Problem Set 9 Solutions

 # 1:

This is a matter of verifying these formulas from the general formula for binomial probabilities:

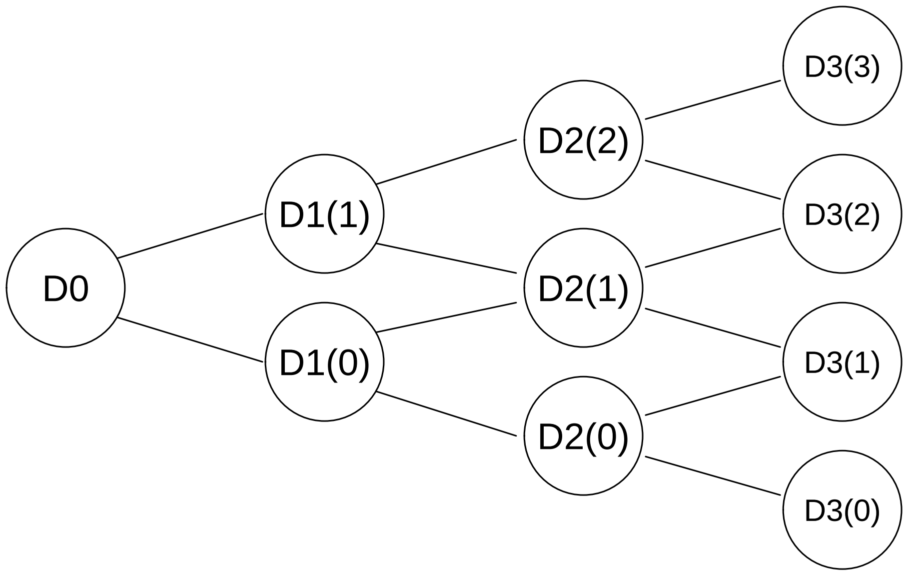
$$\mathrm{bin}(k; m, \tilde{p}) = \frac{m!}{k!(m-k)!}\tilde{p}^k\tilde{q}^{m-k}$$

We compute:

$$\begin{aligned}
\mathrm{bin}(0; 2, \tilde{p}) &= \frac{2!}{0!2!}\tilde{p}^0\tilde{q}^{2-0} = \tilde{q}^2 \\
\mathrm{bin}(1; 2, \tilde{p}) &= \frac{2!}{1!1!}\tilde{p}^1\tilde{q}^{2-1} = 2\tilde{p}\tilde{q} \\
\mathrm{bin}(2; 2, \tilde{p}) &= \frac{2!}{2!0!}\tilde{p}^2\tilde{q}^{2-2} = \tilde{p}^2 \\
\mathrm{bin}(0; 3, \tilde{p}) &= \frac{3!}{0!3!}\tilde{p}^0\tilde{q}^{3-0} = \tilde{q}^3 \\
\mathrm{bin}(1; 3, \tilde{p}) &= \frac{3!}{1!2!}\tilde{p}^1\tilde{q}^{3-1} = 3\tilde{p}\tilde{q}^2 \\
\mathrm{bin}(2; 3, \tilde{p}) &= \frac{3!}{2!1!}\tilde{p}^2\tilde{q}^{3-2} = 3\tilde{p}^2\tilde{q} \\
\mathrm{bin}(3; 3, \tilde{p}) &= \frac{3!}{3!0!}\tilde{p}^3\tilde{q}^{3-3} = \tilde{p}^3
\end{aligned}$$

Note we have used $0! = 1$.

 # 2:
From the 3 step binomial diagram below, in the generalized notation from the general binomial model lecture, we see that $D_1(0)$ and $D_1(1)$ can be written using the 2 step binomial in terms of their 2 step descendents, whild D_0 can be written in terms of them using the risk neutral expecation formula.



