Robust Portfolio Optimization With Uncertainty in Risk Measure

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1 Introduction

A classical problem in computational finance is the financial decision-making under model uncertainty, guaranteed (robust) Portfolio optimization deals with optimal portfolio selection of finitely many risky assets to maximize expected return of the investment subject to controlled "risks". Developed by Harry Markowitz (1952) five decades prior,this approach quantifies the trade-off between the expected return and the risk of portfolios of financial assets using mathematical techniques. The approach is also called Mean Variance Optimization (MVO) as risk is measured by the variance of random portfolio return. Despite being an elegant model, it has been subjected to much skepticism by investment practitioners on its practicality due to the fact that optimal portfolios are often sensitive to changes in the input parameters of the problem(expected returns and the covariance matrix) indicating inputs to the MVO model to be very accurately estimated.

Robust optimization, an emerging branch in the field of optimization, offers vehicles to incorporate estimation risk into the decision making process in portfolio allocation. Generally speaking, robust optimization refers to finding solutions to given optimization problems with uncertain input parameters that will achieve good objective values for all, or most, realizations of the uncertain input parameters.

2 Problem description

The main goal of this project is to present an efficient computational framework for robust portfolio selection in the situation of asset returns described by an ambiguous discrete joint probability distribution. We consider risk measure given by composite objectives of the form $\rho(x,\pi) - \gamma\mu(x,\pi)$, where $\rho(x,\pi)$ is the variance or the expected absolute deviation of the portfolio, $\mu(x,\pi)$ is the portfolio expected return, and γ is a nonnegative parameter. Here, x denotes the portfolio mix and π the discrete distribution of the returns.

In the *nominal* case - i.e., when the probability distribution π is known and given – minimizing the above objectives is equivalent either to a standard Markowitz problem. It is well known that, the optimal portfolio in a standard Markowitz problem can be solved using a quadratic programming problem.

The key point in this project is to consider the return distribution π to be imprecisely known. In particular, we assume that a nominal value η for the distribution is given, but that the actual π is only known to lie in a region at distance no larger than d from its nominal value, where d is a user-definable parameter that quantifies the (lack of) confidence in the nominal probability. To measure the distance among distributions, we use the standard metric given by the *Kullback–Leibler divergence*.

Assume that a nominal return probability distribution η is given, for instance, as a result of estimation from samples. If π, η are two probability vectors in $\mathbb{R}^T$, with η >0 describing the nominal probability, the KL distance between π and η is defined as

$$KL(\pi, \eta) = \sum_{k=1}^{T} \pi_k \log \frac{\pi_k}{\eta_k} \tag{1}$$

We shall henceforth assume that the true probability π is only known to lie within KL distance $d \geq 0$ from η, i.e., $\pi \in$ K(η, d), where

$$K(\eta, d) = \{\pi \in \Pi : KL(\pi, \eta) \leq d\}, \tag{2}$$

Π being the probability simplex given by,

$$\Pi = \{\pi : \pi \succeq 0, 1^\top \pi = 1\}. \tag{3}$$

Specifically, given the ambiguity model K(η,d) for the return distribution, we define the following *worst-case (or robust) measures of risk* for a portfolio with composition x:

$$\Upsilon_{wc} \doteq \max_{\pi \in K(\eta, d)} \quad \rho(x, \pi) - \gamma \mu(x, \pi) \tag{4}$$

for the variance-based measure. The distribution π_{wc} that attains the supremum in the above optimization problem is named the *worst-case distribution*, and the corresponding value function $\Upsilon_{wc}(x)$ is the *worst-case risk* (to uncertainty level d) of the portfolio x. The above optimization problem gives the worst case risk of a portfolio and consequently, our aim is to find the portfolio that mininizes the worst case risk. That is, we now aim at solving the optimization problem, given by :

$$x^* = \underset{x \in \boldsymbol{\chi}}{\operatorname{argmin}} \quad \Upsilon_{wc} \tag{5}$$

where,

$$\boldsymbol{\chi} = \{x : x \succeq 0, 1^\top x = 1\} \tag{6}$$

3 Approach

In our project, we plan to solve the robust portfolio optimization using algorithm taught in the class. Our aim is to implement following algorithms and provide a measure of comparison between the algorithms based on the number of iterations used in respective algorithm to provide an optimal value and also, comparing the time taken by the algorithms. A tentative list of algorithms, that we plan to implement for solving the robust portfolio optimization problem are :

1. Proximal Gradient Method
2. Accelerated Proximal Gradient Method
3. Analytic Centre Cutting Plane Method (ACCP)
4. Interior Point Method
5. Alternating direction method of multipliers (ADMM)

The dataset that will be used in our project is randomly generated for testing the efficiency of the above algorithms. Also, as a comparison of the robust portfolio problem, we will test our algorithms on real data provided by Yahoo Finance®.

References

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