Calculating Biological Quantities

CSCI 2897

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2021, Lecture 4

“Last time“ on CSCI 2897..

1. **How to verify that a function is a solution of an ODE.**
2. **Solving an ODE initial value problem *numerically* by stepping along the solution.**

Lecture 4 Plan

1. **Exponential - Discrete time**
2. **Exponential - Continuous time**
3. **Logistic - Discrete Time**
4. **Logistic - Continuous Time**
5. **Vector fields**
6. **Examples**

Models of population growth

- For any species, at any scale, the number of individuals changes over time in response to:
  - ✓ resource availability
  - ✓ competition
  - predation
  - disease
  - ✓ weather
  - chance events
- Simplest models are called **exponential** and **logistic**.

WildType SARS-CoV-2

↓

B.1.1.7. "Alpha"

↓

"Delta"

Exponential vs Logistic Growth

- Both models assume that **the environment is constant**.
- Both models assume that there are **no interactions with other species**
  - no competing species, predators, parasites, etc.

- The models differ in their assumptions about available resources:
  - The **exponential growth model** assumes that the amount of resources available to each individual is constant, regardless of population size.
  - The **logistic growth model** assumes that fewer resources are available to each individual as the population size increases.

Discrete time exponential growth

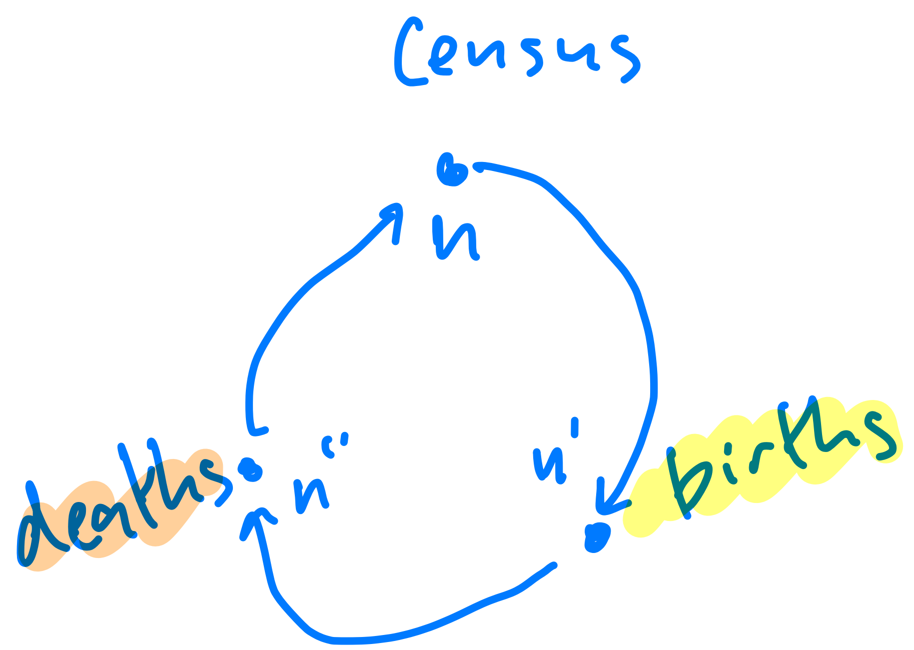
- Let $n(t)$ be the number of individuals at time t.
- Assume that each reproducing parent is replaced by a constant number of individuals R in the next time step.
- This implicitly assumes that all individuals are capable of reproduction, as in a hermaphroditic or asexual species.
    - Can also be applied to species with separate male & female sexes by assuming that the number of offspring is limited by the number of females, and then counting only females.

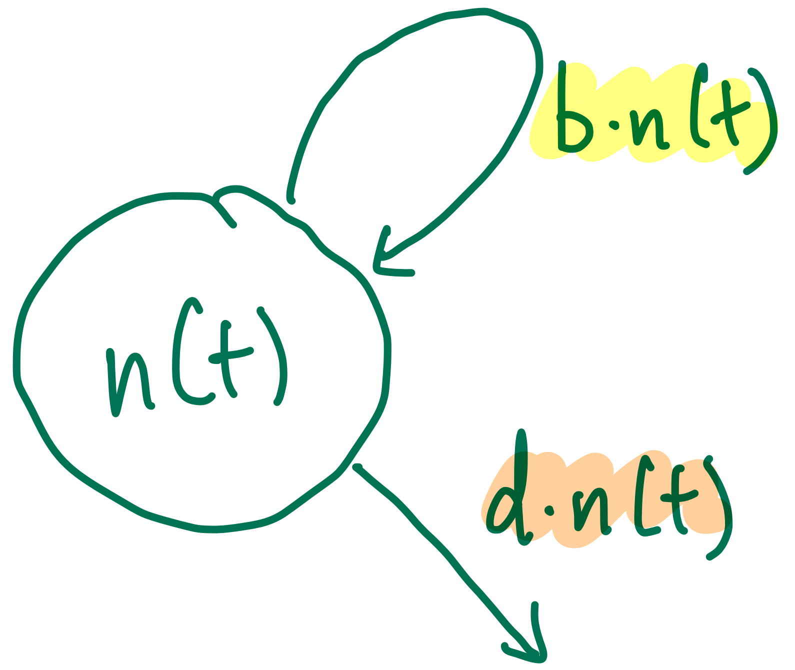
Discrete time exponential growth

- Let $n(t)$ be the number of individuals at time t.
- Assume that each reproducing parent is replaced by a constant number of individuals R in the next time step.
- In this model, we will include just two processes: birth and death.
    - Let b be the number of births per capita per time step
    - Let d be the fraction of the population that dies per time step.

