Calculating Biological Quantities

CSCI 2897

1) HW due tonight 11:59 P.M.
@ Canvas

2) git pull

3) OH
Mon - 9-11
Wed - 4-6
A414, Zoom

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2021, Lecture 5

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Last time on CSCI 2897..

1. **How to verify that a function is a solution of an ODE.**
2. **Solving an ODE initial value problem *numerically* by stepping along the solution.**
3. **Logistic & Exponential Growth**

Lecture 5 Plan

1. **Finding the analytical solution to exponential growth.**
2. **Separation of variables (general)**
3. **Separability**
4. **Finding the analytical solution to logistic growth**

(if time 5. Coding)

Exponential Growth in Continuous Time

- Let $n(t)$ be the population at time t
- Let r be the growth rate of the population
- Then our ODE is $\dfrac{dn}{dt} = r\,n$

rate of change of n = constant × current pop.

(rate of change of n is proportional to n)

- **Separation of Variables** (SoV) is a mathematical technique we can use to solve this ODE.

$\dfrac{dn}{dt}$ = ratio of a little change in n to a little change in t.

Separation of Variables — Exponential Growth

Goal: get all the n terms on the LHS and all the t terms on the RHS.

$$\frac{dn}{dt} = r\ n$$

$$dn = r\ n\ dt$$

$$\frac{dn}{n} = r\ dt$$

$$\int \frac{dn}{n} = \int r\ dt$$

$$\int \frac{1}{n}\, dn = r \int dt$$

$$\ln n = r(t + c)$$

we kept only the constant on the right

$$e \qquad e$$

$$n = e^{rt + rc}$$

$$e^{a+b} = e^a e^b$$

$$n = e^{rt} \underbrace{e^{rc}}_{k}$$

$$\boxed{n(t) = k e^{rt}}$$

Separation of Variables — Exponential Growth

Goal: get all the n terms on the LHS and all the t terms on the RHS.

$$\frac{dn}{dt} = r\,n \quad \rightarrow \quad n(t) = ke^{rt} \longrightarrow \frac{dn}{dt} = ke^{rt}r$$

ODE

solution

Followup: $\underbrace{\text{Verify that what we found is indeed a solution}}_{\text{plug it in}}$ to the ODE.

$$\underbrace{ke^{rt}r}_{\frac{dn}{dt}} = r\underbrace{ke^{rt}}_{n}$$

$$rke^{rt} = rke^{rt}$$

$$\text{LHS} = \text{RHS} \Rightarrow n(t) = ke^{rt} \text{ solves } \frac{dn}{dt} = rn$$

Separation of Variables — General I

e.g. $g(x) = 0$

$g(x) = x$

General Goal: get all the n terms on the LHS and all the t terms on the RHS. $g(x) = \sin x$

RHS only a function of x.

$$\frac{dy}{dx} = g(x)$$

derivative w.r.t. x

$\int g(x)\,dx = G(x) + c$

antiderivative

$$dy = g(x)\,dx$$

$$\int 1\,dy = \int g(x)\,dx$$

$$\boxed{y = G(x) + c}$$

Separation of Variables — General I

General Goal: get all the n terms on the LHS and all the t terms on the RHS.

$$\frac{dy}{dx} = g(x) \qquad \rightarrow \qquad y(x) = \int g(x)\ dx = G(x) + c$$

where $G(x)$ is the antiderivative of $g(x)$.

Followup: Verify that what we found is indeed a solution to the ODE.

$$\frac{d}{dx}\Big(y(x)\Big) = \frac{d}{dx}\Big(G(x) + c\Big)$$

$$= \frac{d}{dx}G(x) + \cancel{\frac{d}{dx}c}^{\,0} = g(x)$$

LHS RHS

$$\boxed{\frac{dy}{dx}} = g(x)$$

$$g(x) = g(x)$$

Separation of Variables — General II

General Goal: get all the n terms on the LHS and all the t terms on the RHS.

$$\frac{dy}{dx} = g(x)\,\underbrace{h(y)}_{\text{new}}$$

definition

$$\frac{1}{h(y)} \stackrel{\text{def}}{=} p(y)$$

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

$$\int p(y)\,dy = G(x) + c$$

$$P(y) = \int \frac{1}{h(y)}\,dy$$

$$\boxed{P(y) = G(x) + c}$$

① if possible, now solve for y

Explicit Solution

$y = \text{RHS}$

② Implicit solution

function of y = RHS

Separation of Variables — Recipe

1. Get your equation into this form: $\dfrac{dy}{dx} = g(x) \times h(y)$
2. Identify $g(x)$ and $h(y)$.
3. Divide both sides by $h(y)$, and multiply both sides by dx.
4. Integrate both sides—don't forget your constant!
5. Solve for $y(x)$ if possible.

