Calculating Biological Quantities

CSCI 2897

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2021, Lecture 2

Lecture 2 Plan

1. **One minute review of the basics:**
    1. Website
    2. Syllabus
    3. Canvas
    4. Slack
2. **Office Hours?**
3. **Asking “modeling” questions**
4. **Some vocabulary**
5. **Steps to modeling a biological problem (1-4)**

Last Time on CBQ…

Sean Taylor (FB Research, Lyft)

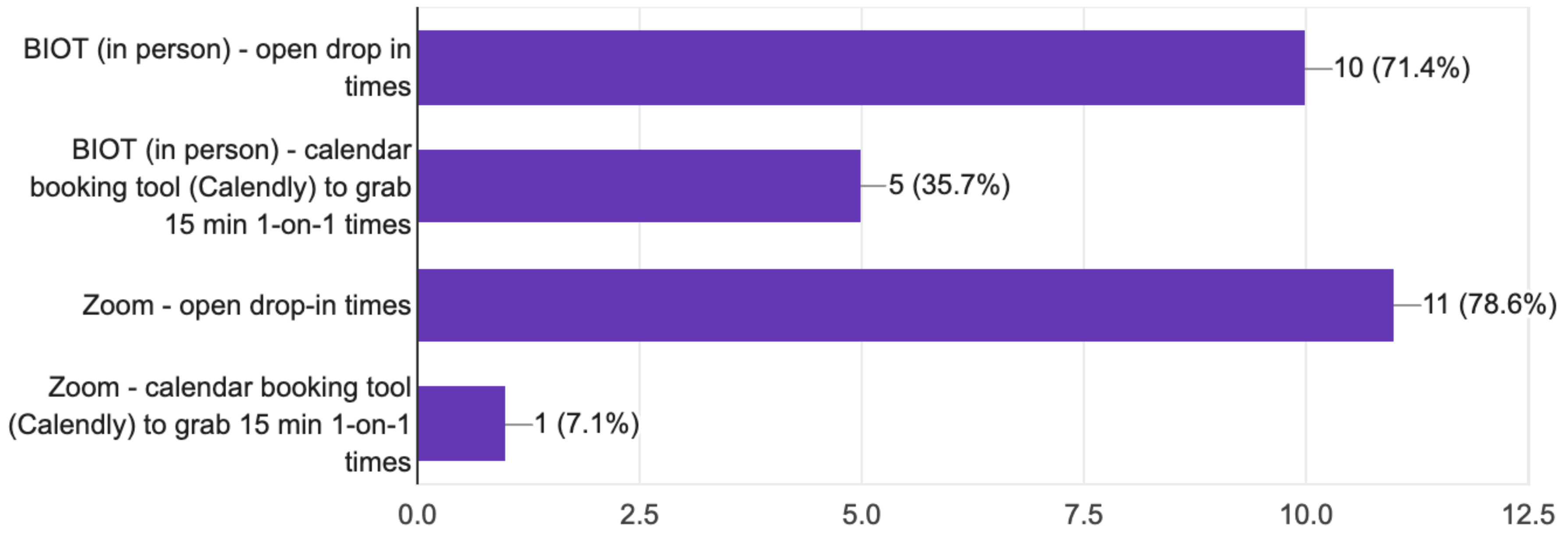
- Website: https://github.com/dblarremore/CSCI2897
    - Homework & reading posted, Code examples, Class notes
- Syllabus: https://github.com/dblarremore/CSCI2897#syllabus
- Canvas: Turn in homework, Lecture links, Check grades
- Slack: **Didn't get the invite? Stick around after class—we'll get you set up!**
- Textbook: See Slack.

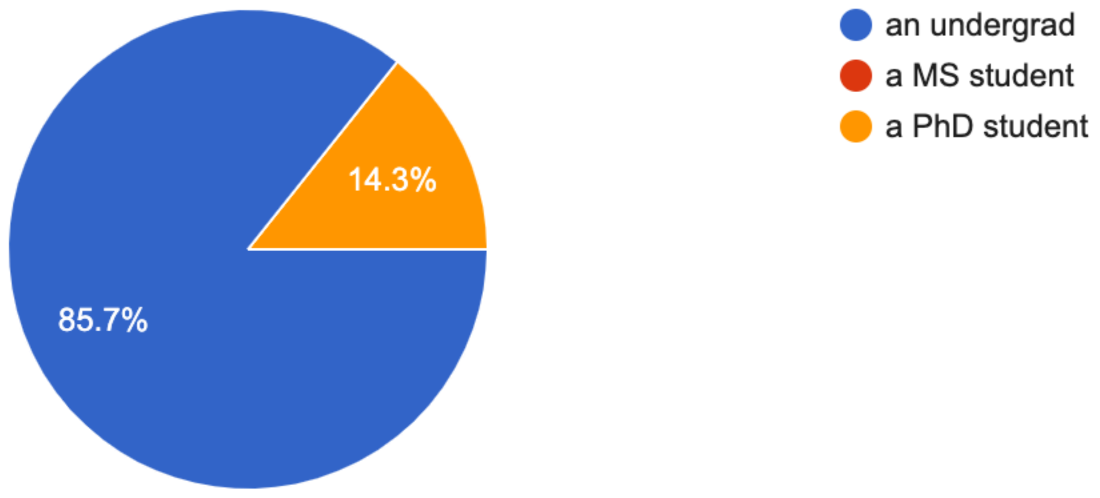
#resources

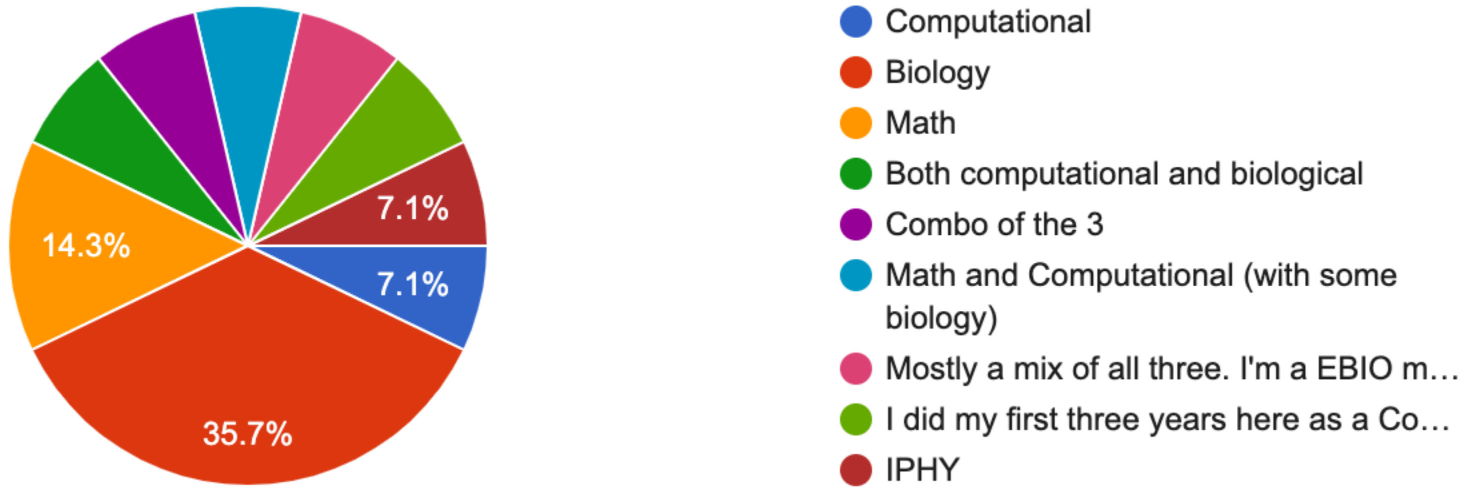
Office Hours?

What is your preference for office hours? (Check any/all)

14 responses







Dynamical Models 101: Ask a question

changes over time

- Think about a problem that puzzles you.
- Draw a "flow diagram" that illustrates the various processes at work.
- *Dynamical* models describe how a system changes over time.

① Aeropress (coffee)

add water

- If I fill it to top, some water drips through.
- Halfway? no drips

What determines the drip rate into the cup? (dynamic)

add water

↓

W — w water in cylinder

↓ drip.

$kf(w - \frac{1}{2})$

coarseness of coffee?

no drip until $w > \frac{1}{2}$?

② How messy is my room?

- accumulation of crap.
- clean up

③ How juicy is meat while being cooked?

- heat/evap
- additions?
- oil? temp?

Deterministic vs Stochastic dynamical models

this course

- **Deterministic** models assume that the future is entirely predicted (i.e. determined) by the model.

