Calculating Biological Quantities

CSCI 2897

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- HW 5 posted.

Last time on CSCI 2897

Definitions: An **Eigenvector** of a square matrix A is a vector x such that $Ax = \lambda x$ for some scalar λ . An **Eigenvalue** is that scalar, λ.

There can be at most n eigenvectors and n eigenvalues for an $n \times n$ matrix.

To compute eigenvalues, we:

1. Write $Ax = \lambda x$ as $Ax - \lambda x = 0$ and then as $(A - \lambda I)x = 0$.
2. If $(A - \lambda I)x = 0$ but $x \neq 0$, this means that $\det(A - \lambda I) = 0$.
3. Write out the characteristic equation: $(a - \lambda)(d - \lambda) - bc = 0$
4. Solve for λ.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\lambda = \frac{tr(A) \pm \sqrt{tr^2(A) - 4\det(A)}}{2}$$

To compute the eigenvectors, for each eigenvalue, we

1. Plug in the λ to $(A - \lambda I)x = 0$, and write out the equations.
2. The equations *should* be redundant. Pick one and determine the relationship between x_1 and x_2. That's your eigenvector!

Why do we care though?

$$\frac{d\vec{n}}{dt} = A\vec{n} \qquad \text{Solution: } \vec{n}(t) = k_1\vec{x_1}e^{\lambda_1 t} + k_2\vec{x_2}e^{\lambda_2 t}$$

Example

$$\frac{dn_1}{dt} = 2n_1 + 3n_2$$

$$\frac{dn_2}{dt} = 2n_1 + n_2$$

rewrite system of ODEs in matrix form

★ $$\frac{d}{dt}\begin{pmatrix}n_1\\n_2\end{pmatrix} = \begin{pmatrix}2&3\\2&1\end{pmatrix}\begin{pmatrix}n_1\\n_2\end{pmatrix}$$

solution to ODEs is in terms of eigenvalues and eigenvectors

$$\begin{pmatrix}n_1\\n_2\end{pmatrix} = k_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-t} + k_2\begin{pmatrix}3\\2\end{pmatrix}e^{4t}$$

we can check that this solves, by plugging into ★

Plug into LHS:

$$\frac{d}{dt}\begin{pmatrix}n_1\\n_2\end{pmatrix} = \frac{d}{dt}\left[k_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-t} + k_2\begin{pmatrix}3\\2\end{pmatrix}e^{4t}\right]$$

$$= k\begin{pmatrix}1\\-1\end{pmatrix}(-1)e^{-t} + k_2\begin{pmatrix}3\\2\end{pmatrix}4e^{4t}$$

Plug into RHS:

$$\begin{pmatrix}2&3\\2&1\end{pmatrix}\begin{pmatrix}n_1\\n_2\end{pmatrix} = \begin{pmatrix}2&3\\2&1\end{pmatrix}\left(k_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-t} + k_2\begin{pmatrix}3\\2\end{pmatrix}e^{4t}\right)$$

$$= k_1\begin{pmatrix}2&3\\2&1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}e^{-t} + k_2\begin{pmatrix}2&3\\2&1\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}e^{4t}$$

$$= k_1(-1)\begin{pmatrix}1\\-1\end{pmatrix}e^{-t} + k_2(4)\begin{pmatrix}3\\2\end{pmatrix}e^{4t}$$

Why do we care though?

$$\frac{d\vec{n}}{dt} = A\vec{n}$$

Solution: $\vec{n}(t) = k_1\vec{x_1}e^{\lambda_1 t} + k_2\vec{x_2}e^{\lambda_2 t}$

Example

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solution to ODEs is in terms of eigenvalues and eigenvectors

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + k_2\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{4t}$$

we can check that this solves, by plugging into ★

Plug solution into LHS of ★

$$\frac{d}{dt}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = -1k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + 4k_2\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{4t}$$

Plug solution into RHS of ★

$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = k_1\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + k_2\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{4t}$$

$Ax_1 = \lambda_1 x_1$

$Ax_2 = \lambda_2 x_2$

$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = k_1(-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + k_2(4)\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{4t}$$

Practice.

Find the eigenvalues & eigenvectors of $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$

① $(A-\lambda I)x = 0 \Rightarrow \det(A-\lambda I) = 0$

$$A-\lambda I = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix}$$

② $\det(A-\lambda I) = (2-\lambda)(1-\lambda) - 3\cdot 2$

$= 2 - \lambda - 2\lambda + \lambda^2 - 6$

set to 0

③ $2 - 3\lambda + \lambda^2 - 6 = 0$

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0$

④ $\Rightarrow \lambda = 4, \lambda = -1$

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could replace these steps with:

$$\lambda = \frac{tr(A) \pm \sqrt{tr^2(A) - 4\det(A)}}{2}$$

To compute eigenvalues, we:

1. Write $Ax = \lambda x$ as $Ax - \lambda x = 0$ and then as $(A - \lambda I)x = 0$.
2. If $(A - \lambda I)x = 0$ but $x \neq 0$, this means that $\det(A - \lambda I) = 0$.
3. Write out the characteristic equation: $(a - \lambda)(d - \lambda) - bc = 0$
4. Solve for λ.

