Problem Set 4 Solutions

 # 1:

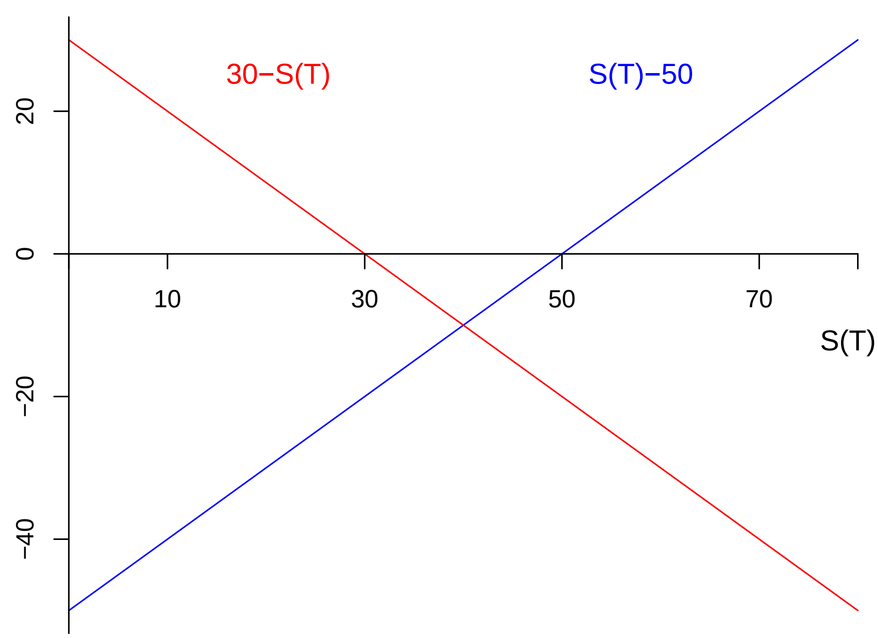
We are asked to plot the payoffs of 2 forward positions, a long forward with a forward price of \$50

$$F(T) = S(T) - 50$$

and a short position with contract price \$30,

$$G(T) = 30 - S(T).$$

Here are the two payoffs plotted with $S(T)$ the horizontal axis:

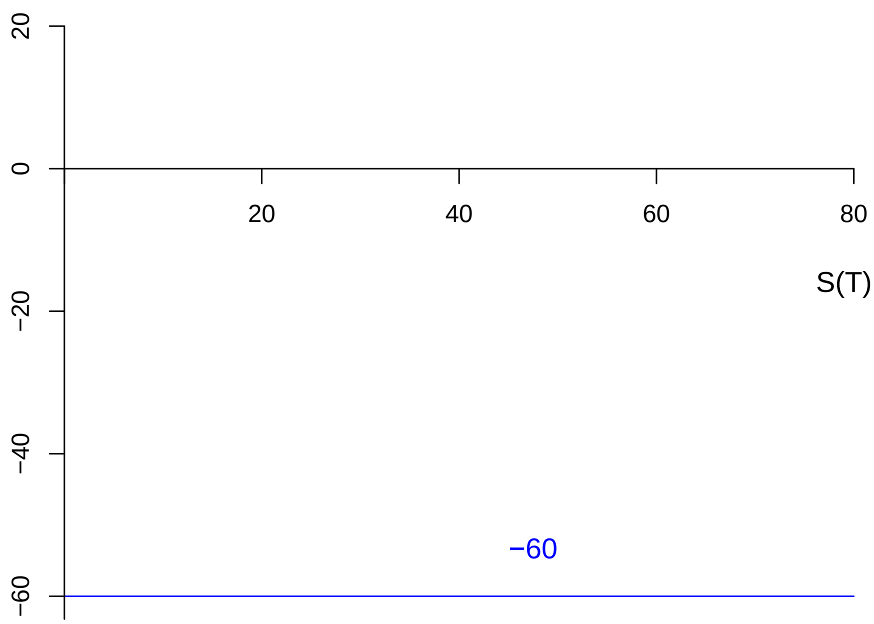


We are now asked to consider a portfolio consisting of an allocation of 3 to the long forward position and 3 to the short position. This portfolio has value at the expiration date

$$\begin{aligned} V(T) &= 3F(T) + 3G(T) \\ &= 3(S(T) - 50) + 3(30 - S(T)) \\ &= -60. \end{aligned}$$

So this portfolio in the end is just a certain obligation to pay \$60 at the expiration time T.

