Problem Set 10 Solutions

 # 1:
The floor function satifsfies the inequalities

$$x - 1 < \lfloor x \rfloor \leq x.$$

So we have

$$\frac{nT - 1}{n} < \frac{\lfloor nT \rfloor}{n} \leq \frac{nT}{n} = T$$

or

$$T - \frac{1}{n} < \frac{\lfloor nT \rfloor}{n} \leq T.$$

By taking limits we have

$$\lim_{n\to\infty} \left(T - \frac{1}{n}\right) < \lim_{n\to\infty} \frac{\lfloor nT \rfloor}{n} \leq T.$$

$$\implies \quad T \leq \lim_{n\to\infty} \frac{\lfloor nT \rfloor}{n} \leq T$$

which shows that

$$\lim_{n\to\infty} \frac{\lfloor nT \rfloor}{n} = T$$

 # 2:
Our task is to show that

$$\begin{aligned} \mathcal{C} &= e^{-rT} \int_{-\infty}^{\infty} \max\{S_0 e^{T(r-\frac{\sigma^2}{2})+\sigma\sqrt{T}z} - K, 0\} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz \\ &= S_0 N(d_+) - K e^{-rT} N(d_-) \end{aligned}$$

where

$$\begin{aligned} d_+ &= \frac{1}{\sigma\sqrt{T}} \left[\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right] \\ d_- &= \frac{1}{\sigma\sqrt{T}} \left[\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T\right] \end{aligned}$$

and N(x) is the normal cumulative distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz$$

by evaluating the integral.

The first step is typically to make the change of variable $x = -z$ in the integral

$$\int_{-\infty}^{\infty} \max\{S_0 e^{T(r-\frac{\sigma^2}{2})+\sigma\sqrt{T}z} - K, 0\} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz.$$

The total effect is

$$\begin{aligned} x &= -z \\ x^2 &= z^2 \\ dx &= -dz \end{aligned}$$

and to reverse the limits to be from ∞ to $-\infty$. However, after absorbing the $-$ from the $-dx$ that appears, the limits are reversed back to their original order. Thus, the total effect of this change of variable is that the Black-Scholes expression becomes

$$\mathcal{C} = e^{-rT} \int_{-\infty}^{\infty} \max\{S_0 e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} - K, 0\} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

so that all that has changed is the sign of the second term in the exponential.

To continue, notice not that the integrand is 0 unless the first argument of the max function is greater than 0, and that if it is, the max function is simply equal to its first component. We wish to translate this condition to a condition on the integration variable x:

$$\begin{aligned} S_0 e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} - K &> 0 \\ \Longrightarrow e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} &> \frac{K}{S_0} \\ \Longrightarrow T(r - \frac{\sigma^2}{2}) - \sigma\sqrt{T}x &> \log\Big(\frac{K}{S_0}\Big) \end{aligned}$$

We state this as an explicit condition on x:

$$\begin{aligned} \sigma\sqrt{T}x &< -\log\Big(\frac{K}{S_0}\Big) + T(r - \frac{\sigma^2}{2}) \\ &= \log\Big(\frac{S_0}{K}\Big) + T(r - \frac{\sigma^2}{2}) \\ \Longrightarrow x &< \frac{\log\Big(\frac{S_0}{K}\Big) + T(r - \frac{\sigma^2}{2})}{\sigma\sqrt{T}} \\ &= d_-. \end{aligned}$$

We may thus replace the ∞ in the upper limit with d_-, Moreover, in this range where x satisfies the condition, the maximum function will evaluate to its first argument. We may thus write

$$\mathcal{C} = e^{-rT} \int_{-\infty}^{d_-} [S_0 e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} - K] \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

The integrand is now just a sum of 2 terms, and we may integrate over each term separately:

$$\begin{aligned}
\mathcal{C} &= e^{-rT} \int_{-\infty}^{d_-} S_0 e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx - e^{-rT} \int_{-\infty}^{d_-} K \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
&= S_0 e^{-rT} \int_{-\infty}^{d_-} e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx - K e^{-rT} \int_{-\infty}^{d_-} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
&= S_0 e^{-rT} \int_{-\infty}^{d_-} e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx - K e^{-rT} N(d_-)
\end{aligned}$$