First, we may write $D_1(1)$ in terms of its 2 level descendents $D_3(3)$, $D_3(2)$, and $D_3(1)$ using the 2 step binomial pricing formula:

$$D_1(1) = \tilde{p}^2 \frac{D_3(3)}{(1+r)^2} + 2\tilde{p}\tilde{q}\frac{D_3(2)}{(1+r)^2} + \tilde{q}^2 \frac{D_3(1)}{(1+r)^2}$$

and similarly $D_1(0)$ may be written in terms of $D_3(2)$, $D_3(1)$, and $D_3(0)$ as

$$D_1(0) = \tilde{p}^2 \frac{D_3(2)}{(1+r)^2} + 2\tilde{p}\tilde{q}\frac{D_3(1)}{(1+r)^2} + \tilde{q}^2 \frac{D_3(0)}{(1+r)^2}$$

Now we can use the risk neutral expectation formula to write D_0 in terms of $D_1(1)$ and $D_1(0)$.

$$\begin{aligned}
D_0 &= \tilde{p}\frac{D_1(1)}{1+r} + \tilde{q}\frac{D_1(0)}{1+r} \\
&= \frac{\tilde{p}}{1+r}\Big[\tilde{p}^2\frac{D_3(3)}{(1+r)^2} + 2\tilde{p}\tilde{q}\frac{D_3(2)}{(1+r)^2} + \tilde{q}^2\frac{D_3(1)}{(1+r)^2}\Big] \\
&+ \frac{\tilde{q}}{1+r}\Big[\tilde{p}^2\frac{D_3(2)}{(1+r)^2} + 2\tilde{p}\tilde{q}\frac{D_3(1)}{(1+r)^2} + \tilde{q}^2\frac{D_3(0)}{(1+r)^2}\Big] \\
&= \tilde{p}^3\frac{D_3(3)}{(1+r)^3} + 2\tilde{p}^2\tilde{q}\frac{D_3(2)}{(1+r)^3} + \tilde{p}\tilde{q}^2\frac{D_3(1)}{(1+r)^3} \\
&+ \tilde{p}^2\tilde{q}\frac{D_3(2)}{(1+r)^3} + 2\tilde{p}\tilde{q}^2\frac{D_3(1)}{(1+r)^3} + \tilde{q}^3\frac{D_3(0)}{(1+r)^3} \\
&= \tilde{p}^3\frac{D_3(3)}{(1+r)^3} + 3\tilde{p}^2\tilde{q}\frac{D_3(2)}{(1+r)^3} + 3\tilde{p}\tilde{q}^2\frac{D_3(1)}{(1+r)^3} + \tilde{q}^3\frac{D_3(0)}{(1+r)^3} \\
&= \mathrm{bin}(3;3,\tilde{p})\frac{D_3(3)}{(1+r)^3} + \mathrm{bin}(2;3,\tilde{p})\frac{D_3(2)}{(1+r)^3} + \mathrm{bin}(1;3,\tilde{p})\frac{D_3(1)}{(1+r)^3} + \mathrm{bin}(0;3,\tilde{p})\frac{D_3(0)}{(1+r)^3} \\
&= E^{\mathrm{bin}(;3,\tilde{p})}\Big[\frac{D_3}{(1+r)^3}\Big]
\end{aligned}$$

 # 3:

We use risk neutral expectations and back propagation to compute the fair price D_0 of this derivative. The risk neutral probabilities are

$$\begin{aligned}
\tilde{p} &= \frac{1+r-d}{u-d} = \frac{1+0.04-0.9}{1.3-0.9} = 0.35 \\
\tilde{q} &= \frac{u-1-r}{u-d} = \frac{1.3-1-0.04}{1.3-0.9} = 0.65.
\end{aligned}$$

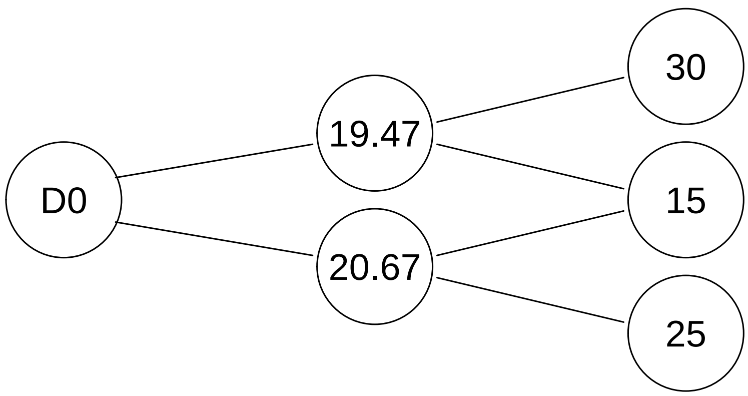
From the risk neutral expectation formula we have

