Problem Set 5: Solutions

 # 1:

The initial margin requirement is \$4000 per contract, so if you enter into 10 contracts, the initial margin you must post is

$$10 \times \$4000 = \$40,000$$

The maintenance margin of \$3400 per contract implies that for 10 contracts your maintenance margin is

$$10 \times \$3400 = \$34,000$$

If the balance in the margin account ever goes below this level, you will have to post enough margin to bring the balance back up to the initial margin level.

So you start off, on the day you enter the futures positions, with a balance of

$$\text{Initial Balance} = \$40,000$$

You have a short position on 10 contracts, and the contract size is 100 ounces. So each day your P&L is

$$\begin{aligned} & 10 \text{ contracts} \times 100 \text{ ounces/contract} \times (K_T(t_{i-1}) - K_T(t_i)) \\ = \ & 1000(K_T(t_{i-1}) - K_T(t_i)) \end{aligned}$$

On the second day, the futures price has moved against you, and you have lost:

$$\begin{aligned} \text{Day 2 P\&L} &= 1000(1230 - 1234) \\ &= -\$4000 \end{aligned}$$

So \$4000 will be deducted from your margin account, leaving a balance of

$$\begin{aligned} \text{Day 2 Closing Margin Balance} &= \$40,000 - \$4000 \\ &= \$36,000 \end{aligned}$$

This is still more than the maintenance margin, so you will not be required to post any more margin as of day 2. On day 3 the futures price has moved against you again:

$$\begin{aligned} \text{Day 3 P\&L} &= 1000(1234 - 1242) \\ &= -\$8000. \end{aligned}$$

\$8000 will be deducted from your margin account, leaving a balance of

$$\begin{aligned}\text{Day 3 Closing Margin Balance} &= \$36,000 - \$8000 \\ &= \$28,000\end{aligned}$$

which is below the maintentance margin of \$34,000. In order to continue trading you will now have to post enough margin to bring the balance back up to \$40,000, the initial margin level. So suppose you post the \$12,000 necessary to do this, bringing your balance back up to \$40,000 to start off with on day 4. Day 4 closes with you booking a profit:

$$\begin{aligned}\text{Day 4 P\&L} &= 1000(1242 - 1238) \\ &= \$4000.\end{aligned}$$

so the exchange will deposit \$4000 in your margin account bringing the balance to

$$\begin{aligned}\text{Day 4 Closing Margin Balance} &= \$40,000 + \$4000 \\ &= \$44,000\end{aligned}$$

On day 5 the futures price moves against you again:

$$\begin{aligned}\text{Day 5 P\&L} &= 1000(1238 - 1245) \\ &= -\$7000.\end{aligned}$$

\$7000 will be deducted from your margin account leaving

$$\begin{aligned}\text{Day 5 Closing Margin Balance} &= \$44,000 - \$7000 \\ &= \$37,000\end{aligned}$$

which is still above the maintenance margin level, so you will not be required to post margin for now.

2:
As a seller of oil, you will want to enter into a short position in futures contracts for price protection. For a unitary hedge of your 50,000 barrel anticipated output in 3 months, you will need to enter a short position in

$$\frac{50,000}{1000} = 50$$

NYMEX oil futures contracts (contract size=1000 barrels) and for 3 months of protection you will want to use futures contracts expiring in at least 4 months.

We now consider the two hypotheticals proposed assuming we have entered into the futures position just described. In both scenarios the hypothetical situation is that the cash oil price has dropped from \$75/barrel to \$60/barrel. We will address the effectiveness of the futures hedging from the point of view of the full hedged position consisting of the cash position long 50,000 barrels of oil, combined with the short position in 50 NYMEX oil futures contracts. This constitutes a unitarily hedged position.

Case 1) is that the cost of carry $c = 0$. In this case we have seen in our lectures that the futures price is the same as the cash price. We have also seen that in this case, for a unitary hedge, there is no price risk at all. That is the fully hedged position enjoys no profit or loss at all. So in this case our hedge has been completely effective, and our loss due to the oil price drop is 0.

