Calculating Biological Quantities

CSCI 2897

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"Last time" on CSCI 2897..

1. **How to verify that a function is a solution of an ODE.**
2. **Solving an ODE initial value problem *numerically* by stepping along the solution.**

Lecture 4 Plan

1. **Exponential - Discrete time**
2. **Exponential - Continuous time**
3. **Logistic - Discrete Time**
4. **Logistic - Continuous Time**
5. **Vector fields**
6. **Examples**

Models of population growth

- For any species, at any scale, the number of individuals changes over time in response to:
  - resource availability
  - competition
  - predation
  - disease
  - weather
  - chance events
- Simplest models are called **exponential** and **logistic**.

Wild Type SARS-CoV-2

↓

B.1.1.7. "Alpha"

↓

"Delta"

Exponential vs Logistic Growth

- ✓ Both models assume that **the environment is constant**.
- ✓ Both models assume that there are **no interactions with other species**
  - no competing species, predators, parasites, etc.

- The models differ in their assumptions about available resources:
  - The **exponential growth model** assumes that the amount of resources available to each individual is constant, regardless of population size.
  - The **logistic growth model** assumes that fewer resources are available to each individual as the population size increases.

Discrete time exponential growth

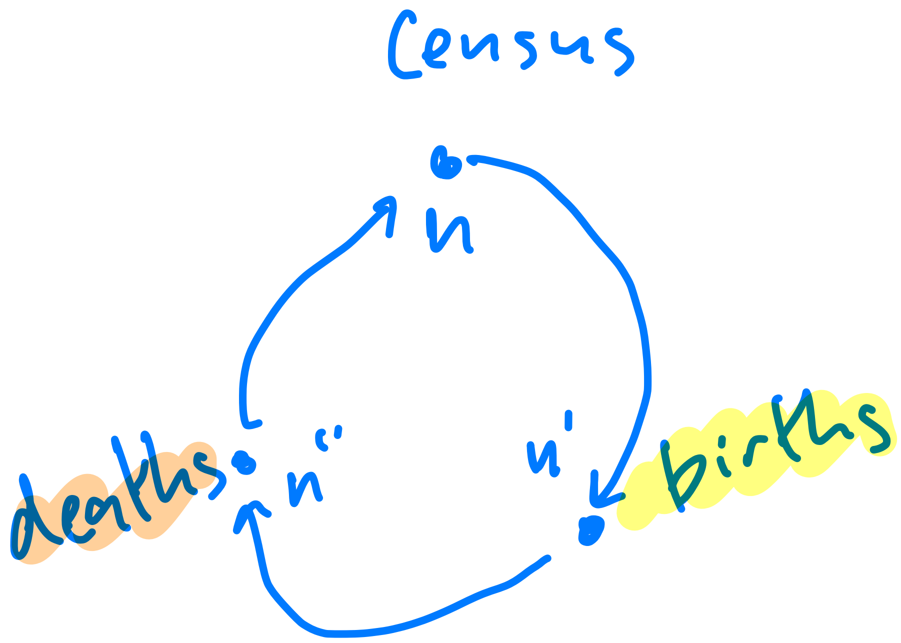
- Let $n(t)$ be the number of individuals at time t.
- Assume that each reproducing parent is replaced by a constant number of individuals R in the next time step.
- This implicitly assumes that all individuals are capable of reproduction, as in a hermaphroditic or asexual species.
    - Can also be applied to species with separate male & female sexes by assuming that the number of offspring is limited by the number of females, and then counting only females.

