Calculating Biological Quantities

CSCI 2897

Prof. Daniel Larremore

daniel.larremore@colorado.edu
@danlarremore

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Today:

1. Linear models with more than one variable
2. Matrices and vectors

Models with more than one dynamic variable

Let's go back to exponential growth in continuous time.

$$\frac{dn}{dt} = rn$$

$$\int \frac{dn}{n} = \int r\,dt$$

$$\ln n = rt + c$$

$$n = e^{rt+c}$$

$$n = e^{rt} \cdot e^{c}$$

$$\boxed{n = ke^{rt}}$$

We know by now that this is called exponential growth because

$$n(t) = ke^{rt}$$

where $k = n(0)$ is the initial condition.

Models with more than one dynamic variable

Now let's imagine that we have two populations, n_1 and n_2

no n_2

$$\frac{dn_1}{dt} = r_1 n_1$$

$n_1(t) = k_1 e^{r_1 t}$

not coupled.

$$\frac{dn_2}{dt} = r_2 n_2$$

$n_2(t) = k_2 e^{r_2 t}$

no n_1

This one is easy too: the populations are totally independent of each other, so we can solve each equation by itself.

Models with more than one dynamic variable

What are the equilibrium solutions for this set of equations?

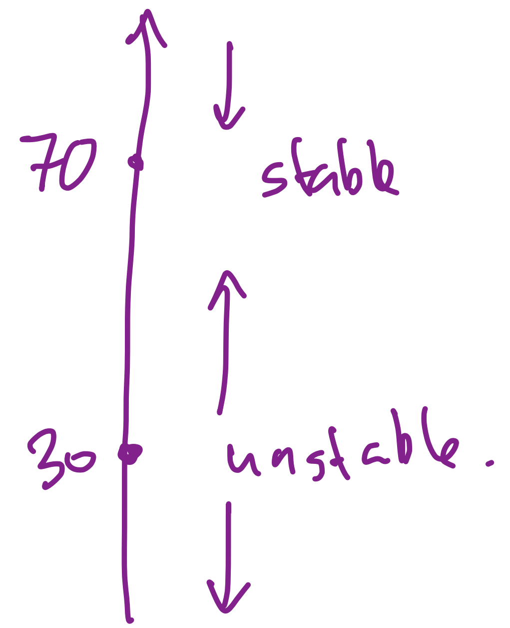
$$\frac{dn_1}{dt} = r_1 n_1 = 0 \longrightarrow n_1 = 0$$

$$\frac{dn_2}{dt} = r_2 n_2 = 0 \longrightarrow n_2 = 0$$

$\boxed{\text{equilibrium } (n_1, n_2) = (0, 0)}$

What can we say about stability of the equilibrium solution(s)?

recall exam q:



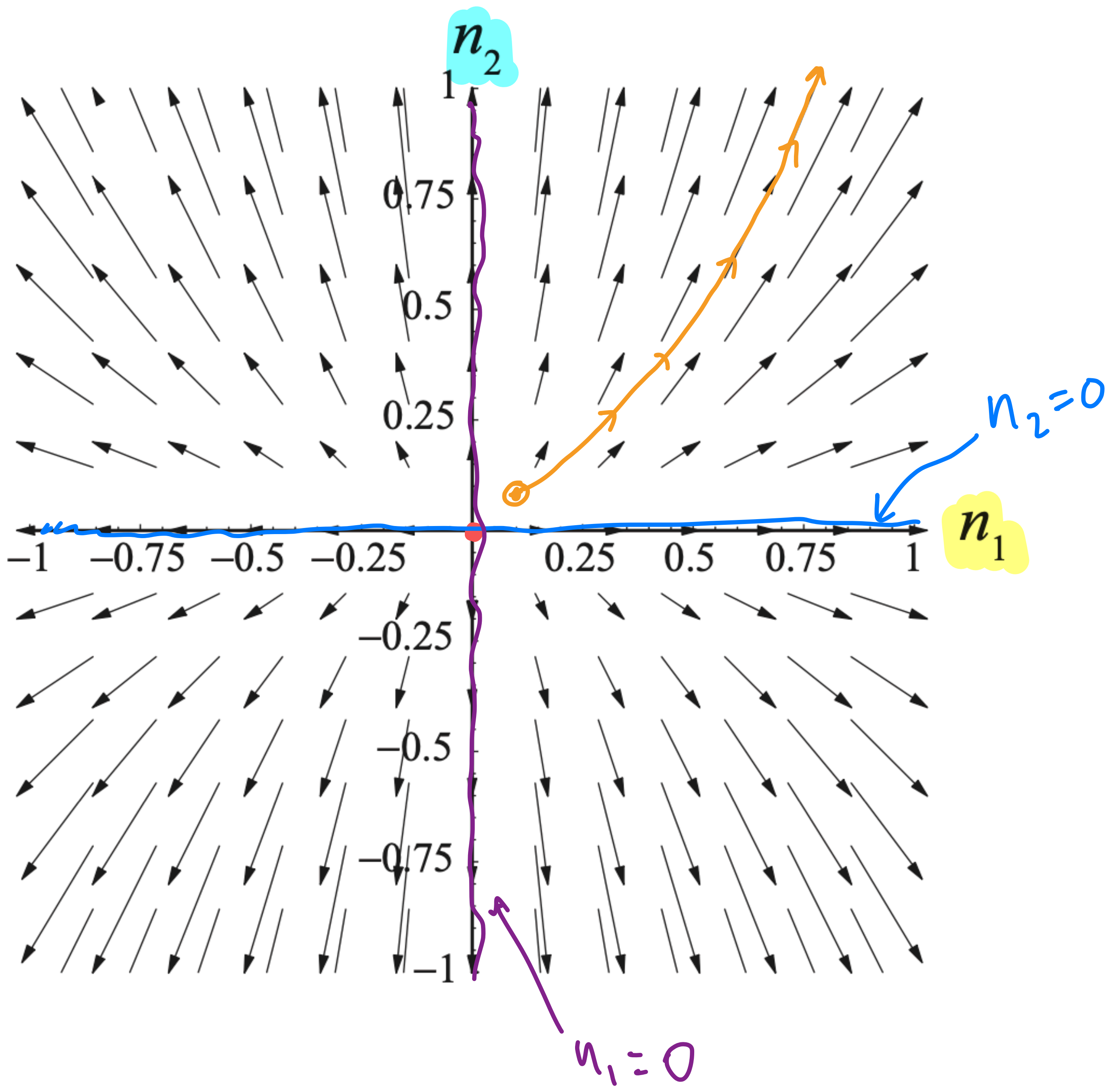
Use vector fields.