Note that we have now recovered the second term from the Black-Scholes formula. The only thing left to do is recover the first term, which means we must show that

$$S_0 e^{-rT} \int_{-\infty}^{d_-} e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = S_0 N(d_+)$$

To do this, multiply all the exponentials together and combine their exponents:

$$\begin{aligned}
& S_0 e^{-rT} \int_{-\infty}^{d_-} e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
= \; & S_0 \int_{-\infty}^{d_-} e^{-rT} e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
= \; & S_0 \int_{-\infty}^{d_-} \frac{e^{-rT} e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} e^{-x^2/2}}{\sqrt{2\pi}} dx \\
= \; & S_0 \int_{-\infty}^{d_-} \frac{e^{-x^2/2-\sigma\sqrt{T}x+T(r-\frac{\sigma^2}{2})-rT}}{\sqrt{2\pi}} dx \\
= \; & S_0 \int_{-\infty}^{d_-} \frac{e^{-\frac{1}{2}[x^2+2\sigma\sqrt{T}x-2T(r-\frac{\sigma^2}{2})+2rT]}}{\sqrt{2\pi}} dx \\
= \; & S_0 \int_{-\infty}^{d_-} \frac{e^{-\frac{1}{2}[x^2+2\sigma\sqrt{T}x-2Tr+2T\frac{\sigma^2}{2}+2rT]}}{\sqrt{2\pi}} dx \\
= \; & S_0 \int_{-\infty}^{d_-} \frac{e^{-\frac{1}{2}[x^2+2\sigma\sqrt{T}x+T\sigma^2]}}{\sqrt{2\pi}} dx \\
= \; & S_0 \int_{-\infty}^{d_-} \frac{e^{-\frac{1}{2}(x+\sigma\sqrt{T})^2}}{\sqrt{2\pi}} dx
\end{aligned}$$

Now we change variables in the integral 1 more time, letting

$$\begin{aligned}
y &= x + \sigma\sqrt{T} \\
\implies dy &= dx
\end{aligned}$$

and the upper limit will be whatever y is when x is d_-. We calculate this limit:

$$
\begin{aligned}
y &= d_- + \sigma\sqrt{T} \\
&= \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T\right] + \sigma\sqrt{T} \\
&= \frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T + \sigma^2 T}{\sigma\sqrt{T}} \\
&= \frac{\log\left(\frac{S_0}{K}\right) + rT - \frac{\sigma^2 T}{2} + \sigma^2 T}{\sigma\sqrt{T}} \\
&= \frac{\log\left(\frac{S_0}{K}\right) + rT + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} \\
&= \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\
&= d_+
\end{aligned}
$$

So, the effect of making this change of variable is

$$
\begin{aligned}
&= S_0 \int_{-\infty}^{d_+} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dx \\
&= S_0 N(d_+)
\end{aligned}
$$

Which is the first term of the Black-Scholes formula, and we have confirmed

$$\mathcal{C} = S_0 N(d_+) - Ke^{-rT} N(d_-)$$

which is the full Black-Scholes formula.

 # 3: Remark: In these calculations we will, for simplicity, assume $t = 0$. This means that T is the time to expiry of the option. To recover the formulas as given in lecture just replace T with $T - t$ in the formulas derived here.

We start with

$$\Delta = \frac{\partial C}{\partial S}.$$

From the Black-Scholes formula

$$C = SN(d_+) - Ke^{-rT} N(d_-)$$

We compute

$$\begin{aligned}\frac{\partial C}{\partial S} &= \frac{\partial}{\partial S}[SN(d_+) - Ke^{-rT}N(d_-)] \\ &= N(d_+) + SN'(d_+)\frac{\partial d_+}{\partial S} - Ke^{-rT}N'(d_-)\frac{\partial d_-}{\partial S}\end{aligned}$$

where we have invoked the product rule and the chain rule. Note the first term is the final answer, according to the formulas we have presented in class, so want we need to do is see that the last 2 terms together add up to 0:

$$SN'(d_+)\frac{\partial d_+}{\partial S} - Ke^{-rT}N'(d_-)\frac{\partial d_-}{\partial S} = 0$$