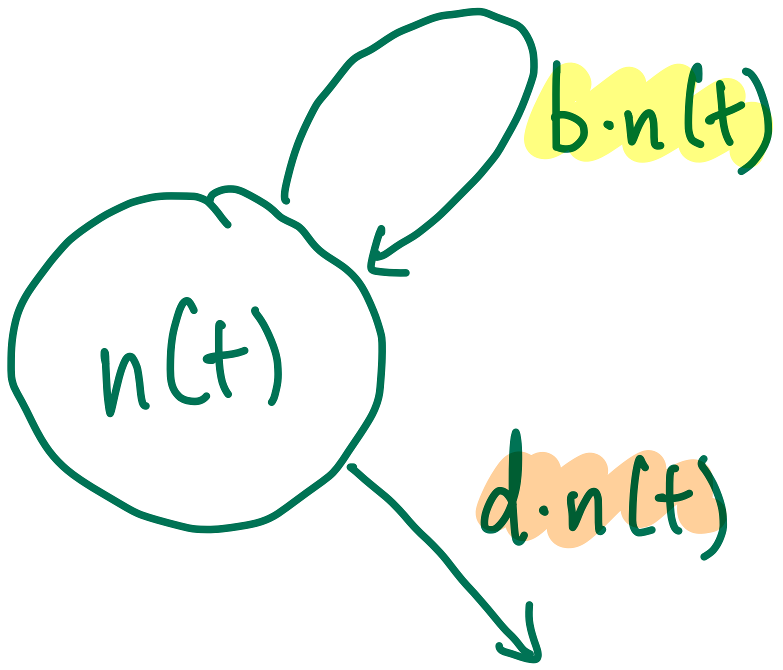
Discrete time exponential growth

- Let $n(t)$ be the number of individuals at time t.
- Assume that each reproducing parent is replaced by a constant number of individuals R in the next time step.
- In this model, we will include just two processes: birth and death.
  - Let b be the number of births per capita per time step
  - Let d be the fraction of the population that dies per time step.

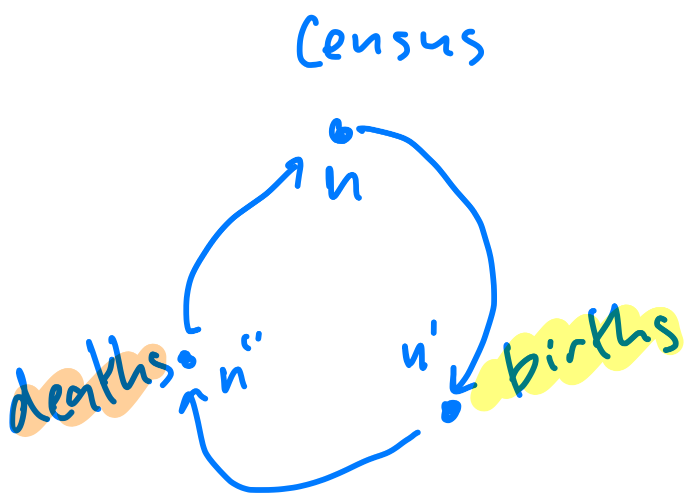
Discrete time exponential growth

- Let $n(t)$ be the number of individuals at time t.
- Let b be the number of births per capita per time step
- Let d be the fraction of the population that dies per time step.
- Let's write down a Life Cycle Diagram and a Flow Diagram for this process.





Discrete time exponential growth



- Let $n(t)$ be the number of individuals at time t.
- Let b be the number of births per capita per time step
- Let d be the fraction of the population that dies per time step.
- Use the life cycle diagram to derive a **recursion** and a **difference equation**.

$$n'(t) = n(t) + b\,n(t) = (1+b)n(t)$$

$$n''(t) = n'(t) - d\,n'(t) = (1-d)n'(t)$$

$$n(t+1) = n''(t) = (1-d)n'(t) = (1-d)(1+b)n(t)$$

$$\boxed{n(t+1) = (1-d)(1+b)n(t)}$$

$$\Delta n = n(t+1) - n(t)$$

$$\Delta n = (1-d)(1+b)n(t) - n(t)$$

$$\Delta n = [\cancel{1} - d + b - db \cancel{-1}]n(t)$$

$$\Delta n = n(t)[b - d - db]$$

birth and death in same time step!

Discrete time exponential growth

- **Recursion**: $n(t+1) = R\ n(t)$

$$n(t+1) = (1-d)(1+b)\,n(t)$$

$$\boxed{R = (1-d)(1+b)}$$

- **Difference**: $\Delta n = (R-1)\ n(t)$

$$R-1 = b-d-db = r$$

- In the biological literature $(R-1)$ is often denoted r. How can we interpret this quantity?

$$\Delta n = r \cdot n(t)$$

\# of new individuals $=$ rate $\cdot\, n(t)$

r new individuals per existing individual.