$$\frac{dy}{h(y)} = g(x)\,dx$$

Separation of Variables — Example I

$e^{a+b} = e^a \cdot e^b$

$g(x)$ $h(y)$

$$\frac{dy}{dx} = xy$$

I.D.

$$\frac{dy}{y} = x\,dx$$

separate

$$\int \frac{1}{y}\,dy = \int x\,dx$$

integrate

solve for y

$$e^{(\ln y)} = e^{\left(\frac{x^2}{2} + c\right)}$$

correct

$$y = e^{\frac{x^2}{2} + c}$$

$$y = e^{\frac{x^2}{2}} \underbrace{e^{c}}_{k}$$

$$\boxed{y = k e^{\frac{x^2}{2}}}$$

Common mistake

$$e^{\ln y} = e^{\frac{x^2}{2}} + e^{c}$$

wrong

Separation of Variables — Example II

$$\frac{dy}{dx} = -\frac{x}{y}$$

$y\,dy = -x\,dx$ separate

$\int y\,dy = -\int x\,dx$ integrate

$\frac{y^2}{2} = -\left(\frac{x^2}{2} + c\right)$

$k = -c$

$\frac{y^2}{2} = -\frac{x^2}{2} + k$

$y^2 = -x^2 + 2k$

$m = 2k$

$y = \pm\sqrt{m - x^2}$

explicit solution.

