Calculating Biological Quantities

CSCI 2897

- HW1 → returned ASAP
- HW2 in → return in < week
- HW3 posted. Due 10/7
  Start early! Team up!

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2021, Lecture 9

Last time on CSCI 2987: Consumer-Resource Models

$$\frac{dn_1}{dt} = f(n_1) - g(n_1, n_2)$$

$$\frac{dn_2}{dt} = \epsilon g(n_1, n_2) - h(n_2)$$

$f(n_1)$: rate of change of the resource via means other than consumption $(n_2 = 0)$.

$g(n_1, n_2)$: rate of consumption of the resource by the consumer.

ϵ : the conversion factor by which resource units $\rightarrow$ consumer units.

$h(n_2)$: rate at which the number of consumers changes without resources $(n_1 = 0)$.

TABLE 3.3
Consumer-resource models. Examples of functions that can be used in the consumer-resource model (3.16), where n_1 refers to the level of resources (e.g., number of prey) and n_2 refers to the level of consumers (e.g., number of predators).

Function	Description
$f(n_1) = \theta$	Inflow of resources at a constant rate
$f(n_1) = -\psi$	Outflow of resources at a constant rate
$f(n_1) = r\, n_1$	Constant per capita growth of resource species
$f(n_1) = rn_1\left(1 - \frac{n_1}{K}\right)$	Per capita growth of resource species declines linearly with resource level (logistic)
$f(n_1) = r\, n_1\, e^{-\alpha n_1}$	Per capita growth of resource species declines exponentially with resource level
$g(n_1, n_2) = a\, c\, n_1\, n_2$	Linear (type I) rate of resource consumption
$g(n_1,n_2) = \frac{a\, c\, n_1}{b + n_1} n_2$	Saturating (type II) rate of resource consumption
$g(n_1,n_2) = \frac{a\, c\, n_1^k}{b + n_1^k} n_2$	Generalized (type III) rate of resource consumption
$h(n_2) = \delta\, n_2$	Constant per capita death rate of consumer
$h(n_2) = (\delta\, n_2)\, n_2$	Per capita death rate of consumer increases linearly with consumer population size

Otto & Day, Table 3.3

Lecture 9 Plan

1. **Let's reverse engineer an equation:**

   what does the equation tell us about the biology?

2. **New math: the "integrating factor" method.**

Working Backward: Equations to Interpretation

By now, we're pretty good at taking a system and working forward, from behavior into a diagram and then into equations.

Sometimes, it's useful to be able to work in reverse, by taking an equation, and then interpreting the equation in terms of biology.

For example, here's the logistic equation with constant hunting or harvesting.

$$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - \theta$$

Working Backward: Equations to Interpretation

However, all 3 of these equations could be described as "logistic growth with constant hunting or harvesting." So what's the difference in *meaning*?

1. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - \theta$$

2. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - Hn(t)$$

3. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - H(n(t) - P)$$

- What is the story being told by each of these?
- What can we say about the "mechanism" of constant harvesting implied by each equation?

Working Backward: Equations to Interpretation

However, all 3 of these equations could be described as "logistic growth with constant hunting or harvesting." So what's the difference in *meaning*?

1. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - \theta$$

(Continuous)

- Harvesting (decreasing n) by a constant amount.
    1) $-\theta$ does not depend on t
    2) $-\theta$ does not depend on n.
- Rate of consumption/harvesting/hunting is independent of population size!
- maybe: $\theta \sim$ death? emigration?

IRL example:

- CO gives out a fixed # of elk tags each year.
- Harvesting same amount of basil every week.

Working Backward: Equations to Interpretation

However, all 3 of these equations could be described as “logistic growth with constant hunting or harvesting.” So what’s the difference in *meaning*?

const H

pop'n size $n(t)$.

$$2.\ \frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - Hn(t)$$

Hunting occurs at a constant "per capita" rate. Scales ↑ ↓ with the pop'n $n(t)$.

Experience / Risk of an individual is different:

- $-\theta$ decreased individual risk as n increases.
- $-Hn(t)$ same individual risk, regardless of n.