Ex:

$r = 0 \Rightarrow$ no growth

$r > 0 \Rightarrow$ growth

$r < 0 \Rightarrow$ decline in population (more death than birth)

Discrete time exponential growth

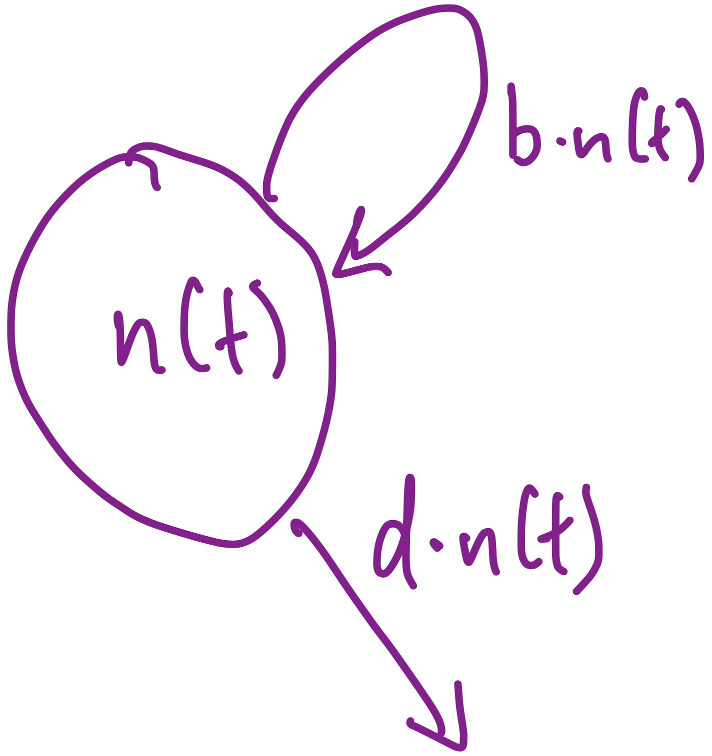
- **Recursion**: $n(t+1) = R\ n(t)$

- **Difference**: $\Delta n = (R-1)\ n(t)$

- In the biological literature $(R-1)$ is often denoted r or r_d .
  - How can we interpret this quantity?
  - r : per-capita change in the number of individuals from one gen. to the next.
  - Sometimes r_d to indicate that this is in $d =$ discrete time.
  - $r_d = R - 1 = (1-d)(1+b) - 1 = b - d - bd$
  - If $R = 1$, then $r_d = 0$, which means no growth—pop. size constant.

Continuous time exponential growth

- What if births and deaths can occur at any time, rather than in specific seasons or time steps?
- Same parameters: per-capita birth rate b and death rate d.
- Using the flow diagram, we can derive the differential equation:



$$\frac{dn}{dt} = b\,n(t) - d\,n(t)$$

$$\boxed{\frac{dn}{dt} = \overbrace{(b-d)}^{r_c}\,n(t)}$$

$$\Delta n = \underbrace{(b - d - db)}_{r_d}\,n(t)$$

Continuous time exponential growth

- What if births and deaths can occur at any time, rather than in specific seasons or time steps?
- Same parameters: per-capita birth rate b and death rate d.
- Using the flow diagram, we can derive the differential equation:
  - $\dfrac{dn}{dt} = bn(t) - dn(t) = r_c\, n(t)$
  - where $r_c = b - d$ is called the per-capita growth rate ($c =$ continuous time).

r_c

x^2

What can we learn from this derivation?

- Notice that r_d became r_c when we took the limit, but
- $r_d =$ $b - d - db$
- $r_c =$ $b - d$
- What the difference, and how can we understand it in terms of modeling?

$-db$

per timestep, per capita birth/death rates

As we make timesteps smaller, b gets smaller, d gets smaller.

→ so $b \times d$ gets really small.

$r_d \approx r_c$ when Δt is v. small!

Alternative:
In cont. time, you can't be born and die in the same instant.
No bd term.

Aside:

- Don't worry! We'll solve this equation (and the next one) numerically and analytically in the next two classes!
- Here is a picture of my dog to tide you over:

Logistic growth in discrete time

- Many factors slow pop. growth, including declining resource availability, increase predation, higher incidence of disease, and so on.
- Logistic model **describes theses processes indirectly** by assuming that the population replacement number R declines with increasing population size.
- We therefore write it as $R(n)$.
- Let's say that when the population size is zero, $R(0) = 1 + r_d$.
  - This is called the **intrinsic rate of growth.**
  - It's what happens when there aren't resource limitations (= prev. model).
- Let's say that $R(n)$ decreases until it becomes 1, at *some* value of n.