Discrete time exponential growth

- Let $n(t)$ be the number of individuals at time t.
- Let b be the number of births per capita per time step
- Let d be the fraction of the population that dies per time step.
- Let's write down a Life Cycle Diagram and a Flow Diagram for this process.





Discrete time exponential growth

- Let $n(t)$ be the number of individuals at time t.
- Let b be the number of births per capita per time step
- Let d be the fraction of the population that dies per time step.
- Use the life cycle diagram to derive a **recursion** and a **difference equation**.

Discrete time exponential growth

- **Recursion**: $n(t+1) = R\ n(t)$

- **Difference**: $\Delta n = (R-1)\ n(t)$

- In the biological literature $(R-1)$ is often denoted r. How can we interpret this quantity?

Discrete time exponential growth

- **Recursion**: $n(t+1) = R\ n(t)$

- **Difference**: $\Delta n = (R-1)\ n(t)$

- In the biological literature $(R-1)$ is often denoted r or r_d .
  - How can we interpret this quantity?
  - r : per-capita change in the number of individuals from one gen. to the next.
  - Sometimes r_d to indicate that this is in $d =$ discrete time.
  - $r_d = R - 1 = (1-d)(1+b) - 1 = b - d - bd$
  - If $R = 1$, then $r_d = 0$, which means no growth—pop. size constant.

Continuous time exponential growth

- What if births and deaths can occur at any time, rather than in specific seasons or time steps?
- Same parameters: per-capita birth rate b and death rate d.
- Using the flow diagram, we can derive the differential equation:

Continuous time exponential growth

- What if births and deaths can occur at any time, rather than in specific seasons or time steps?
- Same parameters: per-capita birth rate b and death rate d.
- Using the flow diagram, we can derive the differential equation:
  - $\dfrac{dn}{dt} = bn(t) - dn(t) = r_c\, n(t)$
  - where $r_c = b - d$ is called the per-capita growth rate ($c =$ continuous time).

What can we learn from this derivation?

- Notice that r_d became r_c when we took the limit, but
- $r_d =$
- $r_c =$
- What the difference, and how can we understand it in terms of modeling?

Aside:

- Don’t worry! We’ll solve this equation (and the next one) numerically and analytically in the next two classes!
- Here is a picture of my dog to tide you over:

Logistic growth in discrete time

- Many factors slow pop. growth, including declining resource availability, increase predation, higher incidence of disease, and so on.
- Logistic model **describes theses processes indirectly** by assuming that the population replacement number R declines with increasing population size.
- We therefore write it as $R(n)$.
- Let's say that when the population size is zero, $R(0) = 1 + r_d$.
    - This is called the **intrinsic rate of growth.**
    - It's what happens when there aren't resource limitations (= prev. model).
- Let's say that $R(n)$ decreases until it becomes 1, at *some* value of n.

Logistic growth in discrete time

- Let's say that when the population size is zero, $R(0) = 1 + r_d$.
    - This is called the **intrinsic rate of growth.**
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- Let's say that $R(n)$ decreases until it becomes 1, at *some* value of n.
- A sketch helps:

Logistic growth in discrete time

- If we write $n(t+1) = R(n)\ n(t)$, we now get
- $n(t+1) =$

Logistic growth in discrete time

- If we write $n(t+1) = R(n)\ n(t)$, we now get
- $n(t+1) = n(t) + r_d\ n(t)\left(1 - \dfrac{n(t)}{K}\right)$
- $\Delta n = r_d\ n(t)\left(1 - \dfrac{n(t)}{K}\right)$

Logistic growth in *continuous time*

- If we also assume that r is a function of n, and that it declines from $r(0) = r_c$ to $r(K) = 0$, then we can also get the ODE:
- $$\frac{dn}{dt} = r_c\, n(t)\left(1 - \frac{n(t)}{K}\right)$$

QUICK QUIZ, HOT SHOT

- $$\frac{dn}{dt} = r_c\, n(t)\left(1 - \frac{n(t)}{K}\right)$$
- Order?
- Linear or nonlinear?
- ODE or PDE?

Understanding an ODE with a *vector field*

- $\dfrac{dn}{dt} = r_c \, n(t)$

Understanding an ODE with a *vector field*

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Examples of logistic growth

- Mable & Otto (2001) — cultivated both haploid & diploid *S. cerevisiae* (yeast) in two separate flasks.
- Diploid yeast cells are *bigger* and thus take up more resources.