1. [30] Suppose we live at a place where days are either sunny, cloudy, or rainy. The weather transition function is a Markov chain with the following transition table:

		Tomorrow will be…		
		Sunny	Cloudy	Rainy
Today is…	Sunny	0.8	0.2	0
	Cloudy	0.4	0.4	0.2
	Rainy	0.2	0.6	0.2

(a) [5] Suppose Day 1 is S (sunny). What is the probability of the following sequence of days: Day2 = C (cloudy), Day3 = C (cloudy), Day4 = R (rainy)?

$$P(C, C, R) = 0.2(0.4)(0.2)$$
$$= 0.016 \quad = 1.6\%$$

(b) [5] Write a simulator with Matlab that can randomly generate sequence of "weathers" from a given initial (any random) weather according to the above probability table (i.e. state transition function).

```
function weather_sequence = simulate_weather(init_state, n_days)
   % Rows: Current state (Sunny, Cloudy, Rainy)
   % Columns: Next state (Sunny, Cloudy, Rainy)
   P = [0.8 0.2 0.0;   % Sunny
        0.4 0.4 0.2;   % Cloudy
        0.2 0.6 0.2];  % Rainy

   % States
   current_state = init_state; % 1=Sunny, 2=Cloudy, 3=Rainy
   weather_sequence = zeros(1, n_days);
   weather_sequence(1) = current_state;

   % Simulate for n_days
   for day = 2:n_days
      % Get transition probabilities for current state
      probs = P(current_state, :);

      % Choose next state based on probabilities
      % next_state = find(rand <= cumsum(probs), 1);
      rand_val = rand;
      probs = cumsum(probs);

      if rand_val <= probs(1)
         next_state = 1;
      elseif (probs(1) < rand_val) && (rand_val <= probs(2))
         next_state = 2;
      elseif (probs(2) < rand_val) && (rand_val <= probs(3))
         next_state = 3;
      end
      weather_sequence(day) = next_state;

      % Update current state
      current_state = next_state;
   end
end
SUNNY = 1;
CLOUDY = 2;
RAINY = 3;

% --- Main Script ---
n_days = 100000;  % Large number for convergence
initial_state = randi(3);  % Random start (1=Sunny, 2=Cloudy, 3=Rainy)

% Run simulation
sequence = simulate_weather(initial_state, n_days);

% Count occurrences
counts = [sum(sequence == SUNNY), ...
          sum(sequence == CLOUDY), ...
          sum(sequence == RAINY)];

% Compute stationary distribution
stationary_dist = counts / n_days;

% Display result
fprintf('Estimated Stationary Distribution:\n');
fprintf('Sunny: %.4f\n', stationary_dist(1));
fprintf('Cloudy: %.4f\n', stationary_dist(2));
fprintf('Rainy: %.4f\n', stationary_dist(3));
```

if 1, 2, 3 were converted to Sunny, Cloudy, rainy

```
> MTE544_HW3
   {'Rainy'}    {'Sunny'}    {'Cloudy'}    {'Rainy'}    {'Cloudy'}

> MTE544_HW3
   {'Rainy'}    {'Rainy'}    {'Sunny'}    {'Sunny'}    {'Cloudy'}

> MTE544_HW3
  Columns 1 through 5

   {'Rainy'}    {'Cloudy'}    {'Rainy'}    {'Cloudy'}    {'Sunny'}

  Columns 6 through 10

   {'Cloudy'}    {'Rainy'}    {'Cloudy'}    {'Cloudy'}    {'Cloudy'}
```

(c) [5] Use your simulator to determine the stationary distribution of this Markov chain. The stationary distribution measures the probability that a random day will be sunny, cloudy, or rainy.

```
Estimated Stationary Distribution:
Sunny: 0.6390
Cloudy: 0.2901
Rainy: 0.0710

Estimated Stationary Distribution:
Sunny: 0.6422
Cloudy: 0.2865
Rainy: 0.0713

Estimated Stationary Distribution:
Sunny: 0.6461
Cloudy: 0.2836
Rainy: 0.0703
```

$$\rightarrow [0.6424, 0.2867, 0.0709]$$

(d) [5] Can you devise a closed-form solution to calculating the stationary distribution by constructing a state transition matrix from the above table?