Case 2) is that the futues basis has dropped from \$2/barrel to \$1.20/barrel. In lecture we have seen that the P&L of a unitarily hedged position is proportional to the change in basis. For our long cash/short futures position it is

$$\text{P\&L of Combined Position} = \text{Position Size} \times \text{Change in Basis}$$

In this case our position size is 50,000 barrels of oil and the basis changes from \$2 to \$1.20, yielding

$$\begin{aligned}\text{P\&L of Combined Position} &= 50,000 \text{ barrels} \times (\$1.20/\text{barrel} - \$2/\text{barrel}) \\ &= -\$40,000.\end{aligned}$$

So our hedged position has lost \$40,000. To asses the effectiveness of the hedge, we must compare this to the loss we would have suffered without hedging. Then we would be exposed to the full drop in the oil price, and our loss would have been

$$\begin{aligned}\text{Loss in Unhedged Cash Position} &= 50,000 \times (\$75 - \$60) \\ &= \$750,000.\end{aligned}$$

By hedging we have reduced a \$750,000 loss to a \$40,000 loss. Conclusion: the hedge has been effective.

 # 3:

Your cash position is short in GBP so to hedge you must take a long futures position. Since you have decided to hedge only half of your total 14 M pound exposure, you must take a long position in

$$\frac{7,000,000}{62,500} = 112$$

CME GBP/USD futures contracts (contract size=62,500 pounds). You want coverage for 4 months, so you should use futures contracts expiring in 5 months.

Your total position can be split into two parts. 7M pounds of short GBP exposure is hedged with an equivalent futures position, and another short 7M pounds is not hedged at all.

Note also that the futures basis initially is

$$\begin{aligned}\text{Basis} &= \text{Cash Price} - \text{Futures Price} \\ &= \$1.30/\text{pound} - \$1.28/\text{pound} \\ &= \$0.02/\text{pound}.\end{aligned}$$

We now consider the two proposed hypothetical situations. The first one is that in 3 months the GBP/USD exchange rate is \$1.25/pound and the futures basis is =\$0.01/pound. The unhedged 7 M pounds that you are short profits from price drop by

$$7,000,000 \times (\$1.30 - \$1.25) = \$350,000.$$

The P&L of the hedged portion of the position is determined by the change in basis. The basis given in this scenario is -\$0.01/pound. Because this position is short in cash and long in futures the P&L is the negative of the position size times the basis change:

$$\begin{aligned}\text{P\&L of hedged position} &= -\text{Position Size} \times \text{Basis Change} \\ &= -7,000,000 \times (-0.01 - 0.02) \\ &= \$210,000.\end{aligned}$$

So the hedged portion of the portfolio profits by $210,000. The P&L of the full position is then the sum of the P&Ls from the two components:

$$\begin{aligned}\text{P\&L of Total Position} &= \$350,000 + \$210,000 \\ &= \$560,000.\end{aligned}$$

In sum, our total position has profited by $560,000.

If we had not hedged, we would simply have a short position in 14 M pounds fully exposed to the change in the spot exchange rate. In this case, this amounts to a profit of

$$14,000,000 \times (1.30 - 1.25) = \$700,000.$$

So the futures hedge has reduced our profits somewhat.

In the second proposed scenario, the market exchange rate has gone *up* by $0.05, rather than down, so this hurts you if you have a short position. For the unhedged 7 M pounds in the situation we're considering, this results in a loss of

$$7,000,000 \times (\$1.35 - \$1.30) = \$350,000$$

However, the hedged portion of the position will enjoy the same profit as in the first scenario, because the change in the basis is assumed to be the same -$0.01-$0.02=-$0.03 resulting once again in a profit of

$$-7,000,000 \times -\$0.03 = \$210,000.$$

The P&L of the total position is then

$$\begin{aligned}\text{P\&L of Total Position} &= -\$350,000 + \$210,000 \\ &= -\$140,000.\end{aligned}$$

So the total position has lost $140,000.

If we had not hedged, the full short position of 14 M pound would have lost

$$14,000,000 \times (1.35 - 1.30) = \$700,000.$$

So as things turned out, the hedge was effective at reducing our loss significantly.

 # 4:
To take a short position using NYMEX oil futures contracts (contract size=1000 barrels) you would enter a short position on

$$\frac{20,000}{1000} = 20 \text{ contracts}$$

choosing contracts expiring in at least 7 months, if you intend to hold the position for 6 months.
With an initial margin requirement of \$4000 per contract, your initial investment will be the margin

$$20 \times \$4000 = \$80,000$$

you must post to open the position.
If, in the next 3 months, the futures price on these contracts drops from \$60/barrel to \$54/barrel then, since you are short, your position profits by

$$\begin{aligned} & \text{Position Size} \times (\$60 - \$54) \\ = \;& 20,000 \times \$6 \\ = \;& \$120,000. \end{aligned}$$

Relative to your initial investment this is a

$$\frac{120,000}{80,000} = 150\%$$

return.
If, instead, the futures price increases from \$60 to \$68, then your short position suffers a loss of

$$\begin{aligned} & 20,000 \times (\$60 - \$54) \\ = \;& \$160,000. \end{aligned}$$

In this case, your initial investment of \$80,000 has been completely wiped out.

