Calculating Biological Quantities

CSCI 2897

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Last time on CSCI 2897..

1. **Haploid models of natural selection**

$$\frac{dp}{dt} = s_c p(t)\big(1 - p(t)\big)$$

$$s_c = (b_A - d_A) - (b_a - d_a)$$

selection coeff.

continuous

S_c positive   A advantage

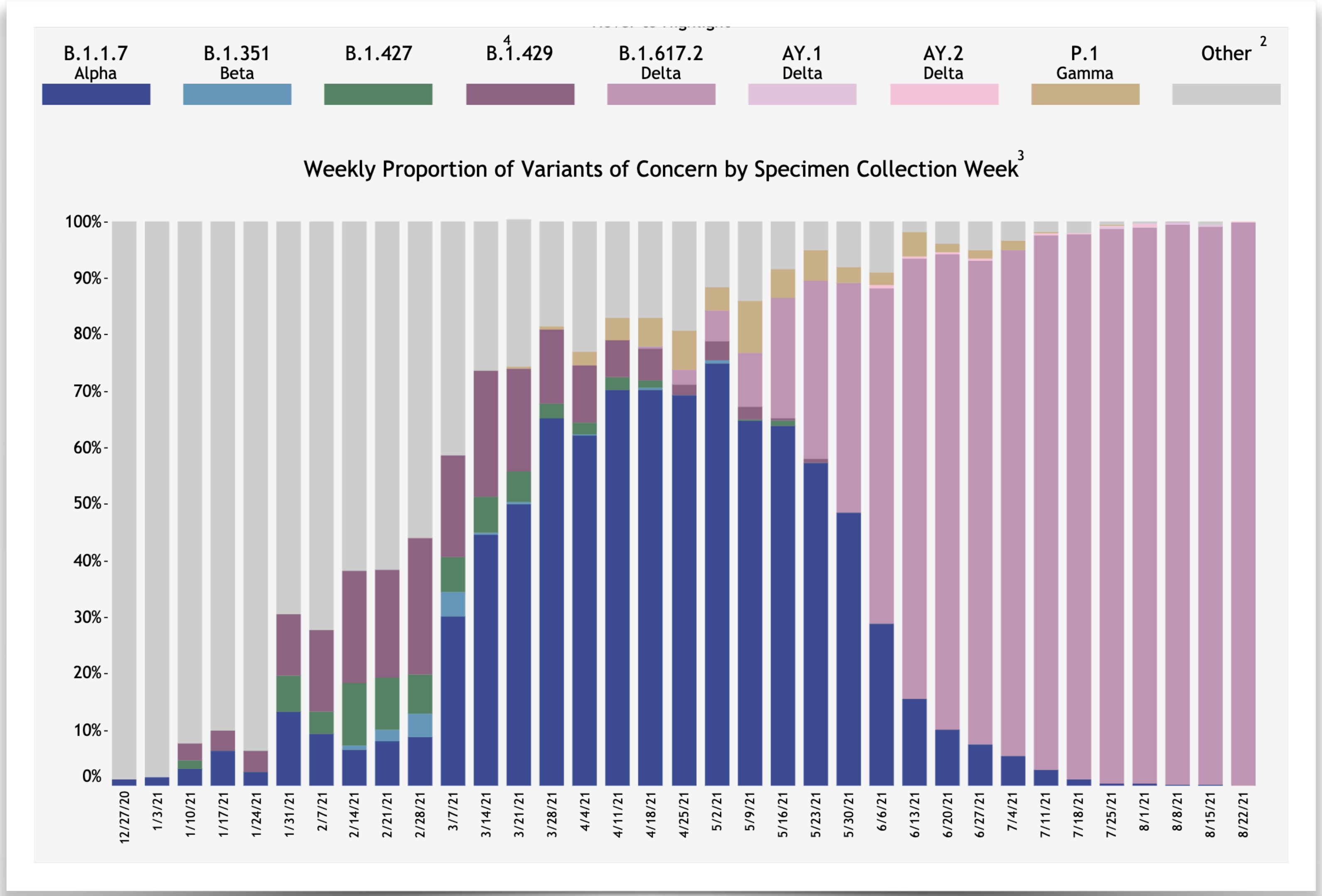
or

negative   a advantage

thought experiment: $d_A = d_a = 0$

$S_c = b_A - b_a$

birthrate of A larger than birthrate of a, $S_c > 0$



Lecture 7 Plan

1. **Equilibrium solutions**
2. **Lotka-Volterra Model of Competition**

Equilibrium

A system at **equilibrium** does not change over time. (Plural: **equilibria**.)

