

# Calculating Biological Quantities

## CSCI 2897

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2021, Lecture 22

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- HW 6 Due Thurs. 11:59 PM.
- Office Hours W 4-6 , M 9-11
- Final Tues (12/14) 1:30-4:00

# Review of Demography Models:

1. What is the **long-term growth rate** of a population?
2. What is the **long-term class structure** of a population?
3. Which **classes contribute most** to the long-term growth rate of a population.

Population **transition matrices** have interesting properties:

1. All entries are  $\geq 0$ .
2. The matrix is square.

When these conditions are met, the **Perron-Frobenius Theorem** tells us that:

1. The eigenvalue with largest magnitude  $\lambda_1$  will *never be negative*.
2. This eigenvalue will also *always be real*.
3. The eigenvector  $\vec{x}_1$  associated with this eigenvalue will *also* be non-negative and real.

# Revisiting the classics — **2 condition models**

**Carbon dating** allows us to estimate when an organism died because:

1. C14 and C12 are in a stable ratio in the environment.
2. C14 is radioactive, and decays with a half-life of 5730 years. *(Dynamics)*
3. Thus measuring the ratio of C14 to C12 in an organism can tell us when the organism died.

**Newton's law of cooling** states that the rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings.

**Buying a house** involves taking out a loan that you pay back over 30 years of constant payments—but the amount that you owe grows in between payments.

# What are “2 condition” scenarios?

In some cases, you need to use a model to answer a question.

However, you don't always get parameters and an initial condition.

Instead, you get **2** conditions — one “initial” and one later. In these scenarios, which I call “2 condition” scenarios, we need to double up our work.

- ✓ 1. Use the changes observed between condition 1 and condition 2 to *learn* the model's parameter.
- ✓ 2. Then use one of the conditions + the parameter to answer the scientific question.

# Carbon Dating

**Carbon dating** allows us to estimate when an organism died because:

1. C14 and C12 are in a stable ratio in the environment.
2. C14 is radioactive\*, and decays (to N14) with a half-life of 5730 years.
3. Thus measuring the ratio of C14 to C12 in an organism can tell us when the organism died.

**Write this as a differential equation.** A “half life” is the amount of time that it takes for the mass to decrease by a factor of 2. How should this show up in the ODE?

Radioactive Decay

$$\frac{dC}{dt} = -kC$$

↓ S.O.V.      ↓ rate

$$C(t) = C_0 e^{-kt}$$

initial condition

Intuition: ↑ half life = ↓ decay rate

Two Conditions:

$$C(0) = C_0$$

$$C(5730) = \frac{1}{2}C_0$$

plug in to our solution!

$$\frac{1}{2}C_0 = C_0 e^{-k \cdot 5730}$$

solve for k.

$$\ln \frac{1}{2} = -k \cdot 5730 \quad \sim \ln \frac{1}{2} = \ln 2$$

$$k = (\ln 2) / 5730$$

$$\rightarrow C(t) = C_0 e^{-\frac{\ln 2}{5730} \cdot t}$$

\* The radioactive decay means that the total mass of the radioactive isotope decreases at a rate directly proportional to that mass.

# Carbon Dating

You are wandering around Boulder looking for a COVID vaccine when you come across the frozen body of a woolly mammoth. Noting C14's half life of 5730 years, you observe that the mammoth has only **20%** of the C14 that it'd have if it were freshly deceased. How old is the mammoth?

$$C(t) = C_0 e^{-\frac{\ln 2}{5730} \cdot t}$$

$$\cancel{\frac{1}{5} C_0} = \cancel{C_0} e^{-\frac{\ln 2}{5730} \cdot t^*}$$

$$C(t^*) = 0.2 C_0$$

$$\ln \frac{1}{5} = -\frac{\ln 2}{5730} \cdot t^*$$

what is  $t^*$ ?

$$\ln 5 = \frac{\ln 2}{5730} t^*$$

$$t^* = \frac{\ln 5}{\ln 2} \cdot 5730$$

$$t^* = 13,304$$

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# Carbon Dating

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\* The radioactive decay means that the total mass of the radioactive isotope decreases at a rate directly proportional to that mass.

