# Calculating Biological Quantities

## CSCI 2897

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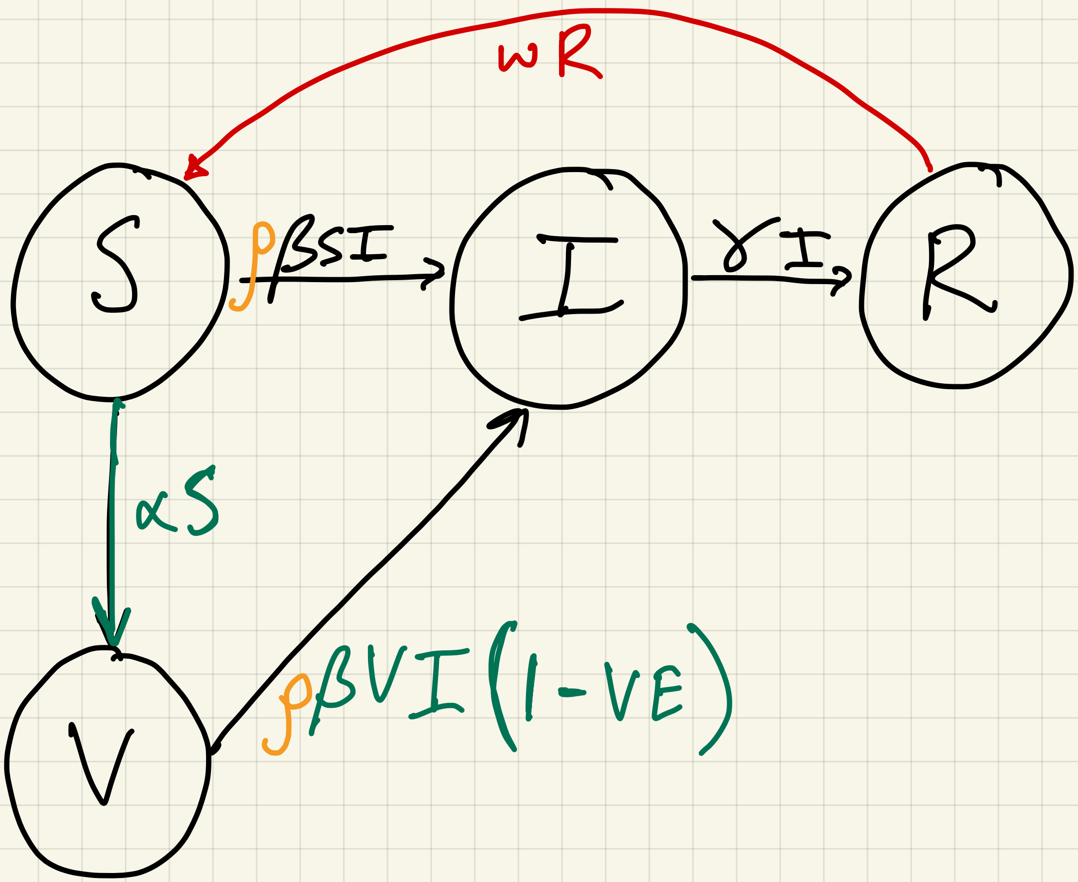
**2021, Lecture 20**

# Why the surge in flu on campuses this year?




① $\rho$ very small for a year.

$$\dot{R} = \gamma I - \omega R$$

$$\dot{R} = -\omega R \longrightarrow R(t) = R_0 e^{-\omega t}$$

$\Rightarrow$ more $S$

② VE is lower this year.

$VE \in [0.25, 0.75]$

$\Rightarrow$ Flow from $V \to I$ looks like flow from $S \to I$

③ High $\rho$ $\longrightarrow$ high $S \to I$, $V \to I$ rates.

$\rho$ - contact parameter  $\rho = 1$ "normal"

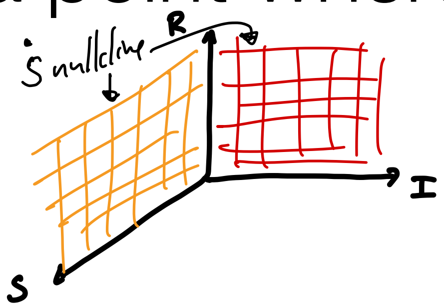
$\rho = \frac{1}{2}$ soc. dist.