To compute the eigenvectors, for each eigenvalue, we

1. Plug in the λ to $(A - \lambda I)x = 0$, and write out the equations.
2. The equations *should* be redundant. Pick one and determine the relationship between x_1 and x_2. That's your eigenvector!

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① Plug in $\lambda = 4$ to $(A-\lambda I)x = 0$

$$\begin{pmatrix} 2-4 & 3 \\ 2 & 1-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so $-2x_1 + 3x_2 = 0$

$2x_1 = 3x_2$

$x_1 = \frac{3}{2}x_2$

$x_2 = \frac{2}{3}x_1$

② $\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$

etc with $\lambda = -1$.

Practice.

Solve $\frac{d}{dt}\begin{pmatrix}n_1\\n_2\end{pmatrix} = \begin{pmatrix}2 & 3\\2 & 1\end{pmatrix}\begin{pmatrix}n_1\\n_2\end{pmatrix}$, $n_1(0) = 7,\ n_2(0) = 3$

$$\frac{d\vec{n}}{dt} = A\vec{n} \longrightarrow \vec{n} = k_1 \vec{x}_1 e^{\lambda_1 t} + k_2 \vec{x}_2 e^{\lambda_2 t}$$

$$\left[\begin{array}{ll} \lambda_1 = -1 & \lambda_2 = 4 \\ \vec{x}_1 = \begin{pmatrix}1\\-1\end{pmatrix} & \vec{x}_2 = \begin{pmatrix}3\\2\end{pmatrix}\end{array}\right]$$

from previous slides.

Solution

$$\vec{n} = k_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-t} + k_2\begin{pmatrix}3\\2\end{pmatrix}e^{4t}$$

[formula for solution]

At $t=0$, $n_1(0) = 7$, $n_2(0) = 3$.

$$\text{so } \vec{n} = \begin{pmatrix}7\\3\end{pmatrix} = k_1\begin{pmatrix}1\\-1\end{pmatrix}\cancelto{1}{e^{-0}} + k_2\begin{pmatrix}3\\2\end{pmatrix}\cancelto{1}{e^{4\cdot 0}}$$

[plug in I.C.]

solve for k_1 and k_2

$$\begin{pmatrix}7\\3\end{pmatrix} = k_1\begin{pmatrix}1\\-1\end{pmatrix} + k_2\begin{pmatrix}3\\2\end{pmatrix}$$

$$\begin{aligned} k_1 + 3k_2 &= 7 \\ -k_1 + 2k_2 &= 3 \end{aligned} \quad \oplus \quad \longrightarrow \begin{pmatrix}1 & 3\\-1 & 2\end{pmatrix}\begin{pmatrix}k_1\\k_2\end{pmatrix} = \begin{pmatrix}7\\3\end{pmatrix}$$

$$\begin{pmatrix}k_1\\k_2\end{pmatrix} = \begin{pmatrix}1 & 3\\-1 & 2\end{pmatrix}^{-1}\begin{pmatrix}7\\3\end{pmatrix}$$

$5k_2 = 10$

$k_2 = 2$ plug back in ... $k_1 + 3\cdot 2 = 7$

$k_1 = 1$

plug in k_1, k_2

$$\boxed{\vec{n} = \begin{pmatrix}1\\-1\end{pmatrix}e^{-t} + 2\begin{pmatrix}3\\2\end{pmatrix}e^{4t}}$$

Linear Multivariable Models

Linear model $\frac{d\overrightarrow{n}}{dt} = M\overrightarrow{n}$

Affine model $\frac{d\overrightarrow{n}}{dt} = M\overrightarrow{n} + \overrightarrow{c}$

Two dimensional case (individual equation form)

$$\frac{dn_1}{dt} = an_1 + bn_2$$

$$\frac{dn_2}{dt} = cn_1 + dn_2$$

$$\frac{dn_1}{dt} = an_1 + bn_2 + c_1$$

$$\frac{dn_2}{dt} = cn_1 + dn_2 + c_2$$

Two dimensional case (matrix vector form)

$$\frac{d}{dt}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Question: what are the equilibria of these systems?