Recall that

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}dz.$$

So, by the fundamental theorem of calculus

$$N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

so that the condition we need to verify is

$$S\frac{e^{-d_+^2/2}}{\sqrt{2\pi}}\frac{\partial d_+}{\partial S} - Ke^{-rT}\frac{e^{-d_-^2/2}}{\sqrt{2\pi}}\frac{\partial d_-}{\partial S} = 0$$

We can cancel out the $\sqrt{2\pi}$ and write the condition

$$Se^{-d_+^2/2}\frac{\partial d_+}{\partial S} = Ke^{-rT}e^{-d_-^2/2}\frac{\partial d_-}{\partial S}$$

From the expressions

$$\begin{aligned}d_+ &= \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right] \\ d_- &= \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T\right]\end{aligned}$$

We see that

$$\frac{\partial d_+}{\partial S} = \frac{\partial d_-}{\partial S}.$$

We may thus cancel those partial derivatives out of the condition, which then reduces to

$$Se^{-d_+^2/2} = Ke^{-rT}e^{-d_-^2/2}. \tag{1}$$

Now, take logarithms, and reduce this condtion further to

$$\log(S) - \frac{d_+^2}{2} = \log(K) - rT - \frac{d_-^2}{2}$$

and by moving the terms around reduce this to

$$\frac{d_+^2 - d_-^2}{2} = \log(S) - \log(K) + rT.$$

To show this condition holds rewrite the expressions for d_+ and d_-:

$$\begin{aligned}
d_+ &= \frac{1}{\sigma\sqrt{T}}\Big[\log\Big(\frac{S}{K}\Big) + rT + \frac{\sigma^2}{2}T\Big] \\
d_+ &= \frac{1}{\sigma\sqrt{T}}\Big[D + \frac{\sigma^2}{2}T\Big] \\
d_- &= \frac{1}{\sigma\sqrt{T}}\Big[\log\Big(\frac{S}{K}\Big) + rT - \frac{\sigma^2}{2}T\Big] \\
d_- &= \frac{1}{\sigma\sqrt{T}}\Big[D - \frac{\sigma^2}{2}T\Big]
\end{aligned}$$

where

$$D = \log\Big(\frac{S}{K}\Big) + rT$$

We may then write

$$\begin{aligned}
\frac{d_+^2 - d_-^2}{2} &= \frac{\Big[D + \frac{\sigma^2}{2}T\Big]^2 - \Big[D - \frac{\sigma^2}{2}T\Big]^2}{2\sigma^2 T} \\
&= \frac{D^2 + 2\frac{\sigma^2}{2}TD + \big(\frac{\sigma^2}{2}T\big)^2 - D^2 + 2\frac{\sigma^2}{2}TD - \big(\frac{\sigma^2}{2}T\big)^2}{2\sigma^2 T} \\
&= \frac{2\sigma^2 TD}{2\sigma^2 T} \\
&= D \\
&= \log\Big(\frac{S}{K}\Big) + rT \\
&= \log(S) - \log(K) + rT
\end{aligned}$$

which was the condition we had set out to verify.

From this condition it follows that the second and third terms in the expression for Δ add up to 0, and we have verified that

$$\Delta = \frac{\partial C}{\partial S} = N(d_+).$$

Next, the gamma:

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta}{\partial S}$$

Using our result for the Δ, this is

$$\frac{\partial}{\partial S} N(d_+) = N'(d_+) \frac{\partial d_+}{\partial S}$$

and writing

$$\begin{aligned} d_+ &= \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right] \\ &= \frac{1}{\sigma\sqrt{T}}\left[\log(S) - \log(K) + \left(r + \frac{\sigma^2}{2}\right)T\right] \end{aligned}$$

we have

$$\frac{\partial d_+}{\partial S} = \frac{1}{\sigma\sqrt{T}S}$$

which finally gives

$$\Gamma = \frac{N'(d_+)}{\sigma\sqrt{T}S}$$

which is the formula given in the lecture.