• equilibrium

Phase portrait
Vector Field

$$\frac{dn_1}{dt} = r_1 n_1$$

$$\frac{dn_2}{dt} = r_2 n_2$$



$r_1 > 0$

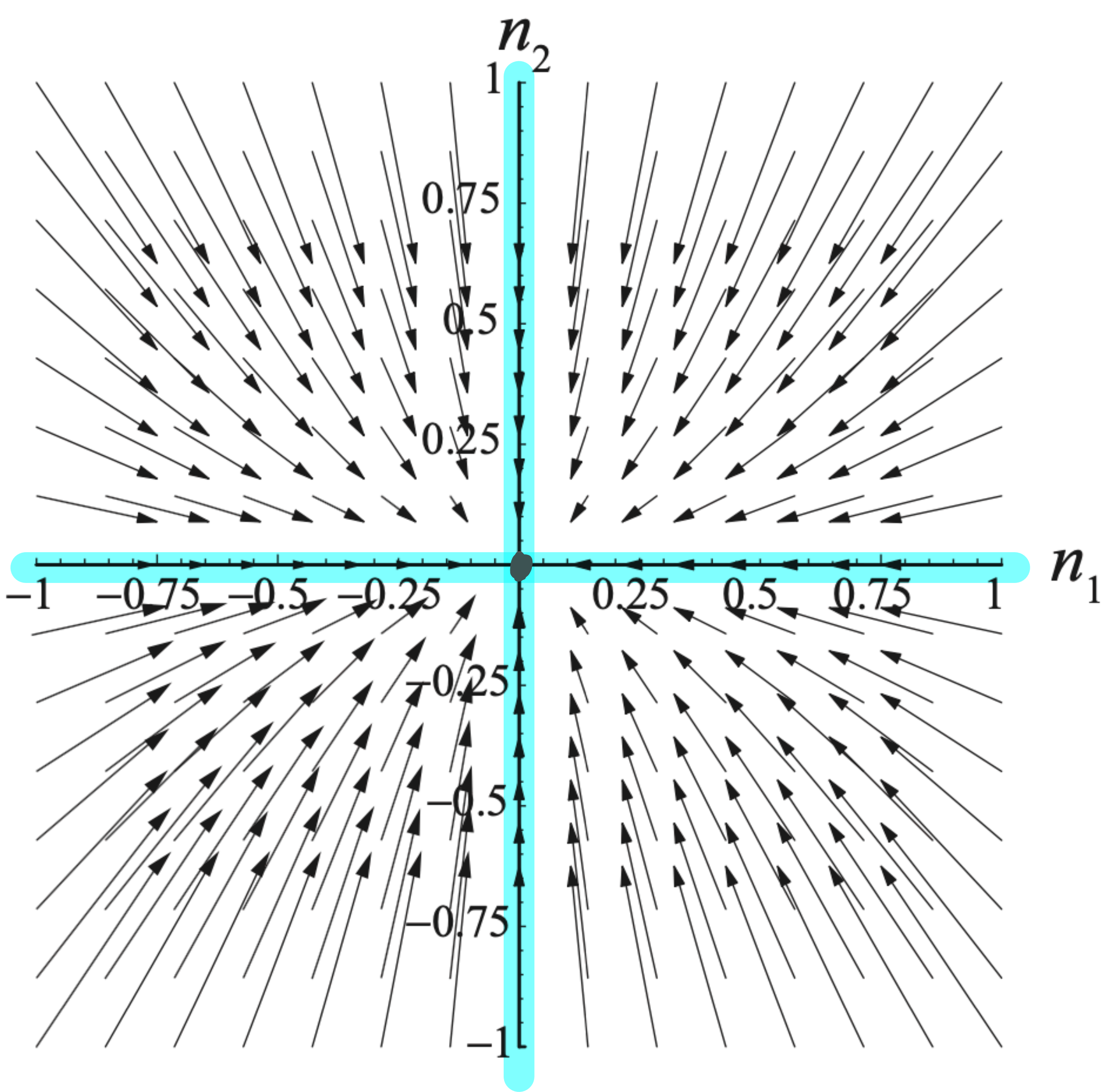
$r_2 > 0$

$r_2 > r_1$

Unstable

$$\frac{dn_1}{dt} = r_1 n_1$$

$$\frac{dn_2}{dt} = r_2 n_2$$



$r_1 < 0$

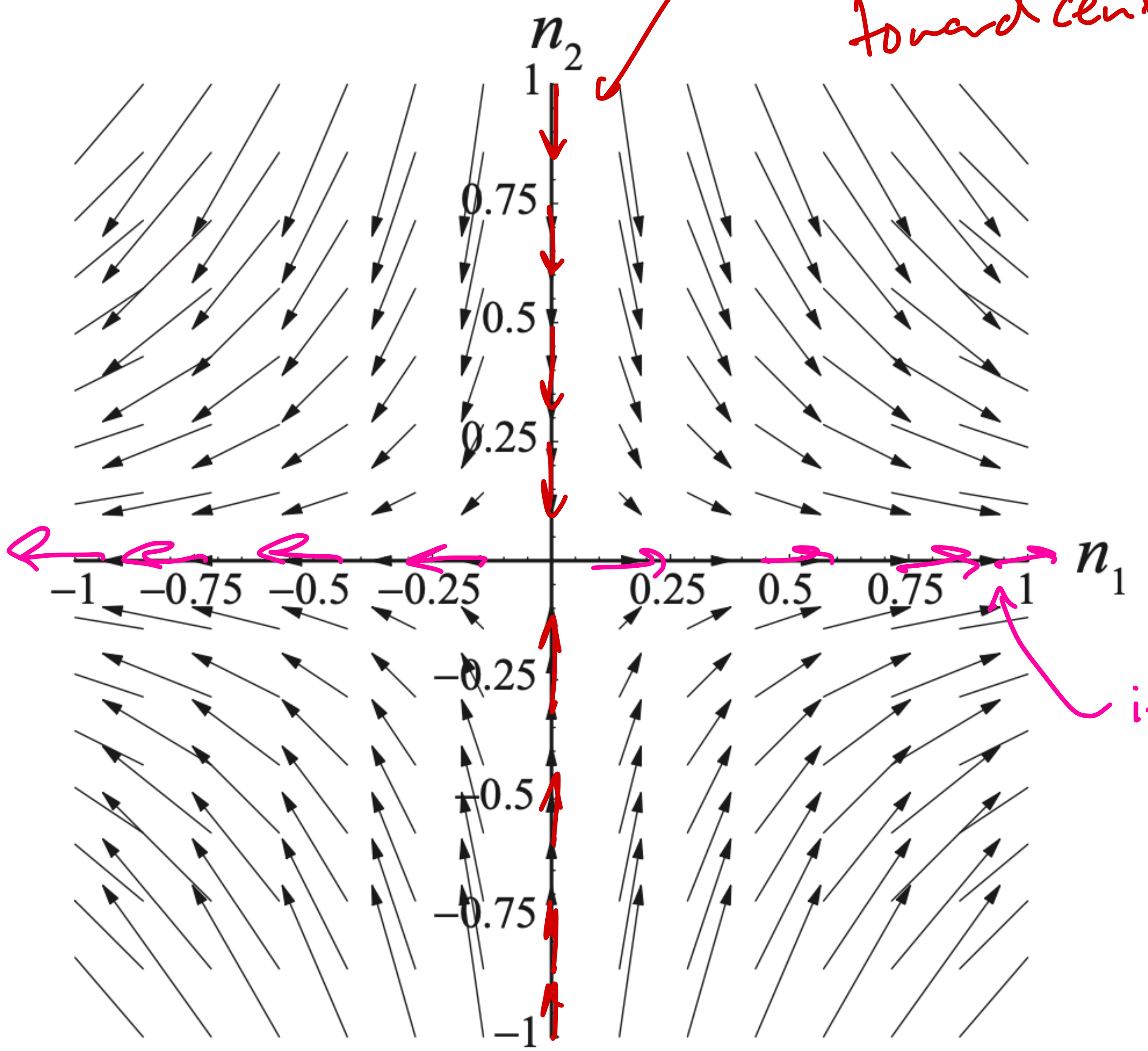
$r_2 < 0$

note: even though we changed the signs of r_1, r_2, the "directions" given by the axes are special.

stable

$$\frac{dn_1}{dt} = r_1 n_1$$

$$\frac{dn_2}{dt} = r_2 n_2$$



now, if you're on the y-axis, move toward center.

if on the x-axis, move away from center.

