Problem Set 1: Solutions

 # 1:

We have 3 different cases: annual compounding, semiannual compounding, and continuous compounding.

Annual compounding case:

The annually compounded interest rate is $r = 6\% = 0.06$.

$$\begin{aligned} P(8) &= (1+r)^8(50,000) \\ &= (1.06)^8(50,000) \\ &= \$79,692 \end{aligned}$$

$$\begin{aligned} \text{Gross Return} &= \frac{79,692}{50,000} \\ &= 1.59 \end{aligned}$$

$$\begin{aligned} \text{Net Return} &= 1.59 - 1 \\ &= 0.59 \\ &= 59\% \end{aligned}$$

Semiannual compounding case:

The semiannually compounded interest rate is $r = 5.5\% = 0.055$.

$$\begin{aligned} P(8) &= \left(1+\frac{r}{2}\right)^{2(8)}(50,000) \\ &= \left(1+\frac{0.055}{2}\right)^{16}(50,000) \\ &= (1.0275)^{16}(50,000) \\ &= \$77,175 \end{aligned}$$

$$\begin{aligned} \text{Gross Return} &= \frac{77,175}{50,000} \\ &= 1.54 \end{aligned}$$

$$\begin{aligned} \text{Net Return} &= 1.54 - 1 \\ &= 0.54 \\ &= 54\% \end{aligned}$$

Continuous compounding case:
The continuously compounded interest rate is $r = 5\% = 0.05$.

$$\begin{aligned} P(8) &= e^{(r)(8)}(50,000) \\ &= e^{0.05(8)}(50,000) \\ &= \$74,591 \end{aligned}$$

$$\begin{aligned} \text{Gross Return} &= \frac{74,591}{50,000} \\ &= 1.49 \end{aligned}$$

$$\begin{aligned} \text{Net Return} &= 1.49 - 1 \\ &= 0.49 \\ &= 49\% \end{aligned}$$

 # 2: For the given payment schedule and interest rates the present value is

$$\begin{aligned} \text{PV} &= \frac{100,000}{\left(1+\frac{r(0.5)}{2}\right)^{2(0.5)}} + \frac{100,000}{\left(1+\frac{r(1)}{2}\right)^{2(1)}} + \frac{200,000}{\left(1+\frac{r(2)}{2}\right)^{2(2)}} + \frac{200,000}{\left(1+\frac{r(3)}{2}\right)^{2(3)}} \\ &= \frac{100,000}{1+\frac{0.02}{2}} + \frac{100,000}{\left(1+\frac{0.03}{2}\right)^{2}} + \frac{200,000}{\left(1+\frac{0.035}{2}\right)^{4}} + \frac{200,000}{\left(1+\frac{0.042}{2}\right)^{6}} \\ &= \frac{100,000}{1.01} + \frac{100,000}{(1.015)^2} + \frac{200,000}{(1.0175)^4} + \frac{200,000}{(1.021)^6} \\ &= \$559,221 \end{aligned}$$

 # 3:
With a coupon rate of $6\% = 0.06$ on a \$10,000 face value bond, the annual coupon, or interest payment is

$$0.06 \times 10,000 = \$600.$$

These are paid semiannually and thus are split into 2 payments of

$$\frac{600}{2} = \$300$$

paid every 6 months.

The bond therefore pays \$300 in 6 months, 1 year, and 18 months, and in 2 years makes a final \$10,300 payment consisting of the last coupon payment plus the face value of \$10,000. The price of the bond is the present value of this stream of payments. With the given 4% semiannually compounded risk free interest rate the bond price is then

$$\begin{aligned} P &= \frac{300}{\left(1+\frac{0.04}{2}\right)^{2(0.5)}} + \frac{300}{\left(1+\frac{0.04}{2}\right)^{2(1)}} + \frac{300}{\left(1+\frac{0.04}{2}\right)^{2(1.5)}} + \frac{10,300}{\left(1+\frac{0.04}{2}\right)^{2(2)}} \\ &= \frac{300}{1.02} + \frac{300}{(1.02)^2} + \frac{300}{(1,02)^3} + \frac{10,300}{(1.02)^4} \\ &= \$10,381 \end{aligned}$$