$y^2 + x^2 = 2k$

$x^2 + y^2 = 2k$

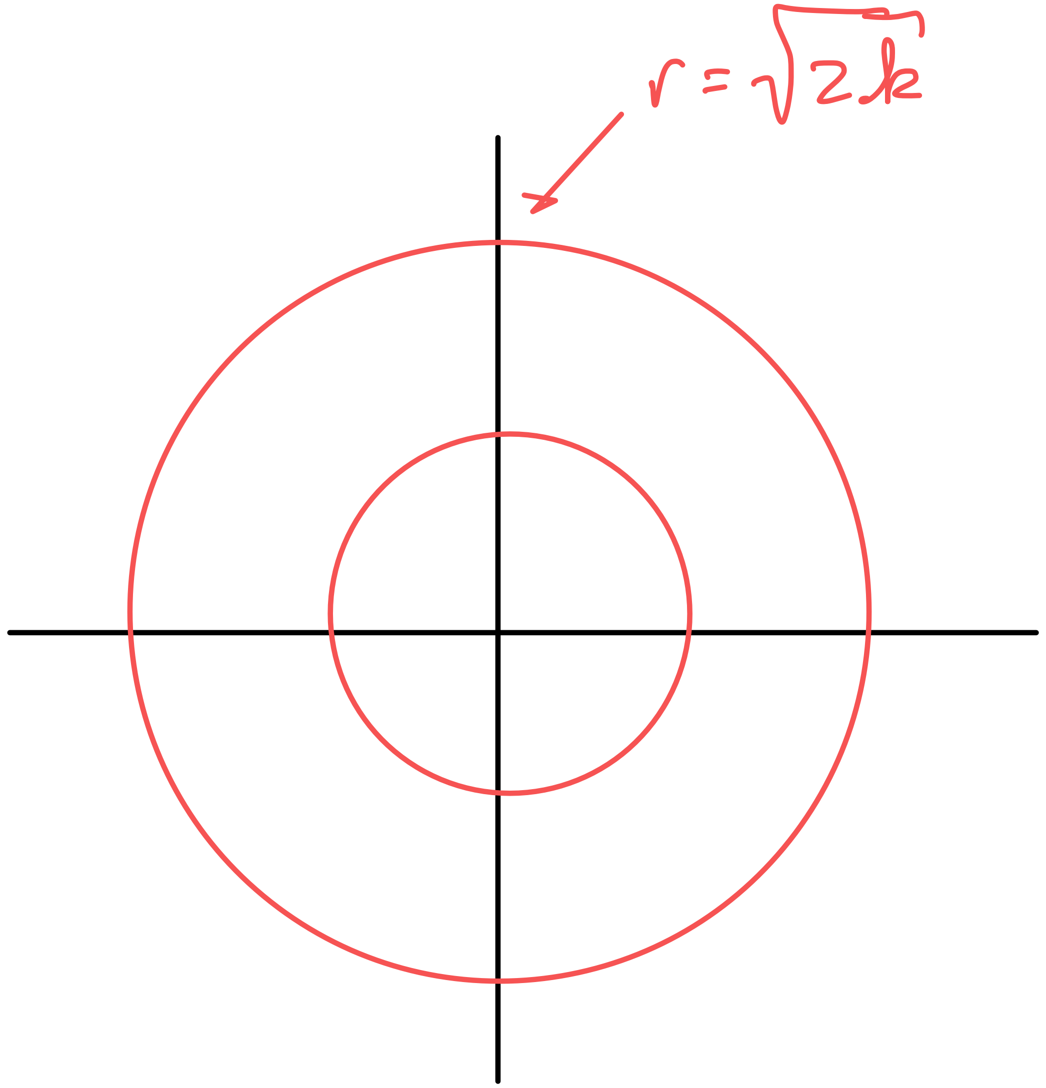
implicit solution

$x^2 + y^2 = r^2$

circle centered @ $(0,0)$

$x^2 + y^2 = r^2$

$r = \sqrt{2k}$



circles.

Separation of Variables — Example III

initial value.

$$\frac{dy}{dx} = \sin 5x, \quad y(0) = 2021$$

① solve

② plug in initial value. AFTER SOLVE.

$\int dy = \int \sin 5x \, dx$

$y = -\frac{\cos 5x}{5} + c$ (General Form 1)

explicit solution!

family

Plug in $x=0$, $y=2021$

$2021 = -\frac{\cos(5\cdot 0)}{5} + c$

$2021 = -\frac{1}{5} + c$

$c = 2021 + \frac{1}{5}$

$c = 2021.2$

$$\boxed{y = -\frac{\cos 5x}{5} + 2021.2}$$

particular solution

Separation of Variables — Example IV

$$\frac{dy}{dx} = x + y \quad \neq \quad \frac{dy}{dx} = g(x)\,h(y)$$

cannot separate!

$$dy = (x+y)\,dx$$

hmmm...

$$\frac{dy}{dx} - y = x$$

$$dy - y\,dx = x\,dx$$

hmm....

Separability

When we can write a first-order ODE in the form $\frac{dy}{dx} = g(x)\ h(y)$,

we call that equation **separable**, or say that it has **separable variables**.

Q: Why do we care?

A: We can solve separable equations using SoV. But if we cannot separate the variables, well… we can't use separation of variables to solve!

Real World Examples: Separability

Suppose that you are collecting data on avian malaria among the local Chickadee population, here in nests around Boulder.

Real World Examples: Separability

Suppose that you are collecting data on avian malaria among the local Chickadee population, here in nests around Boulder. **Suddenly**, a *crazy comp bio professor* leaps out from behind a tree and *shouts at you*:

Real World Examples: Separability

Hint: $\frac{dy}{dx} = g(x)\, h(y)$?

$\ln a^b = b \ln a$

Suppose that you are collecting data on avian malaria among the local Chickadee population, here in nests around Boulder. **Suddenly**, a *crazy comp bio professor* leaps out from behind a tree and *shouts at you*: Which ODEs are separable?!???

✓ 1. $\frac{dy}{dx} = (x+1)^2$ $\qquad dy = (x+1)^2 dx$ ✓

✗ 2. $\frac{dy}{dx} = (x+y)^2$ $\qquad \frac{dy}{dx} = x^2 + y^2 + 2xy$

show that can't separate

show that not $\frac{dy}{dx} = g(x)\, h(y)$

✓ 3. $\frac{dy}{dx} = y^2 e^x \ln x^y = y^2 e^x\, y \ln x = y^3 e^x \ln x = h(y)\, g(x)$

✓ 4. $\frac{dy}{dx} = e^{3x+2y} = e^{3x} \cdot e^{2y} = g(x)\, h(y)$

Revisiting Logistic Growth

$$\frac{dn}{dt} = g(t)\,h(n) \; ?$$

$g(t) \rightarrow 1$

$h(n) \rightarrow rn\left(1 - \frac{n}{K}\right)$

Recall our Logistic Growth Equation:

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right)$$

no function of t on RHS.

