These notes feel really bad. Use at your own risk

1 Products

Suppose that you have $H, K < G$, and consider the product $H \times K$.

Question 1.1. When is $HK \cong H \times K$ (where HK are the h cosets of K).

Well we know that HK is a subgroup precisely when H or K is a subgroup. And then with h_2k_2,

$$h_1k_1k_2^{-1}h_2^{-1} \in HK$$

.

Then if K in particulare is normal, then

$$\begin{aligned} h_1k_1k_2^{-1}h_2^{-1} &= h_1h_2^{-1}h_2k_1k_2^{-1}h_2^{-1} \\ &= h_1h_2^{-1}k_3 \end{aligned}$$

And if H is normal:

$$h_1(k_1k - 2^{-1}h_2^{-1}) == h^{-1}h_3k_1k_2^{-1}.$$

because $khk^{-1} = h^{-1}$.

If H or k is normal, when is

$$HK \cong H \times K?$$

Since these things are both normal, we'll rename them to N and K. $[N, K] \cong e$, then everything in N commutes with everything in K. If N and K are both normal in G, then $[N, K] \subseteq N \cap K$.

So take a generator $(n_1k_1n_1^{-1})k_1^{-1} \in N$.

Theorem 1.2. *If $N, K \trianglelefteq G$ and $N \cap K = e$, then $NK \cong N \times K$.*

Proof.

$$\begin{aligned} \varphi : N \times K &\to NK \\ (n, k) &\mapsto nk. \end{aligned}$$

$\varphi(n_1n_2, k_1k_2) = n_1n_2k_1k_2$ so $\varphi(n_1, k_1) \cdot \varphi(n_2, k_2)$ which equals $n_1n_2k_1k_2$ because everything in N commutes with everything in K, so we have a bijection. □

Lemma 1.3. *If $|G| = pq$ $p < q$, are prime, and G contains normal subgroups $H, |H| = p$, $K, |K| = q$, then G is cyclic.*

Proof. $H \cap K$ is a subgroup of H and K. $|H \cap K| \mid |H|$ and $|H \cap k| \mid |k|$ so $|H \cap K| = 1$.

Thus $HK \cong H \times K$, and moreover $HK = G$ by the counting argument since $|HK| = |G|$. Thus $G \cong H \times K \cong \mathbb{Z}_p \times \mathbb{Z}_q$ generated by $(1, 1)$. □

Now what if only one of them is normal?

Example 1.4. Consider $S_3 \cong D_6$ with presentaiton $\langle s, r \mid s^2 = 1, r^3 = 1,\ sr = r^{-1}s\rangle$.

Then $\langle r^3 \rangle$ is normal (since it is of index 2 in D_6). Then notice that the product $\langle s \rangle \times \langle r \rangle$ is not isomorphic to D_6.

Suppose that we have $G = NH$ with $N \cap H = e$, and $N \trianglerighteq N$. then G/N is a group isomorphic to H. Namely, the sequence

$$1 \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 1$$

Remember: an exact sequence splits if G contains a copy of H where $N \cap H = e$, and

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] & G \arrow[r] & H \arrow[l, bend right, "s"'] \arrow[r] & 1. \end{tikzcd}$$

Example 1.5. In D_6, we have

$$\begin{tikzcd} 1 \arrow[r] & Z_3 \arrow[r] & D_6 \arrow[r] & \mathbb{Z}_2 \arrow[l, bend right, "s"'] \arrow[r] & 1. \end{tikzcd}$$

splits.

Example 1.6. Using $\mathbb{Z}$:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z} \arrow[r, "a \mapsto a^8"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/8\mathbb{Z} \arrow[l, bend right] \arrow[r] & 1. \end{tikzcd}$$

doesn't split.

now from the counting equation:
$G \curvearrowright S$,

$$|S| = |Z| = \sum_{a \in A}^{n} |O_a|$$

where A contains 1 element from each non-trivial orbit.

Definition 1.7. For some prime p, a p-group is a group of order p^n.

Now suppose we have a p-group with $G \curvearrowright S$. so $|Z| = |S| \pmod{p}$
Now we can provide a proof of Cauchy's theorem from earlier.

Proof. $|G| < p$, p a prime such that $p \mid |G|$. Then G contains an element of order p.

Consider the set S of p-tuples in G which multiply to e via the grop operation.

Then $\mathbb{Z}_p$ acts on S by cyclic permutations. Then the cyclic permutations act on the set, and the fixed points are elements of order p or 1.

But then $|Z| \equiv |S| \equiv 0 \pmod{p}$, and it is not zero since there exists an element of order p. So the number of non trivial subgroups is $1 \pmod{p}$. ☐

Claim (Cauchy +)**.** *If G is a finite group, and $p \mid |G|$, where p is prime, $N =$ number of cyclic groups with size p, then $N \equiv 1 \pmod{p}$*

Example 1.8. If $p > 0$ is prime, and $m > 0$, if $|G| = mp$ with $m < p$, then G is not simple.

Then also the number of subgroups with p elements is congruant to $1 \pmod{p}$. Actually, if there's only 1, then it is normal.

Now we state Sylow's theorems.

Definition 1.9. Let $|G| = p^r m$, where $(p, m) = 1$, then a *Sylow-p subgroup* is a subgroup of size p^r

Theorem 1.10 (Sylow I)**.** *G contains a subgroup of size p^r.*

Theorem 1.11 (Sylow II)**.** *All Sylow p subgroups of G are conjugate*

Theorem 1.12. *The number of Sylow p-subgroups is congruant to $1 \pmod{p}$ and divides m.*

Example 1.13. There are no simple groups of size 12. By (III), there must by 1 or 4 Sylow 3 subgroups, so suppose it's 4. Then the number of non-trivial elements in the Sylow-3 subgroup of order 3 is $2 \times 4 = 8$. But then there are 4 elements left for the single subgroup of order 4, but this takes up the rest of the grou, and thus it is normal.

Exercise 1.14. If G is a group of order 33, then G is cyclic. Prove this fact, and generalize for p and q.