The plot of this payoff is simply the plot of the contsant function -60:



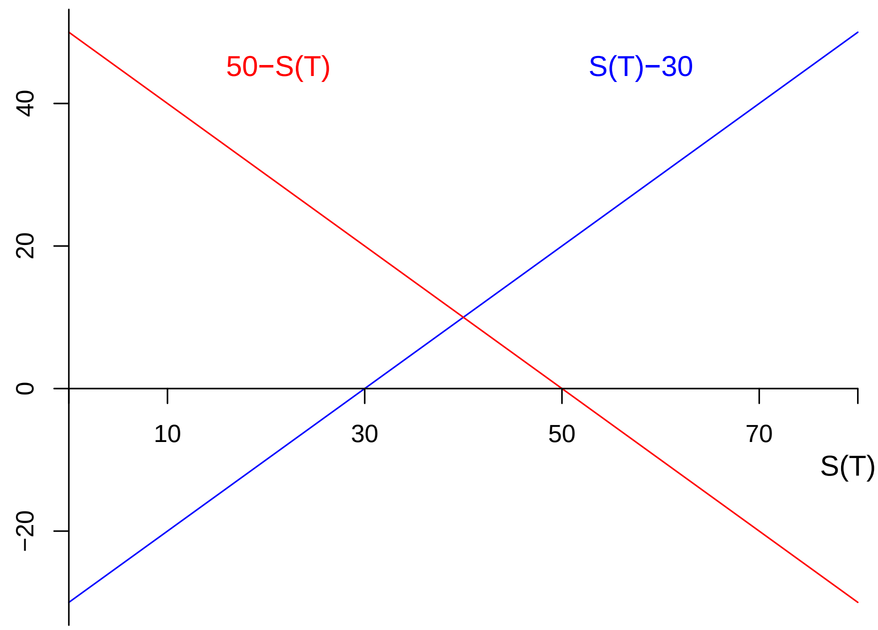
Since the payoff of this portfolio is with certainty -\$60, it's value for any time $t < T$ is simply the discounted value of this payment:

$$V(t) = -e^{-r(T-t)}60$$

Now we are asked to consider the reversal of the previous 2 positions where we are short in the forward with contract price \$50 and long in the forward with contract price \$30. So the 2 payoffs are now

$$\begin{aligned} U(T) &= S(T) - 30 \\ W(T) &= 50 - S(T). \end{aligned}$$

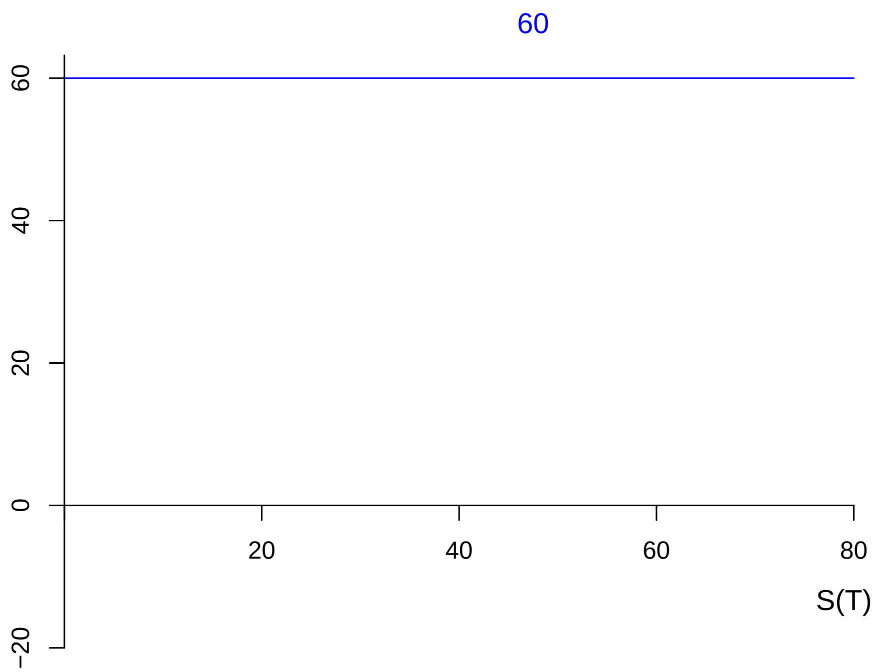
These are the plots:



The portfolio consisting of 3 of each of these positions has value

$$\begin{aligned} V(T) &= 3U(T) + 3G(T) \\ &= 3(S(T) - 30) + 3(50 - S(T)) \\ &= 60. \end{aligned}$$

So plotting the payoff of this combined position reduced to plotting the constant function 60:



Since the payoff of this portfolio is a certain payment of \$60, its value at any earlier time t is just the discounted value of this payment:

$$V(t) = e^{-r(T-t)}60$$

 # 2:

To protect yourself from downside price risk on the 300 shares that you own, you want to lock in the right to sell these shares at the current market price. So you should take the short position on a forward contract on 300 shares of the stock.

Let $S(t)$ be the price of the underlying stock, K the contracted forward price, $r = 0.04$ is the risk free rate. The forward price per share for a forward contract expiring at time T is given by the formula for the fair forward price we have derived in lecture:

$$K = e^{rT} S(0)$$

We have $S(0) = \$150$ and we are entering a T =1 year contract, so

$$\begin{aligned} K &= e^{0.04(1)}(150) \\ &= \$156 \end{aligned}$$

is the fair forward price.

We consider two hypothetical offers: First, if the dealer offered a forward price of $\tilde{K} = 160$. Since $\tilde{K} > K =$ the fair forward price, the offered forward price is too high. You want to sell at this price, so you will enter the short position in this contract.

In order to secure the underlying asset, you will, on a share by share basis, borrow \$150 at the risk free rate, and buy a share of the stock. Your portfolio now consists of

1. the short position in the forward contract;
2. a share of the stock;
3. a debt of \$150 borrowed at the risk free rate of 4%;

We now hold this portfolio for 1 year. At this time the debt has accumulated to

$$e^{0.04}(150) = \$156$$

We may sell the share of stock in the forward contract, and receive \$160 in cash. We may then retire the debt of \$156, and retain a profit of \$4. This is a certain and riskless profit and so this is an arbitrage.

Now consider the second hypothetical offer of a forward price of $\tilde{K} = \$152$. Since $\tilde{K} < K = \$156$ this forward price is too low, so we want to want to enter the long postion in this forward contract.

To construct an arbitrage portfolio, we short the stock, receiving a cash payment of $150 which we invest at the risk free rate, and enter the long position in the forward. We have a porfolio consisisting of

1. the long position in the forward contract;
2. a short position on the stock;
3. a holding of $156 invested at the risk free rate at time 0.

We hold these positions for 1 year. At this time the cash investment has value

$$e^{0.04}(150) = \$156.$$

Under our obligations in the forward contract, we must purchase the underlying stock for $152. We withdraw this amount from our cash investment, puchase the stock under the forward contract, and fulfill our obligations under the short position, thus closing out that position.

We retain $4 from our cash holding, which is a riskless and certain profit.

 # 3:

We are given a risk free rate of $r = 6\% = 0.06$ and that the current cash price of the bond is $S(0) = \$4200$. The forward price for a 1 year forward is $(T = 1)$

$$\begin{aligned} K &= e^{rT}S(0) \\ &= e^{0.06(1)}(4200) \\ &= \$4460 \end{aligned}$$

We now consider 2 hypothetical situations, 6 months later. First, if the bond is now trading for $4250. In this case, the value of the long forward position at time $t = 6$ months=0.5 is

$$\begin{aligned} V(0.5) &= S(0.5) - e^{-r(T-0.5)}K \\ &= 4250 - e^{-(0.06)(1-0.5)}(4460) \\ &= -\$78 \end{aligned}$$

The value of the long position under these circumstances is -$78, we this position has lost.

