Calculating Biological Quantities

CSCI 2897

Prof. Daniel Larremore

daniel.larremore@colorado.edu
@danlarremore

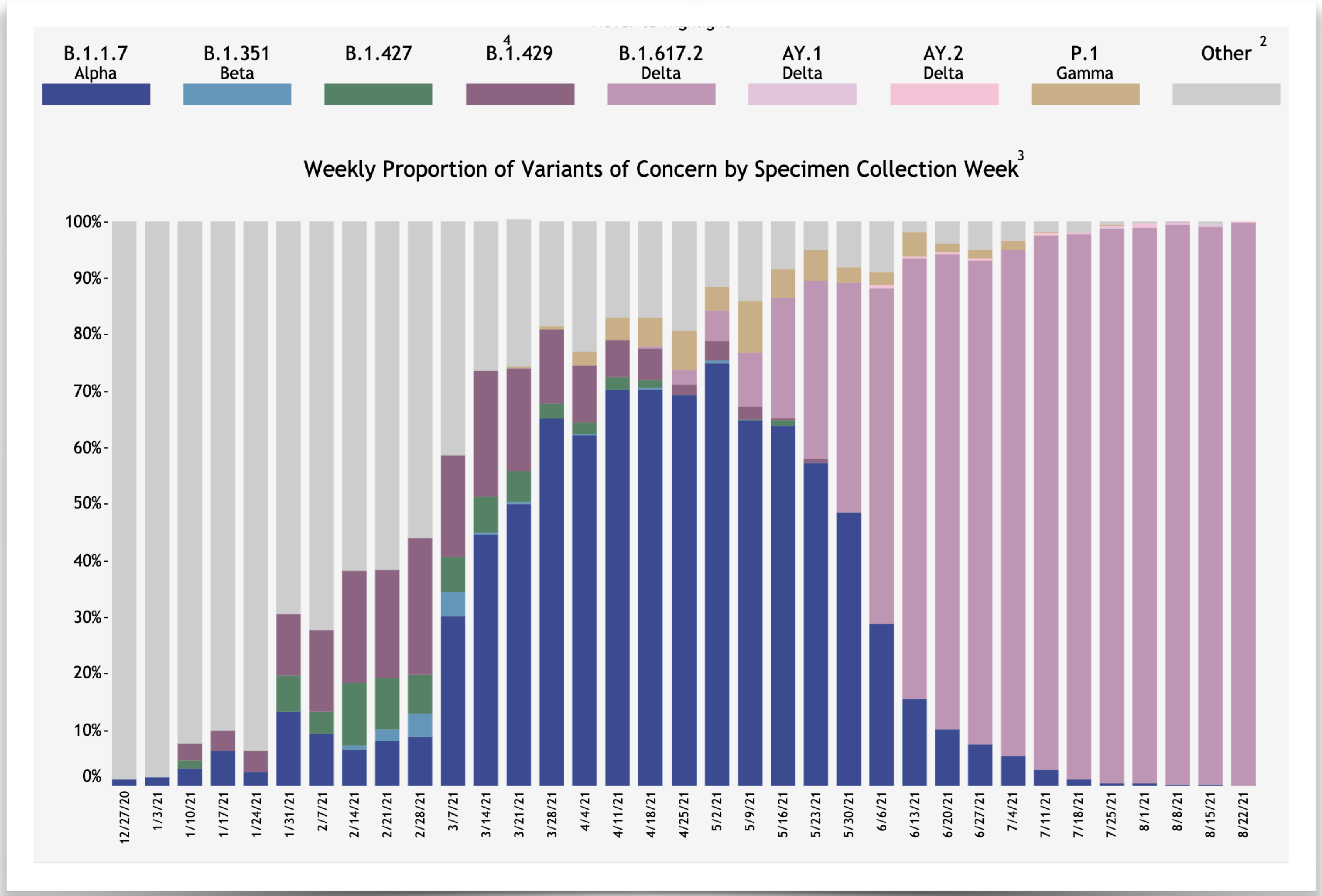
2021, Lecture 7

Last time on CSCI 2897..

1. Haploid models of natural selection

$$\frac{dp}{dt} = s_c p(t)\big(1 - p(t)\big)$$

$$s_c = (b_A - d_A) - (b_a - d_a)$$



Lecture 7 Plan

1. **Equilibrium solutions**
2. **Lotka-Volterra Model of Competition**

Equilibrium

A system at **equilibrium** does not change over time. (Plural: **equilibria**.)

For a discrete time model, at equilibrium, it must be true that:

For a continuous time model, at equilibrium, it must be true that:

Sometimes we call an equilibrium a **steady state**.

Equilibrium

A system at **equilibrium** does not change over time. (Plural: **equilibria**.)

What is the equilibrium / what are the equilibria for our haploid frequency equation?

$$\frac{dp}{dt} = s_c p(t)\big(1 - p(t)\big)$$

Note: we're always solving for equilibrium values of the *variables*, not the *parameters*.

Stability

An equilibrium is **locally stable** if a system near that equilibrium approaches it. This property is called **locally attracting**.

An equilibrium is **globally stable** if a system approaches that equilibrium *regardless* of its initial position.

An equilibrium is **unstable** if a system near the equilibrium moves away from it. This property is called **repelling**.