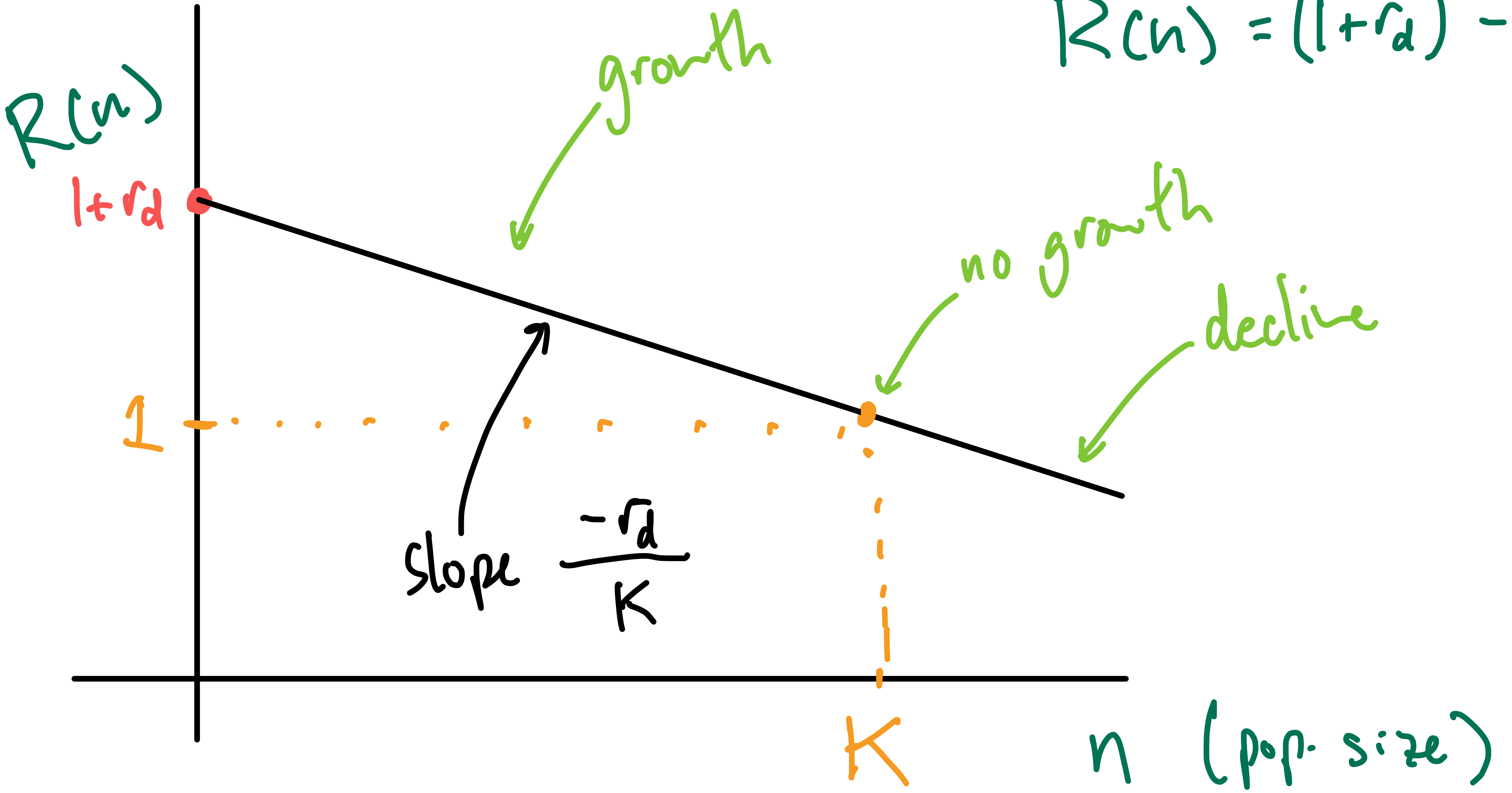
same as exp. growth when $n=0$.