$$\begin{aligned}
D_1(+) &= \tilde{p}\frac{D_2(++)}{1+r} + \tilde{q}\frac{D_2(+-)}{1+r} \\
&= (0.35)\frac{30}{1.04} + (0.65)\frac{15}{1.04} \\
&= \$19.47
\end{aligned}$$

and

$$
\begin{aligned}
D_1(-) &= \tilde{p}\frac{D_2(+-)}{1+r} + \tilde{q}\frac{D_2(--)}{1+r} \\
&= (0.35)\frac{15}{1.04} + (0.65)\frac{25}{1.04} \\
&= \$20.67.
\end{aligned}
$$

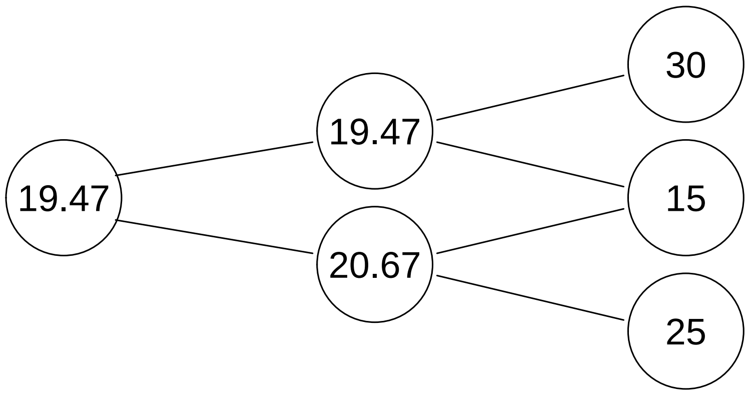
We may write the updated binomial diagram for the derivative values:



Finally, we compute D_0 from its descendents in the same way, from the risk neutral expectation formula:

$$
\begin{aligned}
D_0 &= \tilde{p}\frac{D_1(+)}{1+r} + \tilde{q}\frac{D_1(-)}{1+r} \\
&= (0.35)\frac{19.47}{1.04} + (0.65)\frac{20.67}{1.04} \\
&= \$19.47.
\end{aligned}
$$

The complete binomial diagram for the derivative is then



 # 4:
We will price all 3 derivatives, the call, the put, and the forward, using the risk neutral expectation and back propagation in the 2 step binomial model. We have a risk free interest rate of $r = 7\% = 0.07$. To compute the risk neutral probabilities we need the up and down factors u and d. We may get these from the given binomial diagram for the stock.

$$\begin{aligned} u &= \frac{120}{100} \\ &= 1.2 \end{aligned}$$

and

$$\begin{aligned} d &= \frac{80}{100} \\ &= 0.8. \end{aligned}$$

The risk neutral probabilities are

$$\begin{aligned} \tilde{p} &= \frac{1 + r - d}{u - d} \\ &= \frac{1 + 0.07 - 0.8}{1.2 - 0.8} \\ &= 0.675 \end{aligned}$$

and

$$
\begin{aligned}
\tilde{q} &= \frac{u-1-r}{u-d} \\
&= \frac{1.2-1-0.07}{1.2-0.8} \\
&= 0.325.
\end{aligned}
$$

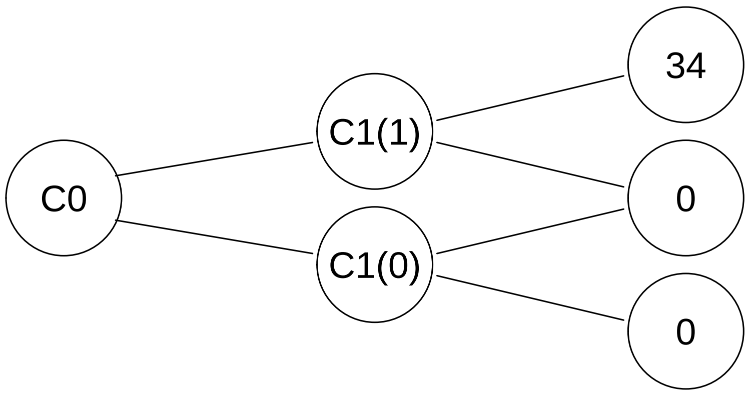
We denote the prices of the call, put, and forward at time t by $C_t(j)$, $P_t(j)$ and $F_t(j)$, in the jth state of the world at time t in the generalized notation. The 3 derivatives all expire at time 2, so their payoffs will populate the last column in the binomial diagram. They are all contingent on the stock values at time 2 S_2 so we refer back to the diagram from the problem statement. The strike/contract price is \$110 for all the instruments. The payoffs, C_2, P_2, and F_2 are