For a discrete time model, at equilibrium, it must be true that:

$\Delta n = 0$ $\quad n(t+1) - n(t) = 0 \quad \leftrightarrow \quad n(t+1) = n(t)$

no change.

For a continuous time model, at equilibrium, it must be true that:

$$\frac{dn}{dt} = 0 \quad \Rightarrow \text{ no change}$$

rate of change

Sometimes we call an equilibrium a **steady state**.

Equilibrium

A system at **equilibrium** does not change over time. (Plural: **equilibria**.)

What is the equilibrium / what are the equilibria for our haploid frequency equation?

$$\frac{dp}{dt} = s_c p(t)\big(1 - p(t)\big)$$

① set $\frac{dp}{dt} = 0$. (no change)

② solve equation for variable. (not for parameters.)

$0 = s_c p(1-p)$

$0 = p(1-p)$

$p = 0 \qquad p = 1$

- If $p=0$, $\frac{dp}{dt} = 0 \Rightarrow p$ will always $= 0$.
- If system is at equilibrium, it will always be there... (unless something else happens) perturbation.

Note: we're always solving for equilibrium values of the *variables*, not the *parameters*.

Stability

An equilibrium is **locally stable** if a system near that equilibrium approaches it. This property is called **locally attracting**.

Suppose @ equilibrium.

→ jiggle/bump system. → go back to equilibrium.

An equilibrium is **globally stable** if a system approaches that equilibrium *regardless* of its initial position.

An equilibrium is **unstable** if a system near the equilibrium moves away from it. This property is called **repelling**.

Suppose @ equilibrium.

→ jiggle/bump system. → do <u>not</u> go back to equilibrium.

Stability

\epsilon = ε "really small"

$\varepsilon > 0$

Are the equilibria for our haploid allele frequency equation stable or unstable?

$$\frac{dp}{dt} = s_c p(t)\big(1 - p(t)\big)$$

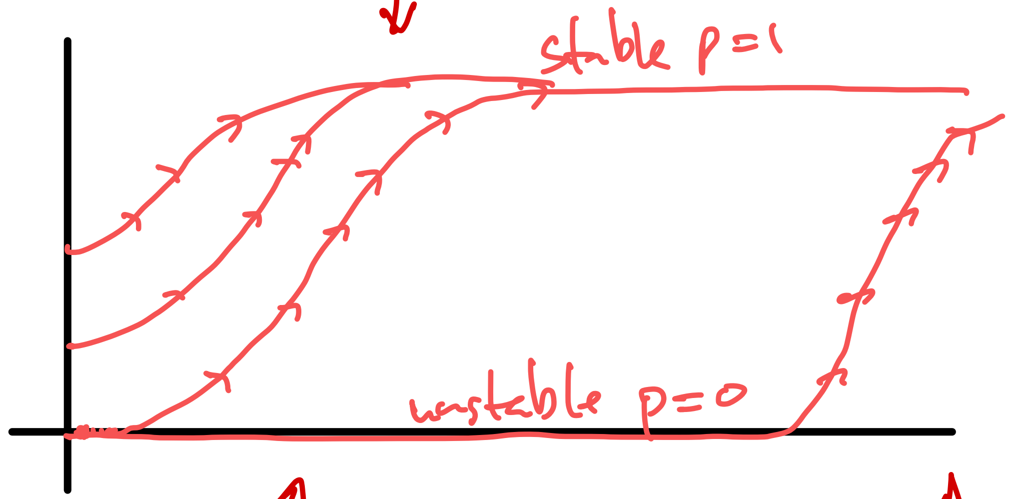
$p = 0$ Let $p = p_{equil} + \varepsilon$ $\longrightarrow$ $\frac{dp}{dt} = \underbrace{\overset{+\text{ or }-}{s_c}\,(\overset{+}{\varepsilon})(\overset{+}{1-\varepsilon})}_{\text{positive or negative?}}$

$s_c > 0 \Rightarrow \frac{dp}{dt} > 0$ unstable.

