Calculating Biological Quantities

CSCI 2897

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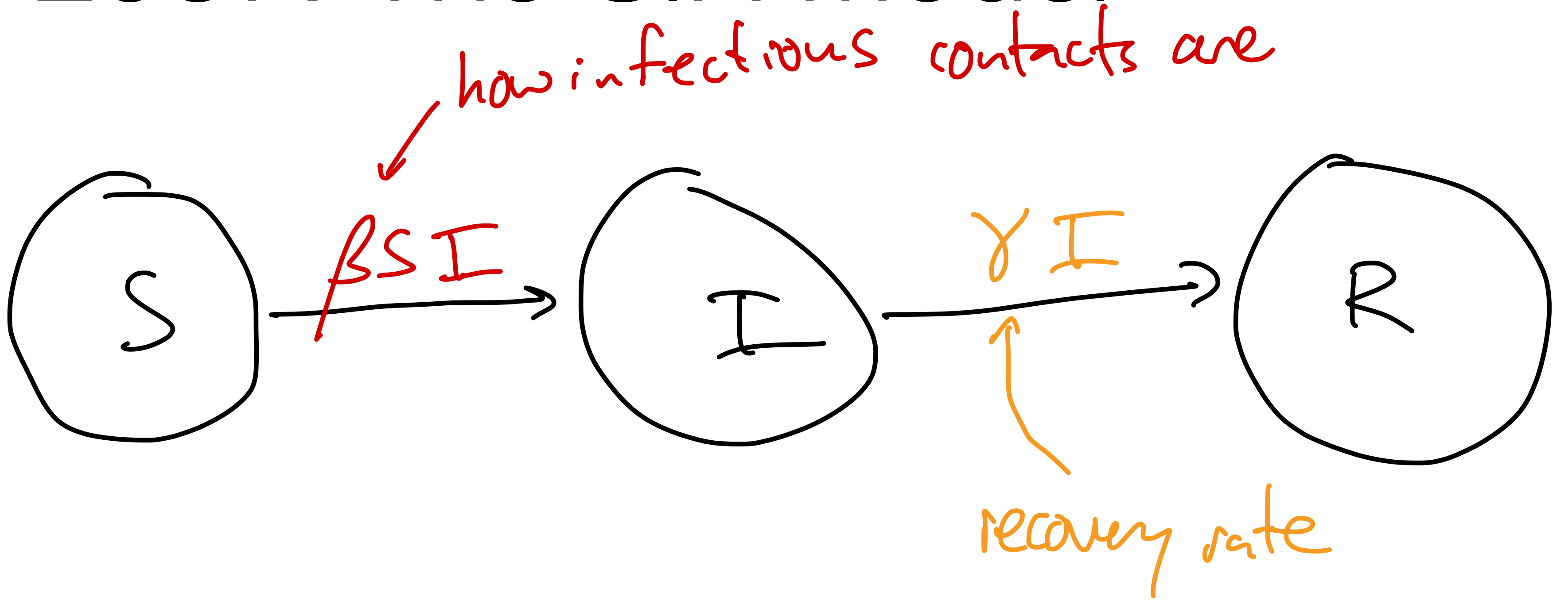
Last time on CSCI 2987: The SIR Model



$$\dot{S} = -\beta S I$$

$$\dot{I} = \beta S I - \gamma I$$

$$\dot{R} = \gamma I$$

+

$= 0$

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$$

$S + I + R = N$

$\Rightarrow \frac{dN}{dt} = 0$

conservation of people!

Let's hop into Jupyter Notebooks to explore a bit.

Rescaling the SIR model: people → population proportion

previously: $S + I + R = N$

now: $\frac{S}{N} + \frac{I}{N} + \frac{R}{N} = 1$

$$\frac{dx}{dt} = \frac{d}{dt}(x) = \frac{d}{dt}\left(\frac{X}{N}\right) = \frac{1}{N}\frac{dX}{dt}$$

$s = \frac{S}{N} \quad i = \frac{I}{N} \quad r = \frac{R}{N}$

$\dot{s} = \frac{\dot{S}}{N} \quad \dot{i} = \frac{\dot{I}}{N} \quad r = \frac{\dot{R}}{N}$

These rescaled equations represent rates of change of the popn fractions in S, I, R.