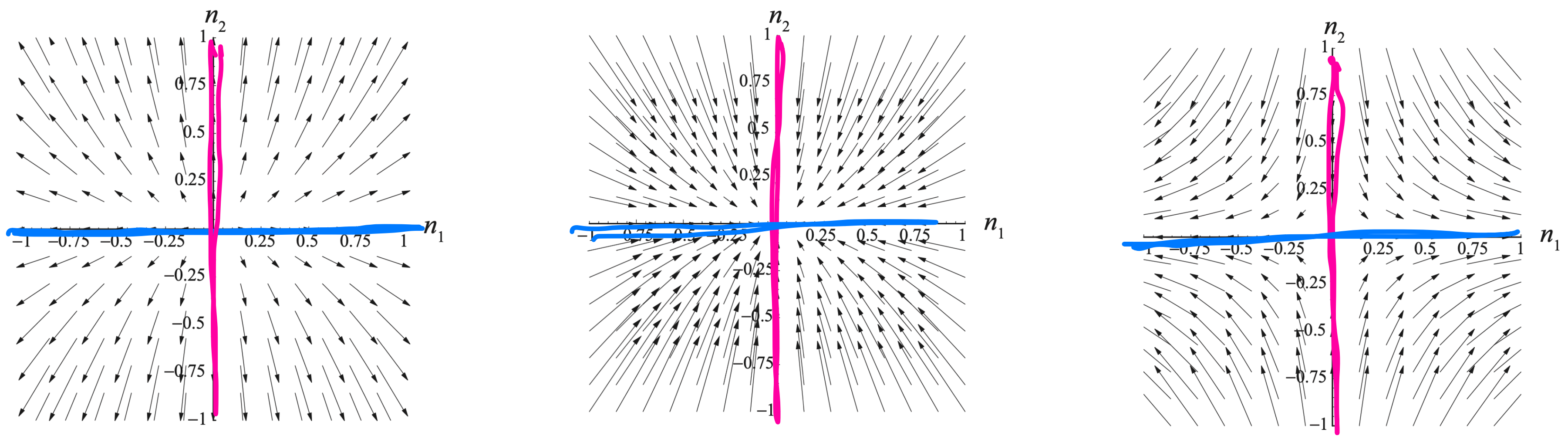
$r_1 > 0$

$r_2 < 0$

Unstable. → must be stable in <u>all</u> directions to be classified stable.

Characteristic directions

Eigen vectors



$$\frac{dn_1}{dt} = r_1 n_1$$

$$\frac{dn_2}{dt} = r_2 n_2$$

eigenvector

Note: for these equations, if you're on either axis, you never leave.

These directions are therefore special:

$(c,0)$ the horizontal axis

$(0,c)$ the vertical axis

for any arbitrary value of c.

Models with more than one dynamic variable - Part 2

Imagine that our 2 populations correspond to 2 strains of bacteria. Suppose that

- a is the rate at which **strain 1** produces **strain 1** daughter cells
- b is the rate at which **strain 2** produces **strain 1** daughter cells by mutation
- c is the rate at which **strain 1** produces **strain 2** daughter cells by mutation
- d is the rate at which **strain 2** produces **strain 2** daughter cells

$$\frac{dn_1}{dt} = \underbrace{a n_1}_{\text{self replication}} + \underbrace{b n_2}_{n_2 \text{ producing } n_1}$$

$$\frac{dn_2}{dt} = \underbrace{c n_1}_{n_1 \text{ producing } n_2} + \underbrace{d n_2}_{\text{self replication}}$$

Models with more than one dynamic variable - Part 2

We are going to rewrite this in a miraculous way

$$\frac{dn_1}{dt} = an_1 + bn_2$$

$$\frac{dn_2}{dt} = cn_1 + dn_2$$

$$\rightarrow \underbrace{\begin{bmatrix} \frac{dn_1}{dt} \\ \frac{dn_2}{dt} \end{bmatrix}}_{\text{vector}} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{matrix}} \underbrace{\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}}_{\text{vector}} \rightarrow \frac{d\vec{n}}{dt} = M\vec{n}$$

linear system of equation — written out $\longrightarrow$ matrix-vector notation

Vectors and Matrices

A vector is a matrix with one row or one column.

A **vector** is a list of elements.

$(1, 1)$ length two

$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ length two

$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ length three

$(2, 4, 6, 19, -2, \pi, 0, 0, 0, 2021)$

- some written vertically
- some written horizontally.

} stay tuned!

A **matrix** is a table of elements.

$\begin{pmatrix} 2 & 0 \\ 1 & 9 \end{pmatrix}$

3 rows $\begin{bmatrix} 1 & 19 \\ 12 & x \\ y & 5 \end{bmatrix}$ 2 columns.

$\begin{bmatrix} 2 & 4 & 6 & 2021 \\ \pi & \gamma & \alpha & 4 \end{bmatrix}$ two rows

4 columns.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

first column

first row

NB: the plural of "matrix" is "matrices."

Vectors in the x-y plane

Remember those characteristic directions from before, $(0,c)$ and $(c,0)$? Those, too, are vectors!

$c = 3$

$c = -2$

$c = -1$

It turns out that points in the x-y plane are also vectors. Why? Because a **vector** is a list of elements.

