Problem Set 1

 # 1:
To take advantage of this mispricing situation: buy stock on the OTC markets for $72 per share, and sell it on the NYSE for $80 per share. Do this as much as you can for as long as you can pocketing $8 per share.

 # 2:
At time 1, in state I of the world, we have

$$\begin{aligned} & 2P_A(1) + P_B(1) \\ = & \ 2(60) + 80 \\ = & \ 200 \\ = & \ P_C(1). \end{aligned}$$

In state II

$$\begin{aligned} & 2P_A(1) + P_B(1) \\ = & \ 2(250) + 40 \\ = & \ 540 \\ = & \ P_C(1). \end{aligned}$$

So, in all states of the world, ie, with probability 1 we have

$$2P_A(1) + P_B(1) = P_C(1).$$

From the Law of One Price we can conclude that at time 0 the same relationship must hold. That is, at time 0, the portfolio consisting of 2 shares of A and 1 share of B must have the same price as 1 share of C or else there will be an arbitrage. But, at time 0

$$\begin{aligned} & 2P_A(0) + P_B(0) \\ = & \ 2(100) + 50 \\ = & \ 250 \\ > & \ 220 \\ = & \ P_C(0). \end{aligned}$$

By the Law of One Price there must be an arbitrage, and it should involve stock C being underpriced relative to stocks A and B.

To exploit this we should short 2 shares of A and 1 share of B and take a long position, ie buy, stock C. Taking the short position immediately yields a cash receipt of

$$2P_A(0) + P_B(0) = 2(100) + 50 = \$250.$$

With this cash, we can purchase 1 share of C for \$220 leaving a cash holding of \$30. We can invest this cash at the risk free rate and hold this position until time 1.

At time 1 we know that in all states of the world 1 share of C has the same price as a portfolio of 2 shares of A and 1 share of B. So, with certainty, we can, at time 1, sell our share of C, use the proceeds to purchase 2 shares of A and 1 share of B, and close out our short positions, leaving us free and clear of all obligations. Meanwhile, we have pocketed the \$30 we invested at time 0. This is a riskless profit, so this shows we have produced an arbitrage.

 # 3:

The clue that there is an arbitrage opportunity here is as follows. The proposed contract allows either a buyer or seller to lock in an oil price of \$67/barrel for a sale to happen in 1 year. I could enter the contract to buy oil for \$67 in 1 year, and invest the discounted value of \$67, which is

$$e^{-0.03}67 = \$65$$

then purchase the oil under the terms of the contract. To this portfolio, consisting of the buyer's position in this contract and an investment of \$65 replicates 1 barrel of oil in 1 year. On the other hand, I could simply purchase a barrel of oil on the markets now for \$64 and hold it for a year, which also replicates 1 barrel of oil in 1 year. Since we have been told we can ignore storage costs for oil, this implies 2 different assets that have the same value in 1 year (however much 1 barrel of oil is worth in a year) are priced differently now, so by the Law of One Price, there must be an arbitrage.

This suggests that the \$67 contract price is too high relatative to the current oil price, so we want to enter the seller's position in the contract and take a long position in oil itself. We build an arbitrage as follows:

1. Borrow \$64 at the risk free rate (3%)
2. Use these funds to buy 1 barrel of oil
3. Enter the seller's position in the contract

Now hold these positions for 1 year.

1 year later we sell the barrel of oil we hold in the contract and receive \$67. The debt is now worth

$$e^{0.03}64 = \$65.95$$

Using the \$67 from selling the oil we can retire this debt and retain a profit of \$1.05.

This is a riskless and certain profit if we execute this strategy: so it is an arbitrage.

 # 4:

For simplicity, assume assets A, B, C, and D are all shares of some stock. At time 1, we have the following relations in the 3 states of the world:

$$\begin{array}{rcl}
\textbf{In state I:} & & \\
P_A(1) + P_B(1) + P_C(1) & = & 20 + 20 + 20 = 60 = 5P_D(1) \\
\textbf{In state II:} & & \\
P_A(1) + P_B(1) + P_C(1) & = & 10 + 12 + 38 = 60 = 5P_D(1) \\
\textbf{In state III:} & & \\
P_A(1) + P_B(1) + P_C(1) & = & 40 + 8 + 12 = 60 = 5_PD(1)
\end{array}$$

So in all states of the world we have

$$P_A(1) + P_B(1) + P_C(1) = 5_PD(1) \tag{1}$$

At time 0 however we have

$$\begin{aligned}
& P_A(0) + P_B(0) + P_C(0) \\
= \; & 15 + 10 + 14 \\
= \; & 39 \\
< \; & 50 \\
= \; & 5P_D(0)
\end{aligned}$$

From the Law of One Price, this implies an arbitrage, and that stock D is overpriced relative to the other 3 stocks.

To implement an arbitrage, we will short 5 shares of D at time 0, resulting in a cash receipt of $50. We will spend $39 out of this to buy 1 share each of A, B, and C, and retain a cash holding of $11 which we invest at the risk free rate. We then hold for until time 1.

From relation (1) we have at time 1 that portfolio consisting of the three stocks is worth the same as 5 shares of D. So we sell that portfolio, purchase 5 shares of D, and exit the short position.

We retain, as a riskless profit, the cash holding from the $11 invested at time 0. This portfolio is therefore an arbitrage.

 # 5:
Law of One Price Extension #1:
A security that pays a sequence of cash flows

$$c_1, c_2, \ldots c_n$$

at times

$$t_1, t_2, \ldots t_n$$

can be regarded as a portfolio of securities each one corresponding to a single payment c_i. This is the critical idea behind discounted cash flow analysis.

Supposing we have a second security that pays a different sequence of cash flows

$$d_1, d_2, \ldots d_n$$

at the same times. If either the payments are all known constants and

$$c_i = d_i$$

for all i, or more generally, if they are random, But

$$\text{Prob}(c_i = d_i) = 1$$

then the Law of One Price says the value of the ith payment at any time before t_i must be the same for both securities, and so, since the value of each security is just the sum of the values of these payments, both securities must have the same price at any time before the payments begin.

Law of One Price Extension #2:
The argument is very similar to the argument for the basic Law of One Price. Suppose instead of the proposed inequality that

$$P_A(0) < P_B(0)$$

holds.

We now short asset B, and thus receive a cash payment of $P_B(0)$. Due to the above inequality, this is enough cash to purchase asset A, leaving a residual cash holding of

$$P_B(0) - P_A(0) > 0$$

We invest this cash holding at the risk free rate, which we denote r.

We now have a portfolio consisting of

1. A holding of asset A
2. A short position in B
3. A cash holding worth $(P_B(0) - P_A(0))e^{rt}$ at time t.

We hold these positions until time T

By assumption we have

$$P_A(T) \geq P_B(T)$$

Thus, at time T we may sell our holding of asset A, receiving $P_A(T)$ in cash, and with this, purchase asset B and close out our short position. We are left with a cash holding of

$$(P_B(0) - P_A(0))e^{rT} + P_A(T) - P_B(T) > 0$$

This is a certain and riskless profit. Therefore, this portfolio is an arbitrage.