frac{dn_1}{dt} = r\, n_1$$

$$\frac{dn_1}{n_1} = r\, dt \xrightarrow[\int]{\text{s.o.v.}} n_1(t) = k\, e^{rt}$$

without n_2 around,

n_1 grows exponentially at rate r.

Consumer-Resource Models: Predator-Prey

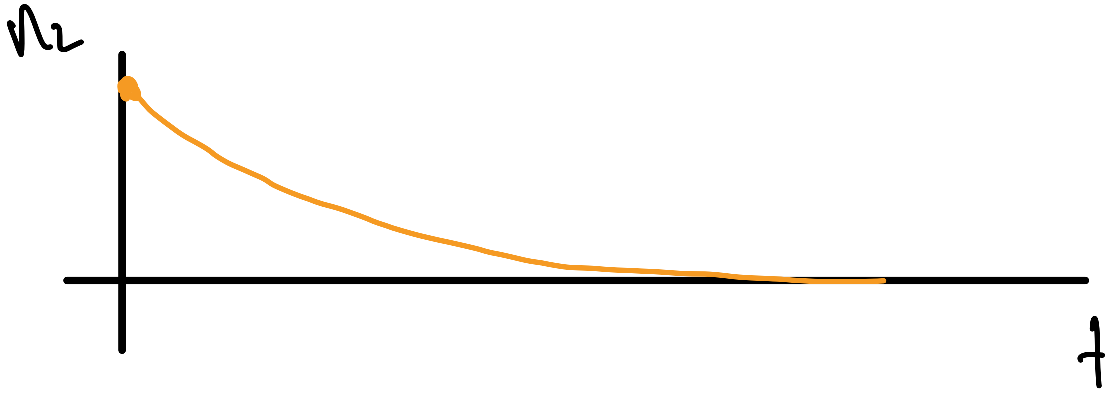
$$\frac{dn_1}{dt} = r\ n_1 - a\ c\ n_1\ n_2$$

What happens to n_2 if there is no n_1?

$\Rightarrow n_1 = 0$

$$\frac{dn_2}{dt} = \epsilon\ a\ c\ \cancel{n_1}^{\,0}\ n_2 - \delta\ n_2$$

$$\frac{dn_2}{dt} = -\delta n_2 \xrightarrow{\text{s.o.v.}} n_2(t) = k e^{-\delta t}$$



what does this mean?

exponential decay at rate $-\delta$

Consumer-Resource Models: Predator-Prey

$$\frac{dn_1}{dt} = r\ n_1 - a\ c\ n_1\ n_2 = 0$$

$$\frac{dn_2}{dt} = \epsilon\ a\ c\ n_1\ n_2 - \delta\ n_2 = 0$$

What is an equilibrium solution to these equations?

$$\frac{dn_1}{dt} = 0 \qquad \frac{dn_2}{dt} = 0$$

$r n_1 - a c n_1 n_2 = 0$

$n_1 (r - a c n_2) = 0$

either $n_1 = 0$ ← 1 $\quad n_1 = 0$ then $n_2 = 0$ $\quad \boxed{(0, 0)}$

or $r - a c n_2 = 0$

$\Rightarrow n_2 = \frac{r}{ac}$ ← 2 $\quad n_2 = \frac{r}{ac}$ then $n_1 = \frac{\delta}{\varepsilon a c}$ $\quad \boxed{\left(\frac{\delta}{a c \varepsilon}, \frac{r}{ac}\right)}$

$\varepsilon a c n_1 n_2 - \delta n_2 = 0$

$n_2 (\varepsilon a c n_1 - \delta) = 0$

either $n_2 = 0$

$\varepsilon a c n_1 - \delta = 0$

$\Rightarrow n_1 = \frac{\delta}{\varepsilon a c}$

Working Backward: Equations to Interpretation

By now, we're pretty good at taking a system and working forward, from behavior into a diagram and then into equations.

Sometimes, it's useful to be able to work in reverse, by taking an equation, and then interpreting the equation in terms of biology.

For example, here's the logistic equation with constant hunting or harvesting.

$$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - \theta$$

Working Backward: Equations to Interpretation

However, all 3 of these equations could be described as "logistic growth with constant hunting or harvesting." So what's the difference in *meaning*?

1. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - \theta$$

2. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - Hn(t)$$

3. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - H(n(t) - P)$$

Working Backward: Equations to Interpretation

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Working Backward: Equations to Interpretation

However, all 3 of these equations could be described as “logistic growth with constant hunting or harvesting.” So what’s the difference in *meaning*?

3. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - H(n(t) - P)$$