IRL examples:

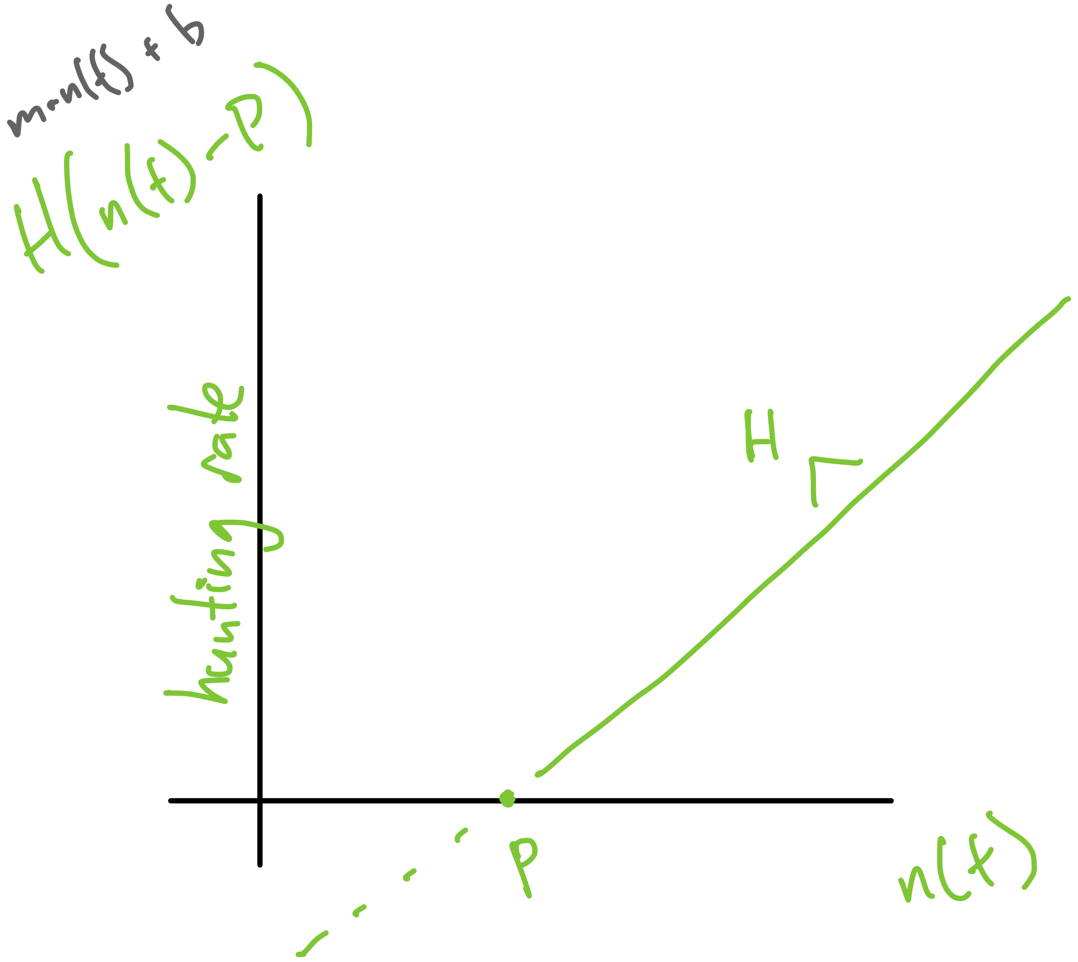
- Harvest a fixed percentage of the ~~basil~~ zucchini.
- Clearing bugs off a plant.
- fish tank filter

Working Backward: Equations to Interpretation

However, all 3 of these equations could be described as “logistic growth with constant hunting or harvesting.” So what’s the difference in *meaning*?

const; pop'n; const.

$$3.\ \frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - H(n(t) - P)$$



- Hunting applies not to entire pop'n n(t), but instead to a subset n(t) − P.
- When n(t) > P ⟶ Hunting
  n(t) < P ⟶ negative hunting? (restocking, replenishing!)
- When n(t) = P, then H(n(t) − P) = 0
  ⟹ no hunting effect n(t) = P.

$$\frac{dn}{dt} = [\text{logistic}] - Hn(t) + HP$$

const. per-capita harvesting

const total population restocking or immigration!