$$
\begin{aligned}
C_2 &= \max(0, S_2 - 110) \\
P_2 &= \max(0, 110 - S_2) \\
F_2 &= S_2 - 110
\end{aligned}
$$

Making reference to the given binomial diagram for the values of S_2 we have, for the call

$$
\begin{aligned}
C_2(2) &= \max(0, 144 - 110) \\
&= 34 \\
C_2(1) &= \max(0, 96 - 110) \\
&= 0 \\
C_2(0) &= \max(0, 64 - 110) \\
&= 0
\end{aligned}
$$

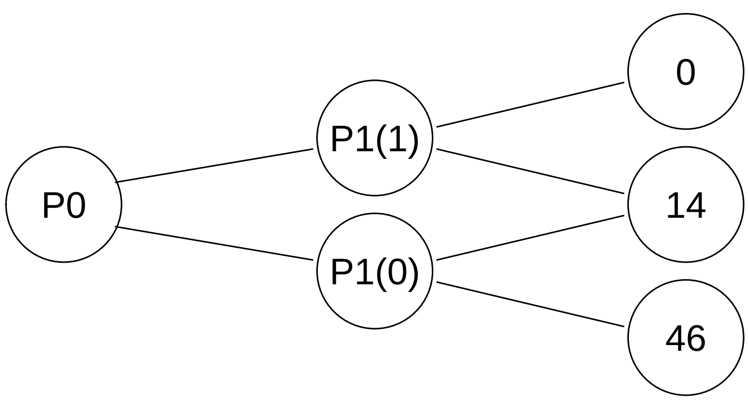
The initial binomial diagram for the call is



For the put we have

$$
\begin{aligned}
P_2(2) &= \max(0, 110 - 144) \\
&= 0 \\
P_2(1) &= \max(0, 110 - 96 \\
&= 14 \\
P_2(0) &= \max(0, 110 - 64) \\
&= 46
\end{aligned}
$$

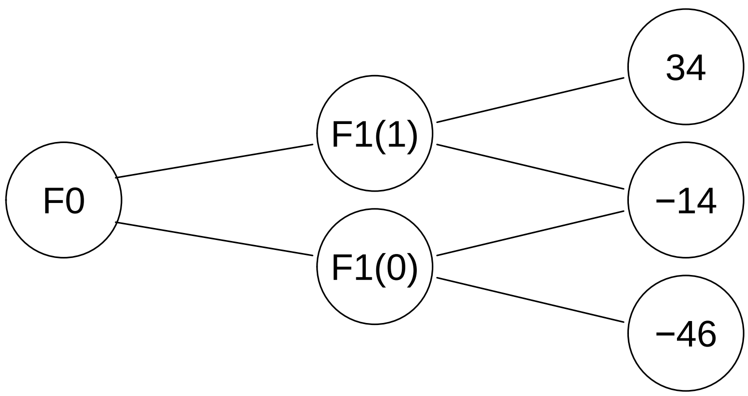
The initial binomial diagram for the put is



And for the forward

$$
\begin{aligned}
F_2(2) &= 144 - 110 \\
&= 34 \\
F_2(1) &= 96 - 110 \\
&= -14 \\
F_2(0) &= 64 - 110 \\
&= -46
\end{aligned}
$$

The initial binomial diagram for the forward is



We will now carry out the back propagation algorithm, fill in the missing values in these diagrams, and price the call, put, and forward.