Finally, the vega

$$\mathcal{V} = \frac{\partial C}{\partial \sigma}$$

From the Black-Scholes formula

$$\begin{aligned} \mathcal{V} &= \frac{\partial}{\partial \sigma}[SN(d_+) - Ke^{-rT}N(d_-)] \\ &= SN'(d_+)\frac{\partial d_+}{\partial \sigma} - Ke^{-rT}N'(D_-)\frac{\partial d_-}{\partial \sigma} \\ &= S\frac{e^{-d_+^2/2}}{\sqrt{2\pi}}\frac{\partial d_+}{\partial \sigma} - Ke^{-rT}\frac{e^{-d_-^2/2}}{\sqrt{2\pi}}\frac{\partial d_-}{\partial \sigma}. \end{aligned}$$

As a consequence of our calculation of the Black-Scholes delta, we have the identity, labelled as equation (1) above

$$Se^{-d_+^2/2} = Ke^{-rT}e^{-d_-^2/2}.$$

Use this as a substitution in the expression for the vega:

$$\begin{aligned}
\mathcal{V} &= S\frac{e^{-d_+^2/2}}{\sqrt{2\pi}}\frac{\partial d_+}{\partial\sigma} - S\frac{e^{-d_+^2/2}}{\sqrt{2\pi}}\frac{\partial d_-}{\partial\sigma} \\
&= S\frac{e^{-d_+^2/2}}{\sqrt{2\pi}}\Big(\frac{\partial d_+}{\partial\sigma} - \frac{\partial d_-}{\partial\sigma}\Big)
\end{aligned}$$

From the expressions for d_+ and d_- we have

$$\begin{aligned}
d_+ &= \frac{1}{\sigma\sqrt{T}}\Big[\log\Big(\frac{S_0}{K}\Big) + \Big(r + \frac{\sigma^2}{2}\Big)T\Big] \\
&= \frac{1}{\sigma\sqrt{T}}\Big[\log\Big(\frac{S_0}{K}\Big) + rT + \frac{\sigma^2}{2}T\Big] \\
&= \frac{\log\Big(\frac{S_0}{K}\Big) + rT}{\sigma\sqrt{T}} + \frac{\sigma}{2}\sqrt{T} \\
\implies \frac{\partial d_+}{\partial\sigma} &= \frac{\partial}{\partial\sigma}\Big[\frac{\log\Big(\frac{S_0}{K}\Big) + rT}{\sigma\sqrt{T}}\Big] + \frac{\sqrt{T}}{2}
\end{aligned}$$

and

$$\begin{aligned}
d_- &= \frac{1}{\sigma\sqrt{T}}\Big[\log\Big(\frac{S_0}{K}\Big) + \Big(r - \frac{\sigma^2}{2}\Big)T\Big] \\
&= \frac{1}{\sigma\sqrt{T}}\Big[\log\Big(\frac{S_0}{K}\Big) + rT - \frac{\sigma^2}{2}T\Big] \\
&= \frac{\log\Big(\frac{S_0}{K}\Big) + rT}{\sigma\sqrt{T}} - \frac{\sigma}{2}\sqrt{T} \\
\implies \frac{\partial d_-}{\partial\sigma} &= \frac{\partial}{\partial\sigma}\Big[\frac{\log\Big(\frac{S_0}{K}\Big) + rT}{\sigma\sqrt{T}}\Big] - \frac{\sqrt{T}}{2}.
\end{aligned}$$

From this it follows

$$\frac{\partial d_+}{\partial\sigma} - \frac{\partial d_-}{\partial\sigma} = \sqrt{T}$$

and so, substituting this in for the expression for vega:

$$\begin{aligned}\mathcal{V} &= S\frac{e^{-d_+^2/2}}{\sqrt{2\pi}}\sqrt{T} \\ &= SN'(d_+)\sqrt{T}\end{aligned}$$

which confirms the formula given in lecture.