Now, we consider the alternative hypothetical we were given that the bond is trading at \$4600 6 months later. Then the value of the long forward position is

$$\begin{aligned} V(0.5) &= S(0.5) - e^{-r(T-0.5)}K \\ &= 4600 - e^{-(0.06)(1-0.5)}(4460) \\ &= \$272 \end{aligned}$$

which is positive, so in this case, the long position has gained.

 # 4:

Recall the forward price on an asset paying a known yield is

$$K = e^{(r-y)T}S(0)$$

where r is the risk free rate, y is the yield rate of the underlying asset, T is the expiration date of the forward and $S(t)$ is the price of the underlying at time t.

To use a cash and carry arbitrage argument to justify this formula, we start by assuming that

$$K > e^{(r-y)T}S(0)$$

and see if we can find an arbitrage.

Since we expect in this case that the forward price is too high, we want to enter the forward as a seller. So we will enter the short position in the forward contract.

We will borrow $e^{-yT}S(0)$ at the risk free rate, and use this cash to purchase an allocation of e^{-yT} in the asset. After taking these steps we now have a portfolio consisting of

1. the short forward position;
2. an allocation of e^{-yT} of the underlying asset;
3. a debt of $e^{-yT}S(0)$ borrowed at the risk free rate.

We will hold these postions until time T, the expiration of the forward contract. During this holding period, all earnings from the asset will be reinvested into further holdings of the asset. As we know, in this case the allocation to the asset at any intermediate time t will be

$$e^{-y(T-t)}$$

At time T our allocation to the asset will be exactly $e^{-y(T-T)} = e^0 = 1$, ie we hold exactly 1 unit of the underlying asset. The value of the debt will be the future value at time T from when we entered into it which is

$$e^{rT} e^{-yT} S(0) = e^{(r-y)T} S(0).$$

At this time, we must fulfill our obligations under the forward contract. So, we must sell the 1 unit of the underlying that we own, receiving the contract price of K in exchange. From the assumed inequality

$$K > e^{(r-y)T} S(0) = \text{Value of Debt}$$

we may now retire the debt from the cash payout from the forward contract and retain as profit

$$K - e^{(r-y)T} S(0) > 0$$

which is a riskless and certain profit, implying this is an arbitrage portfolio.

Now assume the opposite inequality

$$K < e^{(r-y)T} S(0).$$

Since we now expect this forward price is too low, we want to enter into the forward as the buyer, ie the long position.

We will take a short position in e^{-yT} of the underlying asset, and invest the resulting cash payout at the risk free rate. Our portfolio at time 0 now consists of

1. the long forward position;
2. the short position in the underlying asset;
3. a cash holding of e^{-yT} invested at the risk free rate at time 0.

Because we are obligated to repay all dividends earned on the short position back to the rightful owner the short position will grow at the same rate as the long position (with all dividends reinvested) so the short position will be represented in our portfolio value by

$$-e^{-y(T-t)}S(t)$$

at time t. At time T the obligation is woth exactly $S(T)$, exactly one unit of the asset. The value of our cash holding at time T is

$$e^{rT}e^{-yT}S(0) = e^{(r-y)T}S(0) > K$$

where this is just the inequality we assumed at the start. As we are in the long position in the forward contract, we are now obliged to purchase the underlying asset for the contract price K, which we may withdraw from our cash holding thanks to the previous inequality. We may now fulfill the obligations of the short position, and be free of all obligations. We have retained a profit of

$$e^{(r-y)T}S(0) - K > 0.$$

This is a riskless and certain profit, so this is an arbitrage.