Q: How much water is in my coffee maker?

If no random variables → deterministic.

Model: flow out - flow in → deterministic.

- **Stochastic** models assume that random (stochastic) events affect the system.

Q: How much snow @ Eldora? (base height)

Model: Stochasticity in snowfall, temperature.

→ include a random variable — source of stochasticity.

Otto & Day: 7 steps to modeling a biological problem

1. Formulate the question

- what do you want to know?
- Describe that in the form of a specific question.
- Boil the question down → as clear and as well-specified as possible.
- Start with the simplest, biologically reasonable description of the problem.

story

ELI5
Explain it like I'm five.

Otto & Day: 7 steps to modeling a biological problem

1. Formulate the question
2. Determine the basic ingredients

Define: • variables

- constraints? $n \geq 0$ for example.
- interactions between variables.

Decide: time → discrete? (clear clock ticks)

time → continuous?

time scale: how much time between $t=0$, $t=1$?

Define: Parameters • constraints $0 \leq k \leq 1$ → fundamental

→ reasonable

Otto & Day: 7 steps to modeling a biological problem

1. Formulate the question
2. Determine the basic ingredients
3. Qualitatively describe the biological system

- Life cycle diagrams
- Flow diagrams
- Event tables

Otto & Day: 7 steps to modeling a biological problem

LHS left-hand side
RHS right-hand side

1. Formulate the question
2. Determine the basic ingredients
3. Qualitatively describe the biological system
4. Quantitatively describe the biological system

Diagrams (guide) ⟶ equations.

Checks:
- constraints hold?
- units match on LHS, RHS of equations

Big: can the model actually help answer the Q in step 1?

Otto & Day: 7 steps to modeling a biological problem

1. Formulate the question
2. Determine the basic ingredients
3. Qualitatively describe the biological system
4. Quantitatively describe the biological system
5. Analyze the equations
    - solve (analytical)
    - simulate (numerical)
    - analyze

APPM Diff. Eq.

Otto & Day: 7 steps to modeling a biological problem

e.g. exempli gratia → for example

i.e. id est → that is, specifically

Bonus

1. Formulate the question
2. Determine the basic ingredients
3. Qualitatively describe the biological system
4. Quantitatively describe the biological system
5. Analyze the equations
6. Checks & balances

- check against known examples.

e.g. If I don't water for 1 year, soil very dry.

- generalizability
- reflecting. alternatives to this model? repeat earlier steps?

Otto & Day: 7 steps to modeling a biological problem

1. Formulate the question
2. Determine the basic ingredients
3. Qualitatively describe the biological system
4. Quantitatively describe the biological system
5. Analyze the equations
6. Checks & balances
7. Relate the results back to the question

- Did your model help answer the Question?
- Intuitive? Counter-intuitive?
- Insights → tell a story to explain.
- Experiments? Field studies?

1. Formulate the question

- Find a living/biological object/thing/stuff.
- Ask a Q about how it changes over time.

capital delta

↓ \Delta

Δ

lowercase

\delta

δ

1. How does the # of branches on a tree change over time?

pop. growth

2. How does a cat change the # of mice in the yard?

immigration

predation

3. How does # of people w/ COVID-19 change over a month?

interactions between variables

You can tell what the variable is!

2. Determine the basic ingredients

- **Variables**: what entities might change over time?
- Assign a letter to each variable. (Hint: use "intuitive" letters!)
- Write down *fundamental* constraints on your variables.
- Write down *reasonable* constraints on your variables.

#branches $n(t) \geq 0$

mice $m(t) \geq 0$

susceptible $S(t) \geq 0$

infections $I(t) \geq 0$

recovered $R(t) \geq 0$

$S + I + R =$ total population size

You can't have one person in multiple categories at once!

notes:

- $n(t)$ – explicit reminder that n is a variable
- alternative
  - $n(t)$
  - n_t, n_{t+1}
  - n (no t)

conventions

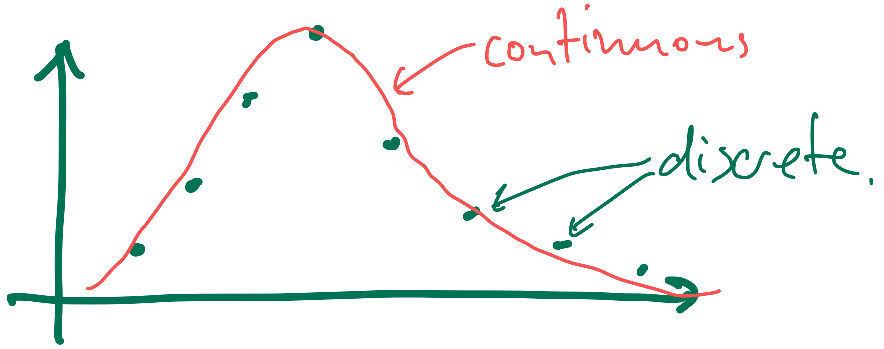
n – population

p – proportions $0 \leq p \leq 1$

$n_1(t)$, $n_2(t)$ two species.

Discrete time vs Continuous time

- **Discrete time models**: "jumpy"
  - assuming is that Δs do not compound within a time step.
  - ↑ holds well Δt is small/reasonable
  - Ex: viral load of SARS-CoV-2.



- **Continuous time models**: "smooth"
  - assumes that variables can change at any point in time.
  - Seems better?

    But: could be unrealistic. Ex: tree might need need a minimum size before branching.
- **Note:**

  Might be easier to work the math in one vs. other!

Be clear about your time scale

- **Time scale**: the unit of time between $t = 0$ and $t = 1$.
  - How much time is in the *tick of the clock?*
- **Discrete time models**:

COVID spread: month, week, day

HIV spread: year?

Animal population: month, decades, seasons

Soil moisture: hours

- **Continuous time models**:

btw…

$\mathbb{R}$ = real numbers

- You'll have to decide whether your variables are discrete or continuous too!

branches $n \geq 0$ int
mice $n \geq 0$ int
discrete

biomass of mice $m \geq 0$ $\mathbb{R}$ continuous

infectious disease: S, I, R $\nearrow$ discrete (people)
$\searrow$ continuous (pop. proportions)

① Often, discrete values get so big that you can model a discretized population as a continuous variable

② Sometimes you can reinterpret a discrete variable as continuous
\# mice $\longrightarrow$ kg of mice

③ Easier math

Recursion Equations

- A **recursion equation** describes the value of a variable in the next time step.

  $n(t+1) =$ "some function of $n(t)$"

- Examples.

looks back two steps: "second order"

- $n(t+1) = n(t) + n(t-1)$ Fibonacci
- Bank Balance
- Excel

1	
$\square + 2$	$= 3$
$\uparrow + 2$	$= 5$
$\uparrow + 2$	$= 7$
	$\vdots$

only looking back one cell / time step.

"First order"

Difference Equations

- A **difference equation** describes the difference between a variable's values in two successive time steps

$$\Delta n \overset{\text{definition}}{=} n(t+1) - n(t) = \text{"some function of } n(t)\text{"}$$

- Examples.

• Excel ex: $\Delta n = 2$

• Bank Interest: $\Delta x = 0.01\, x(t) \xrightarrow{\text{more generic}} \Delta x = r\, x(t)$

parameter (interest rate)

$x(t+1) - x(t) = 0.01\, x(t)$

$x(t+1) = x(t) + 0.01\, x(t)$

Differential Equations

- A **differential equation** describes the rate of change of the variable over time

"slope" "derivative"

$$\frac{dn(t)}{dt} = \text{"some function of } n(t)\text{"}$$

- Examples.