Logistic growth in discrete time

compare to
$y = b + mx$

- Let's say that when the population size is zero, $R(0) = 1 + r_d$.
  - This is called the **intrinsic rate of growth.**
  - It's what happens when there aren't resource limitations (= prev. model).
- Let's say that $R(n)$ decreases until it becomes 1, at *some* value of n.
- A sketch helps:

$$R(n) = (1 + r_d) - \frac{r_d}{K} \cdot n(t)$$



Logistic growth in discrete time

- If we write $n(t+1) = \underbrace{R(n)}_{} \, n(t)$, we now get

$$R(n) = (1+r_d) - \frac{r_d}{K} \cdot n(t)$$

- $n(t+1) = \left[(1+r_d) - \frac{r_d}{K} n(t) \right] n(t)$

$$\underbrace{n(t+1)}_{\text{next}} = \underbrace{n(t)}_{\text{prev}} + \underbrace{r_d \left(1 - \frac{n(t)}{K} \right) n(t)}_{\text{change}}$$

Logistic growth in discrete time

- If we write $n(t+1) = R(n)\ n(t)$, we now get

- $n(t+1) = n(t) + r_d\ n(t)\left(1 - \frac{n(t)}{K}\right)$ recursion

- $\Delta n = r_d\ n(t)\left(1 - \frac{n(t)}{K}\right)$ diff.

Logistic growth in *continuous time*

- If we also assume that r is a function of n, and that it declines from $r(0) = r_c$ to $r(K) = 0$, then we can also get the ODE:

- $$\frac{dn}{dt} = r_c\, n(t)\left(1 - \frac{n(t)}{K}\right)$$

discrete time
$\updownarrow$
recursion or
difference

cont. time
$\updownarrow$
differential
equation.

QUICK QUIZ, HOT SHOT

- $$\frac{dn}{dt} = r_c\, n(t)\left(1 - \frac{n(t)}{K}\right) = r_c\left[n(t) - \frac{n^2(t)}{K}\right]$$

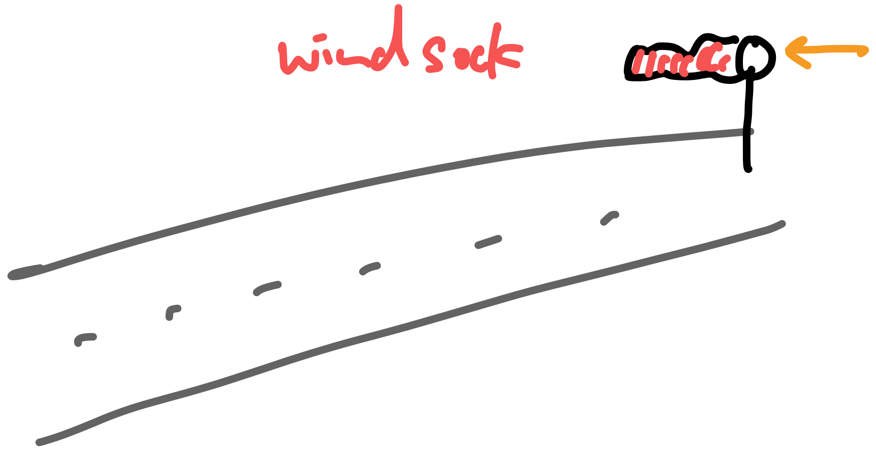
- Order? 1st order → highest deriv is 1st.
- Linear or nonlinear?
- $\boxed{\text{ODE}}$ or PDE?

$\frac{dn}{dt}$ $\frac{\partial n}{\partial t}$

partial differential equation.