Stability

Are the equilibria for our haploid allele frequency equation stable or unstable?

$$\frac{dp}{dt} = s_c p(t)\big(1 - p(t)\big)$$

Bonus

Identify the equilibrium/a of the logistic growth equation, and characterize stability.

$$\dot{n} = r\ n \left(1 - \frac{n}{K} \right)$$

Lotka-Volterra Competition

Imagine that there are two species, with population sizes $n_1(t)$ and $n_2(t)$.

Let's imagine that each one has the property from Logistic Growth where its growth rate R depends on its population size n, so we have $R_1(n_1)$ and $R_2(n_2)$.

What if one species' growth rate depended on the size of the other population?

Specifically, suppose that species i experiences competition *as if its own species had population* $n_i(t) + \alpha_{ij}\, n_j(t)$. (Here, i could be 1 or 2).

Lotka-Volterra Competition

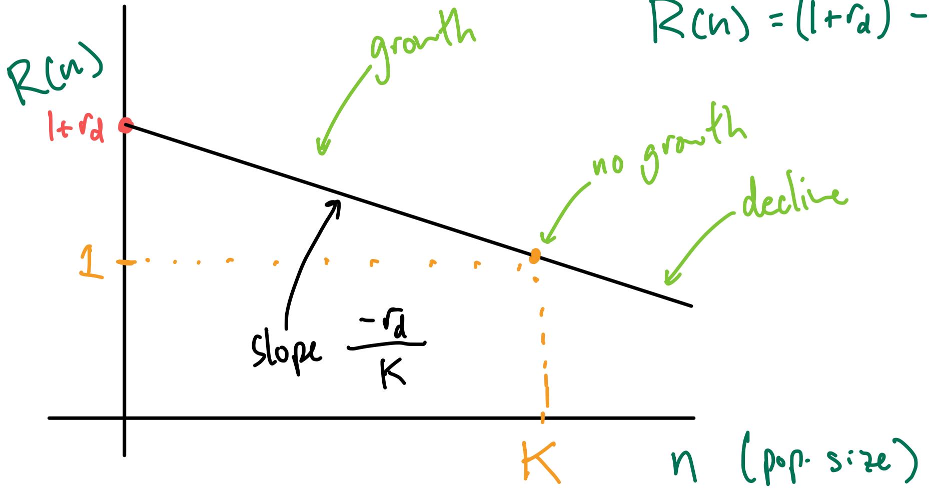
Remember when we derived the Logistic Growth equation?

Logistic growth in discrete time

compare to $y = b + mx$

- Let's say that when the population size is zero, $R(0) = 1 + r_d$.
  - This is called the **intrinsic rate of growth.**
  - It's what happens when there aren't resource limitations (= prev. model).
- Let's say that $R(n)$ decreases until it becomes 1, at *some* value of n.
- A sketch helps:

$R(n) = (1 + r_d) - \frac{r_d}{K} \cdot n(t)$



Logistic growth in discrete time

- If we write $n(t+1) = R(n)\ n(t)$, we now get $R(n) = (1+r_d) - \frac{r_d}{K} \cdot n(t)$
- $n(t+1) = \left[(1+r_d) - \frac{r_d}{K} n(t) \right] n(t)$

$$\underbrace{n(t+1)}_{\text{next}} = \underbrace{n(t)}_{\text{prev}} + \underbrace{r_d \left(1 - \frac{n(t)}{K} \right) n(t)}_{\text{change}}$$

We're now going to modify that equation for $R(n)$.

Lotka-Volterra Competition

Let $R_i = (1 + r_i) + \left(\dfrac{-r_i}{K_i}\right)\Big(n_i(t) + \alpha_{ij} n_j(t)\Big)$

Let's plug in this reproductive factor into *each* of our two update equations:

$$n_1(t+1) =$$

$$n_2(t+1) =$$

Lotka-Volterra Competition

We can write similar equations in continuous time:

$$\frac{dn_1}{dt} =$$

$$\frac{dn_2}{dt} =$$

Lotka-Volterra Competition

Quick check: if the species don't interact, then:

which implies that…

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t) + \alpha_{12}\ n_2(t)}{K_1}\right) =$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t) + \alpha_{21}\ n_1(t)}{K_2}\right) =$$

Interpretation:

Also note: this model is *symmetric* in that relabeling $1 \leftrightarrow 2$ produces the same equations.

Lotka-Volterra…Competition?

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t) + \alpha_{12}\, n_2(t)}{K_1}\right)$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t) + \alpha_{21}\, n_1(t)}{K_2}\right)$$

What if α_{12} is negative? How does an increase in n_2 affect $\frac{dn_1}{dt}$?

Lotka-Volterra…Competition?

α_{12}	α_{21}	Relationship

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t)+\alpha_{12}\, n_2(t)}{K_1}\right)$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t)+\alpha_{21}\, n_1(t)}{K_2}\right)$$

Let's code up the Lotka-Volterra model to explore!

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t) + \alpha_{12}\, n_2(t)}{K_1}\right)$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t) + \alpha_{21}\, n_1(t)}{K_2}\right)$$

with initial conditions

$$n_1(0) = a$$

$$n_2(0) = b$$