Problem Set 6 Solutions

 # 1:

(a) To count the number of paths, it is helpful to verify that there is a 1 to 1 correspondence between sample paths and sequences of heads and tails corresponding to the heads or tails observed in the coin flipping process that generated the sample path.

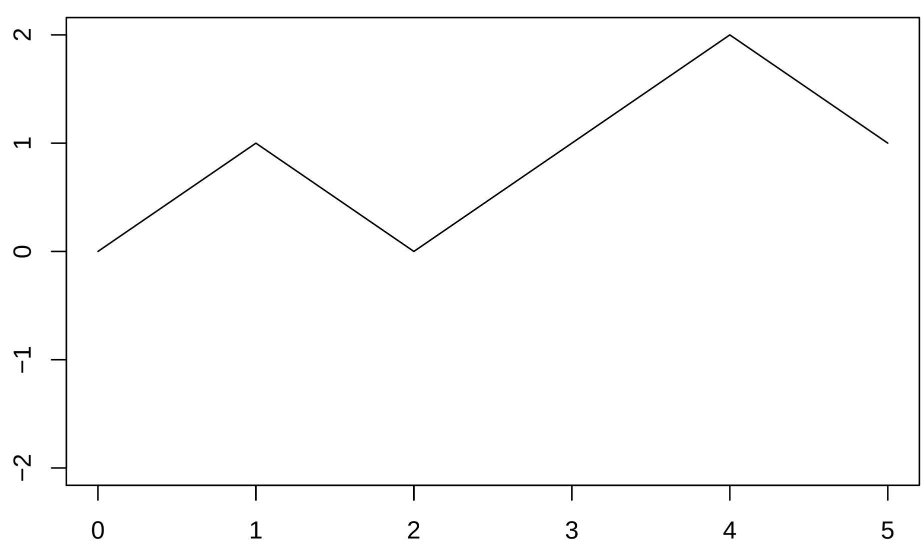
Let us use words consisting of H and T to denote a particular sequence of heads and tails being realized in coin flipping. So, for instance,

$$HHT$$

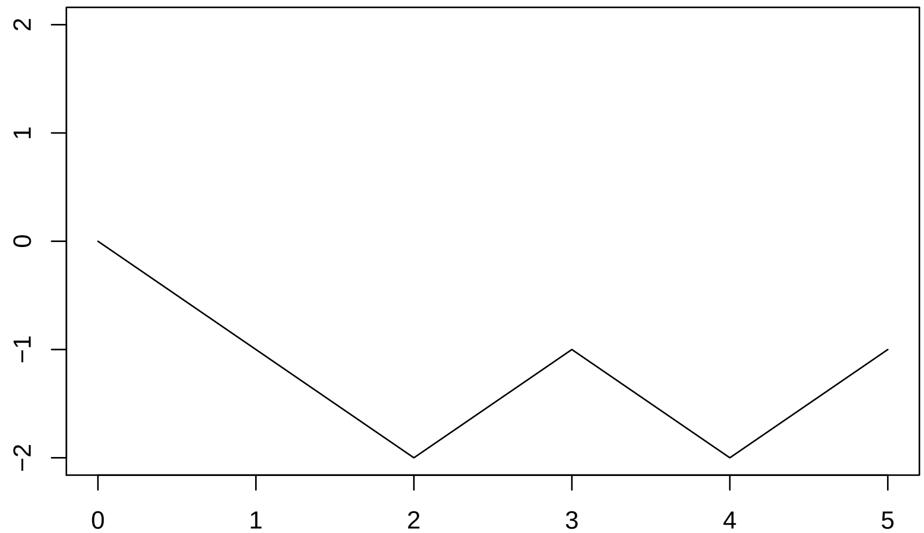
denotes the sequence of getting heads in the first 2 flips followed by tails on the third flip if we flip the coin 3 times. Then there is a 1-1 correspondence between sample paths and a given sequence of this form. For instance the sequences