 # 5: (a) We are given spot rates (which are simply compounded)

$$\begin{aligned} L(0,3) &= 3.5\% = 0.035 \\ L(0,6) &= 5.5\% = 0.055 \end{aligned}$$

To compute the forward rate $L(0,3,6)$ we need the discount factor/zero coupon bond prices $P(0,3)$ and $P(0,6)$:

$$\begin{aligned} P(0,3) &= \frac{1}{1+3L(0,3)} \\ &= \frac{1}{1+3(0.035)} \\ &= 0.90498 \end{aligned}$$

and

$$\begin{aligned} P(0,6) &= \frac{1}{1+6L(0,6)} \\ &= \frac{1}{1+6(0.055)} \\ &= 0.75188 \end{aligned}$$

Out desired forward rate is then

$$\begin{aligned} L(0,3,6) &= \frac{P(0,3)-P(0,6)}{3P(0,6)} \\ &= \frac{0.90498-0.75188}{3(0.75188)} \\ &= 0.06787 \\ &= 6.79\% \end{aligned}$$

(b) Using our given data

$$L(0,6,8) = 9\% = 0.09$$

we calculate, using the relationship between zero coupon bond prices and forward rates

$$\frac{P(0,8)}{P(0,6)} = \frac{1}{1+2L(0,6,8)}.$$

Solving for $P(0,8)$ gives

$$\begin{aligned} P(0,8) &= \frac{P(0,6)}{1+2L(0,6,8)} \\ &= \frac{0.75188}{1+2(0.09)} \\ &= 0.63719. \end{aligned}$$

The 8 year simply compounded spot rate is then

$$\begin{aligned} L(0,8) &= \frac{1-P(0,8)}{8P(0,8)} \\ &= \frac{1-0.63719}{8(0.63719)} \\ &= 0.07117 \\ &= 7.12\% \end{aligned}$$

 # 6: The seller's position in a futures contract is the short position, so if the price drops you profit. In this case the price dropped by \$.80 per contract so the short position profits. The change in the implied futures rate is from

$$100 - 98.5 = 1.5\%$$

to

$$100 - 97.7 = 2.3\%.$$

Recalling that we calculate the P&L of the position using a \$1,000,000 notional, and that the Eurodollar futures rate is a 3 month rate, the short position realizes a profit of

$$(1,000,000)(0.25)(0.023 - 0.015) = \$2000$$

per contract. Since you have 100 contracts, your position profits by

$$100 \times \$2000 = \$200,000$$

and so \$200,000 will be deposited in your margin account.

 # 7: To protect yourself from interest rate risk you want to take the receiver position in a 2 year swap with semiannual payments on a notional of of \$40,000,000.

To calculate the fair swap rate using the discount factors we have been given, we use the formula from the lecture

$$S = \frac{m(1 - d(t_J))}{\sum_{j=1}^{J} d(t_j)}.$$

We have $m = 2$ because we are entering a swap with semiannual payments, and $J = 4$ for the two years the swap will be active, leading to a fair swap rate of

$$\begin{aligned}
S &= \frac{2(1 - d(t_4))}{\sum_{j=1}^{4} d(t_j)} \\
&= \frac{2(1 - d(2))}{d(0.5) + d(1) + d(1.5) + d(2))} \\
&= \frac{2(1 - 0.953)}{0.99 + 0.979 + 0.966 + 0.953} \\
&= 0.0242 \\
&= 2.42\%
\end{aligned}$$

 # 9: (a) Starting at the short end of the curve with simply compounded LIBOR rates for the 1 week, 1 month, 2 month, and 3 month tenors, we are given:

$$\begin{aligned}
L(0, \frac{1}{52}) &= 2\% = 0.02 \\
L(0, \frac{1}{12}) &= 2.2\% = 0.022 \\
L(0, \frac{2}{12}) &= 2.27\% = 0.027 \\
L(0, \frac{3}{12}) &= 2.36\% = 0.0236.
\end{aligned}$$