$s_c < 0 \Rightarrow \frac{dp}{dt} < 0$ stable.

$s_c > 0$

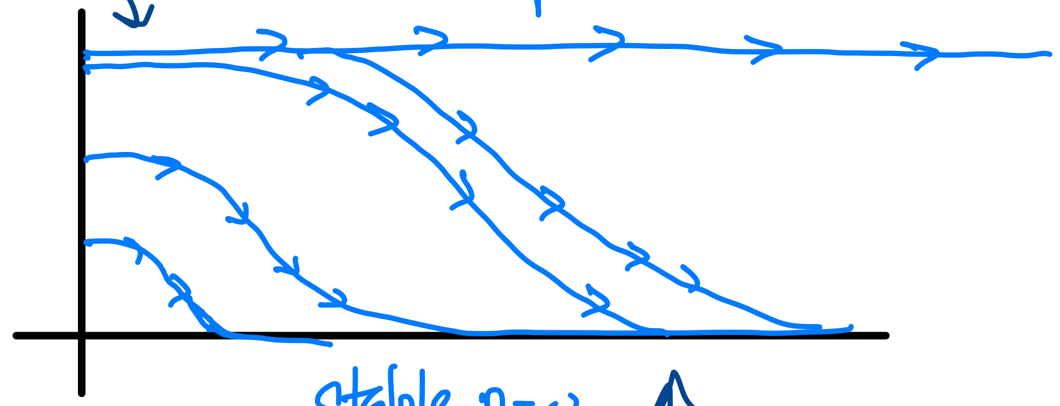
A will eventually dominate!



unless there was never any A to begin with!

$s_c < 0$

unless there was no a to begin with



a will eventually dominate.

$p = 1$ Let $p = p_{equil} - \varepsilon$

$$\frac{dp}{dt} = s_c(1-\varepsilon)\big(1-(1-\varepsilon)\big)$$

$$= \overset{+\text{ or }-}{s_c}\,(\overset{+}{1-\varepsilon})(\overset{+}{\varepsilon})$$

$\frac{dp}{dt} > 0$ when $s_c > 0$ stable

$\frac{dp}{dt} < 0$ when $s_c < 0$ unstable

Bonus

Identify the equilibrium/a of the logistic growth equation, and characterize stability.