Equilibria of linear multivariable models

Three methods:

1. **Solve individual equations.**
2. Solve matrix equation.
3. Nullclines.

$$\frac{dn_1}{dt} = an_1 + bn_2 = 0$$

$an_1 = -bn_2$

$n_1 = -\frac{b}{a} n_2$

$$\frac{dn_2}{dt} = cn_1 + dn_2 = 0$$

$n_1 = 0$

$c\left(\frac{-b}{a}\right)n_2 + dn_2 = 0$

$n_2\left[-\frac{cb}{a} + d\right] = 0$

$n_2 = 0$

$\boxed{\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$

$an_1 + bn_2 + c_1 = 0$

$\boxed{n_1 = \frac{-bn_2 - c_1}{a}}$

$$\frac{dn_1}{dt} = an_1 + bn_2 + c_1 = 0$$

$$\frac{dn_2}{dt} = cn_1 + dn_2 + c_2 = 0$$

$c\left(\frac{-bn_2 - c_1}{a}\right) + dn_2 + c_2 = 0$

$-\frac{cbn_2}{a} - \frac{cc_1}{a} + dn_2 + c_2 = 0$

$n_2\left(-\frac{cb}{a} + d\right) = \frac{cc_1}{a} + c_2 \Rightarrow n_2 = \frac{\frac{cc_1}{a} + c_2}{-\frac{cb}{a} + d}$

Equilibria of linear multivariable models

Three methods:

1. Solve individual equations.
2. **Solve matrix equation.**
3. Nullclines.

def'n of equil.

$$\frac{d}{dt}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \overset{\downarrow}{=} \vec{0}$$

$$M\vec{n} = \vec{0}$$

if M is invertible,

$$M^{-1}M\vec{n} = M^{-1}\vec{0}$$

$$\vec{n} = M^{-1}\vec{0}$$

$$\boxed{\vec{n} = \vec{0}}$$

def'n of equil.

$$\frac{d\vec{n}}{dt} = M\vec{n} + \vec{c} \overset{\downarrow}{=} \vec{0}$$

$$\frac{d}{dt}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$M\vec{n} + \vec{c} = \vec{0}$$

$$M\vec{n} = -\vec{c}$$

$$M^{-1}M\vec{n} = -M^{-1}\vec{c}$$

$$\boxed{\vec{n} = -M^{-1}\vec{c}}$$

Equilibria of linear multivariable models

Three methods:

1. Solve individual equations.
2. Solve matrix equation.
3. **Nullclines.**

A **nullcline** is a line in phase space on which one of the variables is constant (unchanging).