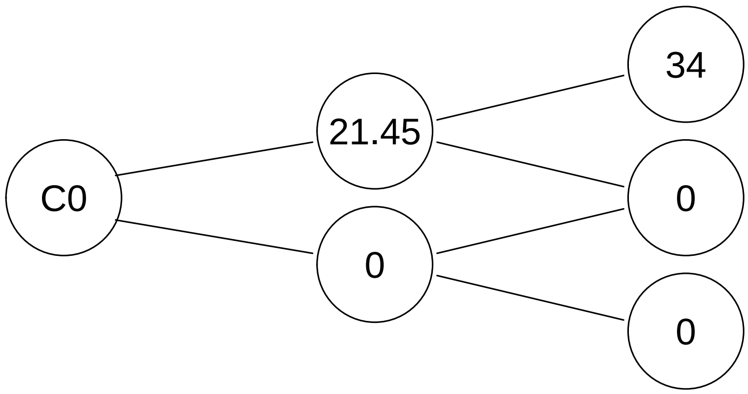
The time 1 values for the call are

$$
\begin{aligned}
C_1(1) &= (0.675)\frac{34}{1.07} + (0.325)\frac{0}{1.07} \\
&= \$21.45
\end{aligned}
$$

and

$$
\begin{aligned}
C_1(0) &= (0.675)\frac{0}{1.07} + (0.325)\frac{0}{1.07} \\
&= \$0
\end{aligned}
$$

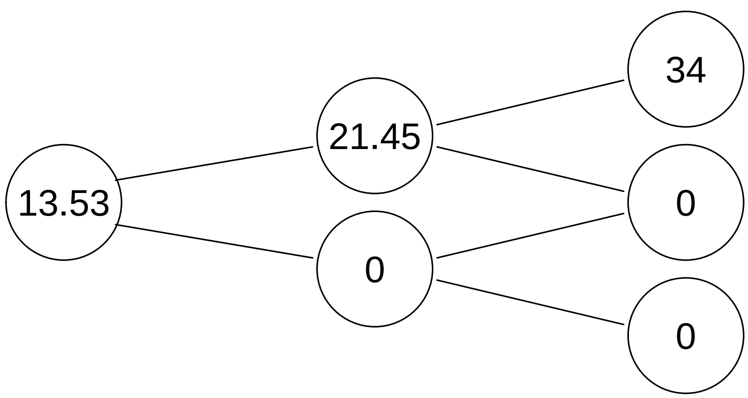
The partially completed binomial diagram for the call is



Finally, we use the risk neutral expectation to compute C_0:

$$\begin{aligned} C_0 &= (0.675)\frac{21.45}{1.07} + (0.325)\frac{0}{1.07} \\ &= \$13.53 \end{aligned}$$

So the finally completed binomial diagram for the call is



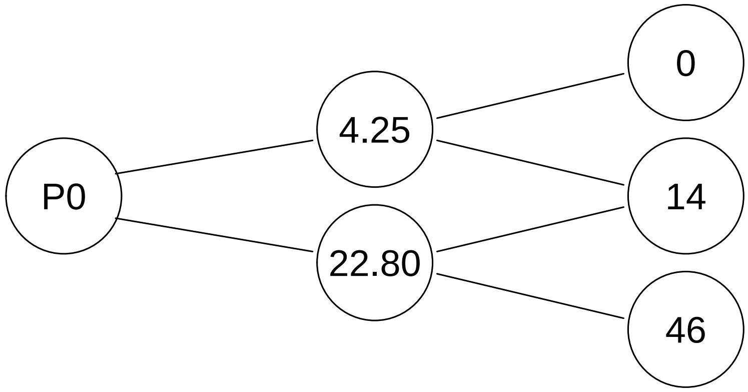
The time 1 values for the put are

$$\begin{aligned} P_1(1) &= (0.675)\frac{0}{1.07} + (0.325)\frac{14}{1.07} \\ &= \$4.25 \end{aligned}$$

and

$$\begin{aligned} P_1(0) &= (0.675)\frac{14}{1.07} + (0.325)\frac{46}{1.07} \\ &= \$22.80 \end{aligned}$$