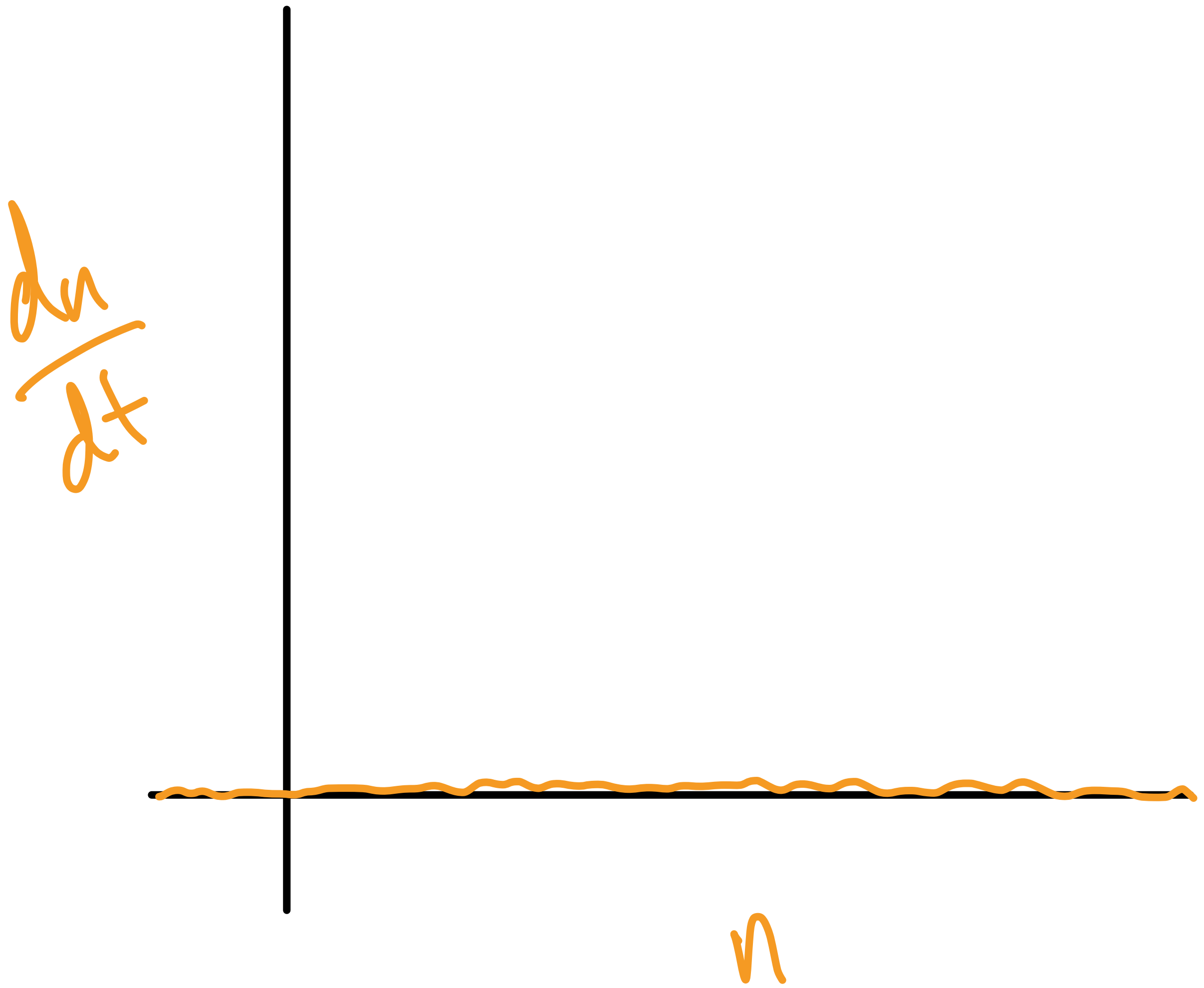
- Continuously compounding interest $\qquad \frac{dn}{dt} = r \cdot n(t)$

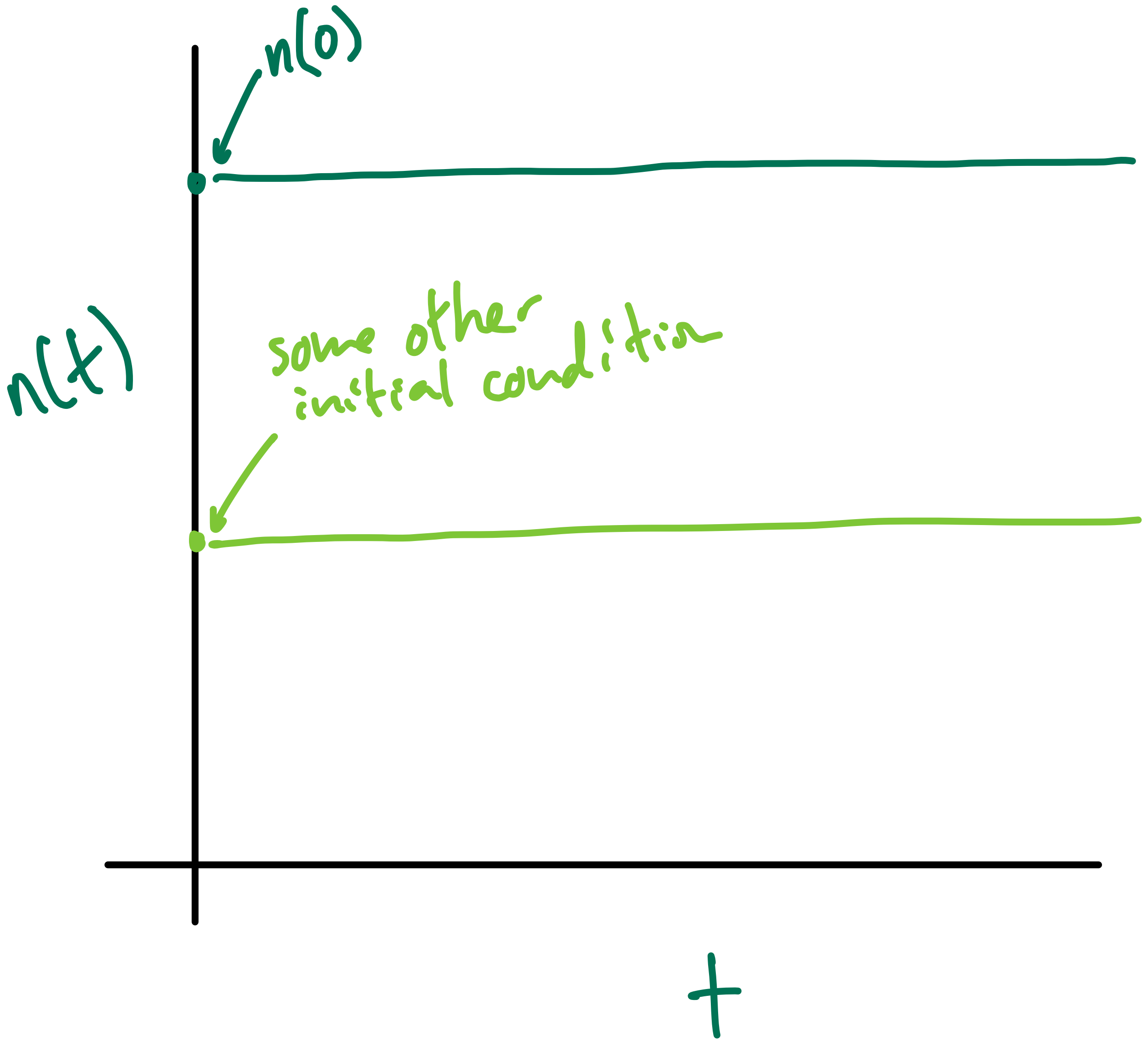
- Newton's law of cooling. $\qquad \frac{dT}{dt} = -k\left(T(t) - T_{room}\right)$

Examples for Intuition

- Ex 1: (A) Sketch the derivative $\dfrac{dn(t)}{dt}$ vs. $n(t)$. (B) Sketch the variable $n(t)$ vs time.