$$\frac{dn_1}{dt} = an_1 + bn_2 = 0$$

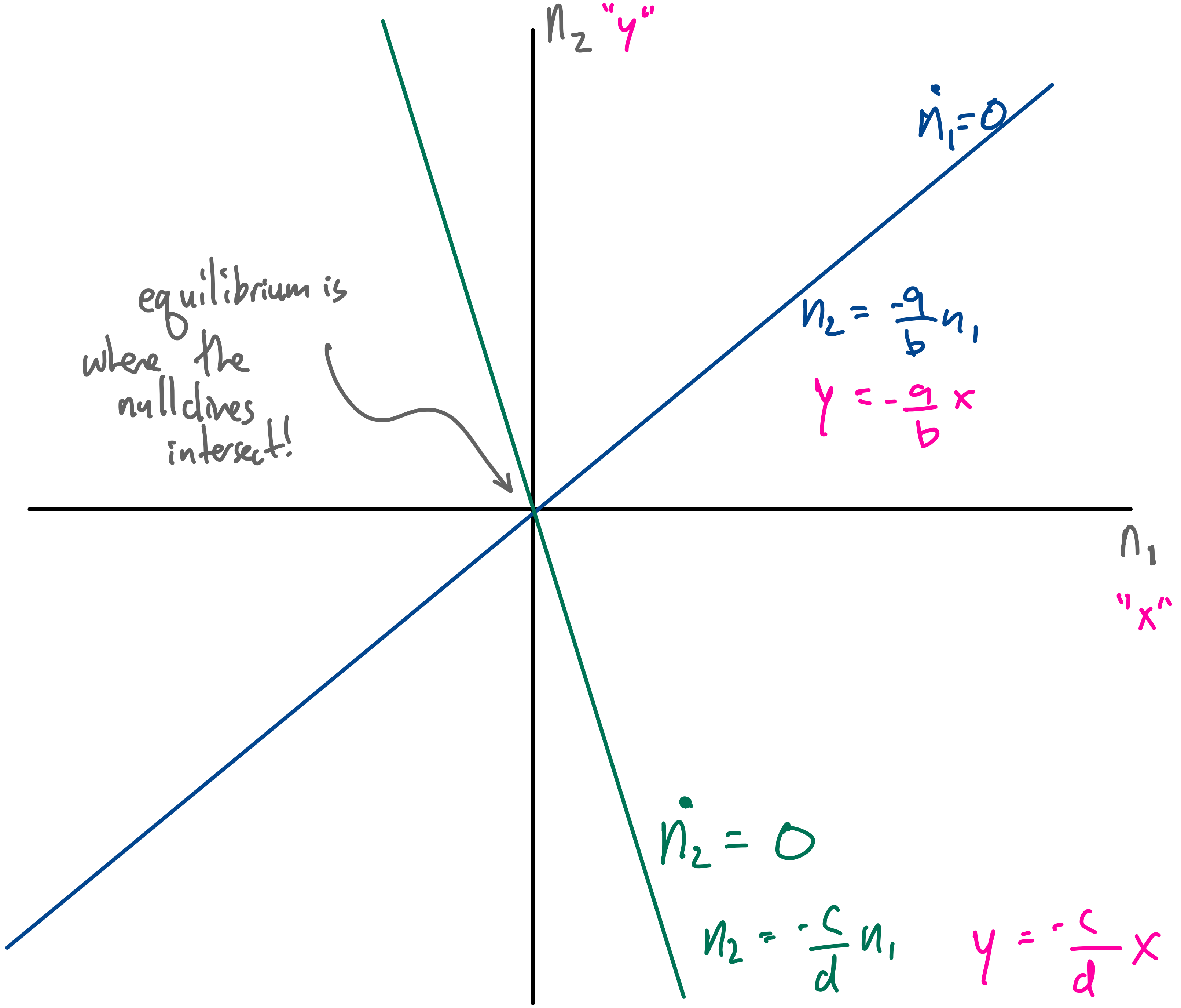
$an_1 + bn_2 = 0$

$n_2 = \underbrace{\frac{-a}{b}}_{\text{slope}} n_1 + \underbrace{0}_{\text{intercept}}$

$$\frac{dn_2}{dt} = cn_1 + dn_2 = 0$$

$cn_1 + dn_2 = 0$

$n_2 = \underbrace{\frac{-c}{d}}_{\text{slope}} n_1 + \underbrace{0}_{\text{int.}}$



Equilibria of linear multivariable models

Three methods:

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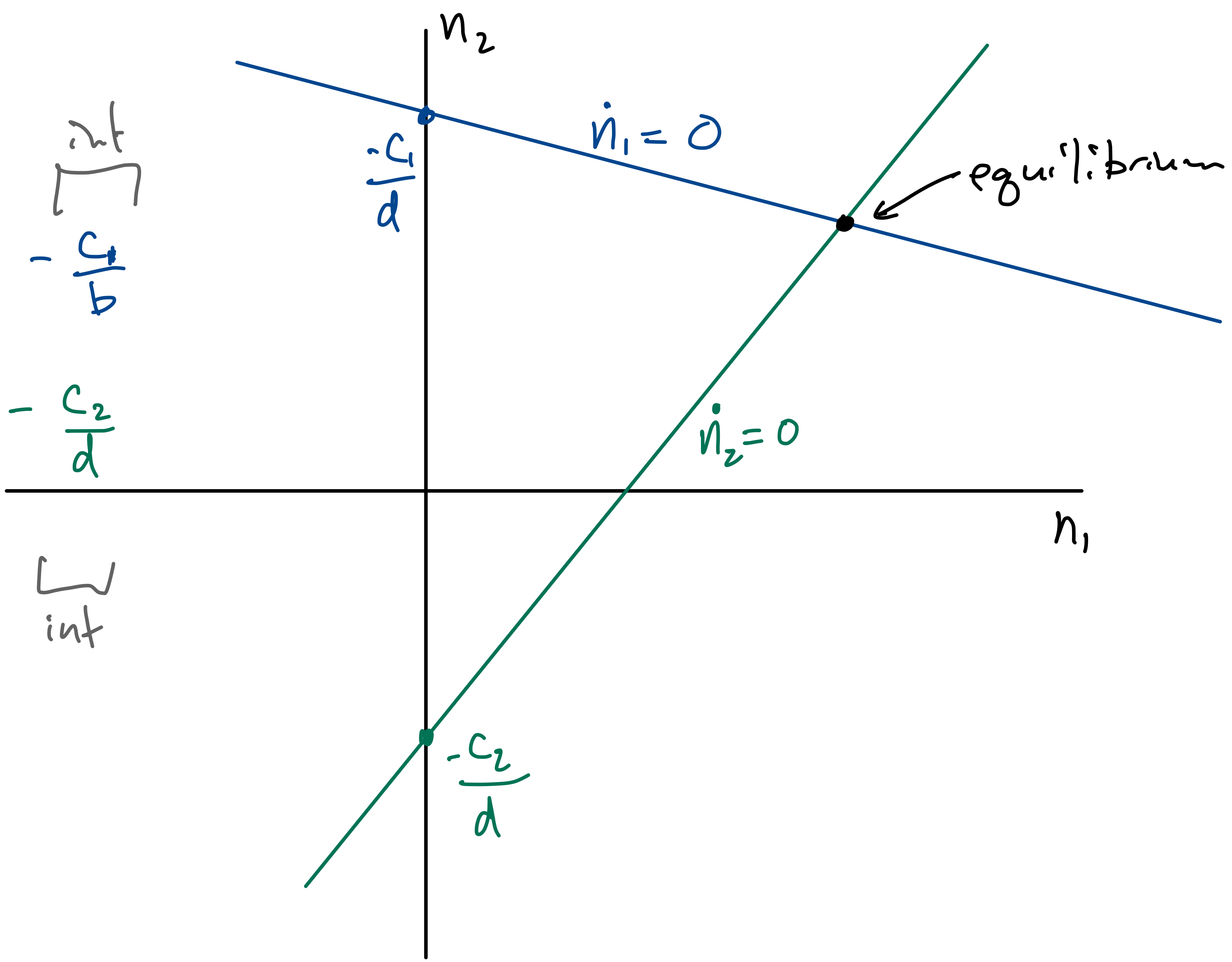
$$\frac{dn_1}{dt} = an_1 + bn_2 + c_1 = 0 \qquad n_2 = \overbrace{-\frac{a}{b}}^{\text{slope}} n_1 \overbrace{- \frac{c_1}{b}}^{\text{int}}$$

