Problem Set 10

Remark: Some of these problems require the computation of the cumulative normal distribution function, $N(x)$. This is something you can do with a good scientific calculator, or you can do it in Excel, or Python, or R, or with Android apps, or (I presume) with Iphone apps. But if you don't have access to this function, just leave terms like $N(a)$ unevaluated, and then you can check the solutions for the final answer.

 # 1: Evaluate the limit

$$\lim_{n\to\infty} \frac{\lfloor nT \rfloor}{n} = T$$

 # 2: Evaluate the integral for the risk neutral expectation of the call payoff to derive the Black-Scholes formula.

Hint: This exercise is pretty mathematically intense, but it is so fundamental to the field, I have to urge everyone to give it a try. Here are some hints to help walk through the process.

The goal is the evaluate the integral for the call price to get the Black-Scholes formula. Denoting the call price by $\mathcal{C}$, what we want to show is that

$$\begin{aligned}
\mathcal{C} &= e^{-rT} \int_{-\infty}^{\infty} \max\{S_0 e^{T(r-\frac{\sigma^2}{2})+\sigma\sqrt{T}z} - K, 0\} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz \\
&= S_0 N(d_+) - K e^{-rT} N(d_-)
\end{aligned}$$

where

$$\begin{aligned}
d_+ &= \frac{1}{\sigma\sqrt{T}} \Big[\log\Big(\frac{S_0}{K}\Big) + \Big(r + \frac{\sigma^2}{2}\Big) T \Big] \\
d_- &= \frac{1}{\sigma\sqrt{T}} \Big[\log\Big(\frac{S_0}{K}\Big) + \Big(r - \frac{\sigma^2}{2}\Big) T \Big]
\end{aligned}$$

and N(x) is the normal cumulative distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz$$

by evaluating the integral.

(i) The first thing to do is to make a change of variable in the integral

$$x = -z.$$

Work through the change of variable in the integral from z to x to get

$$\mathcal{C} = e^{-rT} \int_{-\infty}^{\infty} \max\{S_0 e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} - K, 0\} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

(ii) Now see that because of the maximium function the integrand vanishes unless the first argument of the max is greater than 0. That is, the integrand vanishes unless

$$S_0 e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} - K > 0.$$

Show that this is equivalent to the condition on x

$$x < d_-$$

and from that conclude that

$$\mathcal{C} = e^{-rT} \int_{-\infty}^{d_-} [S_0 e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} - K] \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

(iii) We now have an integral that can be distributed over 2 separate terms. Do this and then simplify the second term to get

$$\mathcal{C} = S_0 e^{-rT} \int_{-\infty}^{d_-} e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx - K e^{-rT} N(d_-)$$

which shows the second term is the second term of the Black-Scholes formula.

(iv) Now we only have to evaluate the first term and show that it equals the first term of the Black-Scholes formula. To do this, multiply all of the exponentials together in the integral, put all the exponents together, and simplify to reduce the first term to

$$S_0 e^{-rT} \int_{-\infty}^{d_-} e^{T(r-\frac{\sigma^2}{2})-\sigma\sqrt{T}x} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = S_0 \int_{-\infty}^{d_-} \frac{e^{-\frac{1}{2}(x+\sigma\sqrt{T})^2}}{\sqrt{2\pi}} dx$$

(v) Now do one more change of variable in the integral that remains:

$$y = x + \sigma\sqrt{T}.$$

Work through this change of variable to finally reduce the first term to

$$\begin{aligned} &= S_0 \int_{-\infty}^{d_+} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dx \\ &= S_0 N(d_+) \end{aligned}$$

and now put all the pieces together to see that the Black-Scholes formula has been recovered.

\# **3:** Derive the Black-Scholes delta, gamma, and vega by taking partial derivatives of the Black-Scholes formula.

\# **4:** Compute the delta, gamma, and vega for a 6 month call option with a strike price of \$60, on an underlying with a volatility of 20% trading at \$72. Assume the risk free interest rate is 4%.

\# **5:** Suppose you hold a position of 50 call options on a stock currently trading at \$110 with a volatility of 15%. Suppose the calls expire in 1 year and have a strike price of \$105. Suppose the risk free interest rate is 4%. What position must you take alongside your option position so as to have a total delta neutral position? Suppose you take this position and hold it for 3 months. What is your P&L if the stock's volatililty has not changed and it is then trading at \$100? \$120?

\# **6:** (a) Consider, at a conceptual level, the example of volatility trading treated in lecture in which a hedged option position profited from an increase in volatility, no matter whether the stock price went up or down. Think about this from the point of view of the delta and the vega and what characteristics a portfolio would need to have for this to work.

(b) Now consider taking a reverse position: setting up a portfolio that will profit if volatility goes *down*. Again, look at this from the point of view of the same concepts, the delta and the vega, that you considered in part (a).

(c) Now, actually set up a portfolio to implement the idea in part (b). Assume the same circumstances as in the example from the volatility trading lecture with 1 difference: now assume the volatility *decreased* from 40% to 10%. Set up a portfolio to profit from this volatility change, and check how your portfolio would fare under the three proposed stock price changes, but with the volatility decrease just described.