These rates imply zero-coupon bond prices/discount factors

$$
\begin{aligned}
P\Big(0, \frac{1}{52}\Big) &= \frac{1}{1+\frac{1}{52}L(0,\frac{1}{52})} \\
&= \frac{1}{1+\frac{1}{52}(0.02)} \\
&= 0.999615532 \\
P\Big(0, \frac{1}{12}\Big) &= \frac{1}{1+\frac{1}{12}L(0,\frac{1}{12})} \\
&= \frac{1}{1+\frac{1}{12}(0.022)} \\
&= 0.99817 \\
P\Big(0, \frac{2}{12}\Big) &= \frac{1}{1+\frac{2}{12}L(0,\frac{2}{12})} \\
&= \frac{1}{1+\frac{2}{12}(0.0227)} \\
&= 0.99623 \\
P\Big(0, \frac{3}{12}\Big) &= \frac{1}{1+\frac{3}{12}L(0,\frac{3}{12})} \\
&= \frac{1}{1+\frac{3}{12}(0.0236)} \\
&= 0.99413
\end{aligned}
$$

Finally, to calculate continuously compounded spot interest rates $y(T)$, we use the above data with the relationship

$$y(T) = \frac{\log(P(0,T))}{T}$$

which gives the interest rates

$$
\begin{aligned}
y\left(\frac{1}{52}\right) &= -\frac{\log(P(0,\frac{1}{52}))}{1/52} \\
&= -\frac{\log(0.999615532)}{1/52} \\
&= 0.019996 \\
&= 2.00\% \\
y\left(\frac{1}{12}\right) &= -\frac{\log(P(0,\frac{1}{12}))}{1/12} \\
&= -\frac{\log(0.99817)}{1/12} \\
&= 0.0219801 \\
&= 2.20\% \\
y\left(\frac{2}{12}\right) &= -\frac{\log(P(0,\frac{2}{12}))}{2/12} \\
&= -\frac{\log(0.99623)}{2/12} \\
&= 0.022663 \\
&= 2.27\% \\
y\left(\frac{3}{12}\right) &= -\frac{\log(P(0,\frac{3}{12}))}{3/12} \\
&= -\frac{\log(0.99413)}{3/12} \\
&= 0.023549 \\
&= 2.35\%
\end{aligned}
$$

NOTE: For very short tenors, when discount factors are very close to 1, it is sometimes necessary to keep many digits of accuracy to calculate rates precisely. This is the reason for the large number of digits kept for the 1 week discount factor.

We move on now to the middle part of the curve, where we use prices for

Eurodollar futures contracts as given. These imply the forward rates

$$\begin{aligned}
L(0, 0.25, 0.5) &= 100 - 97.4 \\
&= 2.6\% \\
L(0, 0.5, 0.75) &= 100 - 97.0 \\
&= 3.0\%
\end{aligned}$$

Using these forward rates, we extend the discount factors:

$$\begin{aligned}
P(0, 0.5) &= P(0, 0.25)\frac{1}{1 + 0.25L(0, 0.25, 0.5)} \\
&= (0.99413)\frac{1}{1 + 0.25(0.026)} \\
&= 0.98771
\end{aligned}$$

and

$$\begin{aligned}
P(0, 0.75) &= P(0, 0.5)\frac{1}{1 + 0.25L(0, 0.5, 0.75)} \\
&= (0.99413)\frac{1}{1 + 0.25(0.03)} \\
&= 0.998036.
\end{aligned}$$

As before, we calculate continuously compounded spot rates from these discount factors:

$$\begin{aligned}
y(0.5) &= -\frac{\log(P(0, 0.5))}{0.5} \\
&= -\frac{\log(0.98771)}{0.5} \\
&= 0.024732 \\
&= 2.47\%
\end{aligned}$$

and

$$\begin{aligned}
y(0.75) &= -\frac{\log(P(0, 0.75))}{0.75} \\
&= -\frac{\log(0.98771)}{0.75} \\
&= 0.026447 \\
&= 2.64\%
\end{aligned}$$

We now move on to the long end of the LIBOR curve, based on swap rates. Recall that the basis for using swap rates to extend discount factors is the equation for the fair swap rate:

$$S(T_J) = \frac{m(1 - P(0, T_J))}{\sum_{j=1}^{J} P(0, T_j)}.$$

As we are assuming semiannual swap payments, $m = 2$. We may solve this equation for $P(0, T_J)$, the longest maturity bond price appearing, and this gives the formula

$$P(0, T_J) = \frac{2 - S(T_J) \sum_{j=1}^{J-1} P(0, T_j)}{2 + S(T_J)} \tag{1}$$

which we use to compute discount factors from swap rates. The formula involves swap rates every 6 months, but we are only given swap rates every year. Following the procedure from lecture, we linearly interpolate the given swap rates, to compute swap rates for maturities of 1.5, 2.5, and 3.5 years. Since these points are midway between the given rates, this amounts to a simple average of the given rates, so that

$$\begin{aligned}
S(1.5) &= \frac{S(1) + S(2)}{2} = \frac{3.0 + 3.6}{2} = 3.3 \\
S(2.5) &= \frac{S(2) + S(3)}{2} = \frac{3.6 + 3.95}{2} = 3.78 \\
S(3.5) &= \frac{S(3) + S(4)}{2} = \frac{3.95 + 4.2}{2} = 4.075
\end{aligned}$$