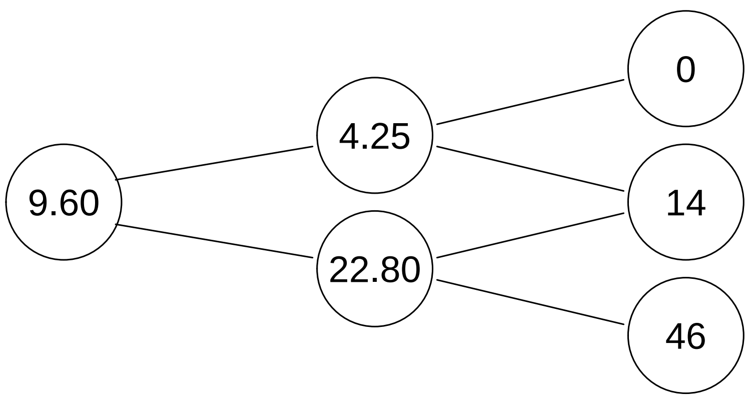
The partially completed binomial diagram for the put is



Using the risk neutral expectation to compute P_0:

$$\begin{aligned} P_0 &= (0.675)\frac{4.25}{1.07} + (0.325)\frac{22.80}{1.07} \\ &= \$9.60 \end{aligned}$$

So the finally completed binomial diagram for the put is



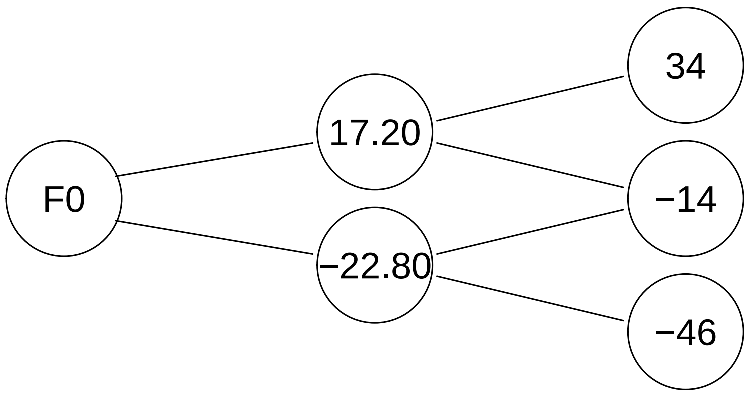
Finally we carry out the algorithm for the forward

$$\begin{aligned} F_1(1) &= (0.675)\frac{34}{1.07} + (0.325)\frac{-14}{1.07} \\ &= \$17.20 \end{aligned}$$

and

$$\begin{aligned} F_1(0) &= (0.675)\frac{-14}{1.07} + (0.325)\frac{-46}{1.07} \\ &= -\$22.80 \end{aligned}$$

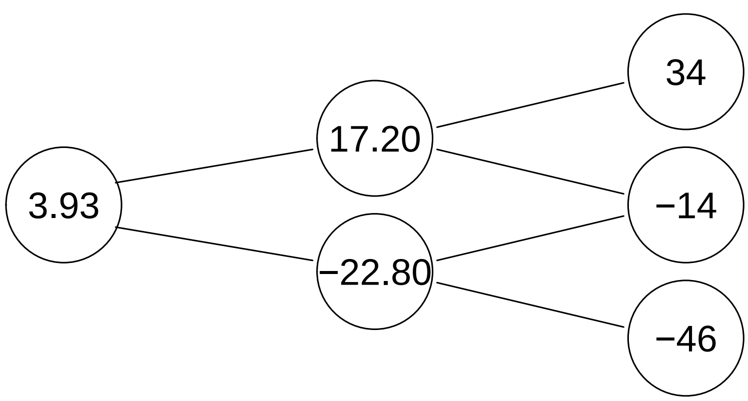
The partially completed binomial diagram for the forward is



Finally, we compute F_0:

$$\begin{aligned} F_0 &= (0.675)\frac{17.20}{1.07} + (0.325)\frac{-22.80}{1.07} \\ &= \$3.93 \end{aligned}$$

So the finally completed binomial diagram for the forward is



The put-call parity relation, in this context, is

$$C_0 - P_0 = F_0$$

We check:

$$\begin{aligned} C_0 - P_0 &= \$13.53 - \$9.60 \\ &= \$3.93 \\ &= F_0. \end{aligned}$$

So we confirm the put-call parity for this model.