Midterm Exam

CSCI 2897 - Calculating Biological Quantities - Larremore - Fall 2021

Instructions

- This exam needs to be taken live, while you're on Zoom for class *or* sitting in the classroom.
- This exam is open note and open textbook.
- Collaboration with others in the class *or* posting this exam to any online service during the exam will be considered cheating. Complete this exam entirely on your own.
- If you have questions, please DM me on Zoom or raise your hand..
- Your completed exam should be submitted at the end of class via Canvas or handed in on paper.
- The Canvas submission window will close at the end of class +10 minutes, i.e. 1:05 P.M.
- Please show your work for the math. One way to do this would be to write on a tablet, for Zoomers. Another way would be to write on paper and then snap a photo. You can drag the photo into MS Word, for instance.

Name (Printed) : Dan Larremore

Honor Code: "On my honor, as a University of Colorado Boulder student I have neither given nor received unauthorized assistance."

By signing my name here, I commit to the Honor Code above:

Efrain's of Boulder [50]

Efrain's of Boulder is undoubtedly the best restaurant in Boulder. Why? Because the food is delicious, the prices are low, the margaritas are strong, and the green chile is *hot*. We like Efrain's.

You are, for the purposes of this exam problem, a chile farmer from Pueblo, Colorado. It is your sacred duty to provide green chiles to the team at Efrain's, and you take your job very seriously. In addition to being a chile farmer, you have taken CSCI 2897, and you set about modeling the growth of your chile crop.

In your first season farming, you conduct numerous experiments and find that, without any harvesting, the growth of your chile crop's mass $\dot{m}(t)$, measured in kilograms per week, follows this differential equation:

$$\dot{m}(t) = \alpha m \left(1 - \frac{m}{\beta}\right)$$

1. (5 points) Name this differential equation and explain what the constants α and β mean.

   This is the differential equation for logistic growth. Here, α is the intrinsic growth rate and β is the carrying capacity. What do these mean? α is the per-capita rate of growth for very small population sizes, before the "logistic" effects take effect and slow the per-capita growth rate. β represents the largest population (here: mass) that the environment can support.

2. (5 points) Suppose that you were trying to decide between (a) expanding your farm, (b) buying fertilizer, (c) planting a faster-growing variety of chile, or (d) planting a variety of chile that uses fewer resources. Which of these potential changes might be modeled as a change to α and which might be modeled as a change to β? Explain your reasoning for each of the four options.

   - Expanding your farm would increase the carrying capacity β by providing more cropland for the chiles to grow and expand into.
   - Buying fertilizer could either increase β, because fertilizer allows more plants in the same area, therefore increasing the carrying capacity, or it could increase α by allowing plants to grow faster. Either answer is ok here, as long as it is justified.
   - Planting a faster-growing variety of chile increases α, the intrinsic growth rate.
   - Planting a variety of chile that uses fewer resources would increase β because it would allow more chiles to live in the previous among of space, thus increasing the carrying capacity.

3. (5 points) You measure $\alpha = 2$ and $\beta = 100$, and plant $m(0) = 1$ kg to start with. According to your model, what will be the steady-state amount of chiles on your farm? Show your work.

   The term "steady state" means the same as "equilibrium" so we can solve for the equilibria.

   $$\dot{m} = 0 \quad \rightarrow \quad 2m\left(1 - \frac{m}{100}\right) = 0$$

   has two solutions: $m = 0$ and $m = 100$. One of these will be the answer. We are given one more piece of information, however: $m(0) = 1$. Because $m = 0$ is unstable and $m = 100$ is stable, the initial condition tells us that we will go to the stable equilibrium. Therefore, the answer is $m = 100$.

Your second season as a farmer has arrived, and with it, great opportunities to use your CSCI 2897 skills for good. With your model for crop growth established, it's time to make sure Efrain's has the chiles they need to keep being Boulder's #1 restaurant. That means it's time to include *harvesting* in your model.

4. (5 points) Write down a new ODE (based on the old one) that includes harvesting chiles at a constant rate of 42 kg per week.

$$\dot{m} = 2m\left(1 - \frac{m}{100}\right) - 42$$

5. (5 points) Choose three of the following options to describe your new ODE with harvesting.
   A. first order
   B. second order
   C. linear
   D. nonlinear
   E. separable
   F. not separable

6. (10 points) What are the two equilibria for your ODE with harvesting? How can there be two equilibria like this, biologically speaking?