$$HTHHT \qquad TTHTH$$

corresponds to the sample paths



and



respectively. It is not hard to see that there is a 1-1 correspondence between such words and sample paths. So to count the number of sample paths it is equivalent to count the number of words made up of the letters H and T of length 5. Since there are 2 choices for the first letter, 2 choices for the second, and so on for all 5, the total number of such words is

$$2 \times 2 \times 2 \times 2 \times 2 = 32$$

To compute the probabilities of different paths, from the previous discussion, this is equivalent to the probablility that a given word, ie sequence of heads and tails is realized in flipping a coin 5 times. Let Y_i for $i = 1 \ldots 5$ be random variables such that Y_i is the result of the ith coin toss. So either $Y_i = H$ or $Y_i = T$ and

$$\begin{aligned} \text{Prob}(Y_i = H) &= \frac{1}{2} \\ \text{Prob}(Y_i = T) &= \frac{1}{2}. \end{aligned}$$

Since each coin flip is independent of all the others, the random variables Y_i are all independent and so the probabilities of any particular sequence of heads and tails turning up just factor. For instance, the probability of observing the sequence $HTHHT$ is

$$\begin{aligned}
& \text{Prob}(Y_1 = H, Y_2 = T, Y_3 = H, Y_4 = H, Y_5 = T) \\
= & \text{Prob}(Y_1 = H)\text{Prob}(Y_2 = T)\text{Prob}(Y_3 = H)\text{Prob}(Y_4 = H)\text{Prob}(Y_5 = T) \\
= & \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\
= & \frac{1}{32}
\end{aligned}$$

Since the probability that any Y_i is either H or T always, this same calculation will produce the same probability no matter what sequence of heads and tails you compute the probability of. So the probability of any particular word of length 5 consisting of H and T is $1/32$, and so the probability of any sample path is $1/32$.

(b) The structure and notation from part (a) can be used for this part. The only thing that has changed are the probabilities of the individual coin flip results. We now have

$$\begin{aligned}
\text{Prob}(Y_i = H) &= p \\
\text{Prob}(Y_i = T) &= q = 1 - p
\end{aligned}$$

for all i. The random variables Y_i are still independent, so when we calculate the probability of a particular sequence of coin toss outcomes, we can still factor it into probabilities for the individual Y_i. So, for instance, to calculate the probability of the sequence $HTHTH$, we proceed as in the fair coin case:

$$\begin{aligned}
& \text{Prob}(Y_1 = H, Y_2 = T, Y_3 = H, Y_4 = T, Y_5 = H) \\
= & \text{Prob}(Y_1 = H)\text{Prob}(Y_2 = T)\text{Prob}(Y_3 = H)\text{Prob}(Y_4 = T)\text{Prob}(Y_5 = H) \\
= & p \times q \times p \times \ q \times p \\
= & p^3 q^2.
\end{aligned}$$

Note that the exponent of p is the number of heads in the sequence. It is easy to see, by the commutativity of multiplication, that we will always be able to group the factors of p together, and the factors of q. There will be as many factors of p as heads in the sequence, and thus as many factors of q as tails. Thus it follows that for any sequence, the probability will be

$$p^{\#\{\text{heads}\}}q^{\#\{\text{tails}\}} = p^{\#\{\text{heads}\}}q^{5-\#\{\text{heads}\}}.$$

The "critical statistic" referred to in the problem statement is #heads, the number of heads in the sequence. The above formula shows that that is the only information you need from the sequence to be able to write down the probability that it is the realized sequence.

(c) It is easy to see that the formula from part (b) easily extends to the case of n coin flips, the only difference being that 5 is replaced with n. Thus, the probability of any sequence is

$$p^{\#\{\text{heads}\}}q^{\#\{\text{tails}\}} = p^{\#\{\text{heads}\}}q^{n-\#\{\text{heads}\}}.$$

