Calculating Biological Quantities

CSCI 2897

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- HW 4 Due tonight 11:59 P.M.

Last time on CSCI 2897:

We learned that **linear algebra** (matrices and vectors) is like regular algebra, but with a few twists:

- Laws:
    - Associative Law $ABC = (AB)C = A(BC)$
    - Left Distributive Law $(X+Y)D = XD + YD$
    - Right Distributive Law $D(X+Y) = DX + DY$
    - Commutative Law for Scalars $kAB = k(AB) = (kA)B = A(kB) = (AB)k$
    - Commutative Law for matrix multiplication? No. Does not usually commute! $AB \neq BA$ unless special circumstances
- Transposes:
    - $(A+B)^T = A^T + B^T$
    - $\left(A^T\right)^T = A$
    - $(AB)^T = B^T A^T$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ etc}$$

- Matrices are machines that turn vectors into other vectors.
    - The identity matrix (ones on the diagonal, zeros elsewhere) reproduces the same vector.
- Trace: sum the diagonals.
- Determinant (2x2): $ad - bc$

Last Time: Determinant (2x2 matrix)

The **determinant** of a matrix is also a scalar. It has a rather peculiar formula:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Practice:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad |A| = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$$

$$A = \begin{pmatrix} 1 & 2 \\ 0.5 & 1 \end{pmatrix} \qquad |A| = 1 \cdot 1 - 2 \cdot 0.5 = 1 - 1 = 0$$

Note: the determinant of a matrix is the same as the determinant of its transpose.

Matrices as Machines II

$$A = \begin{pmatrix} 1 & 2 \\ 0.5 & 1 \end{pmatrix}$$

If a matrix A has determinant $= 0$ $\quad |A| = 0$

its "abilities" as a matrix are diminished $\rightarrow$ line

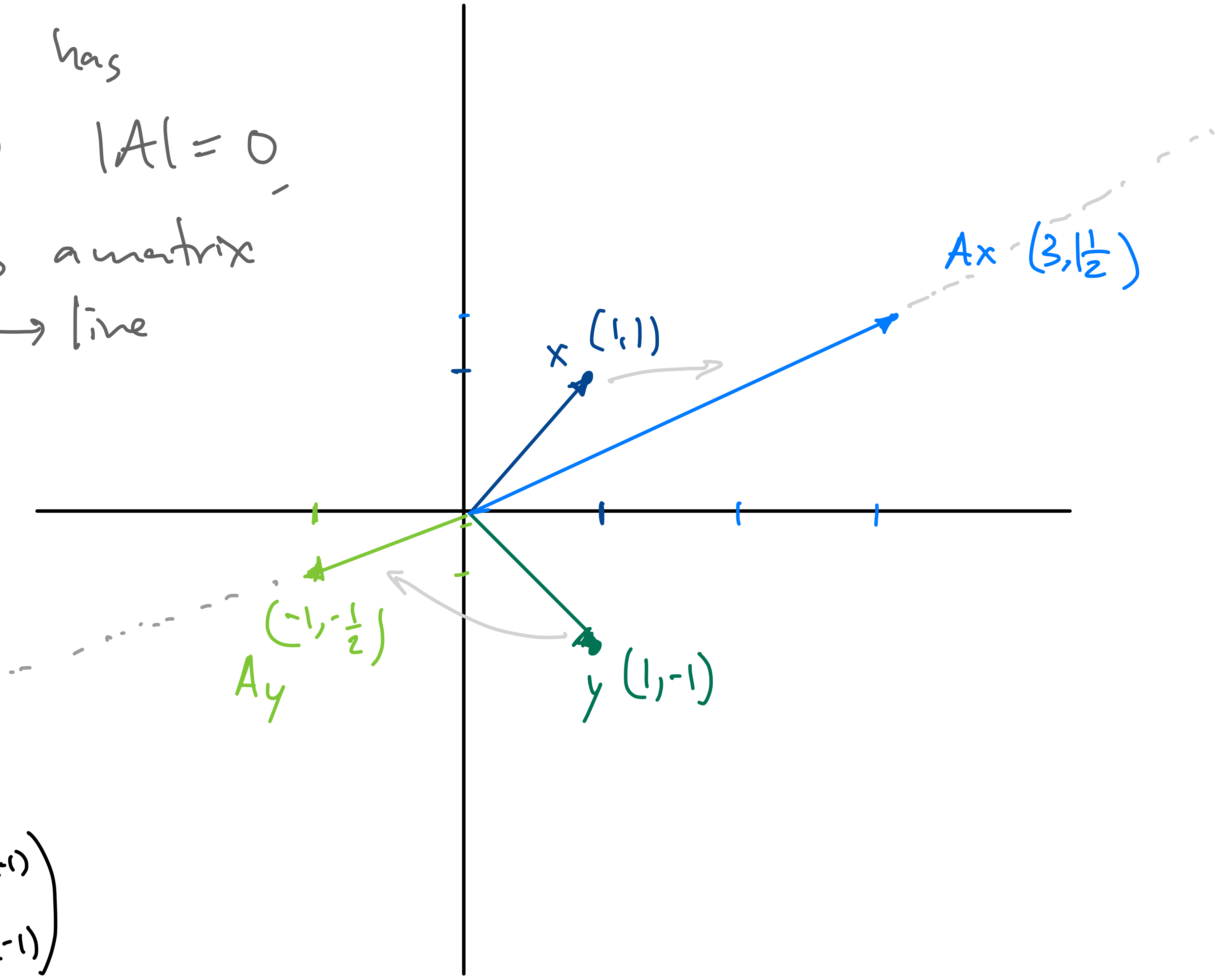
"singular"