$$\frac{dn_2}{dt} = cn_1 + dn_2 + c_2 = 0 \qquad n_2 = \underbrace{-\frac{c}{d}}_{\text{slope}} n_1 \underbrace{- \frac{c_2}{d}}_{\text{int}}$$



Equilibria of linear multivariable models

Rule: A linear or affine model in continuous time has only one equilibrium regardless of the number of variables, provided that the determinant of M is not zero.

- If $\frac{d\vec{n}}{dt} = M\vec{n}$ then $\hat{\vec{n}} = 0$

- If $\frac{d\vec{n}}{dt} = M\vec{n} + \vec{c}$ then $\hat{\vec{n}} = -M^{-1}\vec{c}$

recall:

$\Rightarrow$ • M^{-1} exists

• M is invertible

• $Mx = b$ has a unique solution for x for any b you choose.

• $Mx = 0 \Rightarrow x = 0$.

If det$(M) = 0$, there are an *infinite* number of equilibria.

Stability of Equilibria

Recall that a system grows or decays in the direction of an eigenvector at a rate given by its eigenvalue.

Rule: a system is unstable if it will move away from the equilibrium in at least one direction.

Because moving away = positive eigenvalue, this leads us to conclude:

Stability of equilibria (real eigenvalues):

- If all eigenvalues are negative, the ~~system~~ equilibrium is stable.
- If one or more eigenvalues are positive, the system is unstable.

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \longrightarrow \begin{matrix} \lambda_1 = 4 \\ \lambda_2 = -1 \end{matrix}$$

$$\dot{n} = An \overset{\text{equil!}}{=} 0$$

$$\Rightarrow \hat{n} = 0$$

a) stable

$\boxed{\text{b) unstable}}$

Stability of Equilibria

Stability of equilibria (real eigenvalues):

