Problem Set 7 Solutions

 # 1:
A European put with a strike price of \$40 give the option holder the right, but not the obligation, to sell the underlying asset to the counterparty (the option writer or seller) for \$40. She will only do so if the strike price is better than what is available on markets. So the option will be exercised if and only if the cash price for the underlying asset is less than \$40.

If, on the expiration date, the underlying is trading on the cash market for \$85 then the put will not be exercised, as the market price is more than the strike price. The put thus expires worthless, and the option buyer loses the original premium she paid of \$5.

If the cash price of the underlying at the expiration date is \$10 than the put will be exercised. She must provide to her counterparty an asset worth \$10, but receives \$40 from the sale so she makes a net revenue of

$$\$40 - \$10 = \$30.$$

But she must subtract from this the initial \$5 premium she paid, so her final profit is

$$\$30 - \$5 = \$25.$$

 # 2: A call with a strike price of \$60 gives the option holder the right, but not the obligation to buy the underlying asset for \$60. So it will be exercised if and only if the cash price of the underlying is more than \$60,

We are asked to consider the P&L from the point of view of the option writer. If the underlying is trading for \$80 on the cash market then, since this is more than the strike price of \$60, the call will be exercised. The option writer will be forced to sell an asset worth \$80 for only \$60, and so will loses

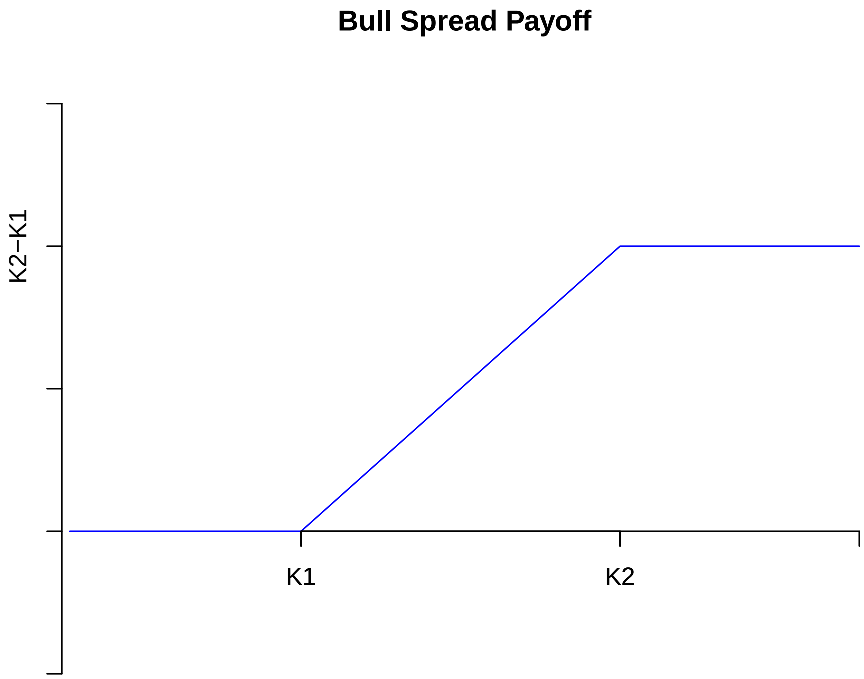
$$\$80 - \$60 = \$20$$

in this transaction. However, his loss is reduced by the \$15 premium he was paid when he sold the option, so his total loss is

$$\$20 - \$15 = \$5.$$

If the underlying is trading for \$30 then the option will not be exercised, and the option writer earns as a profit the original \$15 he was paid when he sold the call.