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Ax = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 1 \\ \frac{1}{2}\cdot 1 + 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1\frac{1}{2} \end{pmatrix}$$

$$y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$Ay = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot(-1) \\ \frac{1}{2}\cdot 1 + 1\cdot(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix}$$



Solving a system of equations

Let's solve these two equations

$$6x + 4y = 12$$

$$2\cdot 3x - 2\cdot 2y = 0\cdot 2 \quad +$$

$$\rightarrow \left(1, \frac{2}{3}\right)$$

$$12x + 0y = 12$$

$$12x = 12$$

$$x = 1$$

$$3\cdot 1 - 2\cdot y = 0$$

$$3 = 2y$$

$$\boxed{y = \frac{3}{2}}$$

$$3x - 2y = 0$$

$$3x = 2y$$

$$x = \frac{2}{3}y$$

$$\rightarrow x = \frac{2}{3}\cdot\frac{3}{2} = 1$$

$$6x + 4y = 12$$

$$6\left(\frac{2}{3}y\right) + 4y = 12$$

$$\frac{12}{3}y + 4y = 12$$

$$8y = 12 \qquad y = \frac{12}{8} = \frac{3}{2}$$

Solving a system of equations

We can also write these equations in “matrix-vector notation.”

$$6x + 4y = 12$$
$$3x - 2y = 0$$

$$\begin{pmatrix} 6 & 4 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

$$M \quad \vec{x} \quad = \quad \vec{b}$$

$$\frac{Mx}{M} = \frac{b}{M}$$ "dividing"

$$M^{-1}Mx = M^{-1}b$$ "multiplying"

$$Ix = M^{-1}b$$

$$x = M^{-1}b$$

goal.

The Inverse Matrix

The **inverse** of a matrix A, denoted A^{-1} is a matrix such that $A^{-1}A = I$

What does our *trick of the transpose* tell us for free?

$$\underbrace{A^T A^{-1^T}} = (A^{-1}A)^T = I^T = \underbrace{I} \qquad A^T A^{-1^T} = I$$

What else can we get "for free" from this equation?

$$A^{-1} A = I$$

$$A A^{-1} = I$$

A and A^{-1} commute.

Whoah! Remember: the *inverse* of a number a, denoted a^{-1}, is a number such that $a \cdot a^{-1} = 1$

Proofs are cool!

A proof is a math story told in simple and clear steps.

Theorem:

$A^{-1}A = I$ if and only if $AA^{-1} = I$.

Proof:

Suppose that $A^{-1}A = I$.

$$\downarrow$$

$$AA^{-1} = I$$

Suppose that $AA^{-1} = I$.

$$\downarrow$$

$$A^{-1}A = I$$

The Inverse Matrix

The **inverse** of a matrix A, denoted A^{-1} is a matrix such that $A^{-1}A = I$

Let's use the inverse to solve the equation from the previous slides.

$$6x + 4y = 12$$
$$3x - 2y = 0$$

$$\underbrace{\begin{pmatrix} 6 & 4 \\ 3 & -2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 12 \\ 0 \end{pmatrix}}_{\vec{b}}$$

$$A^{-1} A \vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = \frac{1}{\det(A)} \begin{pmatrix} -2 & -4 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

$$\det(A) = 6 \cdot -2 - 4 \cdot 3 = -12 - 12 = -24$$

$$\vec{x} = \frac{1}{24} \begin{pmatrix} 2 & 4 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} 12 \\ 0 \end{pmatrix} = \frac{1}{24} \begin{pmatrix} 24 \\ 36 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 3/2 \end{pmatrix} \checkmark$$

The Inverse Matrix (2x2)

There turns out to be a nice formula for an inverse matrix.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let's say that generically, $A^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

If $A^{-1}A = I$ then:

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$w \cdot a + xc = 1$$
$$w \cdot b + x \cdot d = 0$$
$$y \cdot a + z \cdot c = 0$$
$$y \cdot b + z \cdot d = 1$$

goal: solve for w.

$$① \quad wa + xc = 1$$
$$(wb + xd = 0)\left(-\frac{c}{d}\right)$$
$$+ ② \quad w\left(-\frac{bc}{d}\right) + x\left(-\frac{\cancel{d}c}{\cancel{d}}\right) = 0$$

$$wa - w\frac{bc}{d} + 0 = 1$$
$$w\left(a - \frac{bc}{d}\right) = 1 \qquad w = \frac{1}{a - \frac{bc}{d}} \cdot \frac{d}{d}$$
$$w = \frac{d}{ad - bc} \qquad \boxed{w = \frac{d}{\det(A)}}$$

The Inverse Matrix (2x2)

There turns out to be a nice formula for an inverse matrix.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ If $A^{-1}A = I$ then: $\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

etc. (as on prev slide).

$$A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \boxed{\frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1}}$$