VE - vaccine effectiveness

# Last time on CSCI 2897

A **nullcline** is a curve (or surface) in phase space on which one of the variables' rate of change is zero, $\dot{n}_i = 0$. An **equilibrium** is therefore a point where all the nullclines intersect.

e.g. $\dot{S} = -\beta S I \Rightarrow S=0, I=0$




**Linear** model $\dfrac{d\vec{n}}{dt} = M\vec{n}$

$$\hat{\vec{n}} = 0$$

**Affine** model $\dfrac{d\vec{n}}{dt} = M\vec{n} + \vec{c}$

$$\hat{\vec{n}} = -M^{-1}\vec{c}$$

**Rules:**

- A linear or affine model in continuous time has only one equilibrium regardless of the number of variables, provided that the determinant of M is not zero.
- If $\det(M) = 0$, there are an *infinite* number of equilibria.

**Stability of equilibria** (real eigenvalues):

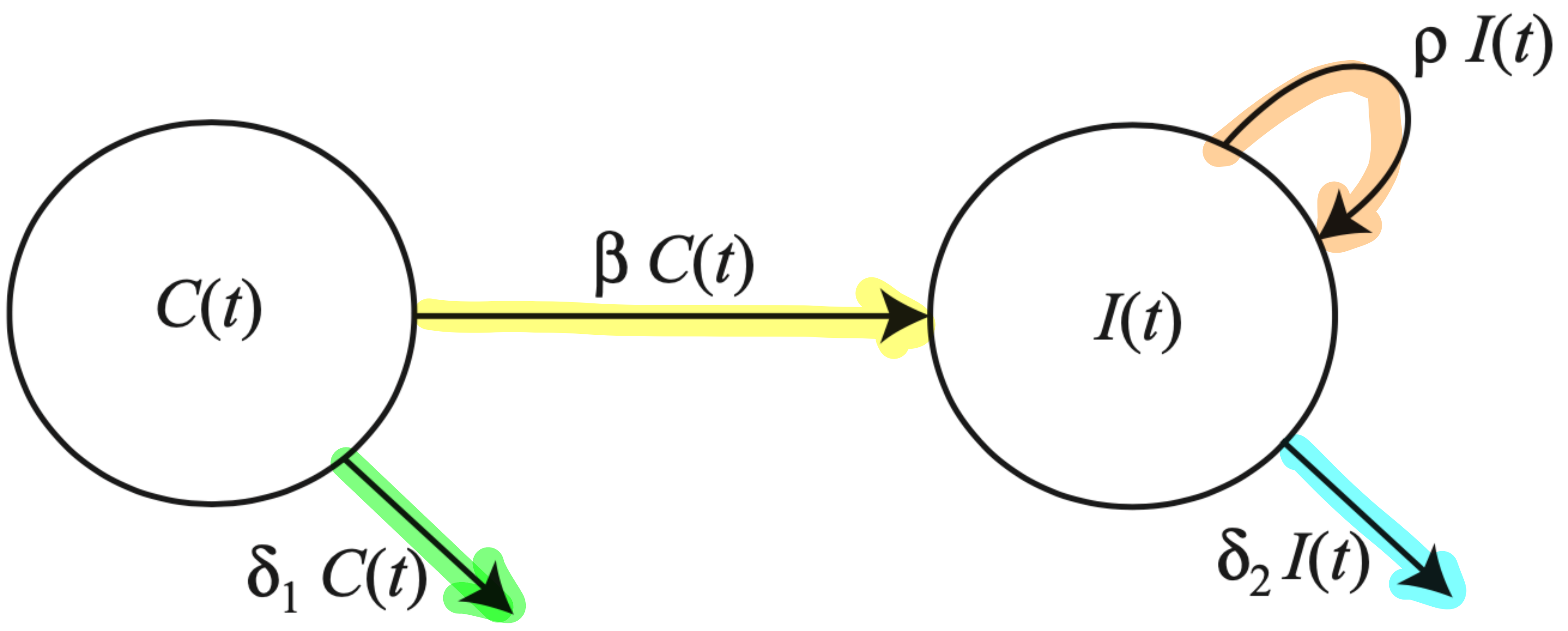
- If all eigenvalues are negative, the system is stable.
- If one or more eigenvalues are positive, the system is unstable.

Otto & Day — Adapted from Glass & Kaplan 1995

# Metastasis of Malignant Tumors

A model for the dynamics of the number of cancer cells lodged in the capillaries of an organ, $C$, and the number of cancer cells that have actually invaded that organ, $I$.