$\bar{\beta} = \beta N$

$$\dot{s} = \frac{1}{N}(-\beta S I) = -\beta \frac{S}{N}\cdot\frac{I}{N} N = -\beta s i \cdot N = -\bar{\beta} s i$$

$$\dot{i} = \frac{1}{N}(\beta S I - \gamma I) = \bar{\beta} s i - \gamma i$$

$$\dot{r} = \frac{1}{N}(\gamma I) = \gamma \frac{I}{N} = \gamma i$$

$\dot{r} = \gamma i$

same?

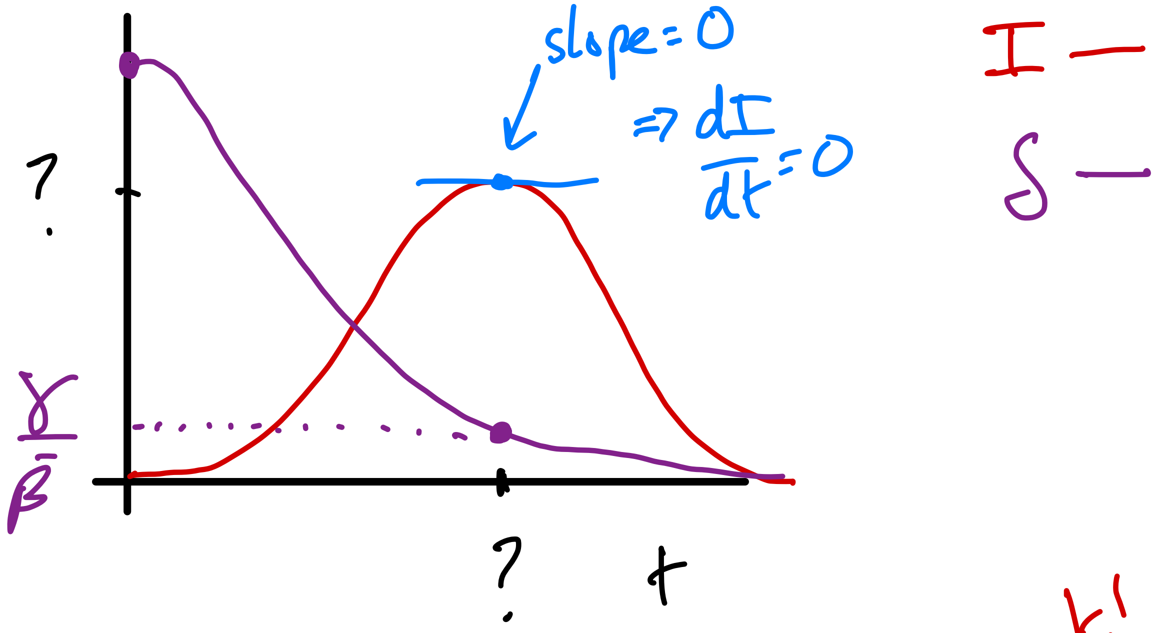
$$\dot{S} = -\bar{\beta} S I$$

$$\dot{I} = \bar{\beta} S I - \gamma I$$

$$\dot{R} = \gamma I$$

Analysis: when does the epidemic peak?

$$\boxed{S^* = \frac{1}{R_0}}$$



$$\frac{dI}{dt} = \underbrace{I}_{I \geq 0} (\bar\beta S - \gamma)$$

could be +, when $S < S^*$
or − when $S > S^*$
(= 0 when $S = S^*$)

Sign of $\dot I$ depends on the sign of $\bar\beta S - \gamma$!

$$S > \frac{\gamma}{\bar\beta} \Rightarrow \text{growth of } I$$

$$S < \frac{\gamma}{\bar\beta} \Rightarrow \text{decline of } I.$$

$$\frac{dI}{dt} = \bar\beta S I - \gamma I \overset{\text{peak!}}{=} 0$$

$$\bar\beta S I = \gamma I$$

$$\bar\beta S = \gamma$$

$$\boxed{S^* = \frac{\gamma}{\bar\beta}}$$

Analysis: how might we model a simple vaccine?