We use these swap rates together with the given rates and equation (1) to extend the discount factors up to the 4 year tenor.

$$
\begin{aligned}
P(0,1) &= \frac{2 - S(1)P(0,0.5)}{2 + S(1)} \\
&= \frac{2 - (0.03)(0.98771)}{2.03} \\
&= 0.97062 \\
P(0,1.5) &= \frac{2 - S(1.5)(P(0,0.5) + P(0,1))}{2 + S(1.5)} \\
&= \frac{2 - (0.033)(0.98771 + 0.97062)}{2.033} \\
&= 0.95188 \\
P(0,2.0) &= \frac{2 - S(2)(P(0,0.5) + P(0,1) + P(0,1.5))}{2 + S(2)} \\
&= \frac{2 - (0.036)(0.98771 + 0.97062 + 0.95188)}{2.036} \\
&= 0.93075 \\
P(0,2.5) &= \frac{2 - S(2.5)(P(0,0.5) + P(0,1) + P(0,1.5) + P(0,2))}{2 + S(2.5)} \\
&= \frac{2 - (0.0378)(0.98771 + 0.97062 + 0.95188 + 0.93075)}{2.0378} \\
&= 0.91008 \\
P(0,3.0) &= \frac{2 - S(3)(P(0,0.5) + P(0,1) + P(0,1.5) + P(0,2) + P(0.2.5))}{2 + S(3)} \\
&= \frac{2 - (0.0395)(0.98771 + 0.97062 + 0.95188 + 0.93075 + 0.91008)}{2.0395} \\
&= 0.88849 \\
P(0,3.5) &= \frac{2 - S(3.5)(P(0,0.5) + P(0,1) + P(0,1.5) + P(0,2) + P(0.2.5) + P(0,3))}{2 + S(3.5)} \\
&= \frac{2 - (0.04075)(0.98771 + 0.97062 + 0.95188 + 0.93075 + 0.91008 + 0.88849)}{2.04075} \\
&= 0.86729
\end{aligned}
$$

$$
\begin{aligned}
& P(0,4) \\
= & \frac{2 - S(4)(P(0,0.5) + P(0,1) + P(0,1.5) + P(0,2) + P(0.2.5) + P(0,3) + P(0,3.5))}{2 + S(4)} \\
= & \frac{2 - (0.042)(0.98771 + 0.97062 + 0.95188 + 0.93075 + 0.91008 + 0.88849 + 0.86729)}{2.042} \\
= & \ 0.84547.
\end{aligned}
$$

Now, from these discount factors, we calculate the continuously compounded spot rates as before:

$$
\begin{aligned}
y(1) &= -\frac{\log(P(0,1))}{1} \\
&= -\frac{\log(0.97062)}{1} \\
&= 0.02982 \\
&= 2.98\% \\
y(2) &= -\frac{\log(P(0,2))}{2} \\
&= -\frac{\log(0.93075)}{2} \\
&= 0.035888 \\
&= 3.59\% \\
y(3) &= -\frac{\log(P(0,3))}{3} \\
&= -\frac{\log(0.88849)}{3} \\
&= 0.03941 \\
&= 3.94\% \\
y(4) &= -\frac{\log(P(0,4))}{4} \\
&= -\frac{\log(0.84547)}{4} \\
&= 0.04197 \\
&= 4.20\%
\end{aligned}
$$

(b) To price the proposed bond note that, from the curve, we need the discount factors $P(0,0.5)$, $P(0,1)$, $P(0,1.5)$, and $P(0,2)$. As we have calculated these in the process of building the curve, we reuse those values from the calcution in (a).

The bond makes \$350 coupon payments every 6 months for the next 2 years, and in 2 years also pays the face value of \$10,000, so the bond price is

$$
\begin{aligned}
B &= P(0,0.5)(350) + P(0,1)(350) + P(0,1.5)(350) + P(0,2)(10350) \\
&= 0.98771(350) + 0.97062(350) + 0.95188(350) + 0.93075(10350) \\
&= \$10,652.
\end{aligned}
$$