# Calculating Biological Quantities

## CSCI 2897

Prof. Daniel Larremore

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daniel.larremore@colorado.edu
@danlarremore

# Last time on CSCI 2897:

The inverse of square matrix $A$ is a matrix called $A^{-1}$ such that

$$A^{-1}A = I$$

and

$$AA^{-1} = I.$$

Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

swap

Then $A^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \dfrac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

negative

# Practice:

1. compute det(A)
2. form new matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
3. multiply

Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then $A^{-1} = \dfrac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Compute the inverses of

$\begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$

$$\det = 1 \cdot 4 - (-2)(-3) = 4 - 6 = -2$$

$$\underbrace{\frac{1}{-2}\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -3/2 & -1/2 \end{pmatrix} = -\begin{pmatrix} 2 & 1 \\ 3/2 & 1/2 \end{pmatrix}}_{\text{all 3 forms are fine!}}$$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\det = 1 \cdot 1 - 0 \cdot 0 = 1$$

$$\frac{1}{1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

$$\det = 2 \cdot 2 - 4 \cdot 1 = 4 - 4 = 0$$

$$\frac{1}{0}\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} = \text{uh oh...}$$

# Things you can do with an inverse matrix.

Let's solve these two equations

$$6x + 4y = 12$$
$$3x - 2y = 0$$

$\downarrow$

$$\begin{pmatrix} 6 & 4 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

$$A^{-1} A \vec{x} = A^{-1}\vec{b}$$

$$I \vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

"solved for $\vec{x}$" ✓

$$A = \begin{pmatrix} 6 & 4 \\ 3 & -2 \end{pmatrix} \qquad A^{-1} = \frac{1}{6(-2) - 4\cdot 3}\begin{pmatrix} -2 & -4 \\ -3 & 6 \end{pmatrix}$$

$$= \frac{1}{-24}\begin{pmatrix} -2 & -4 \\ -3 & 6 \end{pmatrix}$$

$$\vec{x} = \underbrace{\frac{1}{-24}\begin{pmatrix} -2 & -4 \\ -3 & 6 \end{pmatrix}^{2\times 2}}_{A^{-1}} \underbrace{\begin{pmatrix} 12 \\ 0 \end{pmatrix}^{2\times 1}}_{b} = \frac{1}{-24}\begin{pmatrix} -24 \\ -36 \end{pmatrix}^{2\times 1}$$

$$= \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}$$

$x = 1$, $y = 3/2$

# Things you can do with an inverse matrix.

Let's think about the Matrix as Machine idea.