- If all eigenvalues are negative, the system is stable.
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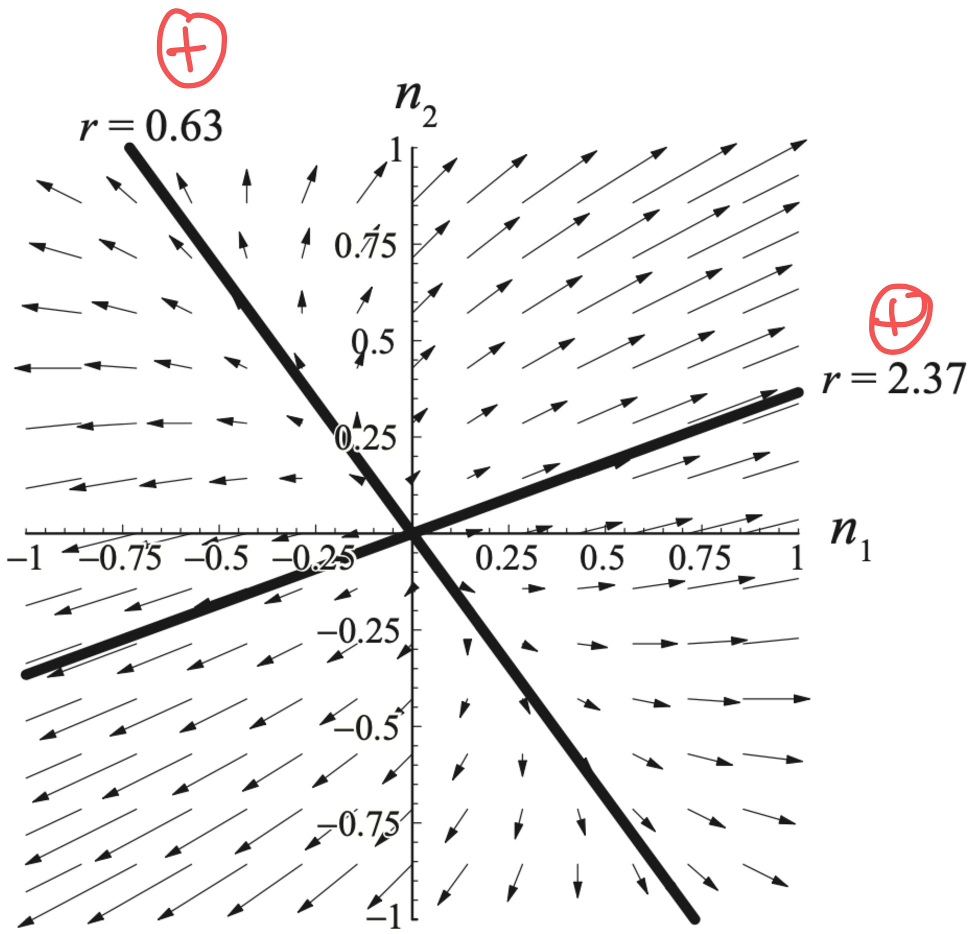
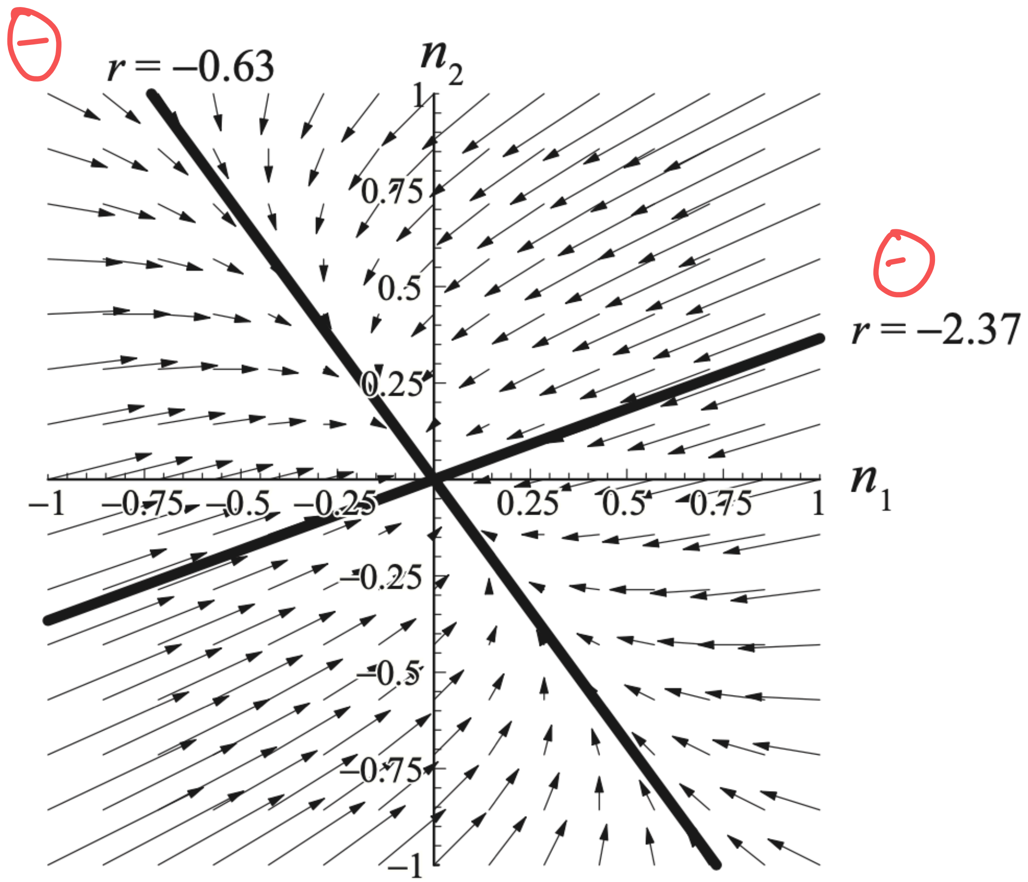
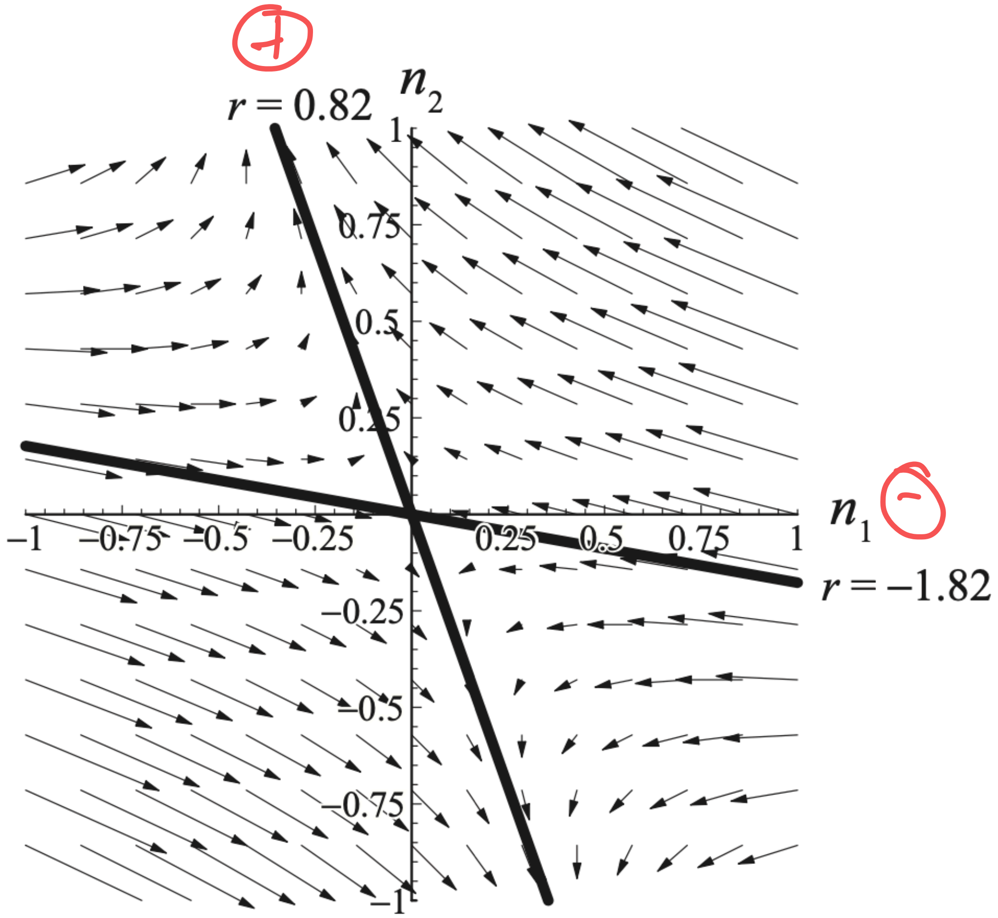