 # 3:
The plot of the payoff of a bull spread with strikes $K_1 < K_2$



In the case that $K_1 = 50$ and $K_2 = 80$, we can read off from the figure that the bull spread payoffs, under the proposed stock prices, will be

Stock Price		Bull Spread Payoff
\$20	$\Longrightarrow$	\$0
\$60	$\Longrightarrow$	\$10
\$90	$\Longrightarrow$	\$30

 # 4:

The maximum function

$$\max\{0, S(T) - K\}$$

can, by definition, be written as a case splitting

$$\max\{0, S(T) - K\} = \begin{cases} S(T) - K & \text{if } \; S(T) - K > 0 \\ 0 & \text{if } \; S(T) - K \leq 0 \end{cases}$$

But one can easily see that this is the same as the alternative form written, since the conditions are equivalent

$$\begin{aligned} S(T) - K > 0 \quad &\equiv \quad S(T) > K \\ S(T) - K \leq 0 \quad &\equiv \quad S(T) \leq K \end{aligned}$$

 # 5:

We wish to prove arbitrage inequality #2

$$P(t) \leq e^{-r(T-t)} K$$

We do this by assuming it is not true, and showing that leads to an arbitrage.

Suppose at time t,

$$P(t) > e^{-r(T-t)} K$$

Construct an arbitrage as follows:

1. sell or write the put, receiving $P(t)$ in cash.
2. invest these proceeds at the risk free rate.

We hold these positions until the expiration of the put.
Rewrite the assumed inequality: $e^{r(T-t)} P(t) > K$.

Note: LHS=value of our investment at time T.

At expiration, 2 events are possible depending on the value of the stock price.

If $S(T) \geq K$:

The put is out of the money and will not be exercised

We retain as a profit our cash investment worth

$$e^{r(T-t)} P(t) > 0$$

If $S(T) < K$:
The put will be exercised, so we pay K in return for the stock.
We retain a cash holding

$$e^{r(T-t)}P(t) - K > 0$$

Our total holdings finally are the retained cash plus the stock.
we have thus earned a riskless profit

$$e^{r(T-t)}P(t) - K + S(T) > 0$$

No matter what happens, we will earn a riskless profit.
Thus, this is an arbitrage portfolio.
We have shown that if the inequality

$$P(t) > e^{-r(T-t)}K$$

is ever observed, we can construct an arbitrage.

Thus this inequality is ruled out.

So we must have

$$P(t) \leq e^{-r(T-t)}K$$

which is the arbitrage inequality we set out to prove.

 # 6:
We wish to prove arbitrage inequality #4

$$P(t) \geq e^{-r(T-t)}K - S(t)$$

We do this by assuming it is not true, and showing that leads to an arbitrage.
Suppose at time t,

$$P(t) < e^{-r(T-t)}K - S(t)$$

Rewrite this inequality as

$$e^{r(T-t)}(P(t) + S(t)) < K$$

Now borrow $P(t) + S(t)$ at the risk free rate. Note that the LHS of the above inequality is the future value of this debt at time T.

Use the proceeds from this loan to purchase the underlying asset and the put. Hold these positions until expiration T.

Because we own the put, with a strike price of K and the underlying, we know we can convert these positions into a minimum cash holding of K. If the underlying is less than K, exercise the put and we will receive K for it. If it's market value is now more than K, sell it for it's market price and let the put expire worthless. Either way we receive a cash receipt of at least K.

The previous inequality implies that our debt is now worth less than K, so we can retire it and retain a positive profit of at least

$$K - e^{r(T-t)}(P(t) + S(t)) > 0$$

where this is just a rewriting of the previous inequality. This is a riskless and certain profit, and this portfolio is an arbitrage.

We have shown that if the inequality

$$P(t) < e^{-r(T-t)}K - S(t)$$

is ever observed, we can construct an arbitrage portfolio.

Thus this inequality is ruled out.

So we must have

$$P(t) \geq e^{-r(T-t)}K - S(t)$$

which is the arbitrage inequality we set out to prove.

 # 7:

The bounds on option prices tell us that the call price $C(t)$ must satisfy the bounds

$$S(t) - e^{-r(T-t)}K \leq C(t) \leq S(t)$$

where $S(t)$ is the underlying stock price, K is the strike price and r is the risk free rate. The data we have is

$$\begin{aligned} C(t) &= \$10 \\ S(t) &= \$65 \\ K &= \$50 \\ r &= 3\% = 0.03. \end{aligned}$$

We have $C(t) < S(t)$, but looking at the LHS of the above inequality

$$\begin{aligned} & S(t) - e^{-r(T-t)}K \\ = \; & 65 - e^{-0.03(0.25)}(50) \\ = \; & \$15.37 \\ > \; & C(t) \end{aligned}$$

So the right inequality is violated, and there must be an arbitrage. The justification of the inequality gives hints as to how to exploit it.