 # 4:

The future value of \$1 paid at time t under m times per year compounding is

$$\left(1+\frac{r_m(t)}{m}\right)^{mt}$$

with a similar formula for k times per year compounding (just replace m with k). We equate these two values and solve for $r_m(t)$:

$$\begin{aligned} \left(1+\frac{r_m(t)}{m}\right)^{mt} &= \left(1+\frac{r_k(t)}{k}\right)^{kt} \\ \Longrightarrow\ 1+\frac{r_m(t)}{m} &= \left(1+\frac{r_k(t)}{k}\right)^{\frac{k}{m}} \end{aligned}$$

by taking (mt)th roots. Now solve for $r_m(t)$:

$$r_m(t) = m\left(1+\frac{r_k(t)}{k}\right)^{k/m} - m$$

 # 5:

Let r_a be the annually compounded interest rate equivalent to a 7% semiannually compounded interest rate. We equate the future value of \$1 in 1 year using the two different interest rates:

$$\begin{aligned} 1+r_a &= \left(1+\frac{0.07}{2}\right)^{2(1)} \\ &= (1.035)^2 \\ \Longrightarrow\ r_a &= (1.035)^2 - 1 \\ &= 0.0712 \\ &= 7.12\% \end{aligned}$$

Now let r_c be the equivalent continuously compounded rate and again equate the future value of \$1 in 1 year:

$$\begin{aligned} e^{r_c} &= \left(1 + \frac{0.07}{2}\right)^{2(1)} \\ &= (1.035)^2 \\ \Longrightarrow r_c &= \log(1.035^2) \\ &= 0.0688 \\ &= 6.88\% \end{aligned}$$

 # 6:

We have 3 distinct compounding cases: semiannual, monthly, and continuous, with an interest rate of 6% in all cases. For each we compute the future value of \$1 and the gross and net returns.

Semiannual compounding case:

The future value of the investment is

$$\begin{aligned} P(1) &= \left(1 + \frac{0.06}{2}\right)^2 (100,000) \\ &= (1.03)^2 (100,000) \\ &= \$106,090. \end{aligned}$$

Thus we have gross and net returns:

$$\begin{aligned} \text{Gross Return} &= \frac{106,090}{100,000} \\ &= 1.0609 \\ \text{Net Return} &= 1.0609 - 1 \\ &= 0.0609 \\ &= 6.09\% \end{aligned}$$

Monthly compounding case:

The future value of the investment is

$$\begin{aligned} P(1) &= \left(1 + \frac{0.06}{12}\right)^{12} (100,000) \\ &= (1.005)^{12} (100,000) \\ &= \$106,168. \end{aligned}$$

Thus we have gross and net returns:

$$\begin{aligned}
\text{Gross Return} &= \frac{106,168}{100,000} \\
&= 1.06168 \\
\text{Net Return} &= 1.06168 - 1 \\
&= 0.06168 \\
&= 6.168\%
\end{aligned}$$

Continuous compounding case:
The future value of the investment is

$$\begin{aligned}
P(1) &= e^{0.06}(100,000) \\
&= \$106,184.
\end{aligned}$$

Thus we have gross and net returns:

$$\begin{aligned}
\text{Gross Return} &= \frac{106,184}{100,000} \\
&= 1.06184 \\
\text{Net Return} &= 1.06184 - 1 \\
&= 0.06184 \\
&= 6.184\%
\end{aligned}$$

For the comparison of the returns with different compounding conventions, recall that with annual compounding, the net return is exactly the interest rate, 6% in this case. We then have

$$6\% < 6.09\% < 6.168\% < 6.184\%$$

which means

$$\begin{aligned}
& \text{annual compounded return} \\
< \; & \text{semiannual compounded return} \\
< \; & \text{monthly compounded return} \\
< \; & \text{continuously compounded return.}
\end{aligned}$$

This reflects that returns increase as the compounding frequency increases when the interest rate stays constant.

 # 7:
Let $r_a(t)$ be the annually compounded interest rate for term t and $r_c(t)$ be the continuously compounded interest rate for term t.

For each discount factor we calculate an annually compounded and continuously compounded rate.

1 year rates:
Annually compounded rate

$$1 + r_a(1) = \frac{1}{d(1)} = \frac{1}{0.966} = 1.0352$$
$$\Longrightarrow r_a(1) = 1.0352 - 1 = 0.0352 = 3.52\%$$

Continuously compounded rate:

$$\begin{aligned} e^{r_c(1)} &= \frac{1}{d(1)} = 1.0352 \\ \Longrightarrow r_c(1) &= \log(1.0352) = 0.0346 = 3.46\% \end{aligned}$$

2 year rates:
Annually compounded rate

$$(1 + r_a(2))^2 = \frac{1}{d(2)} = \frac{1}{0.916} = 1.0917$$
$$\Longrightarrow r_a(2) = \sqrt{1.0917} - 1 = 0.0448 = 4.48\%$$

Continuously compounded rate:

$$\begin{aligned} e^{r_c(2)(2)} &= \frac{1}{d(2)} = 1.0917 \\ \Longrightarrow r_c(2) &= \frac{1}{2}\log(1.0917) = 0.0439 = 4.39\% \end{aligned}$$