 # 2:

(a)(i) Recall the basic property of Brownian motion that if (t_1, t_2) and (t_3, t_4) are two nonoverlapping intervals then $W(t_2)-W(t_1)$ is independent of $W(t_4)-W(t_3)$. In this case the entire sequence consists of Brownian differences over the nonoverlapping intervals $(0, 1)$, $(1, 2)$,...etc (note that successive intervals in this series overlap at the endpoints, but the interiors are disjoint, so the Brownian differences over these intervals will be independent). Thus any 2 distinct elements of this sequence will be independent and it follows that

$$\text{corr}(W(i+1) - W(i), W(j+1), W(j)) \quad = \quad \begin{cases} 1 \;\; \text{if} \;\; i = j \\ 0 \qquad\qquad \text{if} i \neq j \end{cases}$$

(ii) Volatility clustering is the result of dependence between different elements of the series. Since all elements of this series are independent, this series would not exhibit volatility clustering.

(iii) Fat tails characterize distributions whose tails decay slower than the normal distribution. The elements of this series $W(i+1) - W(i)$ are all normally distributed, so they do not have fat tails.

(b)(i) If $|i-j| \geq 2$ then the Brownian differences $W(i+2)-W(i)$, and $W(j+2)-W(j)$ are over nonoverlapping intervals, and so are independent and uncorrelated. But if $|i-j|=1$ then these are differences over overlapping intervals, and so they will nave nonzero correlation. We proceed to compute these correlations using the suggested hint. Since

$$E[W(j+2)-W(j)]=0$$

for any j, the covariance of $W(i+2)-W(i)$ and $W(i+3)-W(i+1)$ is

$$\begin{aligned}
& \operatorname{cov}\big(W(i+2)-W(i), W(i+3)-W(i+1)\big) \\
= \; & E\big[\big(W(i+2)-W(i)\big)\big(W(i+3)-W(i+1)\big)\big] \\
= \; & E\big[\big(W(i+2)-W(i+1)+W(i+1)-W(i)\big) \\
& \times\big(W(i+3)-W(i+2)+W(i+2)-W(i+1)\big)\big] \\
= \; & E\big[\big(W(i+2)-W(i+1)\big)\big(W(i+3)-W(i+2)+W(i+2)-W(i+1)\big) \\
& +\big(W(i+1)-W(i)\big)\big(W(i+3)-W(i+2)+W(i+2)-W(i+1)\big)\big] \\
= \; & E\big[\big(W(i+2)-W(i+1)\big)\big(W(i+3)-W(i+2)\big)\big] \\
& +E\big[\big(W(i+2)-W(i+1)\big)^2\big] \\
& +E\big[\big(W(i+1)-W(i)\big)\big(W(i+3)-W(i+2)\big)\big] \\
& +E\big[\big(W(i+1)-W(i)\big)\big(W(i+2)-W(i+1)\big)\big] \\
= \; & 0+1+0+0 \\
= \; & 1
\end{aligned}$$

where we have used the basic properties of Brownian motion: Brownian differences over disjoint intervals are independent, and the variance of a Brownian difference is the time length of the interval.

We also have

$$\begin{aligned}
\operatorname{Var}\big[\big(W(i+2)-W(i)\big)\big] &= i+2-i=2 \\
\operatorname{Var}\big[\big(W(i+3)-W(i+1)\big)\big] &= i+3-(i+1)=2
\end{aligned}$$

again using the basic properties of Brownian Motion.

We may thus calculate the autocorrelation

$$\begin{aligned}
& \operatorname{corr}\big(W(i+2)-W(i), W(i+3)-W(i+1)\big) \\
= \quad & \frac{\operatorname{cov}\big(W(i+2)-W(i), W(i+3)-W(i+1)\big)}{\sqrt{\operatorname{Var}\big(W(i+2)-W(i)\big)\operatorname{Var}\big(W(i+3)-W(i+1)\big)}} \\
= \quad & \frac{1}{\sqrt{(2)(2)}} \\
= \quad & \frac{1}{2}
\end{aligned}$$

(ii) Because there is a nonzero correlation between successive elements of this series, we would expect to see some, fairly mild, volatility clustering in this series.

(iii) The individual terms in the series $W(i+2)-W(i)$ are still normally distributed, and so will still not exhibit fat tailed distributions.