 # 4:

We have the following data

$$\begin{aligned} K &= 60 \\ S &= 72 \\ \sigma &= 0.2 \\ r &= 0.04 \\ T - t &= 6 \text{ months} = 0.5. \end{aligned}$$

We start by computing d_+:

$$\begin{aligned} d_+ &= \frac{\log\left(\frac{72}{60}\right) + \left(0.04 + \frac{0.2^2}{2}\right)0.5}{0.2\sqrt{0.5}} \\ &= 1.5013. \end{aligned}$$

Thus

$$\Delta = N(d_+) = N(1.5013) = 0.9334.$$

Next the gamma:

$$\begin{aligned} \Gamma &= \frac{N'(d_+)}{S\sigma\sqrt{T-t}} \\ &= \frac{e^{-d_+^2/2}}{S\sigma\sqrt{T-t}\sqrt{2\pi}} \\ &= \frac{e^{-1.5013^2/2}}{72(0.2)\sqrt{0.5}\sqrt{2\pi}} \\ &= 0.01270. \end{aligned}$$

And finally the vega:

$$\begin{aligned}
\mathcal{V} &= S\sqrt{T-t}N'(d_+) \\
&= \frac{S\sqrt{T-t}e^{-d_+^2/2}}{\sqrt{2\pi}} \\
&= \frac{72\sqrt{0.5}e^{-1.5013^2/2}}{\sqrt{2\pi}} \\
&= 6.5811.
\end{aligned}$$

 # 5:
We start by computing $\Delta = N(d_+)$. Our data is

$$\begin{aligned}
S &= 110 \\
K &= 105 \\
T-t &= 1 \\
\sigma &= 0.15 \\
r &= 0.04.
\end{aligned}$$

Then we have

$$\begin{aligned}
d_+ &= \frac{\log\left(\frac{110}{105}\right) + \left(0.04 + \frac{0.15^2}{2}\right)(1)}{0.15\sqrt{1}} \\
&= 0.6518.
\end{aligned}$$

Thus

$$\begin{aligned}
\Delta &= N(0.6518) \\
&= 0.7427 \\
&= 74.27\%.
\end{aligned}$$

So to delta hedge our option position, we need to add to it a short position in

$$0.7427 \times 50 = 37 \text{shares}$$

The combined, delta hedged portfolio consists of the 50 long call postions and a short of 37 shares.

To compute the P&Ls under the proposed scenarios we must find the initial price of the call options. To compute this using the Black-Scholes formula, we need d_- as well as d_+:

$$\begin{aligned} d_- &= \frac{\log\left(\frac{110}{105}\right) + \left(0.04 - \frac{0.15^2}{2}\right)(1)}{0.15\sqrt{1}} \\ &= 0.5018 \end{aligned}$$

Thus the initial value of a call option is

$$\begin{aligned} C &= 110N(0.6518) - 105e^{-0.04(1)}N(0.5018) \\ &= \$11.88, \end{aligned}$$

Hence the initial value of our option position, long in 50 options, is

$$50 \times \$11.88 = \$594.$$

Now we will check the P&L of our delta hedged position in the 2 proposed scenarios.

Scenario #1: S(0.25)=100.

Our short stock position has profited, since the price has dropped, by \$10 per share, so the total profit from the stock position is

$$37 \times \$10 = \$370.$$

To value the option position we need updated values of d_+ and d_-:

$$\begin{aligned} d_+ &= \frac{\log\left(\frac{100}{105}\right) + \left(0.04 + \frac{0.15^2}{2}\right)(0.75)}{0.15\sqrt{0.75}} \\ &= -0.07969 \end{aligned}$$

and

$$\begin{aligned} d_- &= \frac{\log\left(\frac{100}{105}\right) + \left(0.04 - \frac{0.15^2}{2}\right)(0.75)}{0.15\sqrt{0.75}} \\ &= -0.2096. \end{aligned}$$

Substitute into the Black-Scholes formula to price the calls:

$$\begin{aligned} C &= 100N(-0.07969) - 105e^{-0.04(0.75)}N(-0.2096) \\ &= \$4.33. \end{aligned}$$