3 year rates:
Annually compounded rate

$$(1 + r_a(3))^3 = \frac{1}{d(3)} = \frac{1}{0.84} = 1.19$$
$$\Longrightarrow r_a(1) = (1.19)^{1/3} - 1 = 0.0597 = 5.97\%$$

Continuously compounded rate:

$$\begin{aligned} e^{r_c(3)(3)} &= \frac{1}{d(3)} = 1.19 \\ \Longrightarrow r_c(1) &= \frac{1}{3}\log(1.19) = 0.0580 = 5.8\% \end{aligned}$$

4 year rates:
Annually compounded rate

$$\begin{aligned} (1 + r_a(4))^4 &= \frac{1}{d(4)} = \frac{1}{0.763} = 1.311 \\ \Longrightarrow r_a(4) &= (1.311)^{1/4} - 1 = 0.07 = 7\% \end{aligned}$$

Continuously compounded rate:

$$\begin{aligned} e^{r_c(4)(4)} &= \frac{1}{d(4)} = 1.311 \\ \Longrightarrow r_c(4) &= \frac{1}{4}\log(1.311) = 0.0677 = 6.77\% \end{aligned}$$

The described bond makes a \$5000 coupon payment in each of 1,2,3, and 4 years hence, and also pays the par value of \$100,000 in 4 years. The price of the bond is thus the present value of this cash flow stream which we calculate using discount factors:

$$\begin{aligned} P &= d(1)(5000) + d(2)(5000) + d(3)(5000) + d(4)(105,000) \\ &= 0.966(5000) + 0.916(5000) + 0.84(5000) + 0.763(105,000) \\ &= \$93,725 \end{aligned}$$

 # 8:
(a) We have 5 bonds. 1 by 1, we boostrap the yield curve from these bonds in increasing order of maturity. We will denote the continuously compounded yield curve by $y(t)$ expressing yield as a function of the term t.

Bond # 1: The first bond is a pure discount bond with no coupons. Its present price must be the discounted value of it's face value, so:

$$9910 = e^{-0.5y(0.5)}(10,000)$$

Solving this equation for $y(0.5)$ gives

$$\begin{aligned} y(0.5) &= 2\log(\frac{10,000}{9910}) \\ &= 0.01808 \\ &= 1.81\% \end{aligned}$$

Bond # 2: The second bond is a 1 year coupon bond paying a 7% coupon, with semiannual payments. It has a \$10,000 face value, so the payments are

$$\begin{aligned} \text{semiannual coupon} &= \frac{0.07}{2}(10,000) \\ &= \$350. \end{aligned}$$

So the bond pays \$350 in 6 months and \$10,350 in 1 year. The present value equation for its price is

$$10,050 = e^{-0.5y(0.5)}(350) + e^{-(1)y(1)}(10,350).$$

We know $y(0.5) = 0.01808$ from the first bond, so we solve this equation for $y(1)$.

$$\begin{aligned} e^{-(1)y(1)}(10,350) &= 10,050 - e^{-0.5y(0.5)}(350) \\ \Longrightarrow y(1) &= \log\Big(\frac{10,350}{10,050 - e^{-0.5y(0.5)}(350)}\Big) \\ &= \log\Big(\frac{10,350}{10,050 - e^{-0.5(0.01808)}(350)}\Big) \\ &= 0.06454 \\ &= 6.454\% \end{aligned}$$

Bond # 3: The third bond is a 1 year coupon bond paying a 4% coupon, with semiannual payments, and with a \$50,000 face value, the coupons are

$$\begin{aligned} \text{semiannual coupon} &= \frac{0.04}{2}(50,000) \\ &= \$1000. \end{aligned}$$

So the bond pays \$1000 in 6 months and 1 year, and \$51,000 in 18 months. The present value equation for its price is

$$46,500 = e^{-0.5y(0.5)}(1000) + e^{-(1)y(1)}(1000) + e^{-(1.5)y(1.5)}(51,000).$$

We use the values we have calculated for $y(0.5)$ and $y(1)$ and solve the equation for $y(1.5)$.

$$\begin{aligned}
e^{-(1.5)y(1.5)}(51,000) &= 46,500 - e^{-0.5y(0.5)}(1000) - e^{-(1)y(1)}(1000) \\
\implies y(1.5) &= \log\Big(\frac{51,000}{46,500 - e^{-0.5y(0.5)}(1000) - e^{-(1)y(1)}(1000)}\Big) \\
&= \log\Big(\frac{51,000}{46,500 - e^{-0.5(0.01808)}(1000) - e^{-(1)(0.06454)}(1000)}\Big) \\
&= 0.08982 \\
&= 8.982\%
\end{aligned}$$

Bond # 4: Next we have a 2 year coupon bond paying a 5% coupon, with semiannual payments, and with a \$100,000 face value, the coupons are

$$\begin{aligned}
\text{semiannual coupon} &= \frac{0.05}{2}(100,000) \\
&= \$2500.
\end{aligned}$$