$$\frac{dn(t)}{dt} = 0$$



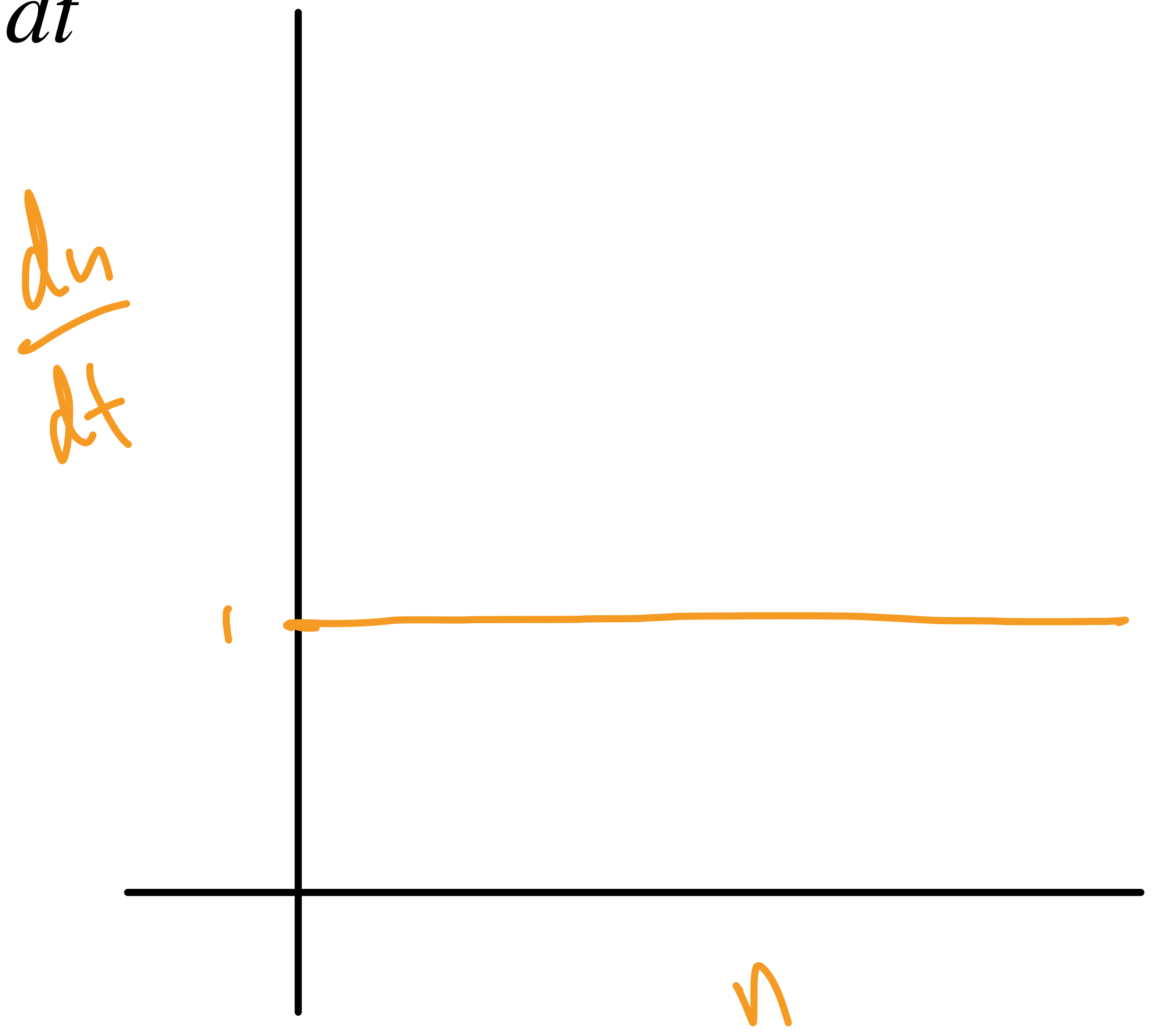


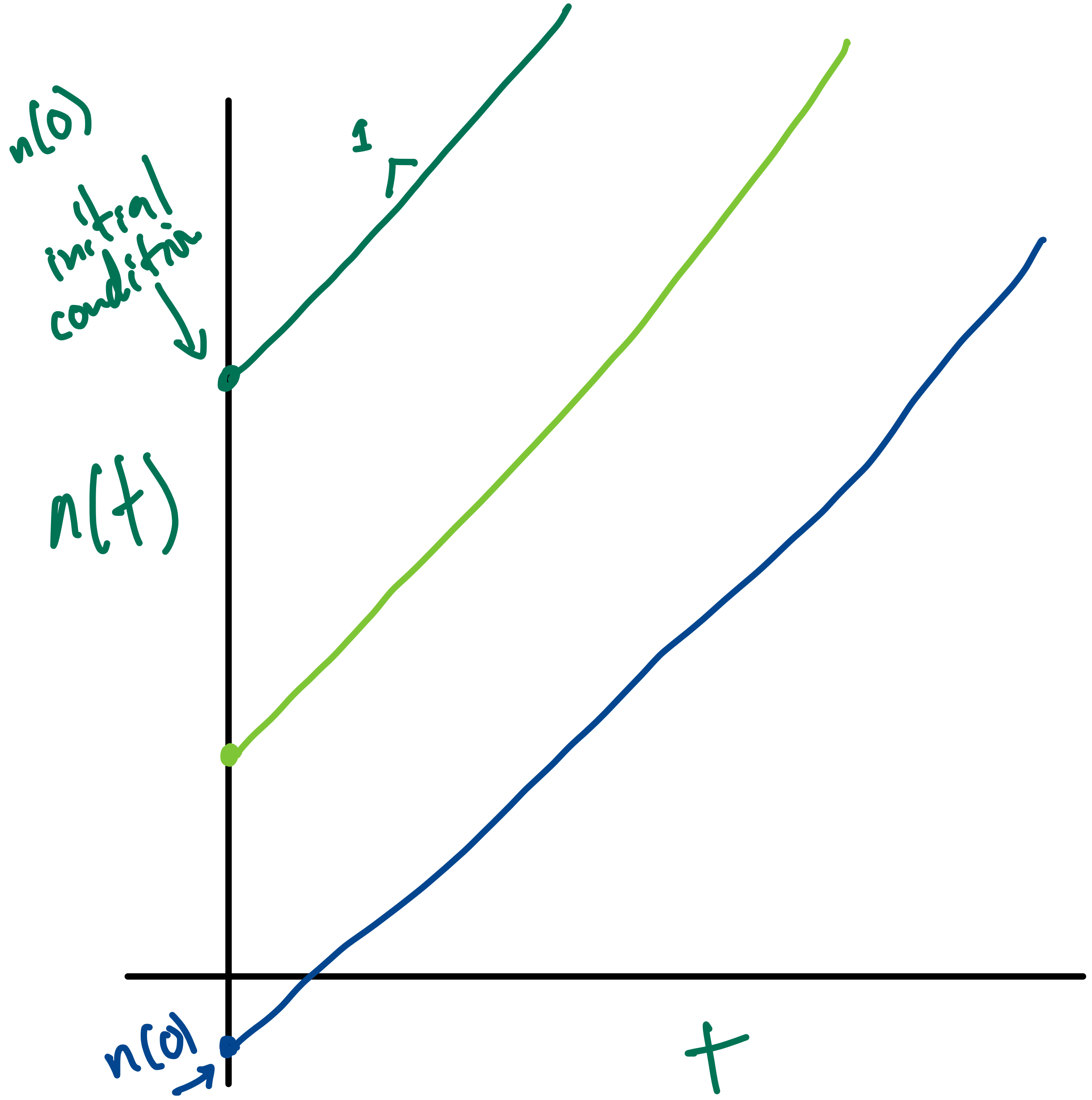
Examples for Intuition

- Ex 2: (A) Sketch the derivative $\frac{dn(t)}{dt}$ vs. $n(t)$. (B) Sketch the variable $n(t)$ vs time.