Suppose that cells are lost from the capillaries by dislodgement or death at a per capita rate $\delta_1$ and that they invade the organ from the capillaries at a per capita rate $\beta$. Once cells are in the organ they die at a per capita rate $\delta_2$, and the cancer cells replicate at a per capita rate $\rho$.




$$\frac{dC}{dt} = -\delta_1 C - \beta C + 0I$$

$$\frac{dI}{dt} = \beta C + \rho I - \delta I$$

# Metastasis of Malignant Tumors

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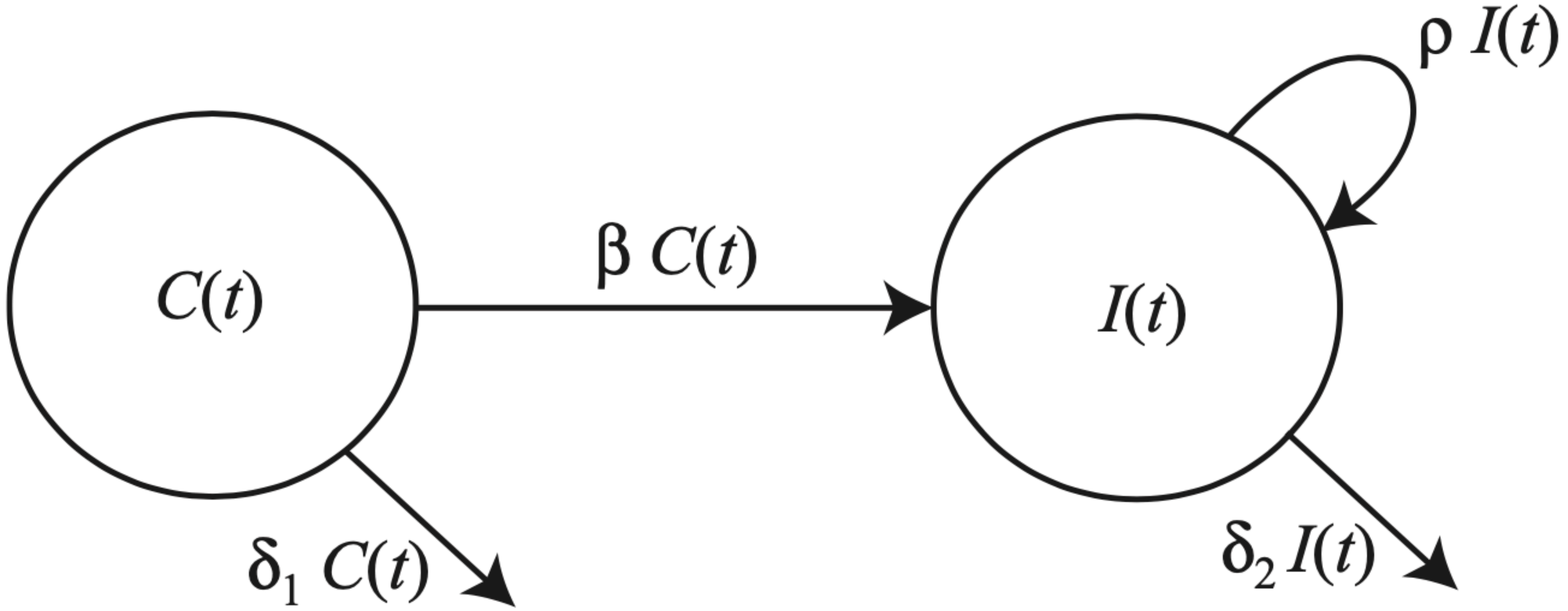



$$\frac{dC}{dt} = \delta_1 C - \beta C$$

$$\frac{dI}{dt} = \beta C - \delta_2 I + \rho I$$

$$\begin{pmatrix} \frac{dC}{dt} \\ \frac{dI}{dt} \end{pmatrix} = M \begin{pmatrix} C \\ I \end{pmatrix}$$

$$M = \begin{pmatrix} -(\delta_1 + \beta) & 0 \\ \beta & \rho - \delta_2 \end{pmatrix}$$

Matrix is driving the dynamics.