[Hint] If we let $X_k \in \mathbb{R}^3$ be the vector of probabilities of weather on k-th day (x_k) , i.e. $X_k = [\Pr(x_k = S) \quad \Pr(x_k = C) \quad \Pr(x_k = R)]$, we can construct a 'state transition matrix' $P \in \mathbb{R}^{3\times3}$ relating X_k and X_{k+1} as $X_{k+1} = PX_k$ using the values in the table above. Stationary distribution means that there exists $\lim_{k\to\infty} X_k = X_{ss}$ such that $X_{ss} = PX_{ss}$. Solving this equation w.r.t. X_{ss} will give us the stationary distribution.

$$X_k = [\Pr(S_k), \Pr(C_k), \Pr(R_k)]^T$$
$$X_{k+1} = [\Pr(S_{k+1}), \Pr(C_{k+1}), \Pr(R_{k+1})]^T$$

$$P = \begin{bmatrix} 0.8 & 0.2 & 0.0 \\ 0.4 & 0.4 & 0.2 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}^T = \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0.0 & 0.2 & 0.2 \end{bmatrix}$$

$$X_{k+1} = PX_k \longrightarrow \begin{cases} 0.8\Pr(S_k) + 0.4(\Pr(C_k)) + 0.2\Pr(R_k) = \Pr(S_{k+1}) \\ 0.2\Pr(S_k) + 0.4(\Pr(C_k)) + 0.6\Pr(R_k) = \Pr(C_{k+1}) \\ 0\Pr(S_k) + 0.2(\Pr(C_k)) + 0.2\Pr(R_k) = \Pr(R_{k+1}) \\ \Pr(S_k) + \Pr(C_k) + \Pr(R_k) = 1 \end{cases}$$

$$\hookrightarrow \begin{bmatrix} \Pr(S_{k+1}) \\ \Pr(C_{k+1}) \\ \Pr(R_{k+1}) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0.0 & 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} \Pr(S_k) \\ \Pr(C_k) \\ \Pr(R_k) \end{bmatrix}$$

when $X_k = X_{k+1} \rightarrow X_{ss}$

$$-0.2\Pr(S) + 0.4(\Pr(C)) + 0.2\Pr(R) = 0 \quad (1)$$
$$0.2\Pr(S) + -0.6(\Pr(C)) + 0.6\Pr(R) = 0 \quad (2)$$
$$0\Pr(S) + 0.2(\Pr(C)) - 0.8\Pr(R) = 0 \quad (3)$$

$$\Pr(S) + \Pr(C) + \Pr(R) = 1 \quad (4)$$

(1), (2), (4) $\rightarrow$ $\underbrace{\Pr(S) = 9/14, \quad \Pr(C) = 2/7, \quad \Pr(R) = 1/14}_{\text{works with (3)}}$

$$X = [P(S), P(C), P(R)] = \underbrace{\left[\tfrac{9}{14}, \tfrac{2}{7}, \tfrac{1}{14}\right]}_{\hookrightarrow \text{ stationary distribution}}$$

(e) [10] From Bayes rule, compute the probability table of yesterday's weather given today's weather. (It is okay to provide the probabilities numerically, and it is also okay to rely on results from (d).)

given todays weather 3 events

Sunny, cloudy, rainy

X : yesterday Z : today

$$P(X|Z) = \frac{P(Z|X)\,P(X)}{P(Z)} = \frac{P(Z|X)\,P(X)}{\Sigma P(Z|X)\,P(X)}$$

$P(Z|X)$ -> table at top

$P(X)$ -> stationary distribution

Example use:

$$p(x=c \mid z=s) = \frac{p(z=s \mid x=c)\,p(x=c)}{\sum_{k=1}^{3} p(z=s \mid x=k)\,p(x=k)}$$

Calculated using matlab

		today: Sunny	Cloudy	Rainy
yesterday:	Sunny	0.8000	0.4500	0
	Cloudy	0.1778	0.4000	0.8000
	Rainy	0.0222	0.1500	0.2000

```
% P(Today | Yesterday)  Rows: Today, Cols: Yesterday
P = [0.8 0.4 0.2;
    0.2 0.4 0.6;
    0.0 0.2 0.2];

% Stationary distribution from part (d)
X_k = [9/14, 2/7, 1/14];

% Compute reverse probabilities using Bayes rule
reverse_probs = zeros(3,3); % rows: Yesterday, cols: Today

for today = 1:3
   denom = sum(P(today,:) .* X_k);
   for yesterday = 1:3
      reverse_probs(yesterday,today) = (P(today,yesterday) * X_k(yesterday)) /
denom;
   end
end

disp('Probability table P(Yesterday | Today):');
disp(reverse_probs);
```

2. [30] Suppose you are a mobile robot who can move on a long straight road (i.e. limited to 1-D motion). Your location x is simply the position along this road. Now suppose that initially, you believe to be at location $x_0 = 5$ m, but you happen to know that this estimate is uncertain. Based on this uncertainty, you model your initial belief by a Gaussian with variance $\sigma_{x_0}^2 = 4$ m². Namely, the probability density function (pdf) of the initial position estimate is given by

$$p(x) = \frac{1}{\sqrt{2\pi\sigma_{x_0}^2}} e^{-\frac{1}{2}\left(\frac{x-x_0}{\sigma_{x_0}}\right)^2} = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-5}{2}\right)^2}$$

