Problem Set 8: Solutions

 # 1:

The riskless portfolio from the lecture had the form

$$V = \Delta S - D$$

To check that V_1 has the same value in both states of the world at time 1 we must check that

$$\begin{aligned} V_1(+) &= V_1(-) \\ \iff \Delta S_1(+) - D_1(+) &= \Delta S_1(-) - D_1(-) \end{aligned}$$

In the lecture we computed

$$\Delta = 0.4.$$

Thus using the value for the assets at time 1 from the lecture we have

$$\begin{aligned} V_1(+) &= \Delta S_1(+) - D_1(+) \\ &= 0.4(65) - 10 \\ &= 16 \end{aligned}$$

and

$$\begin{aligned} V_1(-) &= \Delta S_1(-) - D_1(-) \\ &= 0.4(40) - 0 \\ &= 16 \end{aligned}$$

confirming that V_1 is a riskless portfolio.

The argument for the riskless portfolio is as follows. The portfolio V has the same value in all states of the world at time 1–this is the meaning of "riskless portfolio". While it is uncertain at time 0 whether at time 1 we will be in the $+$ state or the $-$ state, it is not uncertain what the value of the portfolio V will be at time 1 since it has the same value in both states.

The riskless investment has the same property. If I invest some amount of money N at the risk free rate at time 0 I know that at time 1 my investment will be worth $N(1 + r)$ in all states of the world. This means there is a particular value I can invest at the risk free rate at time 0 so that at time 1 the investment is worth the same as $V_1 = 16$, namely whatever value of N that satisfies

$$\begin{aligned} (1+r)N &= 16 \\ \implies N &= \frac{16}{1+r} \\ \implies N &= \frac{16}{1.08} \end{aligned}$$

since $r = 0.08$ in this example. So this investment at the risk free rate agrees with V_1 in all states of the world, that is, with probability 1. By the Law of One Price, these two assets must have the same value at time 0, which means that

$$V_0 = N = \frac{16}{1.08}$$

 # 2:

The stock is an asset which has value S_0 at time 0, and will be worth either uS_0 or dS_0 at time 1, depending on the state of the world. Another asset, also worth S_0 at time 0, is S_0 invested at the risk free rate, and it has value $(1+r)S_0$ at time 1 no matter the state of the world. If the inequalities

$$1 + r < d < u$$

hold, then

$$\begin{aligned} (1+r)S_0 < uS_0 < dS_0 \\ \implies (1+r)S_0 < S_1 \end{aligned}$$

in all states of the world, ie with probability 1. This means that S_0 invested at the risk free rate is an asset that is worth the same thing at the stock at time 0, but is, with certainty, worth less than the stock at time 1. This suggests that either the stock is underpriced or the riskless investment is overpriced. So we attempt to build an arbitrage by shorting the riskless investment and going long the stock.

Shorting the riskless asset means borrowing. So we borrow S_0 at the risk free rate. Using these funds, we buy the stock. At time 1, our investment in the stock is worth either uS_0 or dS_0 but, by the above inequalities, this is more than the value of the debt, $(1+r)S_0$ in either case. So, we know that at time 1 we can sell the stock, retire the debt, and retain a profit of at least

$$dS_0 - (1+r)S_0 = (d - 1 - r)S_0 > 0.$$

This is a riskless and certain profit, and so an arbitrage.

If instead we have the opposite inequalities

$$d < u < 1 + r$$

this suggests that it is the stock that is overpriced, so now we go long the riskless investment and short the stock. So short the stock, yielding S_0 in cash, and invest this cash at the risk free rate. At time 1 our investment is worth

$$\begin{aligned} (1+r)S_0 &> uS + 0 > dS_0 \\ \implies (1+r)S_0 &> S_1 \end{aligned}$$

with probability 1. So we can purchase the stock from the fund in our investment, close out the short position, and earn a riskless profit of at least

$$(1+r)S_0 - uS_0 = (1 + r - u)S_0 > 0$$

so this is again an arbitrage.