Let's level up our ODE game

Warmup: (Sep. of Vars)

$$\frac{dy}{dx} - 3y = 0$$

$$\frac{dy}{dx} = 3y$$

$$\int \frac{dy}{y} = \int 3\,dx$$

$$\ln y = 3x + c$$

$$e^{\wedge} \qquad e^{\wedge}$$

$$y = e^{3x + c}$$

$$y = e^{3x} \cdot \underbrace{e^{c}}_{k}$$

$$\boxed{y = k e^{3x}}$$

$$\int \frac{dy}{3y} = \int dx$$

alt. separation

$$\frac{1}{3} \ln y = x + c$$

$$\ln y = 3x + 3c$$

$$y = e^{3x + 3c}$$

$$y = e^{3x} \cdot \underbrace{e^{3c}}_{k}$$

$$\boxed{y = k e^{3x}}$$

Aside: *remember the product rule*

\mu

$$\frac{d}{dx}\big(\mu(x)\, y(x)\big) = \underbrace{\mu(x) \cdot \frac{dy}{dx}}_{\frac{dy}{dx}\ \text{term}} + \underbrace{\frac{d\mu}{dx} \cdot y(x)}_{y\ \text{term}}$$

$$\int \frac{d}{dx}\big(\mu(x)\, y(x)\big)\, dx = \int \mu(x) \frac{dy}{dx} + \frac{d\mu}{dx} \cdot y(x)\, dx$$

$$\mu(x)\, y(x) = \int \mu(x) \frac{dy}{dx} + \frac{d\mu}{dx}\, y(x)\, dx$$

What if we make it *just* a little different?

dy/dx term

y term

$$\frac{dy}{dx} - 3y = 1$$

$$\frac{dy}{dx} - 3y = 1$$

μ is the "integrating factor"

$$\mu \frac{dy}{dx} - 3\mu y = \mu$$

$$\frac{d}{dx}\left[\mu(x)\cdot y(x)\right] = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y(x)$$

match

$$1\cdot\frac{dy}{dx} + (-3)y(x)$$

match?

mult ODE by μ

$$\mu\cdot\frac{dy}{dx} + (-3\mu)\cdot y(x)$$

mean $\frac{d\mu}{dx} = -3\mu \Rightarrow \mu(x) = ke^{-3x}$

product rule in reverse!

$$\underbrace{ke^{-3x}\frac{dy}{dx} - 3ke^{-3x}y}= ke^{-3x}$$

$$\int \frac{d}{dx}\left[ke^{-3x}\cdot y(x)\right]dx = \int ke^{-3x}\,dx$$

$$\frac{ke^{-3x}\cdot y(x)}{ke^{-3x}} = \frac{\frac{k}{-3}e^{-3x} + c}{ke^{-3x}}$$

$$\boxed{y(x) = -\frac{1}{3} + me^{3x}}$$

$m = \frac{k}{c}$

Recap:

$$\mu \cdot \frac{dy}{dx} - 3y\mu = 1 \cdot \mu$$

- We observed that if there were a function called $\mu(x)$, then (product rule):

$$\frac{d}{dx}\left[\mu(x)y(x)\right] = \mu\frac{dy}{dx} + y\frac{d\mu}{dx}$$

- We compared this to our ODE: one $\dfrac{dy}{dx}$ term and one y term!
- Then, we matched up terms to figure out what $\mu(x)$ should be.
- This required us to solve another ODE (sep. of vars.) which we did.
- Then we integrated both sides and solved for $y(x)$.