unstable



stable



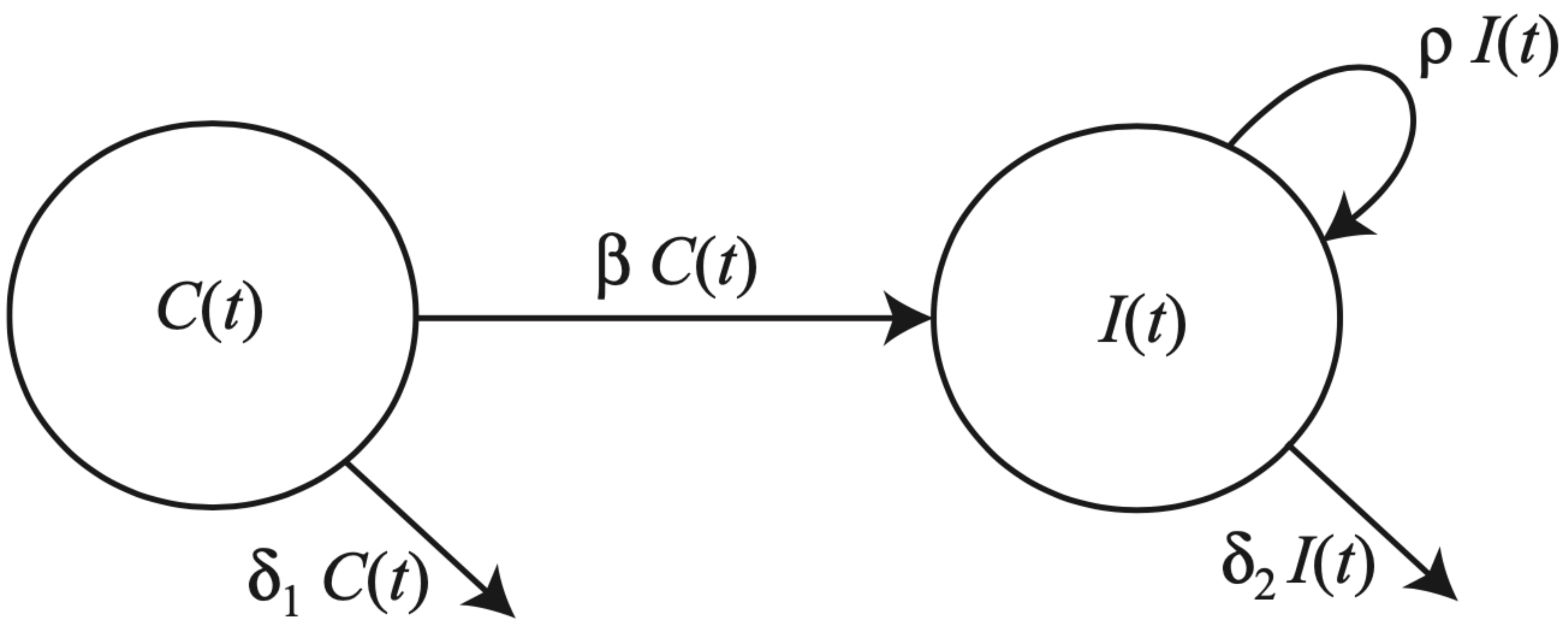
unstable.

Otto & Day — Adapted from Glass & Kaplan 1995

Metastasis of Malignant Tumors

A model for the dynamics of the number of cancer cells lodged in the capillaries of an organ, C, and the number of cancer cells that have actually invaded that organ, I.