So the bond pays \$2500 in 0.5 years, 1 year, 1.5 years, and 2 years, and \$102,500 in 18 months. The present value equation for its price is

$$96,500 = e^{-0.5y(0.5)}(2500)+e^{-(1)y(1)}(2500)+e^{-(1.5)y(1.5)}(2500)+e^{-(2)y(2)}(102,500).$$

We use the values we have calculated for the yield curve up to 1.5 years and solve the equation for $y(2)$.

$$\begin{aligned}
& e^{-(2)y(2)}(102,500) \\
&= 96,500 - e^{-0.5y(0.5)}(2500) - e^{-(1)y(1)}(2500) - e^{-(1.5)y(1.5)}(2500) \\
\implies y(2) &= \log\Big(\frac{102,500}{96,500 - e^{-0.5y(0.5)}(2500) - e^{-(1)y(1)}(2500) - e^{-(1.5)y(1.5)}(2500)}\Big) \\
&= \log\Big(\frac{102,500}{96,500 - e^{-0.5(0.01808)}(2500) - e^{-(1)(0.06454)}(2500) - e^{-(1.5)(0.08982)}(2500)}\Big) \\
&= 0.06785 \\
&= 6.785\%
\end{aligned}$$

Bond # 5: Finally we have a 2.5 year coupon bond paying a 7% coupon, with semiannual payments, and with a \$100,000 face value, the coupons are

$$\begin{aligned}
\text{semiannual coupon} &= \frac{0.07}{2}(100,000) \\
&= \$3500.
\end{aligned}$$

This coupon is paid every 6 months for 2 years, and then the final payment, is 2.5 years is \$102,500. The disounted value equation for price gives

$$\begin{aligned}&98,000 = \\ &e^{-0.5y(0.5)}(3500) + e^{-(1)y(1)}(3500) + e^{-(1.5)y(1.5)}(3500) + e^{-(2)y(2)}(3500) + e^{-(2.5)y(2.5)}(103,500).\end{aligned}$$

As usual, solving the equation for $y(2.5)$, and using our known values for earlier points on the yield curve, gives

$$\begin{aligned}&y(2.5) \\ = \ &\log\left(\frac{103,500}{98,000 - e^{-0.5y(0.5)}(3500) - e^{-(1)y(1)}(3500) - e^{-(1.5)y(1.5)}(3500) - e^{-(2)y(2)}(3500)}\right) \\ = \ &0.07813 \\ = \ &7.813\%\end{aligned}$$

(b) The proposed bond matures in 15 months, with a \$10,000 face value, and with a 3% coupon rate it pays coupons of

$$\frac{0.03}{2}(10,000) = \$150$$

in 3 months and 9 months, and a final payment of \$10,150 in 15 months. Thus, the bond price is

$$\text{Bond Price} = d(0.25)(150) + d(0.75)(150) + d(1.25)(10,150)$$

We compute the discount factors from the spot rates we have calculated in part (a) using the relationship

$$d(t) = e^{-ty(t)}.$$

Since we have yields calculated at tenors 0.5, 1.0, and 1.5 years, we use linear interpolation to determine the values for $y(0.25)$, $y(0.75)$, and $y(1.25)$ which we need.

For the earliest rate $y(0.25)$ we follow the convention of a flat yield curve for any tenor before the shortest calibrated tenor, so

$$y(0.25) = y(0.5) = 0.01808.$$

For the yields at 0.75 and 1.25 note that these are midpoints between the calibrated tenors of 0.5, 1.0, and 1.5. Linear interpolation between these values then gives

$$\begin{aligned} y(0.75) &= \frac{1}{2}y(0.5) + \frac{1}{2}y(1.0) \\ &= \frac{1}{2}(0.01808) + \frac{1}{2}(0.06454) \\ &= 0.04131 \end{aligned}$$

and

$$\begin{aligned} y(1.25) &= \frac{1}{2}y(1.0) + \frac{1}{2}y(1.5) \\ &= \frac{1}{2}(0.06454) + \frac{1}{2}(0.08982) \\ &= 0.07718. \end{aligned}$$

We now have discount factors

$$\begin{aligned} d(0.25) &= e^{-(0.25)(0.01808)} \\ &= 0.9955 \\ d(0.75) &= e^{-(0.75)(0.04131)} \\ &= 0.9695 \\ d(1.25) &= e^{-(1.25)(0.07718)} \\ &= 0.9080 \end{aligned}$$

Thus, we have the price for our bond of

$$\begin{aligned} \text{Bond Price} &= (0.9955)(150) + (0.9695)(150) + (0.908)(10,150) \\ &= \$9511. \end{aligned}$$