Example:

$$\frac{dy}{dx} + y = x, \text{ with } y(0) = 4$$

$$\frac{d}{dx}\left[\mu \cdot y\right] = \mu \cdot \frac{dy}{dx} + \frac{d\mu}{dx} \cdot y$$

$\updownarrow \checkmark$ $\quad \updownarrow$ if $\frac{d\mu}{dx} = \mu$

take ODE, mult. by μ. $\rightarrow \mu \cdot \frac{dy}{dx} + \mu \cdot y = \mu \cdot x$

$$\frac{d\mu}{dx} = \mu \longrightarrow \mu(x) = k e^{x}$$

Plug in μ... to previous equation

$$k e^{x} \frac{dy}{dx} + k e^{x} y = k e^{x} \cdot x$$

MAGIC

$$\int \frac{d}{dx}\left[k e^{x} \cdot y(x)\right] dx = \int k e^{x} x \, dx$$

$$k e^{x} y(x) = k \int \bar{x} e^{\bar{x}} \, d\bar{x}$$

$$y(x) = \frac{k}{k e^{x}}\left[\int \bar{x} e^{\bar{x}} \, d\bar{x}\right]$$

$$y(x) = e^{-x}\left[\int \bar{x} e^{\bar{x}} \, d\bar{x}\right]$$

Example (continued): $\frac{dy}{dx} + y = x$, with $y(0) = 4$

Side Quest

$\int x e^x \, dx$

Integration by Parts!

$u = x \rightarrow du = 1\,dx$

$dv = e^x dx \rightarrow v = e^x$

plug in

$\int u\,dv = u \cdot v - \int v\,du$

$\int x e^x dx = x e^x - \int e^x \cdot 1 \cdot dx$

$= x e^x - e^x + c$

$= e^x(x-1) + c$

side quest complete!

$y(x) = e^{-x}\left[\int \bar{x} e^{\bar{x}} d\bar{x}\right]$

$\downarrow$

$y(x) = e^{-x}\left[e^x(x-1) + c\right]$

$y(x) = x - 1 + ce^{-x}$

plug in to get c.

$x = 0 \quad y = 4$

$4 = 0 - 1 + ce^{-0}$

$4 = -1 + c$

$\boxed{5 = c}$

$\boxed{y(x) = x - 1 + 5e^{-x}}$

One key point: only the *left side* affects $\mu(x)$

(once we get the equation into "standard form")

$$\frac{dy}{dx} + y = x \qquad \mu(x) = e^x$$

$$\frac{dy}{dx} + y = 1998x^2 \qquad \mu(x) = e^x$$

$$\frac{dy}{dx} + y = f(x) \qquad \mu(x) = e^x$$

$$\underbrace{\frac{dy}{dx} + \text{some function} \cdot y}_{\text{used to get } \mu(x)} = \text{whatever}$$

Integrating factors — general version

$$\frac{dy}{dx} + P(x)y = f(x)$$

$$\frac{d}{dx}\left[\mu(x)\cdot y(x)\right] = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y(x)$$

✓

match ODE (mult by $\mu(x)$)

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y(x) = f(x)\mu(x)$$

$$\downarrow$$

$$\frac{d\mu}{dx} = \mu(x)P(x)$$

S.o.V. !

$$\int \frac{d\mu}{\mu(x)} = \int P(x)\,dx$$

$$\ln \mu(x) = \int P(x)\,dx$$

$$\boxed{\mu(x) = e^{\int P(x)dx}}$$

general I.F.

$\downarrow$ method

$$e^{\int P(x)dx}\left[\frac{dy}{dx} + P(x)y\right] = e^{\int P(x)dx}\cdot f(x)$$

$$\int \frac{d}{dx}\left[e^{\int P(x)dx}\cdot y\right] = \int e^{\int P(x)dx}\cdot f(x)$$

$$e^{\int P(x)dx}\cdot y = \int f(x)\,e^{\int P(x)dx}\,dx$$

(2) solve for y

(1) solve RHS

Integrating factors — general version

$$\frac{dy}{dx} + P(x)y = f(x)$$

1st order

Linear, but not separable!

1. Get the equation into this standard form.
2. The integrating factor is $\mu(x) = e^{\int P(x)dx}$. Multiply both sides by $\mu(x)$.
3. Write the LHS as $\dfrac{d}{dx}\left[\mu(x)y(x)\right]$ and integrate both sides with respect to dx
4. Solve for $y(x)$.
5. Plug in the initial condition $y(x_0) = y_0$.
6. 🎉🎉🎉