# Calculating Biological Quantities

## CSCI 2897

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# Last time on CSCI 2987: The integrating factor

$$\frac{dy}{dx} - 3y = 1$$

Integrating Factor

$$\left[\frac{dy}{dx} - 3y\right] e^{-3x} = [1]\, e^{-3x}$$

engineered to get total derivative on LHS.

$$\frac{d}{dx}\left[e^{-3x}y\right] = e^{-3x}$$

$$\int \frac{d}{dx}\left[e^{-3x}y\right]\, dx = \int e^{-3x}\, dx$$

$$e^{-3x}y = \frac{e^{-3x}}{-3} + c$$

$$y = e^{3x}\left(\frac{e^{-3x}}{-3} + c\right)$$

$$y = \frac{1}{-3} + ce^{3x}$$

# Integrating factors — general version

To solve: $\dfrac{dy}{dx} + P(x)y = f(x)$

We multiply through by $\mu(x)$:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$$

Changing just the coloring:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$$

So for all this to work:

$$\frac{d\mu}{dx} = \mu(x)P(x)$$

which means (by SoV):

$$\mu(x) = e^{\int P(x)dx}$$

Plug back in…

By definition:

$$\frac{d}{dx}\left[\mu(x)\ y(x)\right] = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y(x)$$

$$\mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y(x)$$

$$\frac{d}{dx}\left[e^{\int P(x)dx}\ y(x)\right] = e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y(x) = e^{\int P(x)dx}f(x)$$

$\dfrac{d}{dx}\left[e^{\int P(x)dx}\ y(x)\right] =$ cut out the middleman $= e^{\int P(x)dx}f(x)$

Ready to integrate both sides

$$\frac{d}{dx}\left[e^{\int P(x)dx}\ y(x)\right] = e^{\int P(x)dx}f(x)$$

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Changing just the coloring:

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By definition:

$$\frac{d}{dx}\left[\mu(x)\ y(x)\right] = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y(x) \longrightarrow \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y(x)$$

So for all this to work:

$$\frac{d\mu}{dx} = \mu(x)P(x)$$

which means (by SoV):

$$\mu(x) = e^{\int P(x)dx}$$

Plug back in…

$$\frac{d}{dx}\left[e^{\int P(x)dx}\ y(x)\right] = e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y(x) = e^{\int P(x)dx}f(x)$$

$$\frac{d}{dx}\left[e^{\int P(x)dx}\ y(x)\right] = \quad \text{cut out the middleman} \quad = e^{\int P(x)dx}f(x)$$

Ready to integrate both sides

$$\frac{d}{dx}\left[e^{\int P(x)dx}\ y(x)\right] = e^{\int P(x)dx}f(x)$$

# Integrating factors — general version

$$\frac{dy}{dx} + P(x)y = f(x)$$

1. Get the equation into this standard form.
2. The I.F. is $\mu(x) = e^{\int P(x)dx}$. Multiply both sides by $\mu(x)$.
3. Write the LHS as $\dfrac{d}{dx}\left[\mu(x)y(x)\right]$.

   2 & 3 combined: $\dfrac{d}{dx}\left[e^{\int P(x)dx}\, y(x)\right] = e^{\int P(x)dx} f(x)$
4. Integrate both sides with respect to $dx$
5. Solve for $y(x)$.
6. Plug in the initial condition $y(x_0) = y_0$.
7. 🎉🎉🎉

# One key point: only the *left side* affects $\mu(x)$

(once we get the equation into "standard form")

$$\frac{dy}{dx} + y = x \qquad \mu(x) = e^{\int 1\,dx} = e^{x+c} = k e^{x}$$

$$\frac{dy}{dx} + y = 1998x^2 \qquad \mu(x) = e^{\int 1\,dx} = e^{x+c} = k e^{x}$$

$$\frac{dy}{dx} + y = f(x) \qquad \mu(x) = e^{\int 1\,dx} = e^{x+c} = k e^{x}$$

Q: why not include the constant?

A: we would just cancel it out in the next step!

# Today

1. **Wrapping up the Integrating Factor method.**
2. **Measles and the SIR model.**

# Practice makes the master

Solve $x\dfrac{dy}{dx} - 4y = x^5 e^x$, with $y(0) = 1$

Fix this, Dan!