 # 3: In the lecture we found that for the delta hedged portfolio

$$V = \Delta S - D$$

we must have

$$\Delta = \frac{D_1(+) - D_1(-)}{(u-d)S_0}.$$

We may therefore write the value of the porfolio in the 2 states of the world.

$$\begin{aligned} V_1(+) &= \Delta S_1(+) - D_1(+) \\ &= \frac{D_1(+) - D_1(-)}{(u-d)S_0} uS_0 - D_1(+) \\ &= \frac{D_1(+) - D_1(-)}{u-d} u - D_1(+) \\ &= \frac{(D_1(+) - D_1(-))u - (u-d)D_1(+)}{u-d} \\ &= \frac{dD_1(+) - uD_1(+)}{u-d} \end{aligned}$$

and

$$\begin{aligned}
V_1(-) &= \Delta S_1(-) - D_1(-) \\
&= \frac{D_1(+) - D_1(-)}{(u-d)S_0} dS_0 - D_1(-) \\
&= \frac{D_1(+) - D_1(-)}{u-d} d - D_1(-) \\
&= \frac{(D_1(+) - D_1(-))d - (u-d)D_1(+)}{u-d} \\
&= \frac{dD_1(+) - uD_1(+)}{u-d}.
\end{aligned}$$

And we have confirmed that $V_1(+) = V_1(-)$

 # 4:

We have

$$\tilde{p} = \frac{1+r-d}{u-d} \quad \text{and} \quad \tilde{q} = \frac{u-1-r}{u-d}$$

So, compute:

$$\begin{aligned}
\tilde{p} + \tilde{q} &= \frac{1+r-d}{u-d} + \frac{u-1-r}{u-d} \\
&= \frac{1+r-d+u-1-r}{u-d} \\
&= \frac{u-d}{u-d} \\
&= 1
\end{aligned}$$

 # 5:

The first part of this problem is mostly about clarifying terminology and concepts. When we talk about *the* riskless investment this is, in practice, a bit inaccurate. We tend to refer to the riskless asset anytime we invest any amount of cash N at the risk free rate. We could think of *the* riskless asset as one that pays off one particular value at time 1. For instance, we could normalize the riskless asset to be one that pays, risklessly, 1 at time 1. This asset would be worth the discounted value of 1 at time 0 which is

$$\frac{1}{1+r}$$

This would be equivalent to saying the riskless asset is a zero coupon bond, maturing at time 1, with a face value of \$1. In this case, we could think of the general riskless asset as a multiple of this bond, or a bond chosen with a particular notional or face value.

Regardless of formalities, "the" riskless investment corresponds to any investment of some amount of cash N at the risk free rate r. Denoting the value of any such asset by B_t, we have

$$B_0 = N$$

and

$$B_1 = (1+r)N$$

and so the the gross return of any riskless asset is

$$\text{Gross Return} = \frac{B_1}{B_0} = \frac{(1+r)N}{N} = 1+r$$

For the stock, we compute

$$\begin{aligned}
E^{\tilde{P}}[\text{Gross Return of S}] &= E^{\tilde{P}}\left[\frac{S_1}{S_0}\right] \\
&= \tilde{p}\frac{S_1(+)}{S_0} + \tilde{q}\frac{S_1(-)}{S_0} \\
&= \left(\frac{1+r-d}{u-d}\right)\frac{uS_0}{S_0} + \left(\frac{u-1-r}{u-d}\right)\frac{dS_0}{S_0} \\
&= \frac{(1+r-d)u + (u-1-r)d}{u-d} \\
&= \frac{(1+r)(u-d)}{u-d} \\
&= 1+r
\end{aligned}$$

 # 6:

The payoff of a put with strike price K expiring at time 1 is

$$P_1 = \max(0, K - S_1)$$

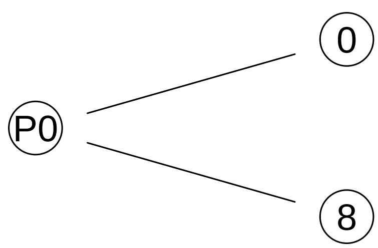
We have $S_1(+) = 88$ and $S_1(-) = 74$. Thus, the values of the put, with strike price \$82 in the 2 states of the world are

$$
\begin{aligned}
P_1(+) &= \max(0, K - S_1(+)) \\
&= \max(0, 82 - 88) \\
&= \max(0, -6) \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
P_1(-) &= \max(0, K - S_1(-)) \\
&= \max(0, 82 - 74) \\
&= \max(0, 8) \\
&= 8.
\end{aligned}
$$

The put values may be represented in the binomial diagram



Our task is to compute P_0. To do this as a risk neutral expectation, we need the risk neutral probabilities $\tilde{p}$ and $\tilde{q}$, and for those we need the up and down factors u and d:

$$
\begin{aligned}
u &= \frac{88}{80} = 1.1 \\
d &= \frac{74}{80} = 0.925.
\end{aligned}
$$