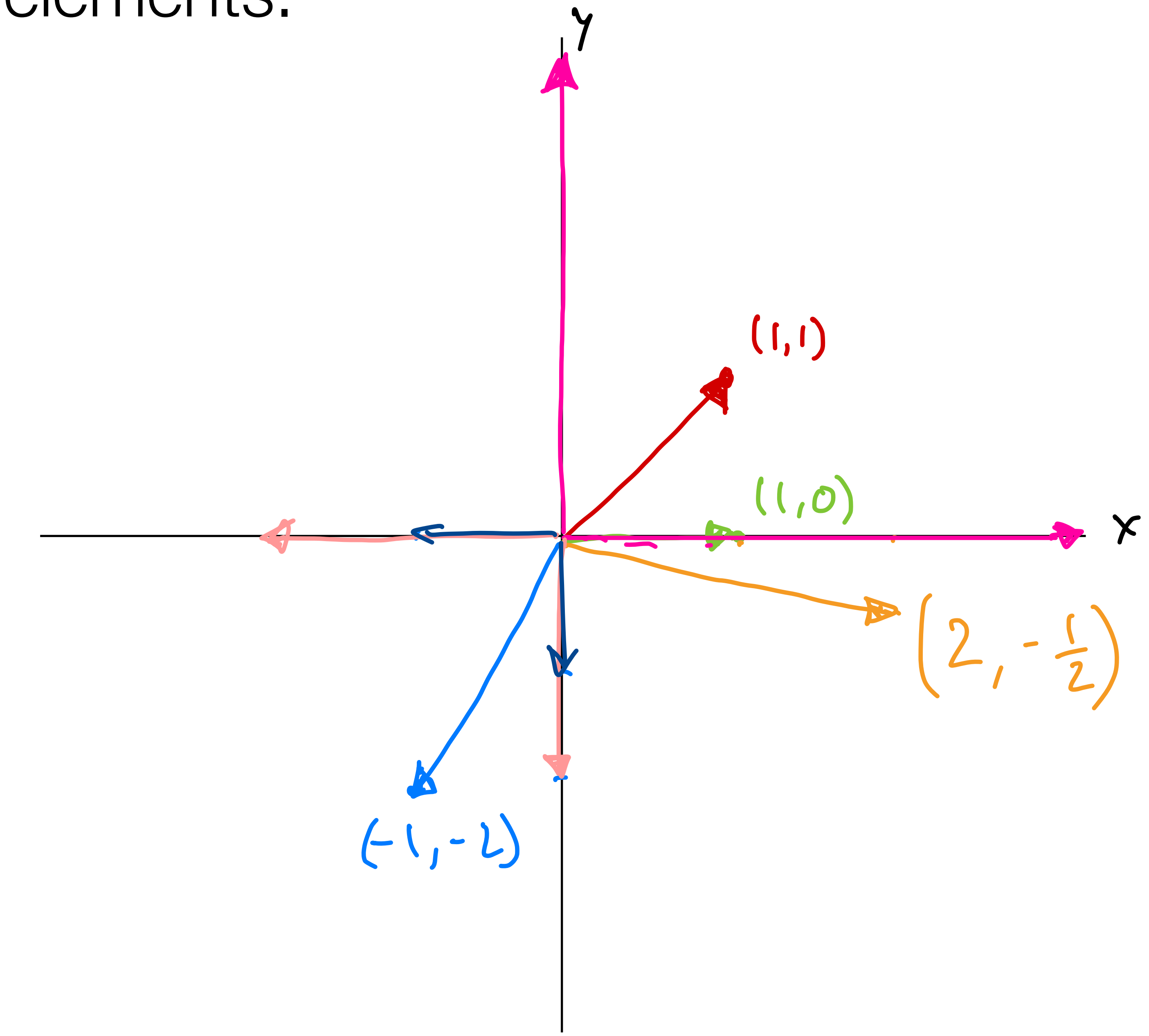
$(1, 1)$

$x \quad y$

$(-1, -2)$

$(1, 0)$

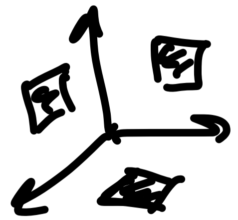
$(2, -\frac{1}{2})$



NB: Because of their use in modeling, we draw vectors as **arrows**, which point in a particular direction, and have a particular magnitude.

Vectors in the x-y-z plane

point $\longleftrightarrow$ vectors.