$$\dot{n} = r\,n\left(1 - \frac{n}{K}\right)$$

$r > 0$

$\dot{n} = 0$

$0 = r\,n\left(1 - \frac{n}{K}\right)$

$0 = n\left(1 - \frac{n}{K}\right)$

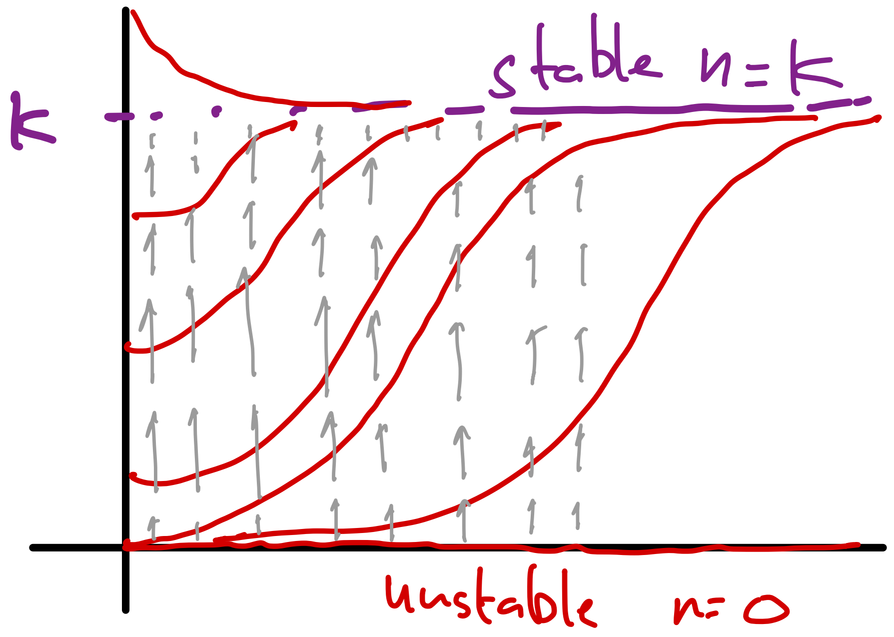
$\boxed{n = 0}$

$n \neq 0$

$0 = \left(1 - \frac{n}{K}\right)$

$1 = \frac{n}{K}$

$\boxed{K = n}$



learn from direction/vector field arrows!

Lotka-Volterra Competition

Imagine that there are two species, with population sizes $n_1(t)$ and $n_2(t)$.

Let's imagine that each one has the property from Logistic Growth where its growth rate R depends on its population size n, so we have $R_1(n_1)$ and $R_2(n_2)$.

What if one species' growth rate depended on the size of the other population?

Specifically, suppose that species i experiences competition *as if its own species had population* $n_i(t) + \alpha_{ij}\, n_j(t)$. (Here, i could be 1 or 2).

my pop feels like (grows like) its size is $n_1(t) + \alpha_{12} n_2(t)$

Lotka-Volterra Competition

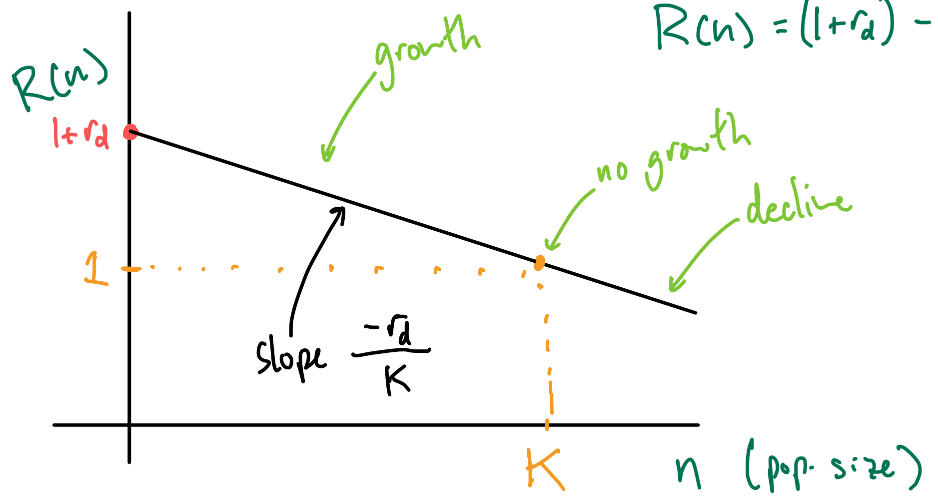
Remember when we derived the Logistic Growth equation?

Logistic growth in discrete time

compare to
$y = b + mx$

- Let's say that when the population size is zero, $R(0) = 1 + r_d$.
  - This is called the **intrinsic rate of growth.**
  - It's what happens when there aren't resource limitations (= prev. model).
- Let's say that $R(n)$ decreases until it becomes 1, at *some* value of n.
- A sketch helps:

$$R(n) = (1+r_d) - \frac{r_d}{K} \cdot n(t)$$



Logistic growth in discrete time

- If we write $n(t+1) = R(n)\ n(t)$, we now get $R(n) = (1+r_d) - \frac{r_d}{K} \cdot n(t)$
- $n(t+1) = \left[(1+r_d) - \frac{r_d}{K} n(t)\right] n(t)$

$$\underbrace{n(t+1)}_{\text{next}} = \underbrace{n(t)}_{\text{prev}} + \underbrace{r_d\left(1 - \frac{n(t)}{K}\right) n(t)}_{\text{change}}$$

We're now going to modify that equation for $R(n)$.