   To find the equilibria, set $\dot{m} = 0$:

$$\begin{aligned} 2m\left(1 - \frac{m}{100}\right) - 42 &= 0 \\ 2m - \frac{2m^2}{100} - 42 &= 0 \\ \frac{2m^2}{100} - 2m + 42 &= 0 \\ \frac{m^2}{50} - 2m + 42 &= 0 \\ m^2 - 100m + 2100 &= 0 \\ (m-70)(m-30) &= 0 \\ \rightarrow m = 70,\ m &= 30 \end{aligned}$$

   Note that you could also use the quadratic formula at any point, as well, if you didn't want to factor.

   There can be two equilibria like this because we know that the growth rates of a logistic model change over time, with low overall growth when the population is small, or when the population is near its carrying capacity. In the middle, when $m = \beta/2$ (half the carrying capacity), the growth rate without harvesting is the largest. There are just two points of balance, where growth and harvesting are equal: one when the total mass of chiles is smaller, and one where it's larger.

7. (5 points) Critique this model. List at least two reasons why it might be a poor description of an actual agricultural crop. Give an example of a different biological system that might be better modeled using an equation like the one you wrote down in problem 4.

There are many reasonable critiques of this model. One is that chiles don't grow, over the course of a season, via a replication process where the number of new chiles depends on the number of existing chiles—that's how yeast or bacteria grow, and how other populations grow over a longer timescale, but it's weird to use such a model for a crop in a season. Another set of critiques all focus on things that are missing: we haven't accounted for anything related to the season, including weather, pests, drought, watering, temperature, and various other factors.

8. (10 points) Using the sign of $\dot{m}$ for various choices of m, explain whether each of your equilibria from problem 6 is stable or unstable. What does this tell you about when you should begin harvesting?

   We can plug any value of m into the Equation from problem 6, and learn whether the amount of chiles would be increasing or decreasing at that point. We know that the amount of chiles is not changing at all whenever $\dot{m} = 0$, which happens at $m = 30$ and $m = 70$. So, plugging in $m = 20$, $m = 50$, and $m = 90$ allows us to see what is happening *between* the equilibria.

   When $m < 30$, $\dot{m} < 0$. When $30 < m < 70$, $\dot{m} > 0$. When $m > 70$, $\dot{m} < 0$. Thus, $m = 30$ is unstable, and $m = 70$ is stable. As a consequence, we should not start harvesting chiles until the crop is at least size $m = 30$; starting before then would result in a total collapse, while starting after then would lead to a steady-state at $m = 70$.

Short Answers [20]

Answer the following in complete sentences.

9. (10 points) Consider the Lotka-Volterra model for competition between species 1 and 2 that we discussed in class.

$$\frac{dn_1}{dt} = n_1\left(1 - \frac{n_1 + \alpha_{12}n_2}{K_1}\right)$$
$$\frac{dn_2}{dt} = n_2\left(1 - \frac{n_2 + \alpha_{21}n_1}{K_2}\right)$$

Explain what α_{21} means. How would setting $\alpha_{21} = 10$ affect the population of species 2?

α_{21} represents the way that the population n_1 affects the growth rate of n_2. It's how population 2 "feels" population 1. When α_{21} is positive, this means that each n_1 makes population 2 feel like it is closer to its own carrying capacity. Thus, if $\alpha_{21} = 10$, then each n_1 makes population 2 feel like it's another 10 population closer to carrying capacity. This slows the growth rate of n_2 and leads to a lower population 2.

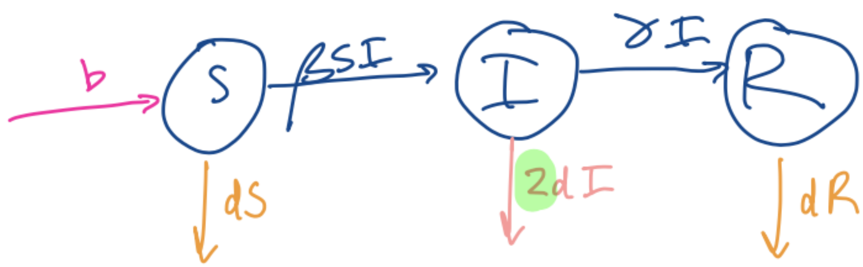
10. (10 points) Suppose that you are studying levels of lipids (fats) in blood samples from humans after they eat a meal. Would a discrete-time or continuous-time model be more appropriate, and why? What might be an appropriate timescale for your model?