# Newton's law of cooling

**Newton's law of cooling** states that the rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings. **Write this as an ordinary differential equation.**

$$\frac{dT}{dt} = -k (T - T_{\text{ext}}) = k (T_{\text{ext}} - T)$$



# Newton's law of cooling

You show up to IRL office hours for CSCI 2897, but there is no one in the office. There is just a French Press.

You touch the French Press. Hmmm. Still warm. You estimate that it is 50 degrees.

You wait 10 minutes. Now the French Press is 40 degrees.

Given that room temperature is 20 degrees, **when was the boiling water** (100 degrees) **poured** into the French Press?

1. Learn the parameter of the model from the clues in the data.
2. Use param. + model  $\longrightarrow$  answer the question!

$$\frac{dT}{dt} = k (T_{\text{ext}} - T)$$

$\uparrow$   
20  
(constant)

$$\text{At } t=0, T=50$$

$$t=10, T=40$$

$$\text{Want: } T(t) = ???$$

Solve using S.O.V.



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Given that room temperature is 20 degrees, **when was the boiling water** (100 degrees) **poured** into the French Press?

$$t=0 \quad T=50$$

$$50 = 20 - b \underbrace{e^{-kt}}_1$$

$$30 = -b$$

$$b = -30$$

$$T(t) = 20 + 30e^{-kt}$$

$$\frac{dT}{dt} = k(T_{\text{ext}} - T)$$

S.o.V.

$$\int \frac{dT}{T_{\text{ext}} - T} = \int k dt$$

$$-\ln(T_{\text{ext}} - T) = kt + c$$

$$\ln(T_{\text{ext}} - T) = -kt - c$$

$$T_{\text{ext}} - T = be^{-kt}$$

$$T = T_{\text{ext}} - be^{-kt}$$

$$b = e^{-c}$$



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Given that room temperature is 20 degrees, **when was the boiling water** (100 degrees) **poured** into the French Press?

$$T(t) = 20 + 30e^{-kt}$$

$\frac{1}{10} \ln \frac{30}{20}$

$$t = 10 \quad T = 40$$

solve for  $k$ .

$$\frac{dT}{dt} = k(T_{\text{ext}} - T)$$



$$40 = 20 + 30e^{-10k}$$

$$\frac{20}{30} = e^{-10k}$$

$$\ln \frac{20}{30} = -10k$$

$$k = -\frac{1}{10} \ln \frac{20}{30} = \frac{1}{10} \ln \frac{30}{20}$$

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You wait 10 minutes. Now the French Press is 40 degrees.

Given that room temperature is 20 degrees, **when was the boiling water** (100 degrees) **poured** into the French Press?

$$T(t) = 20 + 30 e^{-\frac{1}{10} \ln \frac{30}{20} t}$$

$$T = 100$$

$t = ?$   
↑  
solve for  $t$ .

$$-\frac{10 \ln \frac{80}{30}}{\ln \frac{30}{20}} = t$$

$t = -24$  or so  
minutes.

$$\frac{dT}{dt} = k (T_{\text{ext}} - T)$$

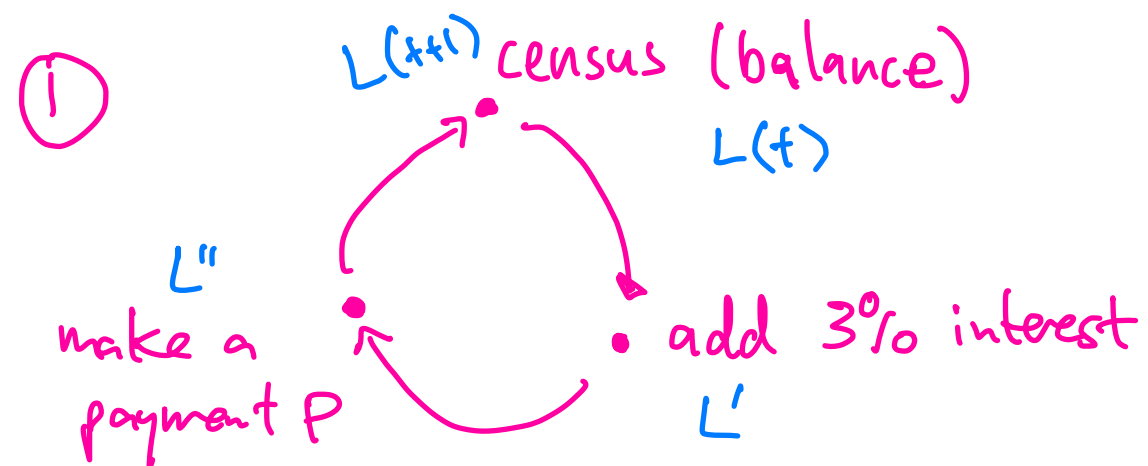


# Home mortgages.

**Buying a house** involves taking out a loan that you pay back over 30 years of constant payments—but the amount that you owe grows in between payments.