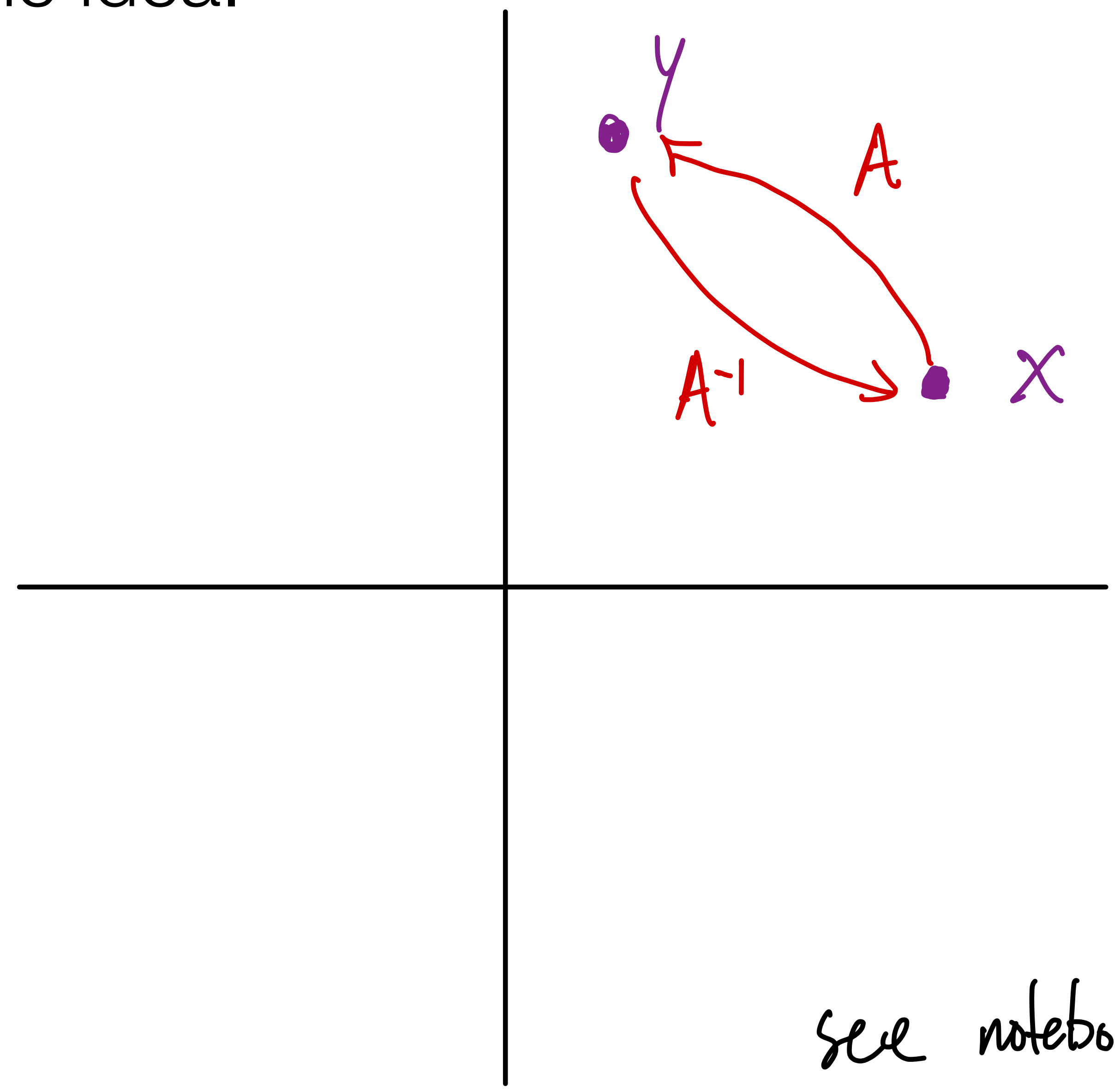
$y = Ax$

What happens if I multiply $A^{-1}y$ ?

$$A^{-1}y = A^{-1}(Ax)$$
$$= A^{-1}Ax$$
$$= Ix$$
$$= x$$

"undo matrix"

Inverse is the ctrl-z of matrices.




see notebook 6.

What is the inverse matrix of $A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

$$\frac{1}{2\cdot 2 - 4\cdot 1}\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} = \frac{1}{0}\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

inverse does not exist!

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

- $\det(A) = 0$
- inverse DNE
- not "invertible"

① 
$$2x + 4y = 2$$
$$x + 2y = 1$$
$$x = 1 - 2y$$
$$2x = 2 - 4y$$
$$x = 1 - 2y$$

no unique solution!
(infinite solutions)

② 
$$2x + 4y = 2$$
$$x + 2y = 2$$
$$x = 2 - 2y$$
$$x = 1 - 2y$$

contradiction!

$$1 - 2y = 2 - 2y$$
$$1 = 2 \ \lightning$$

(no solution)

# Equivalent statements:

1. The matrix $A$ is invertible.
2. $A^{-1}$ exists.
3. For an arbitrary $b$, $Ax = b$ has a unique solution $x$.
4. If $Ax = 0$, this means that $x = 0$.
5. Det$(A) \neq 0$.

1. The matrix $A$ is not invertible.
2. $A^{-1}$ does not exist.
3. For an arbitrary $b$, $Ax = b$ does not have a unique solution $x$.
4. There exists a *nonzero* vector $x$ such that $Ax = 0$.
5. Det$(A) = 0$.

# Characteristic Directions

For any matrix, there are some vectors which are special. For one of these special vectors $\overrightarrow{x}$, computing $\overrightarrow{y} = A\overrightarrow{x}$, produces a $\overrightarrow{y}$ that is just a rescaled version of $\overrightarrow{x}$.

In other words, $\overrightarrow{y} = \lambda\overrightarrow{x}$. This means that $A\overrightarrow{x} = \lambda\overrightarrow{x}$.

**Example 1**: $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}, x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$Ax = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5+2 \\ -9+6 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda x$$

$Ax = -3x$ $\quad Ax = \lambda x, \lambda = -3.$

# Characteristic Directions

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**Example 2**: $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}, x = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$

$$Ax = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} -10 + 18 \\ -18 + 54 \end{pmatrix} = \begin{pmatrix} 8 \\ 36 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 9 \end{pmatrix}$$

$$Ax = \lambda x \qquad Ax = 4x$$

Let’s pop over into our Matrix Machines notebook to see this in action.

# Characteristic Directions

Example 1: $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}$, $x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $Ax_1 = -3x_1$

Example 2: $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$. $Ax_2 = 4x_2$

Definitions: An **Eigenvector** of a square matrix $A$ is a vector $x$ such that $Ax = \lambda x$ for some scalar $\lambda$. An **Eigenvalue** is that scalar, $\lambda$.

There can be at most $n$ eigenvectors and $n$ eigenvalues for an $n \times n$ matrix.

# Finding Eigenvectors and Eigenvalues

What if I give you the matrix $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}$ and ask you for its eigenvalues?

$A\vec{x} = \lambda\vec{x}$

$A\vec{x} - \lambda\vec{x} = \vec{0}$

$(A - \lambda)\vec{x} = \vec{0}$

wrong!

dimensions do not agree

$A\vec{x} - \lambda I\vec{x} = \vec{0}$

$(A - \lambda I)\vec{x} = \vec{0}$

Want $\vec{x} \neq \vec{0}$

Recall that if a matrix $M$ and a non-zero vector $\vec{x}$ have $M\vec{x} = \vec{0}$, then $M$ is singular.