$r \;,\; n\left(1 - \frac{n}{K}\right)$

Is this equation $\boxed{\text{separable}}$?

Revisiting Logistic Growth

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right)$$

Partial Fractions!

$$\frac{1}{n\left(1-\frac{n}{K}\right)} = \frac{1}{n} + \frac{1/K}{1-\frac{n}{K}}$$

$$\int \frac{1}{n\left(1-\frac{n}{K}\right)}\,dn = \int r\,dt$$

$$\ln n - \ln\left(1-\frac{n}{K}\right) = rt + c$$

$$\int \frac{1}{n}\,dn + \int \frac{\frac{1}{K}}{1-\frac{n}{K}}\,dn = rt + c$$

1 $\to$ $\ln n$

2

$$\left.\begin{aligned} u &= 1-\frac{n}{K} \\ du &= -\frac{1}{K}\,dn \\ -K\,du &= dn \end{aligned}\right] \to \int \frac{\frac{1}{\cancel{K}}}{u}\left(-\cancel{K}\,du\right) = -\int \frac{1}{u}\,du = -\ln u$$

$$= -\ln\left(1-\frac{n}{K}\right)$$

Revisiting Logistic Growth

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right)$$

$$\ln n - \ln\left(1 - \frac{n}{K}\right) = rt + c$$

$$\ln \frac{n}{1 - \frac{n}{K}} = rt + c$$

$$\ln \frac{K}{\frac{K}{n} - 1} = rt + c$$

$k = -c$

$$\ln \frac{\frac{K}{n} - 1}{K} = -rt + k$$

$e \qquad e$

$$\ln x - \ln y = \ln \frac{x}{y}$$

$$\frac{K}{K} \cdot \frac{n}{1 - \frac{n}{K}} = \frac{Kn}{K - n} \frac{\frac{1}{n}}{\frac{1}{n}} = \frac{K}{\frac{K}{n} - 1}$$

$$\ln x = -\ln \frac{1}{x} \qquad -\ln a = \ln \frac{1}{a}$$

$$\frac{K}{n} - 1 = Ke^{-rt + k}$$

$$\frac{K}{n} = Ke^{-rt + k} + 1$$

$$\boxed{S(x) = \frac{1}{1 + ae^{-bx}}}$$

logistic function

$$\frac{K}{Ke^{-rt + k} + 1} = n$$

$$\frac{K}{Ke^{-rt} m + 1} = n$$

$$\boxed{n(t) = \frac{K}{1 + Kme^{-rt}}}$$

- HW1 grades back soon — Sana is efficient!

- NO in-person class this Thursday -or- next Tuesday.
  (I will post recordings to watch at home instead.)

- YES office hours this Wednesday 4-6p
  NO office hours next Monday 9-11a

- HW2 due next Thurs.

M	T	W	R	F
13	14	15	16	17
	today	O.H.	online (Canvas)	
20	21	22	23	24
no O.H.	online (Canvas)	O.H.	class HW	

Revisiting Logistic Growth

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right)$$

$\boxed{\text{ODE}}$

leads to a solution

K, the carrying capacity

$$n(t) = \frac{K}{1 + CKe^{-rt}}$$

C, our constant of integration, To be determined by our initial condition.

r, the intrinsic growth rate.

Revisiting Logistic Growth

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right) \quad \rightarrow \quad n(t) = \frac{K}{1 + CKe^{-rt}}$$

$$\boxed{n(t) = \frac{K}{1 + \frac{K - n(0)}{n(0)} e^{-rt}}}$$

$$\boxed{C = \frac{K - n(0)}{K n(0)}}$$

- What happens when $t = 0$?

$$n(0) = \frac{K}{1 + CK}$$

initial value of pop.

What happens when $t \rightarrow \infty$?

$$n(t) = \frac{K}{1 + CKe^{-rt}} \longrightarrow \frac{K}{1 + 0} \text{ as } t \rightarrow \infty = \boxed{K}$$

$$\lim_{t \to \infty} e^{-rt} = \lim_{t \to \infty} \frac{1}{e^{rt}} = 0$$

as time gets really large, population $\longrightarrow K$, carrying capacity.

Examples of logistic growth

- Mable & Otto (2001) — cultivated both haploid & diploid *S. cerevisiae* (yeast) in two separate flasks.
- Diploid yeast cells are *bigger* and thus take up more resources.