Heat equation. $\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}$

Understanding an ODE with a *vector field*



- $\dfrac{dn}{dt} = r_c\, n(t)$

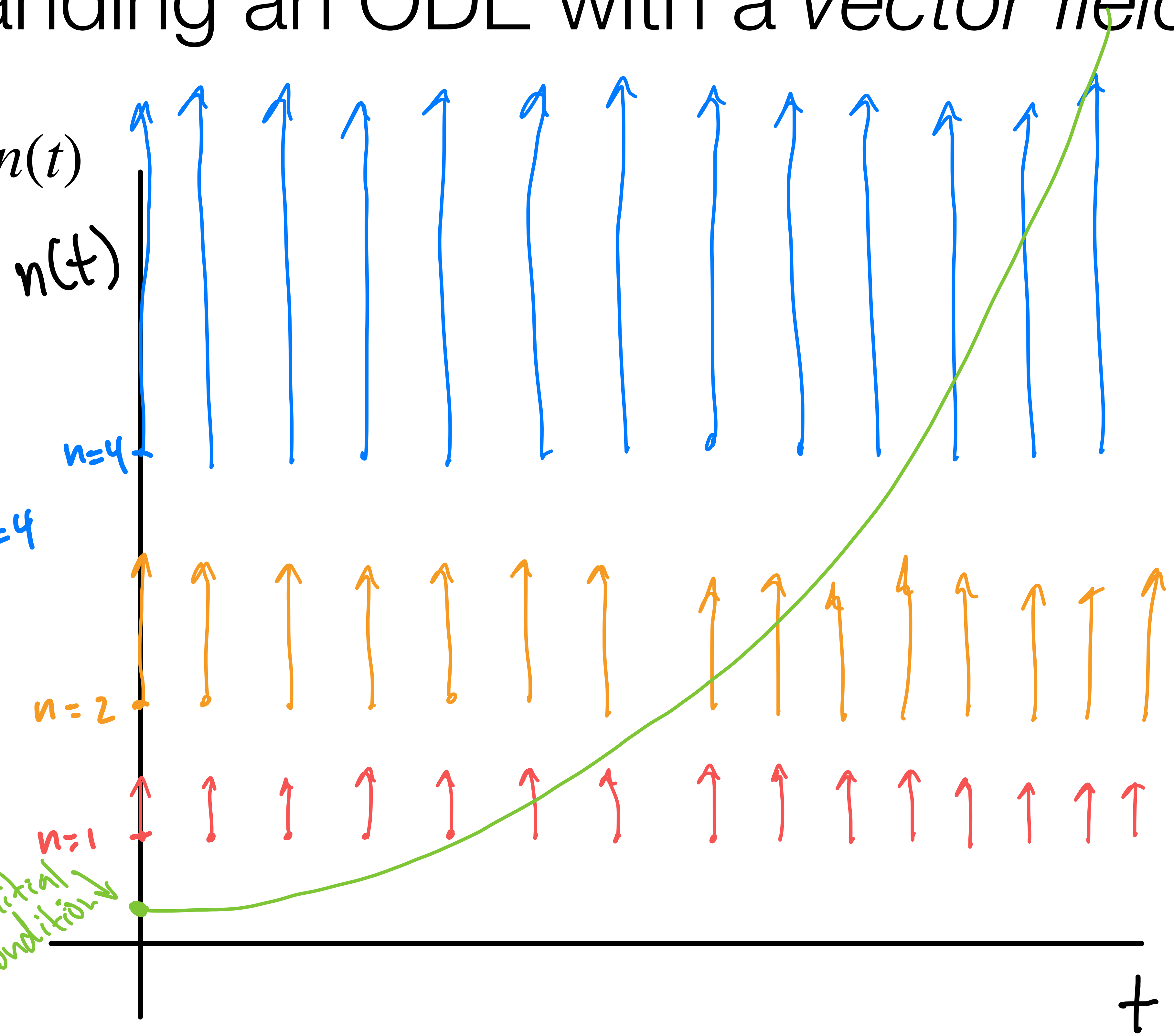
$r_c > 0$
(growth)

$\dfrac{dn}{dt} = 4$

$r_c = 1$

$\dfrac{dn}{dt} = 1 \cdot 2 = 2$

$\dfrac{dn}{dt} = 1 \cdot 1 = 1$



bigger arrow
=
stronger wind
=
steeper slope

vectors (arrows)
express direction
and magnitude
of growth/change.