Various answers are ok here, as long as they are justified. A continuous time model might reflect the fact that blood lipids are changing in continuous time, and not tied to any sort of "clock" that is ticking and updating lipid values. In other words, the process itself is fundamentally continuous, and therefore we should model it as such. On the other hand, a discrete time process might be fine, as well, and could even be easier to work with. This is because one cannot take a continuous blood sample, and instead, samples are taken at intervals. Because discrete time models update our variable values on intervals, it might be reasonable to match the model to discretely collected data, for convenience. Any justified answers are good.

The time scale should match the problem, meaning that timescales of seconds to minutes to hours are all reasonable. Days to weeks or longer are not.

Math and Modeling [30+]

11. (10 points) When modeling the spread of an infectious disease in class, we assumed a constant population. What if the population could change? Draw a flow diagram for the spread of an infectious disease in a population where people are born at a constant rate b, susceptible and recovered people die at a per-capita rate d, and infected people die at a per-capita rate $2d$. Assume that babies are born susceptible. Then use your diagram to write differential equations for $\dot{S}$, $\dot{I}$, and $\dot{R}$.



$$\dot{S} = -\beta SI + \mathbf{b} - \mathbf{dS}$$
$$\dot{I} = \beta SI - \gamma I - \mathbf{2dI}$$
$$\dot{R} = \gamma I - \mathbf{dR}$$

12. (10 points) Showing all your work, solve the following initial value problem using the integrating factor method.

$$3\frac{dy}{dx} = x^2y + x^2, \qquad y(0) = 2.$$

$$3\frac{dy}{dx} - x^2 y = x^2$$
$$\frac{dy}{dx} - \frac{x^2}{3} y = \frac{x^2}{3}$$
$$\frac{dy}{dx} + \left(-\frac{x^2}{3}\right) y = \frac{x^2}{3}$$
$$\mu(x) = e^{\int -\frac{x^2}{3} dx} = e^{-\frac{x^3}{9}}$$

$$\int \frac{d}{dx}\left[e^{-\frac{x^3}{9}} \cdot y\right] dx = \int \frac{x^2}{3} e^{-\frac{x^3}{9}} dx$$
$$u = -\frac{x^3}{9} \qquad du = -\frac{x^2}{3} dx$$
$$e^{-\frac{x^3}{9}} y = -\int e^u du$$
$$e^{-\frac{x^3}{9}} y = -e^{-x^3/9} + C$$
$$y = -1 + Ce^{x^3/9}$$
$$2 = -1 + C \cdot 1 \Rightarrow C = 3$$
$$\boxed{y = -1 + 3e^{x^3/9}}$$

13. (10 points) Explain what it means for a function to "solve" a differential equation.

A function solves a differential equation if, when the function and its derivatives (as needed) are substituted back into the differential equation, the left-hand-side = the right-hand side.

14. (Extra Credit) Show that your answer in Problem 12 solves the differential equation you set out to solve.

We want to show that $y = -1 + 3e^{\frac{x^3}{9}}$ solves $3\frac{dy}{dx} = x^2 y + x^2$. We'll need to substitute in for $\frac{dy}{dx}$, so we can take that derivative now.

$$
\begin{aligned}
y &= -1 + 3e^{\frac{x^3}{9}} \\
\frac{d}{dx}y &= \frac{d}{dx}\left[-1 + 3e^{\frac{x^3}{9}}\right] \\
\frac{dy}{dx} &= 3e^{\frac{x^3}{9}}\frac{x^2}{9}3 \\
\frac{dy}{dx} &= x^2 e^{\frac{x^3}{9}}
\end{aligned}
$$

We can plug this, and our equation for y, into the ODE:

$$
\begin{aligned}
3\frac{dy}{dx} &= x^2 y + x^2 \\
3x^2 e^{\frac{x^3}{9}} &= x^2\left(-1 + 3e^{\frac{x^3}{9}}\right) + x^2 \\
3x^2 e^{\frac{x^3}{9}} &= -x^2 + 3x^2 e^{\frac{x^3}{9}} + x^2 \\
3x^2 e^{\frac{x^3}{9}} &= 3x^2 e^{\frac{x^3}{9}}
\end{aligned}
\tag{1}
$$

and that checks out!