If $\vec{x} \neq 0$, but $\underbrace{(A - \lambda I)}_{\text{singular!}}\vec{x} = 0$

$\Rightarrow \det(A - \lambda I) = 0$

$A - \lambda I = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -5-\lambda & 2 \\ -9 & 6-\lambda \end{pmatrix}$

$\det(A - \lambda I) = \det\left( \begin{matrix} -5-\lambda & 2 \\ -9 & 6-\lambda \end{matrix} \right) = (-5-\lambda)(6-\lambda) - 2(-9)$

# Finding Eigenvectors and Eigenvalues

$$\det(A-\lambda I) = (-5-\lambda)(6-\lambda) - 2(-9)$$

$$= -30 - 6\lambda + 5\lambda + \lambda^2 + 18$$

$$= \lambda^2 - \lambda - 12$$ characteristic polynomial

remember the goal:

$$\det(A-\lambda I) = 0$$

$$\lambda^2 - \lambda - 12 = 0$$ characteristic equation

$$(\lambda-4)(\lambda+3) = 0$$

$$\lambda = 4, \quad \lambda = -3$$

For a 2x2 matrix $\rightarrow$ quadratic

For a 3x3 $\rightarrow$ cubic

etc.

# Finding Eigenvalues

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\qquad A - \lambda I = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$

To compute eigenvalues, we:

1. Write $Ax = \lambda x$ as $Ax - \lambda x = 0$ and then as $(A - \lambda I)x = 0$. lin. alg.
2. If $(A - \lambda I)x = 0$ but $x \neq 0$, this means that $\det(A - \lambda I) = 0$. def'n of singular matrix
3. Write out the characteristic equation: $(a - \lambda)(d - \lambda) - bc = 0$
4. Solve for $\lambda$.

$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$$

$$\Rightarrow \boxed{\lambda = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}^2(A) - 4\det(A)}}{2}}$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a\lambda^2 + b\lambda + c = 0$ quadratic formula

What if we also want eigenvectors?

# Finding Eigenvectors and Eigenvalues

Given the matrix $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}$ and $\lambda = 4$, what's the matching eigenvector?

① Plug into $(A-\lambda I)\vec{x} = \vec{0}$, solve for $\vec{x}$.

$$A-\lambda I = \begin{pmatrix} -5-4 & 2 \\ -9 & 6-4 \end{pmatrix} = \begin{pmatrix} -9 & 2 \\ -9 & 2 \end{pmatrix}$$

$$(A-\lambda I)\vec{x} = \vec{0} \qquad \begin{pmatrix} -9 & 2 \\ -9 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-9x_1 + 2x_2 = 0$$

② solve for relationship $\quad x_2 = \frac{9x_1}{2}$

③ plug in 1 (simple!)

let $x_1 = 1$

$\Rightarrow x_2 = \frac{9}{2}$

$$\vec{x} = \begin{pmatrix} 1 \\ 9/2 \end{pmatrix}$$

④ rescale for interpretability.

$$\vec{x} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$$

# Finding Eigenvectors and Eigenvalues

Given the matrix $A = \begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}$ and $\lambda = -3$, what's the matching eigenvector?

1. Plug into $(A - \lambda I)\vec{x} = \vec{0}$
2. solve for $x_1$, $x_2$ relationship
3. plug in $x_1 = 1$ or $x_2 = 1$
4. rescale as needed.

1. $A - \lambda I = \begin{pmatrix} -5-(-3) & 2 \\ -9 & 6-(-3) \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -9 & 9 \end{pmatrix}$

2. $\begin{pmatrix} -2 & 2 \\ -9 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

so $-2x_1 + 2x_2 = 0 \Rightarrow x_1 = x_2$

$-9x_1 + 9x_2 = 0$

3. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \underset{\substack{\uparrow \\ \text{because} \\ x_2 = x_1}}{=} \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} \underset{\substack{\uparrow \\ \text{set } x_1 = 1}}{=} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

4. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

no rescaling needed!

# Finding Eigenvalues & Eigenvectors

To compute eigenvalues, we:

1. Write $Ax = \lambda x$ as $Ax - \lambda x = 0$ and then as $(A - \lambda I)x = 0$.
2. If $(A - \lambda I)x = 0$ but $x \neq 0$, this means that $\det(A - \lambda I) = 0$.
3. Write out the characteristic equation: $(a - \lambda)(d - \lambda) - bc = 0$
4. Solve for $\lambda$.

To compute the eigenvectors, for each eigenvalue, we

1. Plug in the $\lambda$ to $(A - \lambda I)x = 0$, and write out the equations.
2. The equations *should* be redundant. Pick one and determine the relationship between $x_1$ and $x_2$. That's your eigenvector!

# Why do we care though?

$$\frac{d\overrightarrow{n}}{dt} = A\overrightarrow{n}$$

It turns out the answer is $\overrightarrow{n}(t) = k_1\overrightarrow{x_1}e^{\lambda_1 t} + k_2\overrightarrow{x_2}e^{\lambda_2 t}$

$$\frac{dn_1}{dt} = 2n_1 + 3n_2$$

$$\frac{dn_2}{dt} = 2n_1 + n_2$$

$$\frac{d}{dt}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + k_2\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{4t}$$