$$\frac{dn(t)}{dt} = 1$$

$\frac{dn}{dt}$ is always 1

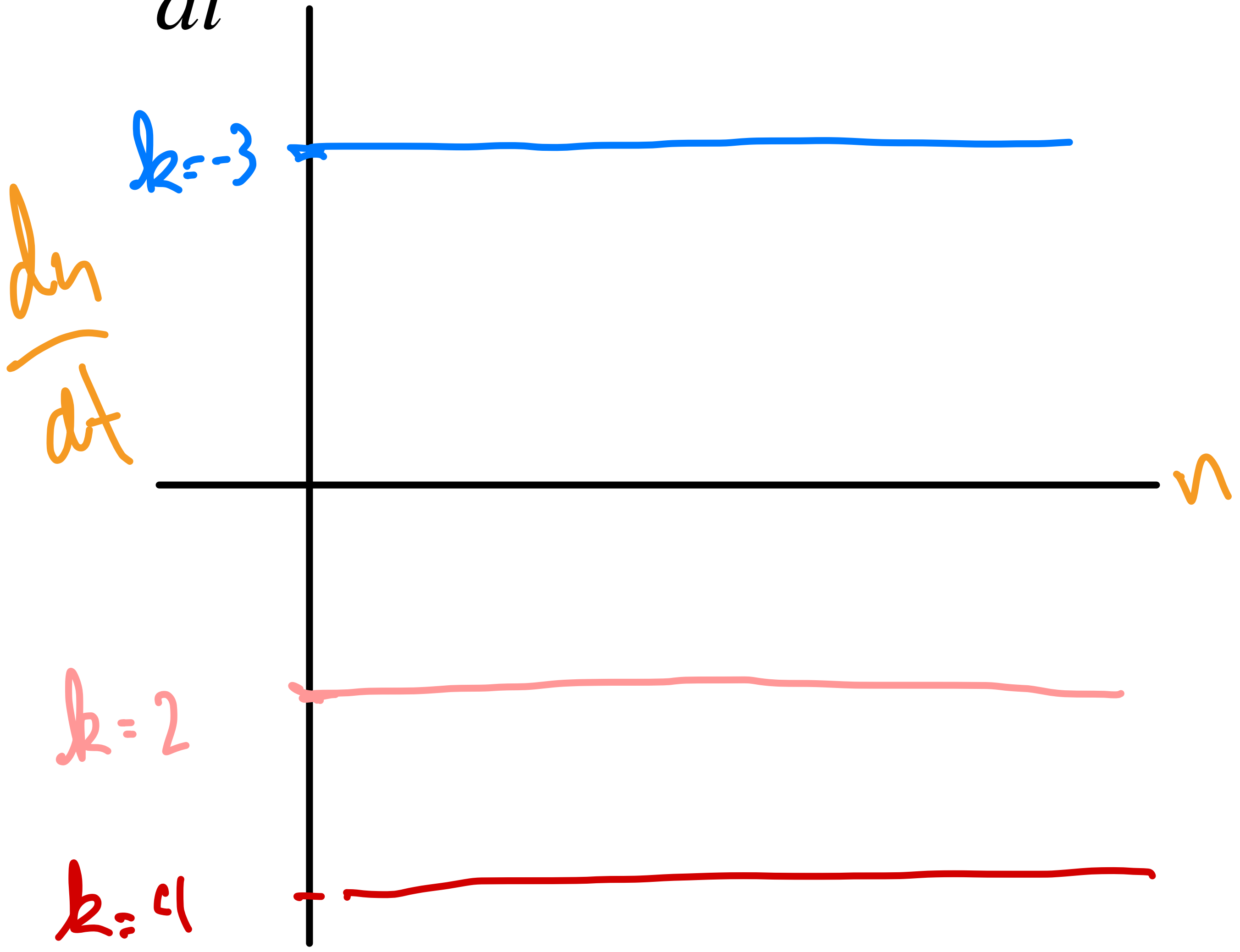


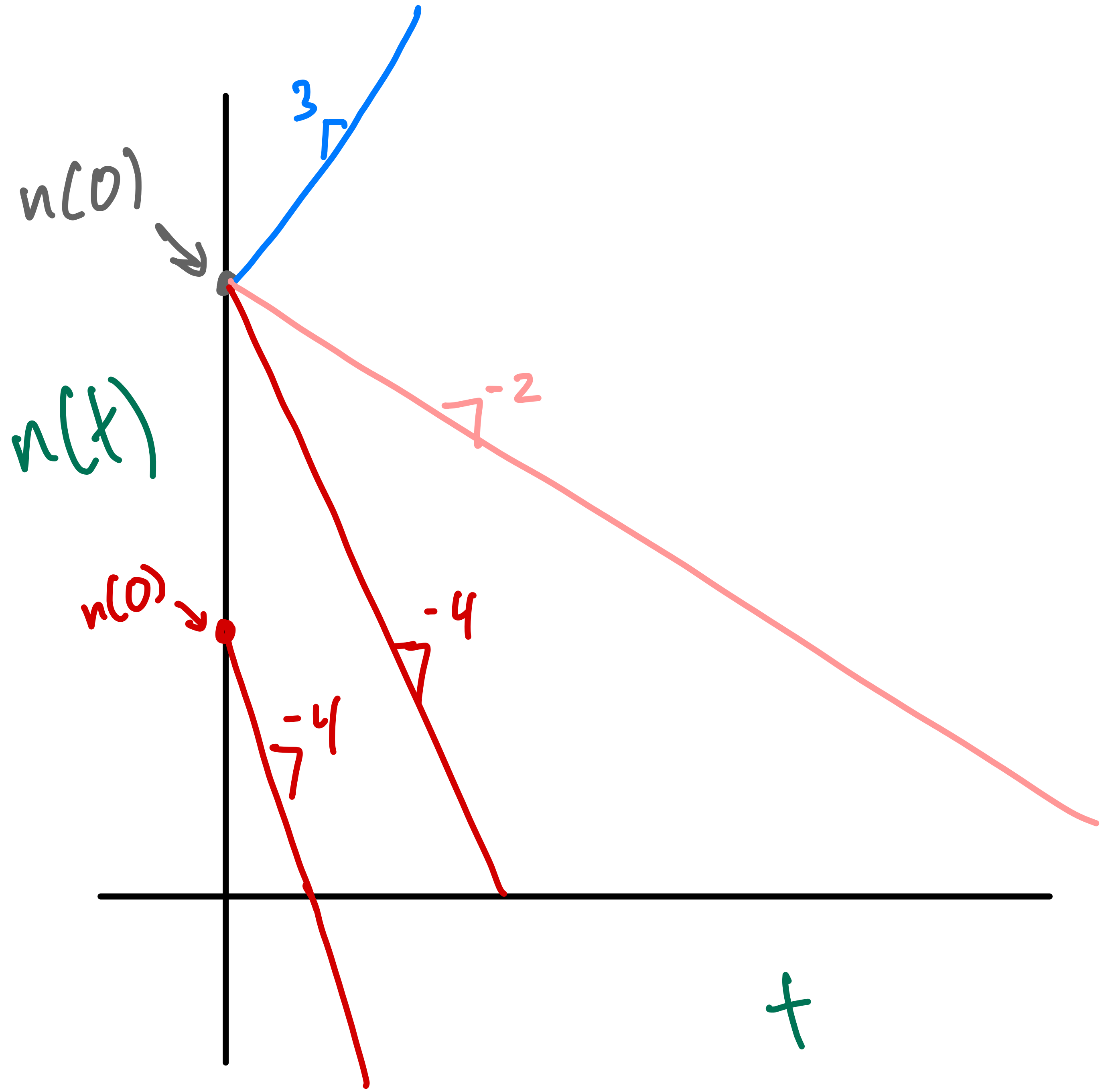


Examples for Intuition

- Ex 3: (A) Sketch the derivative $\frac{dn(t)}{dt}$ vs. $n(t)$. (B) Sketch the variable $n(t)$ vs time.