Suppose that cells are lost from the capillaries by dislodgement or death at a per capita rate δ_1 and that they invade the organ from the capillaries at a per capita rate β. Once cells are in the organ they die at a per capita rate δ_2, and the cancer cells replicate at a per capita rate ρ.



$$\frac{dC}{dt} = -\delta_1 C - \beta C + 0 I$$

$$\frac{dI}{dt} = \beta C + \rho I - \delta_2 I$$

Otto & Day — Adapted from Glass & Kaplan 1995

Metastasis of Malignant Tumors

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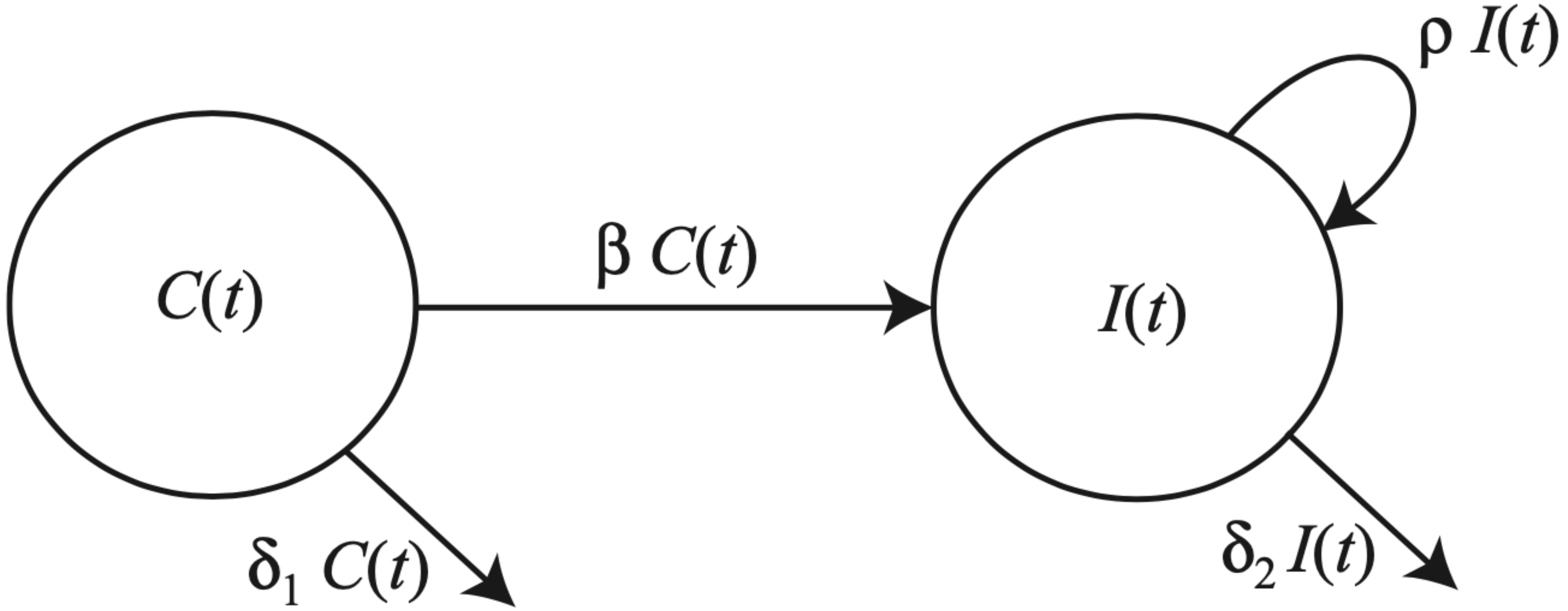

$$\frac{dC}{dt} = \delta_1 C - \beta C$$

$$\frac{dI}{dt} = \beta C - \delta_2 I + \rho I$$

$$\begin{pmatrix} \frac{dC}{dt} \\ \frac{dI}{dt} \end{pmatrix} = M \begin{pmatrix} C \\ I \end{pmatrix}$$

$$M = \begin{pmatrix} -(\delta_1 + \beta) & 0 \\ \beta & \rho - \delta_2 \end{pmatrix}$$

Otto & Day — Adapted from Glass & Kaplan 1995

Metastasis of Malignant Tumors

$$\begin{pmatrix} \frac{dC}{dt} \\ \frac{dI}{dt} \end{pmatrix} = M \begin{pmatrix} C \\ I \end{pmatrix} \qquad M = \begin{pmatrix} -(\delta_1 + \beta) & 0 \\ \beta & \rho - \delta_2 \end{pmatrix}$$

Suppose that cells are lost from the capillaries by dislodgement or death at a per capita rate δ_1 and that they invade the organ from the capillaries at a per capita rate β. Once cells are in the organ they die at a per capita rate δ_2, and the cancer cells replicate at a per capita rate ρ.

1. Identify the equilibrium or equilibria.
2. Determine the stability.