# Metastasis of Malignant Tumors

$$\begin{pmatrix} \frac{dC}{dt} \\ \frac{dI}{dt} \end{pmatrix} = M \begin{pmatrix} C \\ I \end{pmatrix} \qquad M = \begin{pmatrix} -(\delta_1 + \beta) & 0 \\ \beta & \rho - \delta_2 \end{pmatrix}$$

Suppose that cells are lost from the capillaries by dislodgement or death at a per capita rate $\delta_1$ and that they invade the organ from the capillaries at a per capita rate $\beta$. Once cells are in the organ they die at a per capita rate $\delta_2$, and the cancer cells replicate at a per capita rate $\rho$.

1. Identify the equilibrium or equilibria.
2. Determine the stability.

① If $\det(M) \neq 0 \Rightarrow$ equilibrium $\begin{pmatrix} C \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Check $\det(M) = -(\delta_1 + \beta)(\rho - \delta_2) - 0 \cdot \beta$

$$-\underbrace{(\delta_1 + \beta)}_{\text{cannot be zero, because } \delta_1 \text{ and } \beta > 0}\underbrace{(\rho - \delta_2)}_{\text{can be 0 if } \rho = \delta_2}$$

Can this be zero?

Conclusion: If $\rho \neq \delta_2$, then the equilibrium occurs at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

② Stability. $\lambda_1, \lambda_2 = \dfrac{tr(M) \pm \sqrt{tr^2(M) - 4\det}}{2}$

$\rightarrow \lambda_1 = -(\delta_1 + \beta) \leftarrow$ always neg!

$\lambda_2 = \rho - \delta_2 \leftarrow$ sometimes negative.

Stability of the no-cancer equilibrium depends on $\delta_2$ and $\rho$

if $\rho > \delta_2 \rightarrow$ unstable (growth > death)

$\rho < \delta_2 \rightarrow$ stable (death > growth)

$\rho = \delta_2 \rightarrow$ infinite equilibria (see ①)

# Definitions & Tricks: Diagonal and Triangular Matrices

Eigenvalues are *especially* easy to find when a matrix is diagonal or triangular.

Definition: a **diagonal matrix**

$$\begin{pmatrix} M_{11} & & & 0 \\ & M_{22} & & \\ & & \ddots & \\ 0 & & & M_{nn} \end{pmatrix}$$

A matrix $M$ is diagonal if and only if $M_{ij} = 0$ for all $i \neq j$.

Definition: a **triangular matrix**

$M_{ij} = 0$ for all $i > j$

$M_{ij} = 0$ for all $i < j$

$$\begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ & M_{22} & \cdots & M_{2n} \\ & 0 & \ddots & \vdots \\ & & & M_{nn} \end{pmatrix}$$

Upper Triangular Matrix

$$\begin{pmatrix} M_{11} & & & \\ M_{21} & M_{22} & 0 & \\ \vdots & & \ddots & \\ M_{n1} & \cdots & \cdots & M_{nn} \end{pmatrix}$$

Lower Triangular Matrix

4 nice facts.

1. if $D$ is diagonal, $D = D^T$
2. if $U$ is upper triangular $\Rightarrow U^T$ is lower triangular
3. if $L$ is lower triangular, $\Rightarrow L^T$ is upper triangular.
4. if $M$ is both lower triangular and upper triangular, then... $M$ is diagonal.

# Definitions & Tricks: Diagonal and Triangular Matrices

Eigenvalues are *especially* easy to find when a matrix is diagonal or triangular.

~~Definition: a **diagonal matrix**~~ If $M$ is

- diagonal

  or

- upper triangular

  or

- lower triangular

~~Definition: a **triangular matrix**~~

then, its eigenvalues are the entries on the diagonal of $M$.

Ex:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 4 & \pi \end{pmatrix} \rightarrow \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = \pi \end{array}$$

Ex:

$$\begin{pmatrix} 19 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & -2000 \end{pmatrix}$$

my eigenvalues

## Definition:

- A <u>symmetric</u> matrix $M$ has entries such that $M_{ij} = M_{ji}$ for all $i, j$.
- Symmetric matrices <u>must</u> be square.
- $$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & & \\ M_{31} & & \end{pmatrix} \qquad M_{21} = M_{12}$$

  symmetry
- $$M = M^T$$

# Complex Eigenvalues

Unreal! Sometimes we can have eigenvalues which are *complex numbers*.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightarrow \quad \lambda_1, \lambda_2 = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}^2(A) - 4\mathrm{det}(A)}}{2}$$

diff. a, b, c than our matrix.