$$\frac{dn(t)}{dt} = -k$$



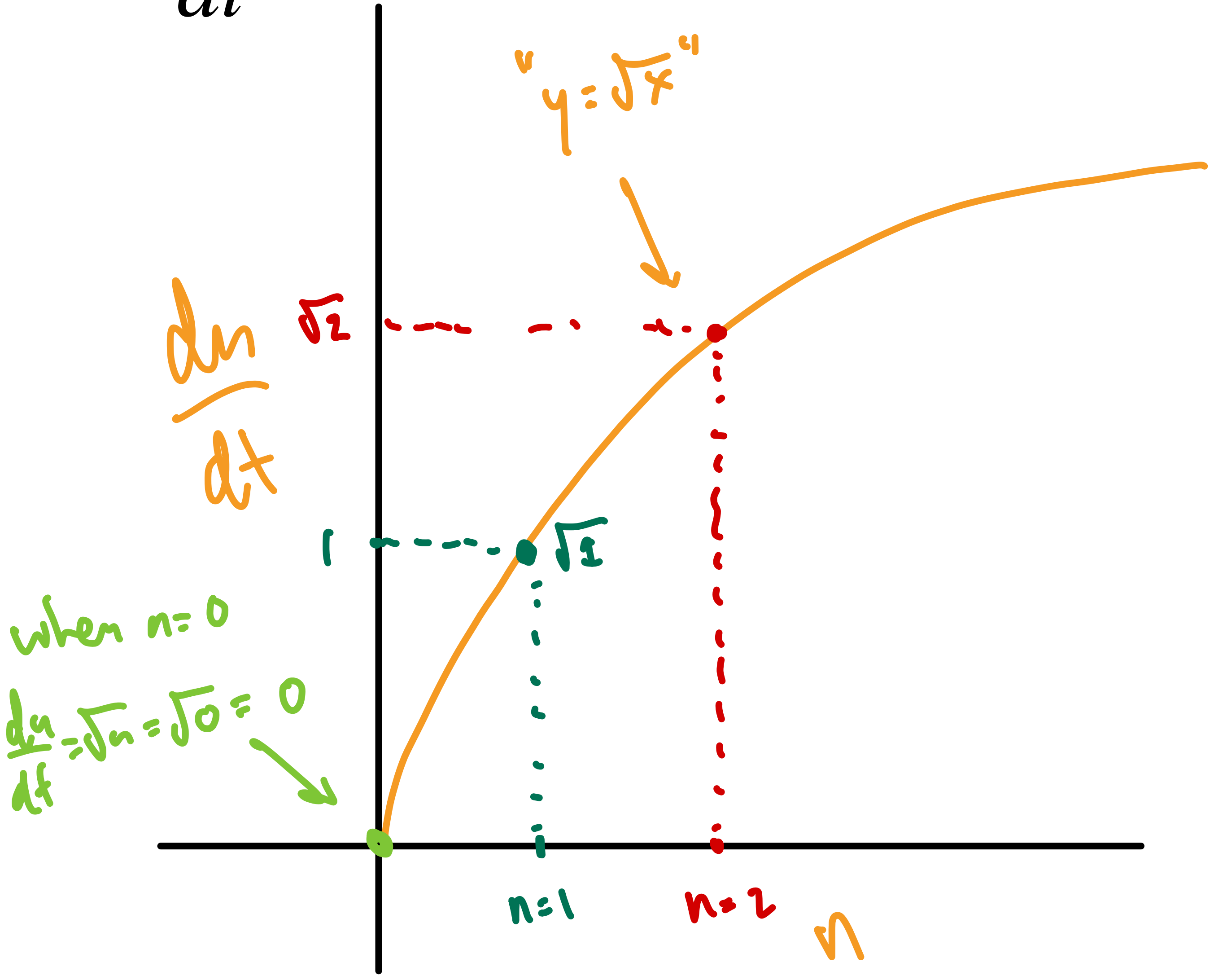


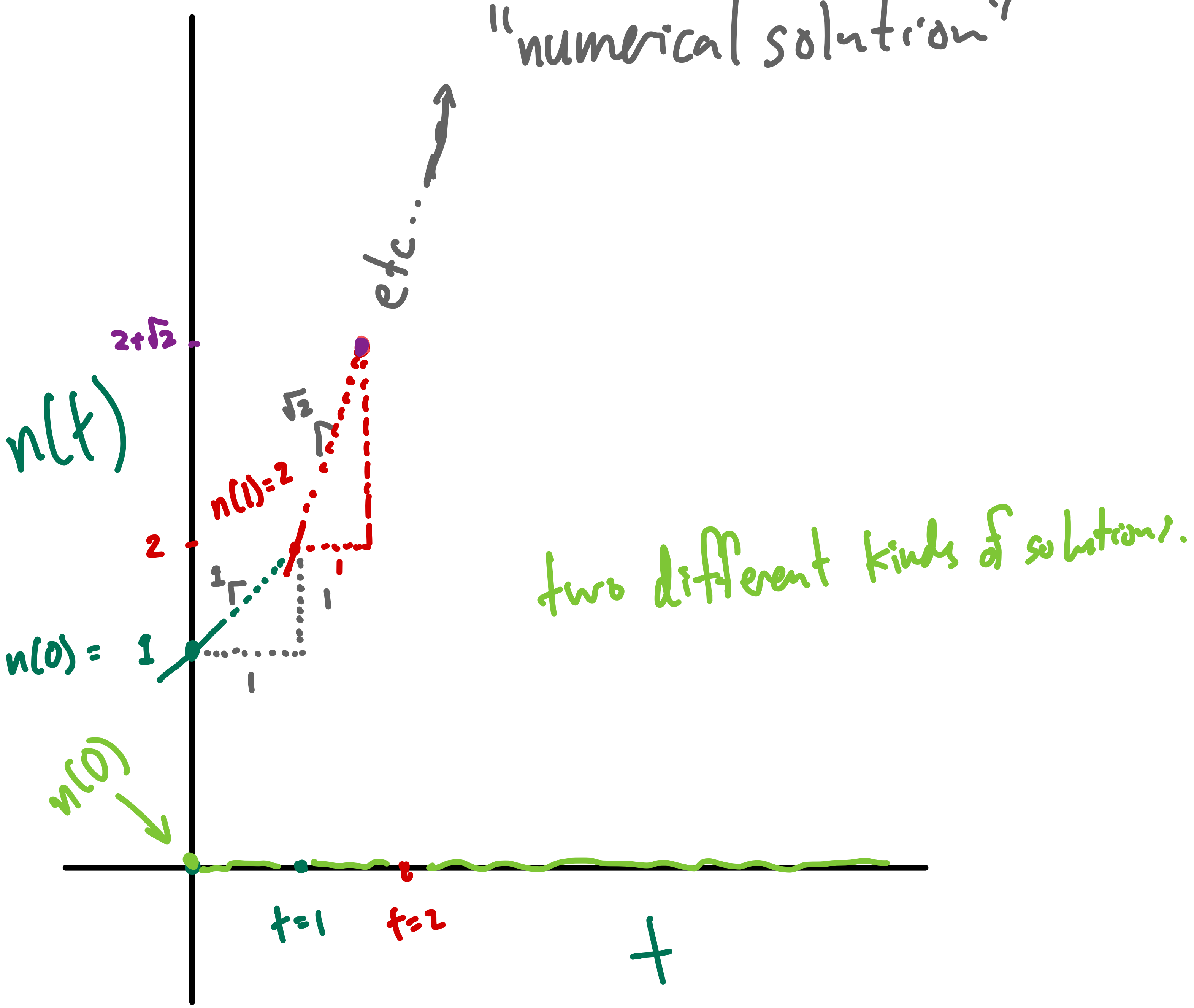
Examples for Intuition

- Ex 4: (A) Sketch the derivative $\frac{dn(t)}{dt}$ vs. $n(t)$. (B) Sketch the variable $n(t)$ vs time.

$y = \sqrt{x}$

$$\frac{dn(t)}{dt} = \sqrt{n(t)}$$





Parameters

- The **parameters** of the model are quantities that influence the dynamics but remain fixed over time.
- Examples:

$$\frac{dn}{dt} = -k$$

$$n(t+1) = n(t) + r\,n(t)$$

- Branches per day per existing branch b
- Fraction of mice eaten by the cat d (death)
- # of mice born per day per existing mouse b (birth)
- Rate of contacts per potential interaction per day. c
- Probability of transmission, per exposure a

Parameters

- The **parameters** of the model are quantities that influence the dynamics but remain fixed over time.
- When we fix parameters and look at a trajectory of the equation, that's called **forward simulation** or **forward integration**. Model + Parameters → Data

see $\frac{dn}{dt} = \sqrt{n}$

- When we have data and a model, and we determine the values of the parameters that best fit the data, that's **parameter inference.** Model + Data → Parameters

What is the rate of spread of Delta? → learn spreading parameters.

- Note: parameters' units need to match the kind of model we're using.
- Note: parameters may have *reasonable* ranges in addition to *fundamental* ranges.

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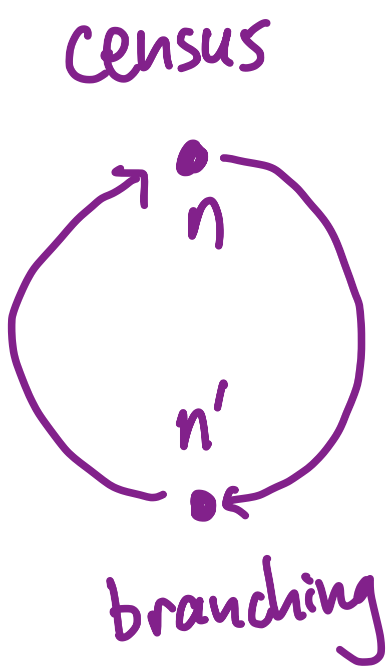
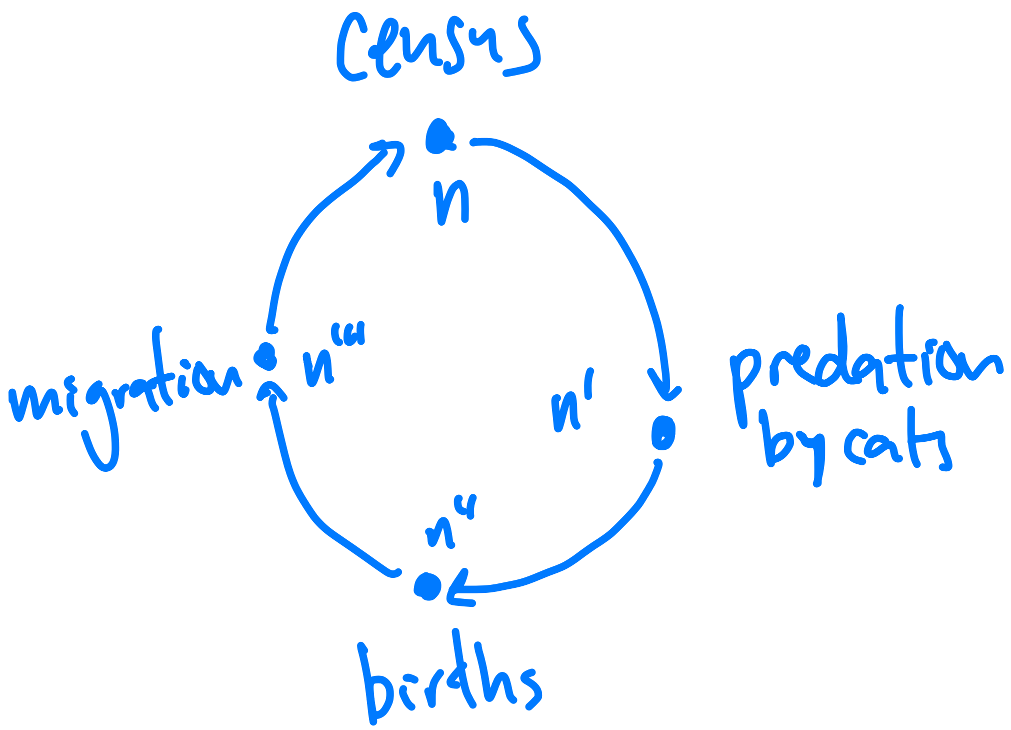
Diagrams: Life Cycle

- Keep track of the events occurring during a single time step *and their order*.