$\boxed{V > 1 - \frac{1}{R_0}}$

- Suppose we vaccinate a fraction V of the population.

$$S \xrightarrow{\beta S I} I \xrightarrow{\gamma I} R \qquad V \qquad S+I+R+V=1$$

- How many people do we need to put in V so that $\underbrace{\dot{I} < 0}_{\text{no epidemic!}}$?

from last slide:	$S > \frac{\gamma}{\beta} \Rightarrow$ growth of I
	$S < \frac{\gamma}{\beta} \Rightarrow$ decline of I. (*)

$$S = 1 - I - R - V < \frac{\gamma}{\beta}$$

total pop. equation

(*)

$$1 - V < \frac{\gamma}{\beta}$$

$$-V < \frac{\gamma}{\beta} - 1$$

no epidemic IF vax fraction exceeds $1 - \frac{\gamma}{\beta}$

$$\boxed{V > 1 - \frac{\gamma}{\beta}}$$

"herd immunity"

$V > 87\%$ (students)

$V > 90\%$ (staff, fac.)

Analysis: how does vaccination create "herd immunity"?

$v > 1 - \frac{\gamma}{\bar{\beta}}$ means $\dot{I} < 0$ which means epidemic dies out.

vaccination does this by decreasing the proportion of the pop'n that is susceptible!

Analysis: the basic reproductive number

R_0 : # of new infections caused by each existing infection (over the entire duration of the first infection) in an otherwise completely susceptible population.

"R naught"

this R is not Recovered. It's R as in "Reproduction."

$\dot{S} = -\beta S I$

$\dot{I} = \beta S I - \gamma I$

new infections (per time) $= \beta S I$

new infections per time per infected person $= \frac{\beta S I}{I} = \beta S$

How many total new infections are created over the course of a single infection?

Duration of a typical infection: $\frac{1}{\gamma}$

$\Rightarrow R_0 = \frac{\beta S}{\gamma}\Big|_{S=1} = \frac{\beta}{\gamma}$

$$\boxed{R_0 = \frac{\beta}{\gamma}}$$

Linearization and Stability

The big question with an infectious agent is: **will we get an epidemic?**

Recall: $I=0$ is an equilibrium.

Imagine: $I = 0 +$ "tiny tiny nudge"

Does an epidemic take off?

yes → $I=0$ was an unstable equilibrium

no → $I=0$ was a stable equilibrium

$$\dot{I} = \beta S I - \gamma I$$

$I =$ very small $= \varepsilon$ $\qquad \dot{I} = \dot{\varepsilon}$

$S = 1 - I = 1 - \varepsilon$

$$\dot{\varepsilon} = \beta(1-\varepsilon)\varepsilon - \gamma\varepsilon$$

$$\dot{\varepsilon} = \beta\varepsilon - \cancel{\beta\varepsilon^2} - \gamma\varepsilon$$

ignore ε^2 terms. too small!

$$\dot{\varepsilon} = \beta\varepsilon - \gamma\varepsilon$$

$$\dot{\varepsilon} = (\beta - \gamma)\varepsilon \qquad \dot{x} = cx$$

$$\varepsilon(t) = k e^{(\beta-\gamma)t}$$

Linearization and Stability

The big question with an infectious agent is: **will we get an epidemic?**

$$\varepsilon(t) = k\, e^{(\beta-\gamma)t}$$

When does $\varepsilon(t)$ grow?

$\beta - \gamma > 0$ or $\beta > \gamma$ or $\frac{\beta}{\gamma} > 1$ $\xrightarrow{e}$ $R_0 > 1$

(unstable)

When does it shrink?

$\beta - \gamma < 0$ or $\beta < \gamma$ or $\frac{\beta}{\gamma} < 1$ $\xrightarrow{e}$ $R_0 < 1$

(stable)

Summary: The SIR Model

$$\dot{S} = -\beta SI$$

$$\dot{I} = \beta SI - \gamma I$$

$$\dot{R} = \gamma I$$

where $S + I + R = 1$

$$\implies \dot{S} + \dot{I} + \dot{R} = 0$$

Equilibrium when:

$$I = 0$$

Epidemic peak:

$$S^* = \frac{\gamma}{\beta}$$

Herd Immunity (vaccination)

$$v > 1 - \frac{\gamma}{\beta}$$

Basic Reproduction Number

$$R_0 = \frac{\beta}{\gamma}$$