Suppose that you take out a loan of \$200,000, to be paid back after 30 years of monthly payments. Each month you pay down the loan by  $P$ , but you accumulate interest at an annual rate of 3%. (compounded monthly)

1. Write a **life-cycle diagram** for your loan balance. (One cycle = one month)
2. Write a **recursion** for your loan balance.
3. What is the **initial condition** of the recursion? What is the **final condition**?
4. What is your **monthly payment  $P$** ?



② 
$$L'(t) = L(t) + L(t) \frac{0.03}{12}$$
$$= L(t) \left[ 1 + \frac{0.03}{12} \right]$$

$$L''(t) = L'(t) - P$$

$$\begin{aligned} L(t+1) &= L''(t) \\ &= L'(t) - P \\ &= L(t) \left[ 1 + \frac{0.03}{12} \right] - P \end{aligned}$$

$$L(t+1) = L(t) \left[ 1 + \frac{0.03}{12} \right] - P$$

# Home mortgages.

$$n(t+1) = R n(t) - P$$

Suppose that you take out a loan of \$200,000, to be paid back after 30 years of monthly payments. Each month you pay down the loan by  $P$ , but you accumulate interest at an annual rate of 3%.

1. Write a **life-cycle diagram** for your loan balance. (One cycle = one month)
- ✓ 2. Write a **recursion** for your loan balance.
- ✓ 3. What is the **initial condition** of the recursion? What is the **final condition**?
4. What is your **monthly payment  $P$** ?  $R = 1 + \frac{0.03}{12}$

$$L(t+1) = L(t) \left[ 1 + \frac{0.03}{12} \right] - P \quad \rightarrow \quad \underline{L(t+1) = R L(t) - P}$$

$$L(0) = 200,000 \equiv L_0$$

$$L(360) = 0$$

$$L(0) = L_0$$

$$L(1) = L_0 R - P$$

$$L(2) = (L_0 R - P) R - P$$

$$L(3) = ((L_0 R - P) R - P) R - P = L_0 R^3 - P R^2 - P R - P$$

$$L(t) = L_0 R^t - P R^{t-1} - P R^{t-2} \dots - P R - P$$

$$L(t) = L_0 R^t - P \sum_{n=0}^{t-1} R^n \quad \text{geometric series!}$$



# Home mortgages.

Suppose that you take out a loan of \$200,000, to be paid back after 30 years of monthly payments. Each month you pay down the loan by  $P$ , but you accumulate interest at an annual rate of 3%.

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$$L(t) = L_0 R^t - P \sum_{n=0}^{t-1} R^n$$

geometric series!

$$L(t) = L_0 R^t - P \left[ \frac{1 - R^t}{1 - R} \right]$$

partial sum formula.

$$0 = L_0 R^{360} - P \left[ \frac{1 - R^{360}}{1 - R} \right]$$

$$L(360) = 0$$
$$L_0 R^{360} = P \left[ \frac{1 - R^{360}}{1 - R} \right]$$

$$P = L_0 R^{360} \frac{1 - R}{1 - R^{360}}$$

$$L_0 = 200,000, \quad R = 1 + \frac{0.03}{12}$$

# Home mortgages.

Suppose that you take out a loan of \$200,000, to be paid back after 30 years of monthly payments. Each month you pay down the loan by  $P$ , but you accumulate interest at an annual rate of 3%.

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# Calculating Biological Quantities

## Biology

population growth

competition, mutualism, commensalism, parasitism

predation

infectious disease

vaccination

variants

population genetics

demography

## Models

exponential growth

logistic growth

Lotka-Volterra

SIR (etc)

affine systems

linear systems

## Math & Methods

recursions

differential equations

equilibria

stability

linear algebra

flow & lifecycle diagrams