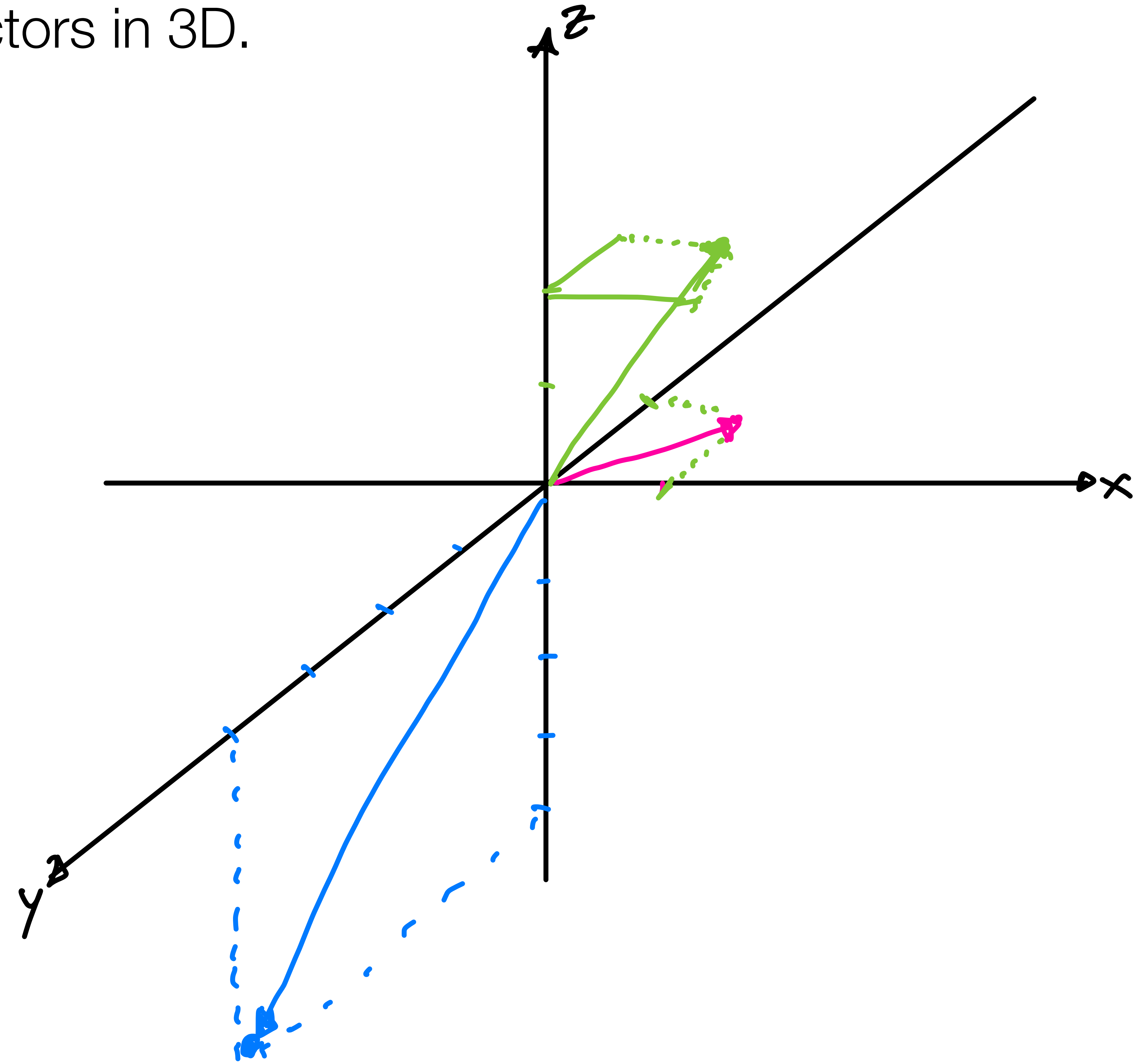


We can also draw vectors in 3D.

$(1, -1, 0)$

$(1, -1, 2)$

$(0, 4, -4)$



LaTeX: \vec{v} $\vec{v}$

Row vectors and Column vectors

In general, the **dimension** of a vector is:
or
matrix

(# of rows) × (# of columns)
height width

Because a vector has either 1 row or 1 column, we get two definitions for free:

1. a vector that's a row is called a **row vector**
2. a vector that's a column is called a **column vector**.

Examples:

row vector: $[3, 2, 1]$ 1×3 vector or 1×3 matrix

column vector $\begin{bmatrix} 2 \\ 5 \\ \pi \\ 9 \end{bmatrix}$ 4×1 vector or 4×1 matrix

To denote a vector, we sometimes write a little arrow $\overrightarrow{v}$

LaTeX: \vec{v}

Matrix and Vector addition

Rule: you can add two matrices or two vectors **only if** they have the same dimensions.

Examples:

① $\underset{2\times1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \underset{2\times1}{\begin{bmatrix} 19 \\ 20 \end{bmatrix}} = \begin{bmatrix} 2+19 \\ 1+20 \end{bmatrix} = \underset{2\times1}{\begin{bmatrix} 21 \\ 21 \end{bmatrix}}$

② $\underset{1\times3}{(3, 2, -1)} + \underset{1\times3}{(12, 12, 12)} = \underset{1\times3}{(15, 14, 11)}$

③ $\underset{2\times2}{\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}} + \underset{2\times2}{\begin{pmatrix} 5 & 5 \\ 5 & x \end{pmatrix}} = \underset{2\times2}{\begin{pmatrix} 6 & 7 \\ 7 & x \end{pmatrix}}$

④ $\underset{2\times2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + \underset{2\times1}{\begin{pmatrix} 2 \\ 2 \end{pmatrix}} = \text{NOPE.}$

$2\times2 \neq 2\times1$

Matrix and Vector scalar multiplication

Rule: you can **multiply** a matrix or a vector~~s~~ **by a constant**.