On our long option position we have lost

$$50 \times (11.88 - 4.33) = \$377$$

Thus the P&L on the combined, delta hedged position is

$$\begin{aligned}\text{Total P\&L} &= \$370 - \$377 \\ &= -\$7\end{aligned}$$

Scenario #2: S(0.25)=$120.
The short stock position has now lost

$$37 \times (120 - 110) = \$370.$$

We compute d_+ and d_- to evaluate the call price

$$\begin{aligned}d_+ &= \frac{\log\left(\frac{120}{105}\right) + \left(0.04 + \frac{0.15^2}{2}\right)(0.75)}{0.15\sqrt{0.75}} \\ &= 1.3238\end{aligned}$$

and

$$\begin{aligned}d_- &= \frac{\log\left(\frac{120}{105}\right) + \left(0.04 - \frac{0.15^2}{2}\right)(0.75)}{0.15\sqrt{0.75}} \\ &= 1.1939.\end{aligned}$$

The call price is thus

$$\begin{aligned}C &= 120N(1.3238) - 105e^{-0.04(0.75)}N(1.1939) \\ &= \$18.82.\end{aligned}$$

So on the option position we have profited:

$$50 \times (18.82 - 11.88) = \$347$$

The P&L on the combined position is

$$\begin{aligned}\text{Total P\&L} &= -\$370 + \$347 \\ &= -\$23\end{aligned}$$

6:
(a) The idea in the lecture example is to set up a position that will profit if the volatility increases, no matter what the stock price does. Since option prices are very sensitive to the price of the underlying asset (that is what the delta measures) the portfolio has to be modified to insulate the options from loss if the stock price moves adversely. The delta hedge accomplishes this by building a portfolio whose overall delta is 0 or close to 0.

So we delta hedge, and thus mostly eliminate the risk of changes in the underlying price. But now, how do we profit from a volatility increase? For this, consider the vega

$$\begin{aligned}\mathcal{V} &= S\frac{e^{-d_+^2/2}}{\sqrt{2\pi}}\sqrt{T} \\ &= SN'(d_+)\sqrt{T}\end{aligned}$$

The vega, which is the sensitivity of an option to the volatility, is always positive, so if you hold a position in options (no matter whether calls or puts) and the volatility goes up, your position profits–all else being equal. Which is the point of the delta hedge, to make "all else equal" by removing the sensitivity to the underlying asset.

So if we want to profit from an increase in volatility, one way to do it is to set up a long option position that is delta hedged.

(b) If, instead, we want to profit from a decrease in volatility (that is, short volatility) then consideration of the same sensitivities, the delta and the vega, implies we can do this with a short position in options, once again, delta hedged to remove risk from changes in the underlying price itself.

(c) To profit from a drop in volatility, we want to take a short position in calls. So we will short, or sell, 1000 calls on this stock, and delta hedge by taking a long position in the stock itself.

We need the call price and the delta. We have the following data initially:

$$\begin{aligned}S(0) &= \$100 \\ K &= \$100 \\ T &= 1 \\ \sigma &= 0.4 \\ r &= 0.05.\end{aligned}$$

We start by calculating d_+ and d_-:

$$
\begin{aligned}
d_+ &= \frac{\log\left(\frac{100}{100}\right) + \left(0.05 + \frac{0.4^2}{2}\right)(1)}{0.4\sqrt{1}} \\
&= 0.325
\end{aligned}
$$

and

$$
\begin{aligned}
d_+ &= \frac{\log\left(\frac{100}{100}\right) + \left(0.05 - \frac{0.4^2}{2}\right)(1)}{0.4\sqrt{1}} \\
&= -0.075.
\end{aligned}
$$

We compute the Black-Scholes call price:

$$
\begin{aligned}
C &= 100N(0.325) - 100e^{-0.05(1)}N(-0.075) \\
&= \$18.02.
\end{aligned}
$$