Tree Branches

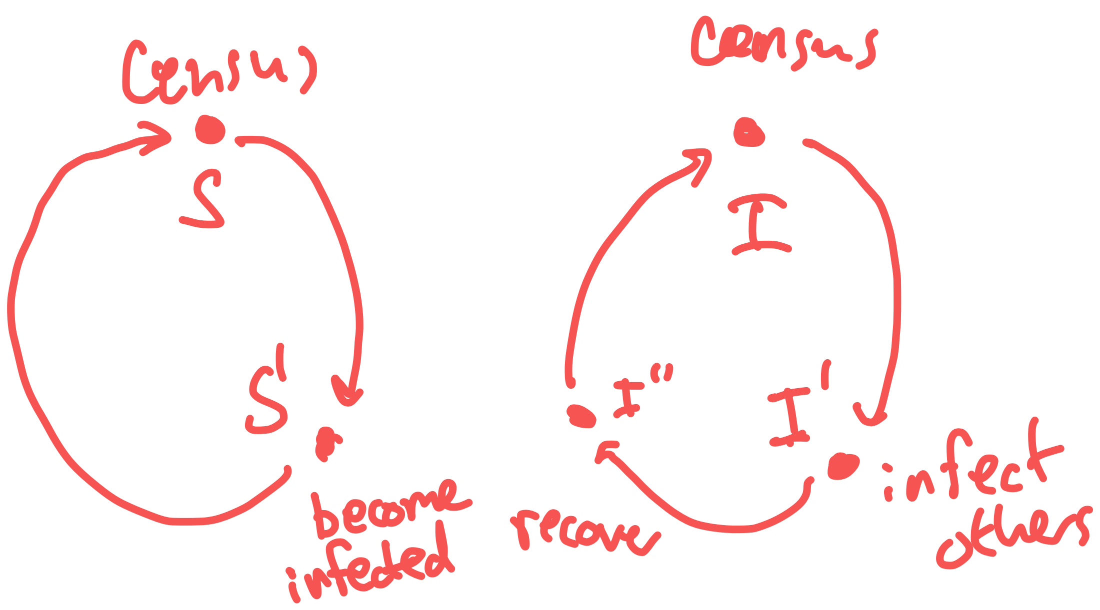


Mice in the Yard



(order matters)

I. D.

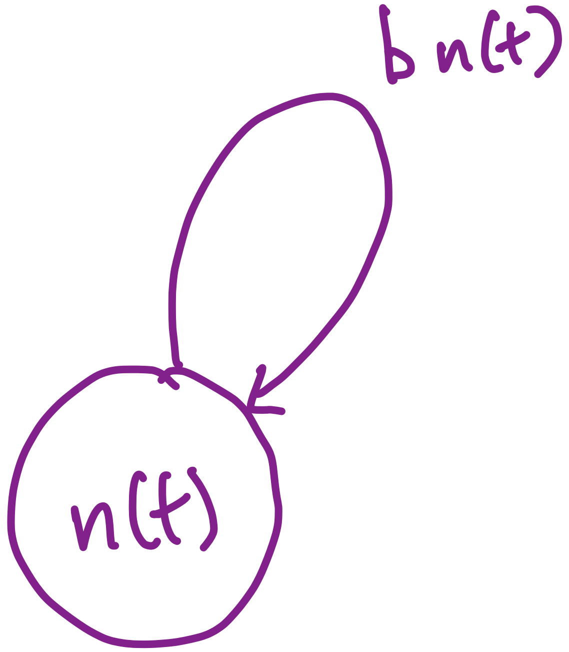


track multiple variables

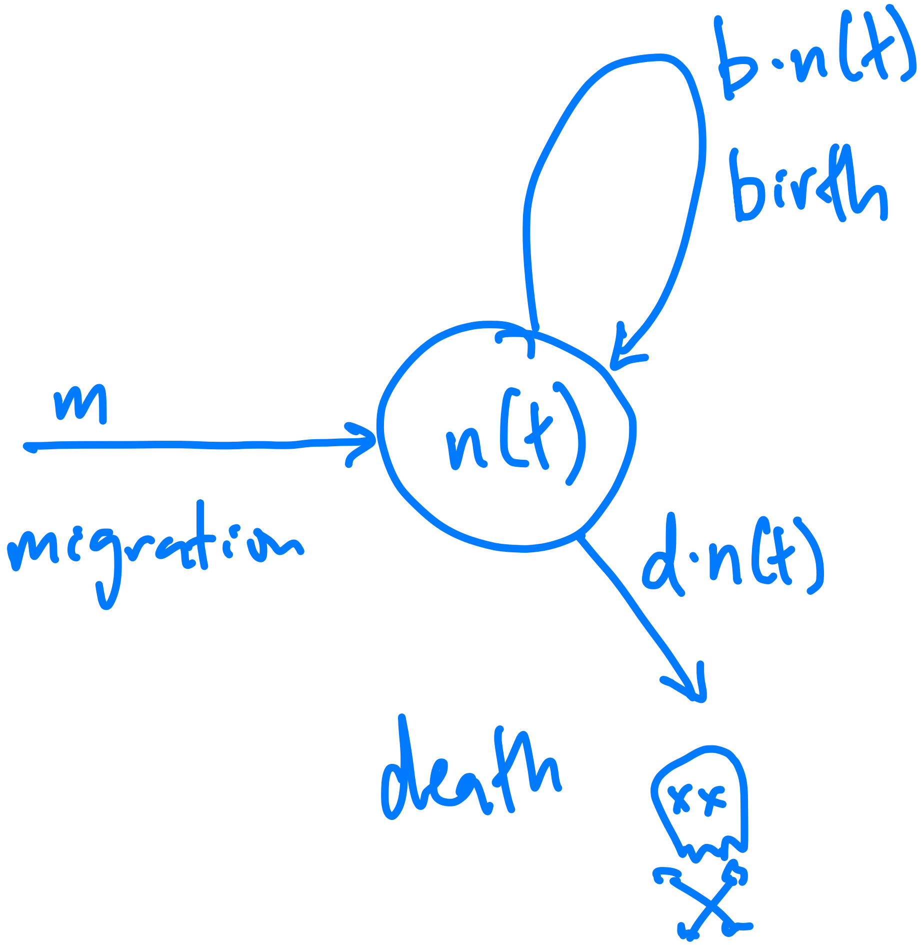
Diagrams: Flow

- Keep track of the events occurring during a single time step *and their order*.

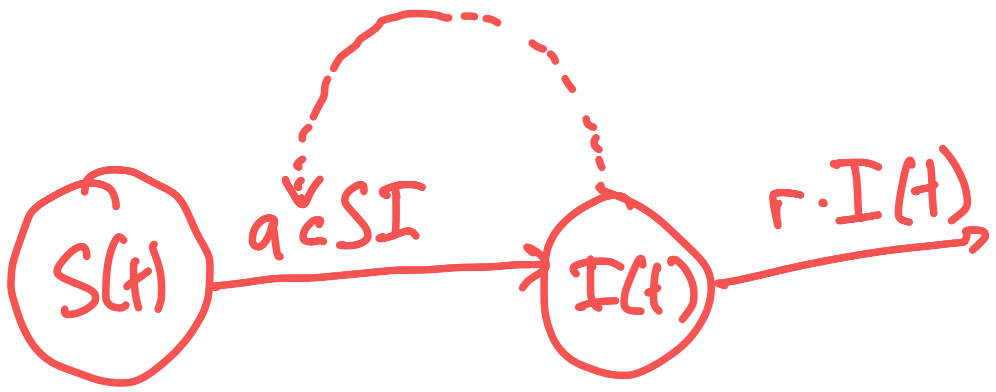
Branches



Mice



I. D.



Diagrams: Table of Events I.D.