Examples:

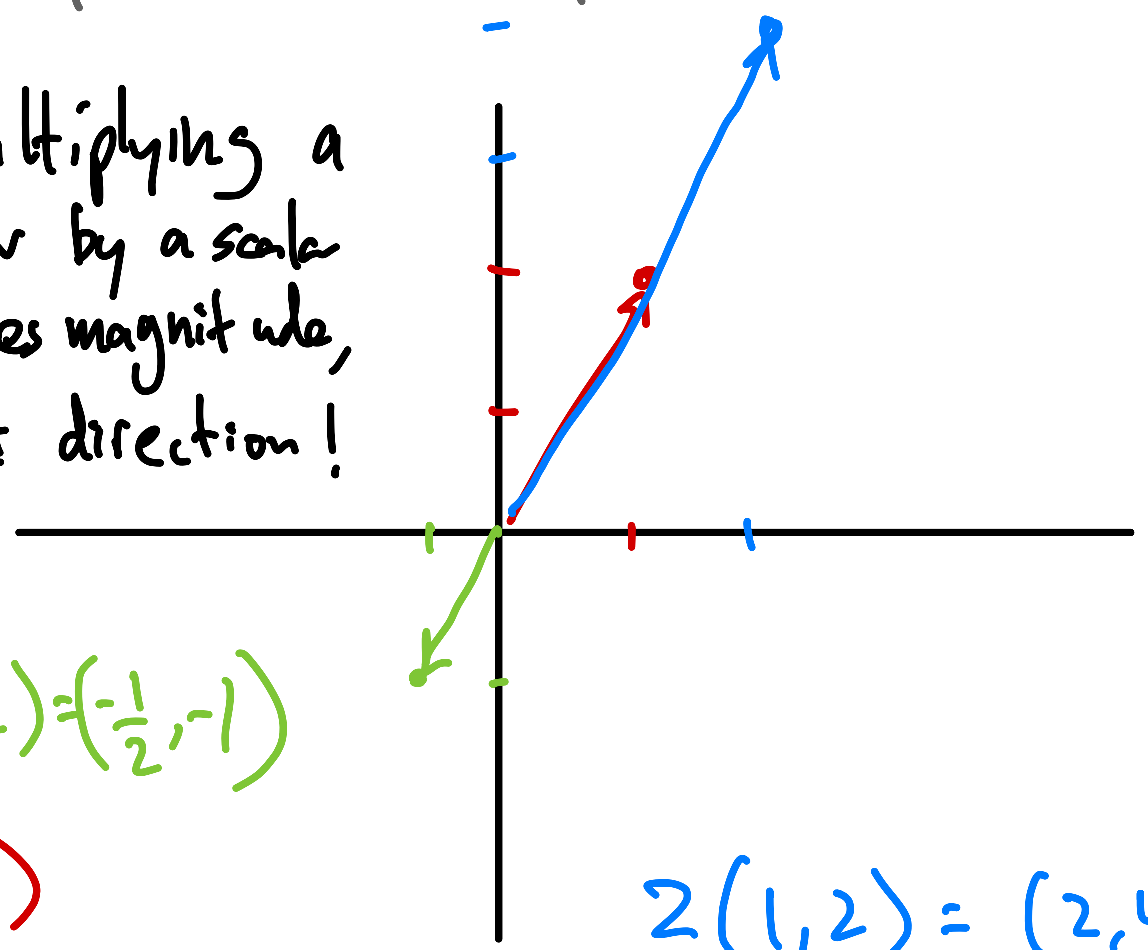
1. $$2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$

2. $$\pi\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}2\pi & 0\\0 & \pi\end{pmatrix}$$

3. $$(-1)[2, 1, 0] = [-2, -1, 0]$$

4. $$\begin{pmatrix}2\\4\\6\end{pmatrix} = 2\begin{pmatrix}1\\2\\3\end{pmatrix} = -2\begin{pmatrix}-1\\-2\\-3\end{pmatrix}$$

5. multiplying a vector by a scalar changes magnitude, but <u>not</u> direction!



$-\frac{1}{2}(1,2) = (-\frac{1}{2},-1)$

$(1,2)$

$2(1,2) = (2,4)$

Matrix and Vector subtraction

Because we can

1. multiply a matrix or vector by -1, and
2. add it to another matrix or vector,

this means that we we can do subtraction too*!

① $\underset{2\times 1}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} - \underset{2\times 1}{\begin{pmatrix} 2 \\ 0 \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} 2-2 \\ 1-0 \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

② $\underset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} - \underset{2\times 2}{\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} 1-4 & 2-3 \\ 3-2 & 4-1 \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} -3 & -1 \\ 1 & 3 \end{pmatrix}}$

*Reason: $b - a = b + (-1)a$

Vector-vector multiplication

Rule: we can multiply a **row vector** by a **column vector** provided that they have the same number of elements.

Formula: Step **across the row vector** and **down the column vector**, multiplying each pair of elements. Then **add the products**.

technically a 1×1 matrix. we will treat the answer like a scalar in this class.

① $\underset{1\times2}{(2, 4)} \underset{2\times1}{\begin{pmatrix} 3 \\ 1 \end{pmatrix}} = 2\cdot3 + 4\cdot1 = 6 + 4 = 10$

② $\underset{1\times2}{(1, 0)} \underset{2\times1}{\begin{pmatrix} 5 \\ 9 \end{pmatrix}} = 1\cdot5 + 0\cdot9 = 5+0 = 5$

③ $\underset{1\times3}{(2, 3, 1)} \underset{3\times1}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = 2\cdot x + 3y + 1\cdot z = 2x+3y+z$

④ $(a, b)\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = an_1 + bn_2$

NB: This kind of vector-vector multiplication **produces a scalar**.