We also have a risk neutral rate $r = 5\% = 0.05$. The risk neutral probabilities are

$$
\begin{aligned}
\tilde{p} &= \frac{1 + r - d}{u - d} = \frac{1 + 0.05 - 0.925}{1.1 - 0.925} = 0.71 \\
\tilde{q} &= \frac{u - 1 - r}{u - d} = \frac{1.1 - 1 - 0.05}{1.1 - 0.925} = 0.29.
\end{aligned}
$$

With these, we calculate the fair put premium P_0 as a risk neutral expectation of the discounted value of its payoff:

$$\begin{aligned}
P_0 &= E^{\tilde{P}}\left[\frac{P_1}{1+r}\right] \\
&= \tilde{p}\frac{P_1(+)}{1.05} + \tilde{q}\frac{P_1(-)}{1.05} \\
&= (0.71)\frac{0}{1.05} + (0.29)\frac{8}{1.05} \\
&= \$2.21
\end{aligned}$$

 # 7:

We carry through the prescribed calculations one by one.

For the factors u and d we have

$$\begin{aligned}
u &= \frac{S_1(+)}{S_0} \\
&= \frac{58}{50} \\
&= 1.16
\end{aligned}$$

and

$$\begin{aligned}
d &= \frac{S_1(-)}{S_0} \\
&= \frac{44}{50} \\
&= 0.88.
\end{aligned}$$

We have a risk free rate of $r = 3\% = 0.03$ so checking that $0.88 < 1.03 < 1.16$ confirms the arbitrage inequalities.

The risk neutral probabilities are

$$\begin{aligned}
\tilde{p} &= \frac{1+r-d}{u-d} = \frac{1+0.03-0.88}{1.16-0.88} = 0.54 \\
\tilde{q} &= \frac{u-1-r}{u-d} = \frac{1.16-1-0.03}{1.16-0.88} = 0.46.
\end{aligned}$$

We now compute the fair premium D_0 for the derivative given. To use the riskless portfolio whose value is

$$V_t = \Delta S_t - D_t$$

we use the general form for Δ derived in the second lecture on the 1 step binomial model:

$$\Delta = \frac{D_1(+) - D_1(-)}{(u-d)S_0}$$

From the binomial diagram we have $D_1(+) = 20$, and $D_1(-) = 10$. We also have $S_0 = 50$, and we have calculated u and d above. So, Δ can be computed:

$$\begin{aligned} \Delta &= \frac{20 - 10}{(1.16 - 0.88)50} \\ &= 0.7143. \end{aligned}$$

The value of the riskless portfolio at time 1

$$V_1 = \Delta S_1 - D_1$$

should have the same value in both states of the world, but we check this

$$\begin{aligned} V_1(+) &= \Delta S_1(+) - D_1(+) \\ &= 0.7143(58) - 20 \\ &= \$21.43 \end{aligned}$$

and

$$\begin{aligned} V_1(-) &= \Delta S_1(-) - D_1(-) \\ &= 0.7143(44) - 10 \\ &= \$21.43 \end{aligned}$$

confirming that this is a riskless portfolio. It follows that V_0 must be the discounted value of this common value at time 1:

$$\begin{aligned} V_0 &= \frac{21.43}{1.03} \\ &= \$20.81 \end{aligned}$$

We therefore have

$$\begin{aligned} V_0 &= \Delta S_0 - D_0 \\ \Longrightarrow 20.81 &= \Delta S_0 - D_0 \\ &= 0.7143(50) - D_0 \end{aligned}$$

and to find D_0 we solve this equation:

$$\begin{aligned} D_0 &= 0.7143(50) - 20.81 \\ &= \$14.91 \end{aligned}$$

Now we compute D_0 using the risk neutral expectation.

$$\begin{aligned} D_0 &= E^{\tilde{P}}\left[\frac{D_1}{1+r}\right] \\ &= \tilde{p}\frac{D_1(+)}{1+r} + \tilde{q}\frac{D_1(-)}{1+r} \\ &= (0.54)\frac{20}{1.03} + (0.46)\frac{10}{1.03} \\ &= \$14.95 \end{aligned}$$

The difference, a 4 cent discrepancy, can be attributed to roundoff error.