Once we have an expression for the forward contract, we can use the equation (1) from the lecture

$$V(t; K, T) = (K_T(t) - K)e^{-r(T-t)}$$

To use this formula, we use the formula for the fair forward price that we have deduced in this problem, only replacing T, the time to expiry, with $T-t$ and $S(0)$ with $S(t)$, so that

$$K_T(t) = e^{(r-y)(T-t)}S(t).$$

We thus have, for the value of a forward contract on an asset paying a known yield y with contract price K

$$\begin{aligned} V(t) &= (e^{(r-y)(T-t)}S(t) - K)e^{-r(T-t)} \\ &= e^{-y(T-t)}S(t) - e^{-r(T-t)}K \end{aligned}$$

 # 5:

To use a replication argument we must, as usual, construct a portfolio that replicates the payoff of a forward

$$S(T) - K$$

at expiration, where $S(t)$ is the price of the underlying asset and K is the contract price.

The payment term K may be replicated, as usual, by a debt equal to the discounted value of the forward price K.

However, the asset $S(T)$ cannot be replicated simply with a holding of the asset $S(t)$ at some time $t < T$. If we hold the asset from time t to T, then at time T we will hold the asset plus the future value of all income earned from the asset. Thus to replicate $S(T)$ we must subtract the value of the income received from the asset.

To replicate the term $S(T)$ at time 0, it is exactly I, the present value of the income earned by holding the asset, that must be subtracted, so that the portfolio

$$S(0) - I$$

will exactly replicate $S(T)$ at time T. So the portfolio

$$S(0) - I - e^{-rT}K$$

replicates the forward payoff, and is the value of the forward contract at time 0. To determine the fair forward price, we set this value to 0:

$$\begin{aligned} S(0) - I - e^{-rT}K &= 0 \\ \Longrightarrow e^{-rT}K &= S(0) - I \\ \Longrightarrow K &= e^{rT}(S(0) - I) \end{aligned}$$

confirming the formula found in lecture.

 # 6:

The coupons are 9% of $100,000, so $9000, but these are split into 2 semi-annual payments of $4500 each. Since we are given that a coupon payment was just made, in the next 2 years the bond will pay this coupon payment in each of the next 6 months, 1 year, 18 months, and 2 years. This represents a known income paid by the bond. Therefore, to price a forward on this bond we can use the formulas for forwards on assets paying a known income. For the forward price this is

$$K = e^{rT}(S(0) - I)$$

where r is the risk free interest rate, T is the time until expiration of the forward, $S(t)$ is the price the bond is trading at at time t and I is the present value of the income that will be paid by the bond over the life of the forward contract.

We are given $r = 6\% = 0.06$, $S(0) = \$100,000$, and $T = 2$ since we are considering a 2 year forward contract. I must be calculated by a discounted cash flow calculation:

$$\begin{aligned} I &= e^{-0.06(0.5)}(4500) + e^{-0.06(1)}(4500) + e^{-0.06(1.5)}(4500) + e^{-0.06(2)}(4500) \\ &= \$16,709. \end{aligned}$$

Thus, the forward price is

$$\begin{aligned} K &= e^{(0.06)(2)}(100,000 - 16,709) \\ &= \$93,910. \end{aligned}$$

Your cash position is long the bond, so for price protection you want to take the short position in this forward.

We now suppose we have entered this short forward position, and consider the 2 proposed hypothetical situations 6 months later. First consider the case that the bond is then trading at \$98,100. The value of the short position in the forward contract at this time is

$$-V(0.5) = I(0.5) + e^{-r(T-0.5)}K - S(0.5)$$

where $I(0.5)$ is the present value, 6 months later, of the remaining coupon payments during the life of the forward contract. At that time there are 3 coupon payments left during the remaining 18 months of the futures contract, 6 months, 1 year, and 18 months hence, so

$$\begin{aligned} I(0.5) &= e^{-0.06(0.5)}(4500) + e^{-0.06(1)}(4500) + e^{-0.06(1.5)}(4500) \\ &= \$12,718. \end{aligned}$$

For the bond price we have, under this proposal, $S(0.5) = \$98,100$. The forward position thus has value

$$\begin{aligned} -V(0.5) &= 12,718 + e^{-0.06(2-0.5)}(93,910) - 98,100 \\ &= \$445. \end{aligned}$$

The forward position has a positive value, and so has profited.