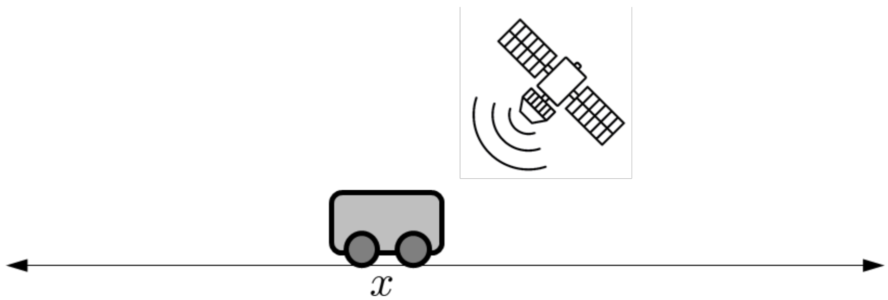

Figure 1. 1D mobile robot with GPS sensor

To find out more about your location, you query a GPS receiver. For a given value of x, the GPS receiver provides a measurement z which is known to also follow Gaussian distribution with its mean at the actual position x and its error variance of $\sigma_z^2 = 1$ m².

(a) [5] Write the probability density function (pdf) of the measurement $\underbrace{p(z|x)}$.

↳ Probability of z given x

$$p(x) = \frac{1}{2\sqrt{2\pi}} \exp\left(\frac{-1}{2}\left(\frac{x-5}{2}\right)^2\right)$$

$$z \sim N(x, \sigma_z^2) = N(x, 1)$$

$$\hookrightarrow p(z|x) = \frac{1}{\sqrt{2\pi(1)}} \exp\left(\frac{-1}{2}\left(\frac{(z-x)^2}{1}\right)\right)$$

$$p(z|x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-x)^2}$$

(b) [15] Using Bayes rule, find the mean and the variance of the posterior pdf $p(x|z)$. (**Hint**: A product of two Gaussian pdf's is also a Gaussian pdf.)

$$p(x|z) = \frac{p(z|y)\,p(z)}{p(x)} = \eta\, p(z|x)\,p(x)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-x)^2} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-5}{2}\right)^2\right)$$

$$= \frac{1}{4\pi} \exp\Bigg(\underbrace{\frac{-1}{2}(z-x)^2 - \frac{1}{2}\left(\frac{x-5}{2}\right)^2}\Bigg)$$

$$= \frac{-1}{2}(z-x)^2 - \frac{1}{8}(x-5)^2$$

$$= \frac{-1}{2}x^2 + zx - \frac{z^2}{2} - \frac{1}{8}x^2 + \frac{10x}{8} - \frac{25}{8}$$

$$= \frac{-5}{8}x^2 + (zx + 1.25x) - \frac{z^2}{2} - \frac{25}{8}$$

$$= \frac{-5}{8}\left(x^2 - \frac{zx + 1.25x}{5/8}\right) - \frac{z^2}{2} - \frac{25}{8}$$

$$= \frac{-5}{8}\left(x^2 - \left(\frac{4z}{5} + 1\right)x\right) - \frac{z^2}{2} - \frac{25}{8}$$

$$= \frac{-5}{8}\left(x - \left(\frac{4z}{5} + 1\right)\right)^2 - \frac{z^2}{2} - \frac{25}{8} - \frac{2}{5}z^2 - z - \frac{5}{8}$$

$$= \frac{-5}{8}\left(x - \left(\frac{4z}{5} + 1\right)\right)^2 - \frac{3}{5}z^2 - 15 - z$$

$$\underbrace{\frac{1}{\sqrt{10\pi}}}_{\eta} \frac{1}{\sqrt{2\pi}\left(\sqrt{\frac{4}{5}}\right)} e^{\left(\frac{-1}{2}\left(\frac{\left(x - \left(\frac{4}{5}z + 1\right)\right)^2}{4/5}\right)\right)} \underbrace{e^{\left(-\frac{3}{5}z^2 - 15 - z\right)}}_{\eta} = p(x|z)$$

$$\boxed{\sigma^2 = 4/5 \quad \mu = \frac{4}{5}z + 1}$$

(c) [10] If the measurement from the GPS tells you that your initial location is $z_0 = 6$ m, what would be the most likely value of x (given the GPS measurement and our initial belief of x)?

most likely $x = \mu = \frac{4}{5}(6) + 1$

$= 29/5$