Understanding an ODE with a *vector field*

- $$\frac{dn}{dt} = r_c\, n(t)\left(1 - \frac{n(t)}{K}\right)$$

$r_c = 1$ $\quad$ $K = 10$

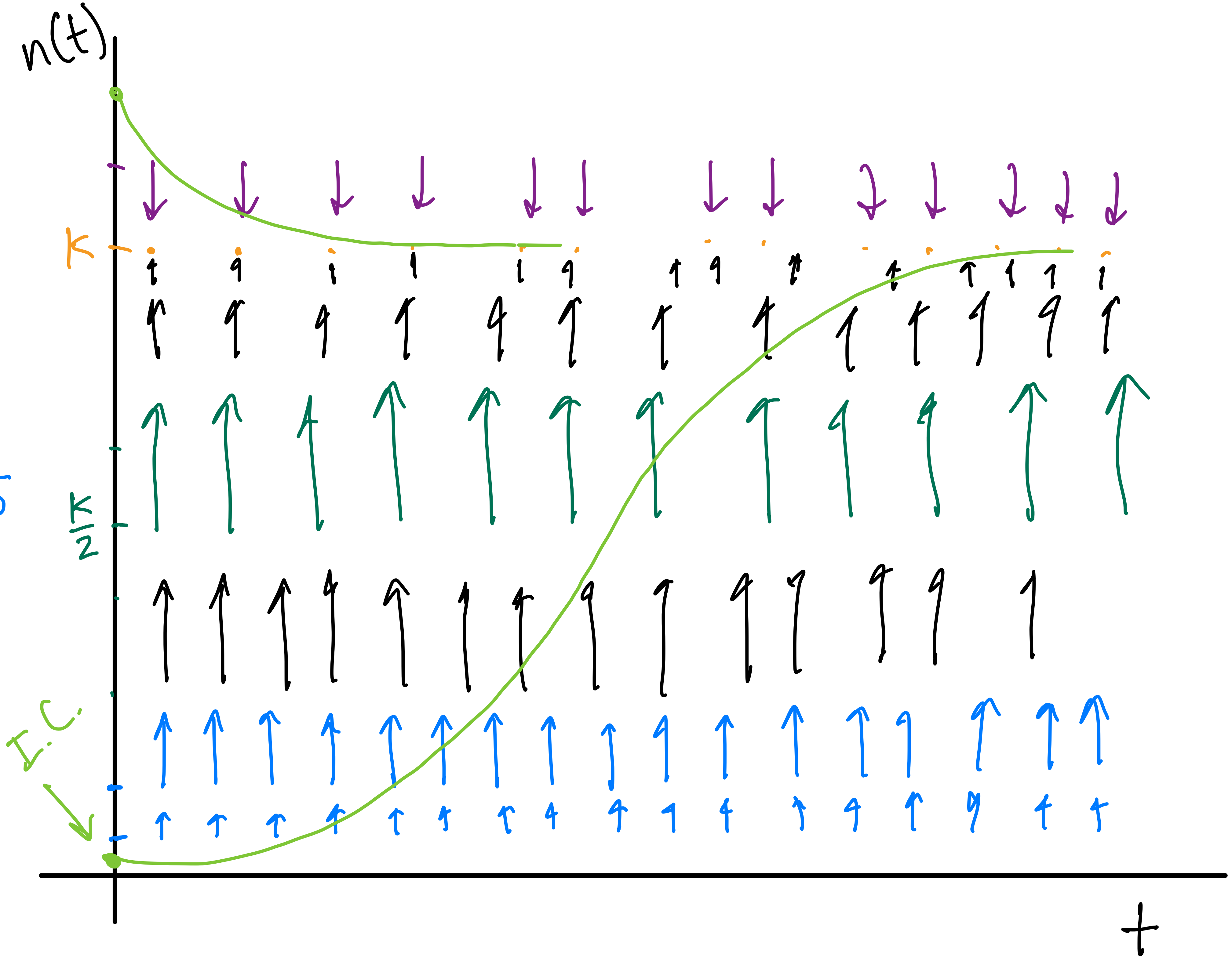
$n = 1 \rightarrow \frac{dn}{dt} = 1 \cdot 1\left(1 - \frac{1}{10}\right) = \frac{9}{10}$

$n = \frac{1}{2} \rightarrow \frac{dn}{dt} = 1 \cdot \frac{1}{2}\left(1 - \frac{1/2}{10}\right) = \frac{19}{40}$

$n = \frac{K}{2} = 5 \quad 1 \cdot 5\left(1 - \frac{5}{10}\right) = \frac{5}{2}$

$n = K \quad \frac{dn}{dt} = 1 \cdot 10\left(1 - \frac{10}{10}\right) = 0$

$n = K+1 \quad \frac{dn}{dt} = 1 \cdot 11\left(1 - \frac{11}{10}\right) = -\frac{11}{10}$