The other proposal is $S(0.5) = \$102,000$. In this case the forward position has value

$$\begin{aligned} -V(0.5) &= 12,718 + e^{-0.06(2-0.5)}(93,910) - 102,000 \\ &= -\$3455. \end{aligned}$$

In this case the forward position has lost money.

 # 7:

Your cash position is short Euro, because you need to make a payment. Therefore, to protect yourself from price risk you must enter the long position in the forward contract. In this case you will want to enter a long position on a 6 month forward contract on an underlying asset of 15 M Euros.

The data we are given is

$$\begin{aligned} r_d &= \text{USD risk free rate} = 5\% = 0.05 \\ r_f &= \text{Euro risk free rate} = 2\% = 0.02 \\ S(0) &= \$1.28/\text{Euro} \end{aligned}$$

where $S(t)$ is the spot Euro/USD exchange rate. The forward rate for a $T =$ 6 month forward contract is

$$\begin{aligned} K &= e^{(r_d - r_f)T} S(0) \\ &= e^{(0.05-0.02)(0.5)}(1.28) \\ &= \$1.2993/\text{Euro}. \end{aligned}$$

Now we consider 2 hypothetical offers. Suppose we were offered a forward exchange rate of \$1.29/Euro. This is less than $K = 1.2993$. We follow the steps outlined in the lecture for FX forwards.

We will borrow 1000 Euros at the Euro risk free rate $r_f = 2\%$. In 6 months this debt will have accumulated to

$$e^{0.02(0.5)}(1000) = 1010 \text{ Euros}$$

We will need to buy this many Euros in 6 months to pay back the debt, so we enter into the forward contract to buy 1010 Euros in 6 months at the offered exchange rate of \$1.29/Euro.

We now convert these borrowed Euros to USD, which yields

$$1000 \times 1.28 = \$1280$$

using the current exchange rate of \$1.28/Euro. We now invest these dollars for 6 months at the USD risk free rate of $r_d = 5\% = 0.05$. In 6 months our cash holding is worth

$$e^{0.05(0.5)}(1280) = \$1312$$

At this time we are required by the forward contract to purchase 1010 Euros at an exchange rate of \$1.29/Euro. We will thus exchange for these an amount in Dollars equal to

$$1010 \times 1.29 = \$1303.$$

So we withdraw \$1303 form our cash holding, use it to purchase 1010 Euros under the forward contract, and then retire our Euro debt. We will retain a profit of

$$\$1312 - \$1303 = \$9$$

which is a riskless and certain profit, implying that this was an arbitrage portfolio.

Now we consider the second proposed hypothetical of an offered forward rate of \$1.31/Euro. This is now more than the fair forward exchange rate of \$1.2993/Euro. In this case we will follow the opposite procedure as for the previous hypothetical: we will borrow in USD, convert to Euro and invest, and then convert back to USD and pay back the debt.

So we borrow \$1000 at the USD risk free rate of 5%. We convert these borrowed Dollars to Euros at the prevailing exchange rate of \$1.28/Euro yielding

$$\frac{1000}{1.28} = 781 \text{ Euros.}$$

We will invest these at the Euro risk free rate for 6 months, yielding an investment worth

$$e^{0.02(0.5)}(781) = 789 \text{ Euros.}$$

In 6 months we will need to convert these Euros back to USD so we enter the forward contract to sell 805 Euros at the offered forward rate of \$1.31/Euro. After doing this, our holding of USD cash will be worth

$$789 \times 1.31 = \$1033.$$

Meanwhile, our US debt is now worth

$$e^{0.05(0.5)}(1000) = \$1025.$$

We now retire the debt, using the US cash from converting back our Euro investment. We retain a profit of

$$\$1033 - \$1025 = \$8$$

which is a riskless arbitrage profit.