If $\mathrm{tr}^2(A) - 4\mathrm{det}(A) < 0$, then $\lambda_1, \lambda_2$ will be **complex numbers**.

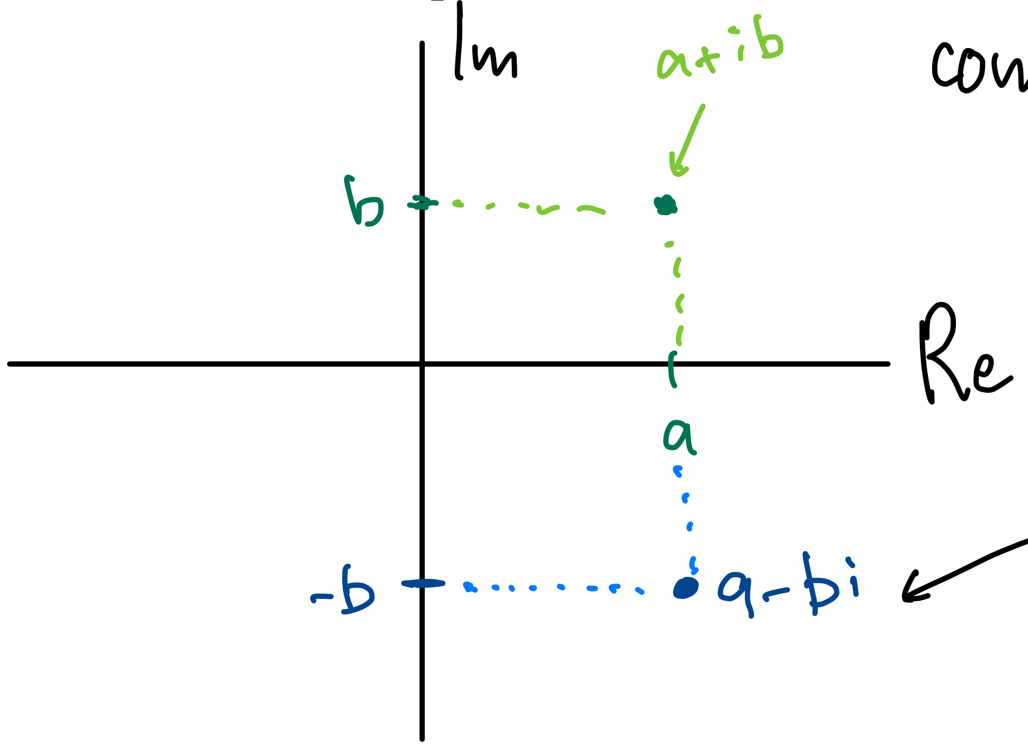
A **complex number** is a number $c = a + bi$, where $a$ and $b$ are real and $i = \sqrt{-1}$ is "imaginary."

In our formula above, what's the **real part**? And the **imaginary part**?

$c = a + bi$

real part — Re

imaginary part — Im




complex plane

complex conjugate: $a - bi$

# Complex Eigenvalues

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If $\text{tr}^2(A) - 4\text{det}(A) < 0$, then $\lambda_1, \lambda_2$ will be **complex numbers**.

A complex number is a number $c = a + bi$, where $a$ and $b$ are real and $i = \sqrt{-1}$ is "imaginary."

$$a = \frac{\text{tr}(A)}{2}, \quad \text{and} \quad b = \frac{\sqrt{-\text{tr}^2(A) + 4\text{det}(A)}}{2}$$

and therefore $\lambda_1 = a + bi, \quad \lambda_2 = a - bi$

① if $\lambda_1$ complex $\Rightarrow$ $\lambda_2$ complex

② $\lambda_1$ and $\lambda_2$ are complex conjugates if complex

Fundamental Thm of Algebra.

**Notice: either both eigenvalues are complex, or both are real.**

# Euler's Equation

$$e^{\lambda t} = e^{(a+bi)t}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We will not derive this miraculous equation, but come to office hours if you are excited or puzzled by this!

$e^x$

exponential

$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

exponential (Taylor's version)

For extra magic, set $\theta = \pi \ldots$

$$e^{i\pi} = \underbrace{\cos\pi}_{-1} + i\underbrace{\sin\pi}_{0}$$

$$\boxed{e^{i\pi} + 1 = 0}$$

5 fundamental constants.