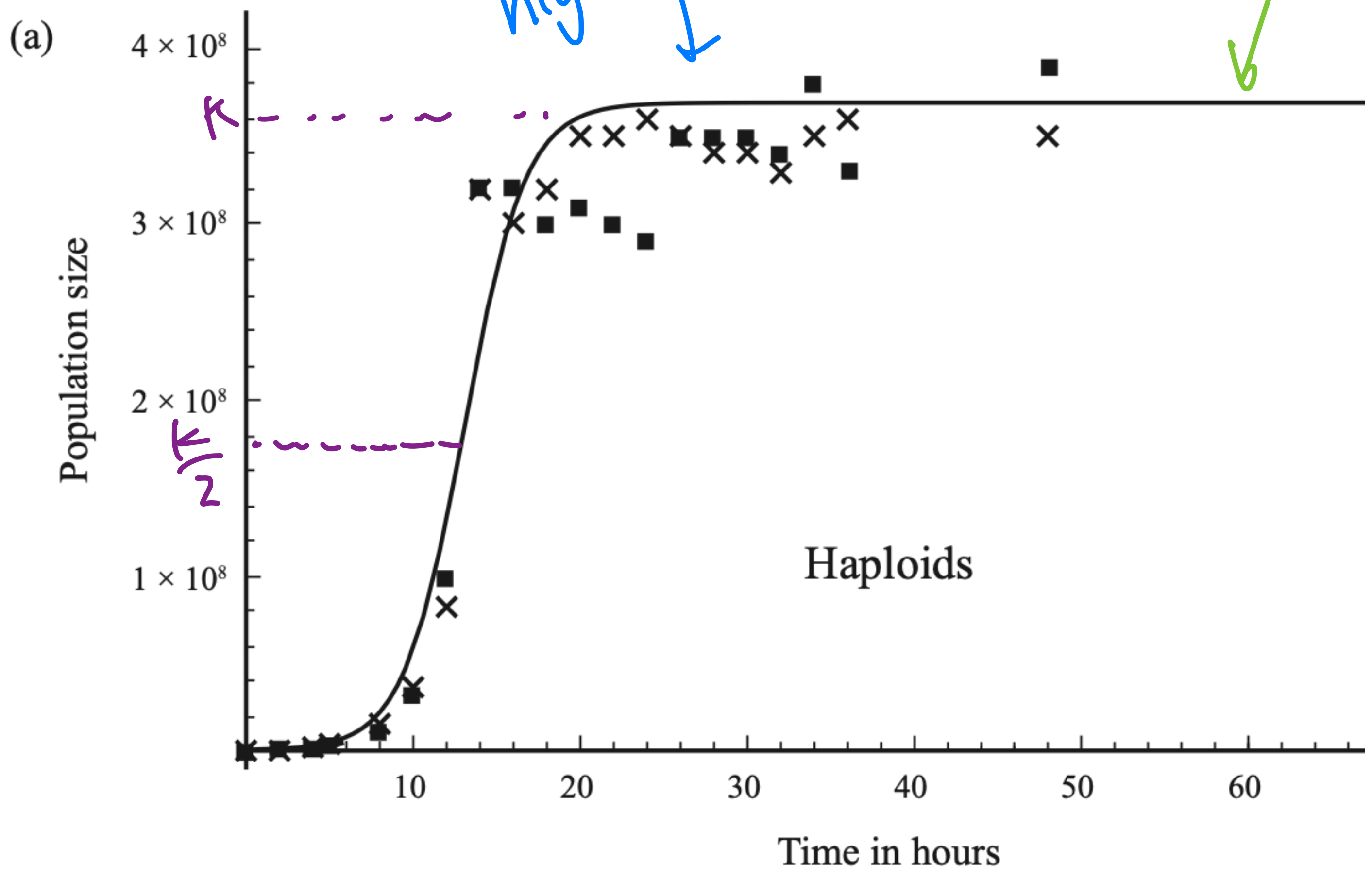
Examples of logistic growth

K is called "carrying capacity" (steady state)

- Mable & Otto (2001) — cultivated both haploid & diploid *S. cerevisiae* (yeast) in two separate flasks.
- Diploid yeast cells are *bigger* and thus take up more resources.

logistic curve

higher K



lower K