And we compute the Black-Scholes delta:

$$
\begin{aligned}
\Delta &= N(0.325) \\
&= 0.627 \\
&= 62.7\%.
\end{aligned}
$$

The delta hedged position will consist of 1000 short call positions and a long position on

$$0.627 \times 1000 = 627 \text{ shares.}$$

Now we suppose we hold these positions for 6 months, and that now the volatility has decreased to 10%=0.1. We consider 3 scenarios for the stock price $S(0.5)$ at this time: \$90, \$110, and \$100.
Scenario 1: S(0.5)=\$90
Our long stock position has lost

$$627 \times (100 - 90) = \$6270.$$

For the option position, we compute the updated call price:

$$
\begin{aligned}
d_+ &= \frac{\log\left(\frac{90}{100}\right) + \left(0.05 + \frac{0.1^2}{2}\right)(0.5)}{0.1\sqrt{0.5}} \\
&= -1.1011
\end{aligned}
$$

and

$$d_+ = \frac{\log\left(\frac{90}{100}\right) + \left(0.05 - \frac{0.1^2}{2}\right)(0.5)}{0.1\sqrt{0.5}}$$
$$= -1.1718.$$

And the Black-Scholes call price:

$$\begin{aligned} C &= 90N(-1.1011) - 100e^{-0.05(0.5)}N(-1.1718) \\ &= \$0.42. \end{aligned}$$

So our short option position has profited by

$$1000 \times (18.02 - 0.42) = \$17,600.$$

The P&L of the full hedged position is

$$\begin{aligned} \text{Total P\&L} &= \$17,600 - \$6270 \\ &= \$11,330 \end{aligned}$$

Scenario 2: S(0.5)=$110
Now our long stock position has profited:

$$627 \times (110 - 100) = \$6270.$$

For the option position:

$$d_+ = \frac{\log\left(\frac{110}{100}\right) + \left(0.05 + \frac{0.1^2}{2}\right)(0.5)}{0.1\sqrt{0.5}}$$
$$= 1.7368$$

and

$$d_+ = \frac{\log\left(\frac{110}{100}\right) + \left(0.05 - \frac{0.1^2}{2}\right)(0.5)}{0.1\sqrt{0.5}}$$
$$= 1.6661.$$

And the Black-Scholes call price:

$$\begin{aligned} C &= 110N(1.7368) - 100e^{-0.05(0.5)}N(1.6661) \\ &= \$12.60. \end{aligned}$$

The short option position profits by

$$1000 \times (18.02 - 12.60) = \$5420$$

Thus the P&L of the total position is

$$\begin{aligned}\text{Total P\&L} &= \$5420 + \$6270 \\ &= \$11,690.\end{aligned}$$

Scenario 3: S(0.5)=$100
Now we break even on the stock position, so the total P&L is purely from the option position. Once again, we price the calls from the Black-Scholes formula:

$$\begin{aligned}d_+ &= \frac{\log\left(\frac{100}{100}\right) + \left(0.05 + \frac{0.1^2}{2}\right)(0.5)}{0.1\sqrt{0.5}} \\ &= 0.3889\end{aligned}$$

and

$$\begin{aligned}d_+ &= \frac{\log\left(\frac{100}{100}\right) + \left(0.05 - \frac{0.1^2}{2}\right)(0.5)}{0.1\sqrt{0.5}} \\ &= 0.3182.\end{aligned}$$

And the Black-Scholes call price:

$$\begin{aligned}C &= 100N(0.3889) - 100e^{-0.05(0.5)}N(0.3182) \\ &= \$4.19.\end{aligned}$$

So the P&L of the position is

$$\begin{aligned}\text{Total P\&L} &= 1000 \times (18.02 - 4.19) \\ &= \$13,830.\end{aligned}$$

Taking a close look reveals that the options lost money in all 3 scenarios and thus the short option position was profitable in all 3 cases. So we did not actuallly need the delta hedge to protect us from outright loss. Notice, though, that in spite of the wide variability in the profitability of the short option position in the 3 different cases, the P&L of the combined position was remarkably consistent in all 3 cases. This demonstrates that the delta hedge still mitigated a significant risk factor, exposure to the underlying asset.

The consistent loss of the calls in all 3 cases is due to the volatility drop and to the time decay of options.