1. $\dfrac{dy}{dx} + P(x)y = f(x)$

1. Get the equation into this standard form.
2. The integrating factor is $\mu(x) = e^{\int P(x)dx}$. Multiply both sides by $\mu(x)$.
3. Write the LHS as $\dfrac{d}{dx}\left[\mu(x)y(x)\right]$ and integrate both sides with respect to $dx$
4. Solve for $y(x)$.
5. Plug in the initial condition $y(x_0) = y_0$.
6. 🎉🎉🎉

① into standard form

$$\frac{dy}{dx} - \frac{4}{x}y = x^4 e^x$$

② this means $P(x) = -\dfrac{4}{x}$

so the I.F. is $\mu(x) = e^{\int -\frac{4}{x}dx}$

$$= e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}$$

③ Rewriting the ODE:

$$\int \frac{d}{dx}\left[x^{-4}\cdot y(x)\right]dx = \int \cancel{x^4} e^x \cancel{x^{-4}}\, dx$$

3... Integrating both sides, we get

$$x^{-4}y(x) = e^x + c$$

④ Solving for $y$, we get:

$$y(x) = e^x x^4 + cx^4 = (e^x + c)x^4$$

⑤ $y_0 = (e^{x_0} + c)x_0^4 \longrightarrow \dfrac{y_0}{x_0^4} = e^{x_0} + c$

$$c = \left(\frac{y_0}{x_0^4} - e^{x_0}\right)$$

$$\boxed{y(x) = x^4\left(e^x + \frac{y_0}{x_0^4} - e^{x_0}\right)}$$

# Final thoughts 1 — linear first order ODEs

**Remember that** a linear first-order ODE can be written in this form:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Today we worked with equations that were in the form

$$\frac{dy}{dx} + P(x)y = f(x)$$

What is the relationship between $P$, $f$, and $a_1$, $a_0$?

$$P(x) = \frac{a_0(x)}{a_1(x)} \qquad f(x) = \frac{g(x)}{a_1(x)}$$

# Final thoughts 2 — linear first order ODEs

**Remember that** a linear first-order ODE can be written in this form:

$$a_1(x)\frac{dy}{dx}+a_0(x)y = g(x)$$

Solve using S.o.V.

When $g(x) = 0$, we call the linear first order ODE **homogeneous**.

When $g(x)$ is anything other than $0$, we call the ODE **nonhomogeneous**.

I.F. method.

# Final thoughts 2 — linear first order ODEs

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When $g(x)$ is anything other than $0$, we call the ODE **nonhomogeneous**.

Remember: people name ideas and create vocabulary/jargon when they think those things are important.

- It's a nice thing to **learn another field's vocabulary**, because **you learn what they think is important.** Try to see this as an opportunity, rather than a barrier—don't let vocab push you out or exclude you.
- **Be gentle with outsiders** from other fields when they try to learn your vocabulary too.

# Measles and the SIR model

- **Measles** is an infectious disease caused by *Measles morbillivirus* (MeV).
- MeV is a single-stranded RNA virus, that infects only humans.
- Measles causes fever, cough, runny nose, inflamed eyes, and a rash.
- It is spread by contact—coughing, sneezing, or secretions.

# Measles and the SIR model

AIDS
COVID-19 Malaria HIV
SARS-CoV-2 Plasmodium parasite

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- Case fatality rate is dependent on care (0.3% in the U.S, but up to 20% in some places.)
- Immunosuppressive — leads to complications.
- Around 0.1% of cases lead to encephalitis, with potential brain damage.

# Measles and the SIR model

"well mixed"

- To model an infectious disease like measles, we use a **compartmental model** call the "SIR" model:

**S:** susceptible **I:** infected (infectious) **R:** recovered / removed

- Rules:

1. Each person is a member of only one compartment in each time step.
2. People can move between compartments according to these rules:

$S \times S \longrightarrow$ remain $S$

$I \times I \longrightarrow$ remain $I$

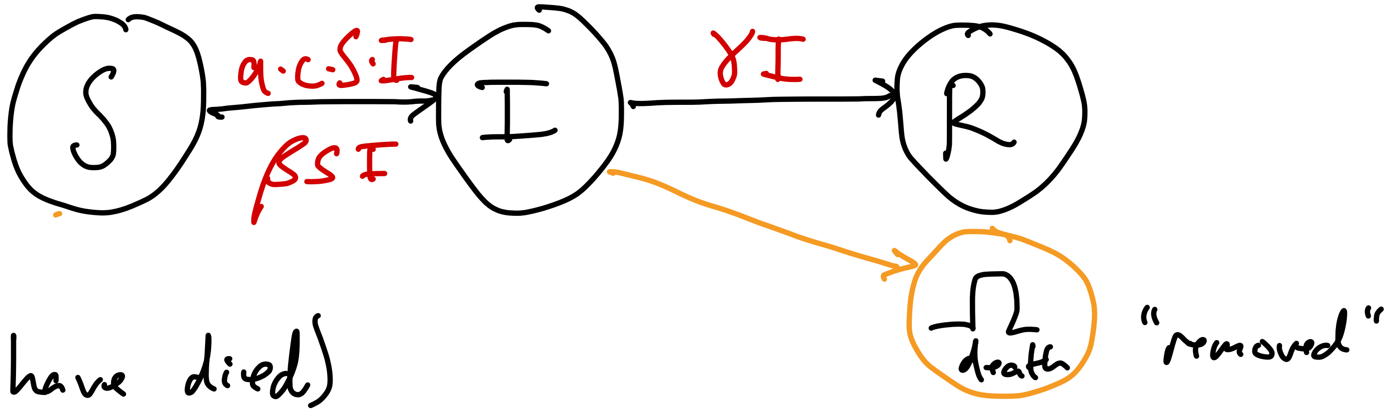
$R \times R \longrightarrow$ remain $R$

$S \times I \longrightarrow$ $S$ becomes $I$ w.p. $a$

- Rate at which people come into contact $= c$
- $I$ people recover at a rate $\gamma$ per person per time

$$S \xrightarrow{a \cdot c \cdot S \cdot I} I \xrightarrow{\gamma I} R$$

# The canonical SIR model:




① $S + I + R = N$ total pop. size.

Assume $N$ constant. (Includes those who may have died)

② Immunity: • must be infected before you get immunity (R)

• immunity lasts forever (can't get out of R)

$$\boxed{\begin{aligned} \frac{dS}{dt} &= -\beta S I \\ \frac{dI}{dt} &= \beta S I - \gamma I \\ \frac{dR}{dt} &= \gamma I \end{aligned}}$$

→ What if $I = 0$?

$$\frac{dS}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0 \quad \text{(equilibrium)}$$

$$(S, I, R)_{equil} = (S, 0, N-S)$$

If no infections, the process stops.

• small pox.

• New Zealand COVID-19

→ What if $S = 0$?

$$\frac{dS}{dt} = 0 \qquad \frac{dI}{dt} = -\gamma I \quad \rightarrow \quad I(t) = k e^{-\gamma t}$$

If no more susceptibles, Infectious population decreases exponentially.

# What's one thing we can do with a set of ODEs?

Equilibria! $\longrightarrow$ ALL derivatives $= 0$

$$\frac{dS}{dt} = -\beta S I = 0 \qquad ③ \quad -\beta S I = 0 \;\; (\text{eq.}) \qquad -\beta S \cdot 0 = 0 \qquad 0 = 0$$

$$\frac{dI}{dt} = \beta S I - \gamma I = 0 \qquad ② \quad \beta S I - \gamma I = 0 \qquad \beta S \cdot 0 - \gamma \cdot 0 = 0 \qquad 0 = 0$$

$$\frac{dR}{dt} = \gamma I = 0 \qquad ① \quad \gamma I = 0 \quad \Rightarrow \boxed{I = 0}$$

no new info!

when $I = 0$, no $\frac{dR}{dt}$

and no $\frac{dI}{dt}$

$\Rightarrow I = 0$

IDK anything about S or R

except $S + R = N$

$$(S, I, R)_{eq} = (S, 0, N - S)$$

"only" equilibrium

# Breakout:

$$S \xrightarrow{\beta SI} I \xrightarrow{\gamma I} R$$

$$\frac{dS}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

I claim that: $\dfrac{dS}{dt} + \dfrac{dI}{dt} + \dfrac{dR}{dt} = 0$

Can you think of at least one way to show why this is true using math?

$$-\beta SI + \beta SI - \gamma I + \gamma I = 0$$

Can you explain why this is true in words?

For every outflow, there's a matching inflow (see diagram).

$$N = S + I + R$$

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$$

$$0 = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$$

Pop'n isn't changing