Vector-vector multiplication

"dot product"

This kind of vector-vector multiplication **produces a scalar**.

Question: how can we **multiply two column vectors?**

A: u can't

A: flip the first one.

$$\begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rightarrow \overset{1\times 3}{(3, 1, 7)} \overset{3\times 1}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} = 3\cdot 1 + 1\cdot 0 + 7\cdot 1 = 10$$

$$(1, 0, 1) \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} = 1\cdot 3 + 0\cdot 1 + 1\cdot 7 = 10$$

The Transpose

To take the **transpose** of a **row-vector**, flip it & write it as a column vector.
To take the **transpose** of a **column-vector**, flip it & write it as a row vector.

$$\underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}^T}_{2\times} = \underbrace{(2, 1)}_{1\times 2} \qquad \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}^{T^T}}_{2\times 1} = \underbrace{(a, b)^T}_{1\times 2} = \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{2\times 1}$$

$$\underbrace{(5, 1, 19, \gamma)^T}_{1\times 4} = \underbrace{\begin{bmatrix} 5 \\ 1 \\ 19 \\ \gamma \end{bmatrix}}_{4\times 1}$$

Question: what happens to the *dimensions* of a vector when we take its transpose?

$(\text{rows}, \text{columns})^T \mapsto (\text{"columns"}, \text{"rows"})$ swap dimensions!

The Transpose - Part 2

$$\mathrm{diag}(A) = \mathrm{diag}(A^T)$$

To take the **transpose** of a matrix, think of its columns as column vectors, and then write them as row vectors. The first column becomes the first row.

① $\begin{bmatrix} 1 & 0 \\ x & y \\ \pi & e \end{bmatrix}^T = \begin{bmatrix} 1 & x & \pi \\ 0 & y & e \end{bmatrix}$

3×2 $\quad$ 2×3

② $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The diagonal of a matrix is a vector of elements from the matrix's diagonal.

Question: what happens to the *dimensions* of a matrix when we take its transpose?

swap # rows # cols

Matrix-vector multiplication

cols in matrix = # rows in vector.

rows in matrix → # rows in answer

Suppose we have a **2x2 matrix** and a **2x1 vector**.
We can define matrix-vector multiplication as follows:

1. Multiply the 1st row of the matrix by the vector.
2. Multiply the 2nd row of the matrix by the vector.
3. Stack the answers in a new vector.

Example:

③ $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} an_1 + bn_2 \\ cn_1 + dn_2 \end{pmatrix}$$

① $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 1\cdot 5 + 2\cdot 10 \\ 3\cdot 5 + 4\cdot 10 \end{pmatrix} = \begin{pmatrix} 25 \\ 55 \end{pmatrix}$$

② $$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 1\cdot 5 + 0\cdot 5 \\ 2\cdot 5 + (-1)\cdot 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

Matrix-vector multiplication

Suppose we have a **3x3 matrix** and a **3x1 vector**.

1. Multiply the 1st row of the matrix by the vector.
2. Multiply the 2nd row of the matrix by the vector.
3. Multiply the 3rd row of the matrix by the vector.
4. Stack the answers in a new vector.

Example:

$$\begin{pmatrix}2 & 1 & 4\\ 0 & 1 & 3\\ 1 & 5 & 7\end{pmatrix}\begin{pmatrix}1\\ 0\\ 2\end{pmatrix} = \begin{pmatrix}2\cdot 1 + 1\cdot 0 + 4\cdot 2\\ 0\cdot 1 + 1\cdot 0 + 3\cdot 2\\ 1\cdot 1 + 5\cdot 0 + 7\cdot 2\end{pmatrix} = \begin{pmatrix}10\\ 6\\ 15\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 2\end{pmatrix}$$

Matrix-vector multiplication

Suppose we have a **NxN matrix** M and a **Nx1 vector** $\overrightarrow{x}$.

Let's write a formula for the ith element of the resulting vector, $\overrightarrow{v} = M\overrightarrow{x}$

comes from. i^{th} row of matrix

$$\boxed{v_i = \sum_j M_{ij} \cdot x_j} = (M_{row\ i}) \cdot \vec{x} = M_{i1}x_1 + M_{i2}x_2 + \dots + M_{iN}x_N$$

subscript tells me which element I'm looking at

columns in M = # of rows in x

Rule: To multiply a matrix and a vector, *what must be true of their dimensions?*

Models with more than one dynamic variable - Part 2

We are going to rewrite this in a miraculous way

$$\frac{dn_1}{dt} = an_1 + bn_2$$

$$\frac{dn_2}{dt} = cn_1 + dn_2$$

$$\underset{\text{vector}}{\begin{pmatrix} \frac{dn_1}{dt} \\ \frac{dn_2}{dt} \end{pmatrix}} = \underset{\text{vector}}{\begin{pmatrix} an_1 + bn_2 \\ cn_1 + dn_2 \end{pmatrix}} = \underset{\text{matrix}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \times \underset{\text{vector.}}{\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}}$$

$$\frac{d\vec{n}}{dt} = M\vec{n}, \text{ where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$