Lotka-Volterra Competition

my actual population. how pop of n_2 gets experienced by n_1.

$$\text{Let } R_i = \underbrace{(1+r_i)}_{\text{intrinsic}} + \underbrace{\left(\frac{-r_i}{K_i}\right)}_{\text{slope}} \underbrace{\left(n_i(t) + \alpha_{ij} n_j(t)\right)}_{\text{what I feel my population is.}}$$

$$n(t+1) = R(n)\, n(t)$$

"indexed by i"

$i = 1, 2$

Let's plug in this reproductive factor into *each* of our two update equations:

influence of pop 2 on pop 1

$$n_1(t+1) = \left[1 + r_1 - \frac{r_1}{K_1}\left(n_1(t) + \alpha_{12} n_2(t)\right)\right] \cdot n_1(t)$$

$$n_2(t+1) = \left[1 + r_2 - \frac{r_2}{K_2}\left(n_2(t) + \alpha_{21} n_1(t)\right)\right] \cdot n_2(t)$$

influence of pop 1 on pop 2

Lotka-Volterra Competition

$$\frac{dn}{dt} = r n \left(1 - \frac{n}{K}\right)$$

We can write similar equations in continuous time:

$$\frac{dn_1}{dt} = r_1 n_1(t) \left[1 - \frac{n_1(t) + \alpha_{12} n_2(t)}{K_1} \right]$$

$$\frac{dn_2}{dt} = r_2 n_2(t) \left[1 - \frac{n_2(t) + \alpha_{21} n_1(t)}{K_2} \right]$$

α_{12} and α_{21} need not be the same.

- α_{12} = impact that 1 feels due to 2
- α_{21} = impact that 2 feels due to 1.

Lotka-Volterra Competition

Quick check: if the species don't interact, then: $\alpha_{12} = 0$ $\alpha_{21} = 0$

which implies that…

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t) + \cancel{\alpha_{12}\, n_2(t)}}{K_1}\right) = r_1 n_1 \left(1 - \frac{n_1}{K_1}\right)$$ Log. Growth

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t) + \cancel{\alpha_{21}\, n_1(t)}}{K_2}\right) = r_2 n_2 \left(1 - \frac{n_2}{K_2}\right)$$ Log. Growth

Interpretation: If species don't interact $\longrightarrow$ Log. Growth

Also note: this model is *symmetric* in that relabeling $1 \leftrightarrow 2$ produces the same equations.

Lotka-Volterra…Competition?

$\alpha_{12} = -1$ $\quad K = 10$

$r = 1$

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t) + \alpha_{12}\, n_2(t)}{K_1}\right)$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t) + \alpha_{21}\, n_1(t)}{K_2}\right)$$

$\frac{dn_1}{dt}$ when $n_1 = 1$ $\quad n_2 = 1$

$\frac{dn_1}{dt}$ when $n_1 = 1$ $\quad n_2 = 2$

What if α_{12} is negative? How does an increase in n_2 affect $\frac{dn_1}{dt}$?

↑ growth rate of n_1 !

$$\frac{dn_1}{dt} = 1 \cdot 1 \left(1 - \frac{1 + (-1) \cdot 1}{10}\right)$$

$$= \left(1 - \frac{(1-1)}{10}\right) = 1$$

$$\frac{dn}{dt} = 1 \cdot 1 \left(1 - \frac{1 + (-1) \cdot 2}{10}\right)$$

$$= \left(1 - \frac{1-2}{10}\right) = \left(1 - \frac{-1}{10}\right) = \frac{11}{10}$$

Lotka-Volterra…Competition?

α_{12}	α_{21}	Relationship
0	0	none
+	+	competitive
+	–	parasitic
–	+	parasitic
–	–	mutualistic
–	0	commensal
0	–	comensal

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t)+\alpha_{12}\, n_2(t)}{K_1}\right)$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t)+\alpha_{21}\, n_1(t)}{K_2}\right)$$

Let's code up the Lotka-Volterra model to explore!

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t) + \alpha_{12}\, n_2(t)}{K_1}\right)$$

notebook 3.

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t) + \alpha_{21}\, n_1(t)}{K_2}\right)$$

with initial conditions

$$n_1(0) = a$$

$$n_2(0) = b$$