Short the stock, which immediately pays \$65 in cash. With this cash, purchase the call for \$10, and invest the remaining \$55 at the risk free rate.

3 months later, when the call expires, our cash investment is worth

$$e^{0.03(0.25)}(55) = \$55.41$$

Because we have a long position in a call option with a strike price of \$50, the most we have to pay to buy the stock is \$50: either $S(T) > \$50$ in which case we exercise the option and buy the stock for \$50, or $S(T) \leq \$50$ in which case we can buy it for even less. Either way, have enough cash from our investment to buy the stock and close out the short position. We will have retained a profit of at least

$$\$55.41 - \$50 = \$5.41$$

which is a riskless profit.

 # 8:

From the option inequalities we know the put price $P(t)$ must satisfy the bounds

$$e^{-r(T-t)}K - S(t) \leq P(t) \leq e^{-r(T-t)}K$$

Our data is

$$\begin{aligned} P(t) &= \$12 \\ S(t) &= \$60 \\ K &= \$80 \\ r &= 6\% = 0.06. \end{aligned}$$

While there is no problem with the right inequality, for the left we have

$$\begin{aligned} & e^{-r(T-t)}K - S(t) \\ = \ & e^{-0.06(0.5)}(80) - 60 \\ = \ & \$17.64 \\ > \ & P(t) \end{aligned}$$

so the left inequality is violated. Once again, the justification of this inequality gives hints as to how to construct an arbitrage.

Borrow \$72 at the risk free rate. With these funds purchase the stock and the put. 5 months later the debt has accumulated to

$$e^{(0.06)(0.5)}(72) = \$74.19$$

Because we have a long position in a put with a strike of \$80, we know we will at least get paid \$80 for selling it. So we are certain to have enough money to retire the debt and be left with a profit of at least

$$\$80 - \$74.19 = \$5.81$$

which is a riskless, arbitrage profit.

 # 9:

We check the data we have against the put-call parity relation which is

$$C(t) - P(t) = S(t) - e^{-r(T-t)}K$$

Our data is

$$\begin{aligned} C(t) &= \$35 \\ P(t) &= \$10 \\ S(t) &= \$75 \\ K &= \$60 \\ T - t &= 3 \text{ months} = 0.25 \\ r &= 5\% = 0.05. \end{aligned}$$

And we have

$$\begin{aligned} C(t) - P(t) &= \$35 - \$10 \\ &= \$25 \end{aligned}$$

and

$$
\begin{aligned}
S(t) - e^{-r(T-t)}K &= 75 - e^{-0.05(0.25)}(60) \\
&= \$15.75
\end{aligned}
$$

And we see that put-call parity is violated. There must therefore be an arbitrage somewhere.

Since

$$C(t) - P(t) > S(t) - e^{-r(T-t)}K$$

this suggests that the call is overpriced relative to the put and the stock.

Start by selling the call, for which we receive $C(t)$ =\$35. Now borrow \$50 at the risk free rate. With the \$85 funds on hand now, buy the underlying (\$75) and the put (\$10). Hold these positions for 3 months when the call and the put expire.

At the expiration of the two options, one of two things happens depending on whether the stock price is more or less than the strike price of \$60. If it is more, the call (which we sold) will be exercised and we will have to sell the stock for \$60. If it is less, then we will exercise the put, in which we are long, and receive \$60 for the stock we own. So, no matter what, at expiration we can sell the stock for \$60. At this time, the debt is now worth

$$e^{(0.05)(0.25)}(50) = \$50.63.$$

With our cash holding we can retire this debt and retain a profit of

$$\$60 - \$50.63 = \$9.37$$

which is a certain and riskless profit.