- Discrete-time models with **multiple events** per time step and **multiple variables**.

Interaction	# events	Result of Event
$S \times S$	$c\,S\,S$	—
$I \times I$	$c\,I\,I$	—
$S \times I$	$c\,S\,I$	I: $+a$; S: $-a$

"law of mass action"

Pros and Cons?

- See Otto & Day, Chapter 2.4

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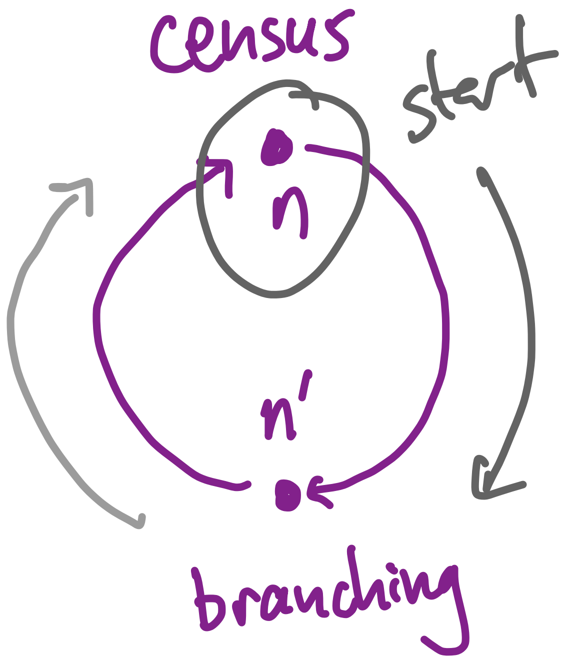
Example: tree branching

- Use the life cycle diagram to derive a recursion, and use that to create a difference equation.



$$n'(t) = n(t) + b\,n(t)$$

$$n(t+1) = n'(t)$$

$$n(t+1) = n(t) + b\,n(t)$$

$$\boxed{n(t+1) = (1+b)\,n(t)}$$

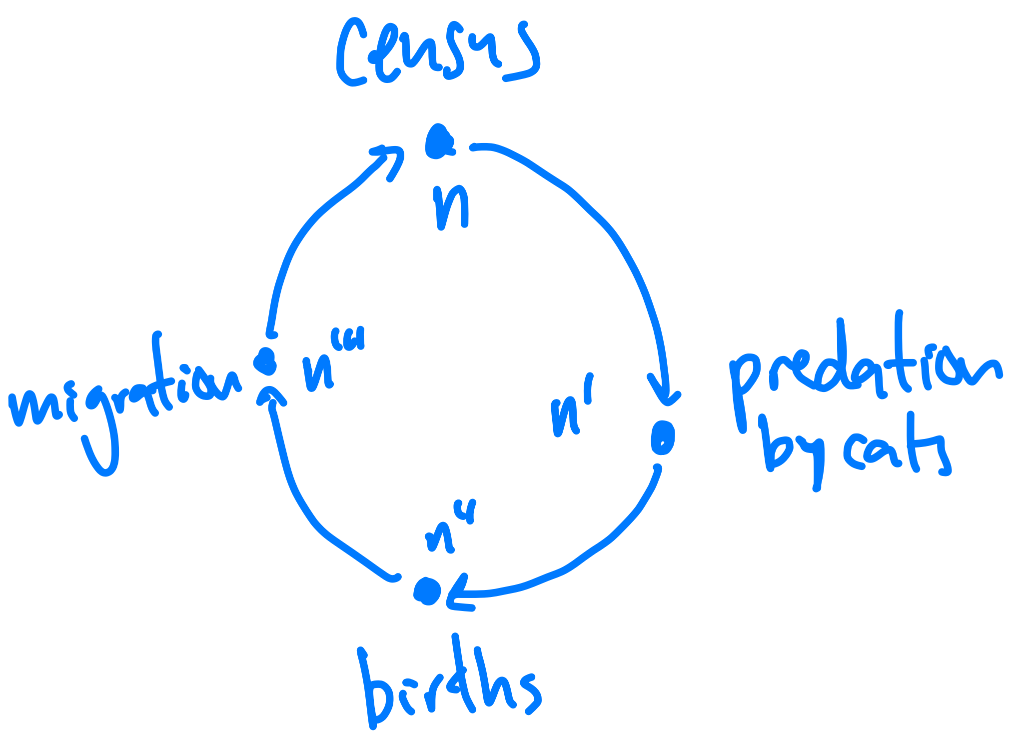
$$\Delta n = n(t+1) - n(t)$$

$$\Delta n = n(t+1) - n(t) = b\,n(t)$$

$$\boxed{\Delta n = b\,n(t)}$$

Example: mouse model

- Use the life cycle diagram to derive the stages of the recursion.



$$n'(t) = n(t) - d\,n(t) = (1-d)\,n(t)$$

$$n''(t) = n'(t) + b\,n'(t)$$

$$n'''(t) = n''(t) + m$$

$$n(t+1) = n'''(t)$$

$$n(t+1) = n''(t) + m$$

$$n(t+1) = \left[n'(t) + b\,n'(t)\right] + m$$

$$= (1+b)\,n'(t) + m$$

$$\boxed{n(t+1) = (1+b)(1-d)\,n(t) + m}$$

Recipes: recursion & difference equations from life cycle diagrams

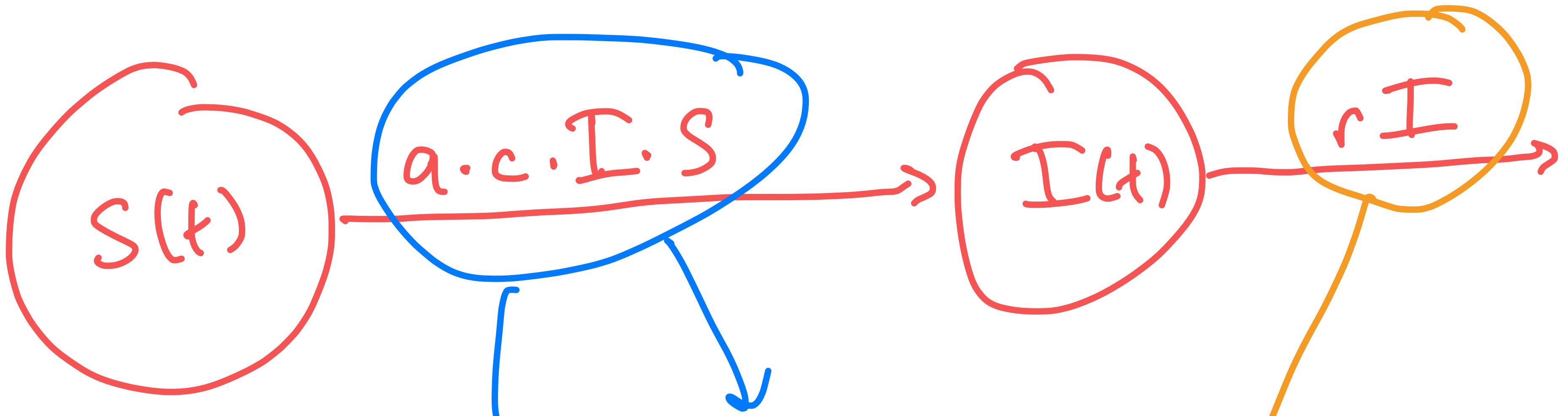
1. Use $n'(t), n''(t), n'''(t)$ etc to denote the variable's value after each life cycle event.
2. Set $n(t+1)$ to the value of n after the final event in the cycle.
3. Substitute, and get $n(t+1)$ in terms of $n(t)$ by eliminating $n'(t)$ etc.

Bonus

4. Subtract $n(t)$ from both sides and simplify to get the difference equation
   $\Delta n = n(t+1) - n(t) = \ldots$

Example: COVID-19

- Use the flow diagram to create the recursion equations for COVID-19 spread.



$$S(t+1) = S - acIS$$

$$I(t+1) = I + acIS - rI$$

inflow: +

outflow: −

Recipes: differential equations from flow diagrams

$$\frac{d(n(t))}{dt} = \ldots$$

the flow rates along arrows *entering* the circle

+ the flow rates along arrows leaving & returning to the circle

– the flow rates along arrows exiting the circle

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