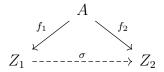
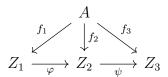
## Assignment 1

- 1.  $\mathcal{C}^A$ : The objects are things of the form  $f: A \to Z$ 
  - Morphisms between objects are things like  $\sigma \in \operatorname{Hom}_{\mathcal{C}}(Z_1, Z_2)$  that make



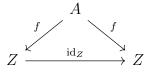
commute.

– Let two morphisms  $\varphi \in \text{Hom}(Z_1, Z_2)$  and  $\psi \in \text{Hom}(Z_2, Z_3)$  be given. Then we have



and we know that the left triangle and the right triangle commute. But then  $f_3 = \psi \circ f_2$ , and  $f_2 = \varphi \circ f_1$ , so  $f_3 = \psi \circ \varphi \circ f_1$ , which is precisely what we wanted.

– The identity is the identity on Z inherited from C, namely the morphism  $\mathrm{id}_Z \in \mathrm{Hom}_{\mathcal{C}}(Z,Z)$  such that

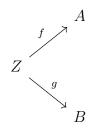


commutes. Notice that the identity here obeys the comp. rule on both sides. Namely, the diagrams

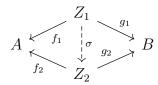
$$Z \xrightarrow{\operatorname{id}_{Z}} Z \xrightarrow{\psi} G F \xrightarrow{f} Z \xrightarrow{\operatorname{id}_{Z}} Z$$

Both commute and then also  $id_Z \circ \varphi = \varphi$  and  $\psi \circ id_Z = \psi$ .

 $C_{A,B}$ : – The Objects are things that look like

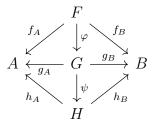


– Morphisms between objects are maps  $\sigma \in \operatorname{Hom}_{\mathcal{C}}(Z_1, Z_2)$  such that the diagram



commutes. Notice that the terminal objects in this category are precisely products.

– We check now composition. Let  $\varphi \in \text{Hom}(F,G), \psi \in \text{Hom}(G,H)$  be given. Then we have:



But we know that the upper part of the diagram commutes since  $\varphi$  is a morphism, and likewise for the bottom part of the diagram with  $\psi$ . We want to show that the diagram

$$A 
\downarrow^{f_A} \qquad F 
\downarrow^{\psi \circ \varphi} \qquad B 
\downarrow^{h_B} \qquad B$$

commutes, but this is immediate since we know that  $f_A = g_A \circ \varphi$ , but also  $g_A = h_A \circ \psi$ , so  $f_A = h_A \circ \psi \circ \varphi$ , which is what we wanted to show. (Technically this is only half, but the B side follows identically).

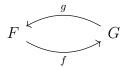
– The identity morphism is obviously just the identity inherited from C, namely, it is  $\mathrm{id}_Z \in \mathrm{Hom}_{\mathcal{C}}(Z,Z)$ , the morphism which causes the diagram

$$A \xrightarrow{z_A} Z \xrightarrow{z_B} B$$

$$Z \xrightarrow{z_B} B$$

to commute.

2. Proof. Suppose that we have  $F, G \in \text{Obj}(\mathcal{C})$  that are both final objects in  $\mathcal{C}$ . Then, since F, G are final, then there exists unique morphisms  $f \in \text{Hom}_{\mathcal{C}}(F, G)$ ,  $g \in \text{Hom}_{\mathcal{C}}(G, F)$ , and our goal is to show that the diagram



commutes and namely that f undoes g, and visa-versa.

In order to see this, recall that there must exist identity morphisms on F and G, so then the diagram

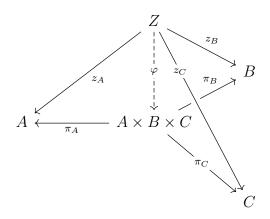
$$F \xrightarrow{f} G$$

$$\downarrow^{g} \downarrow^{\operatorname{id}_{G}}$$

$$F \xrightarrow{f} G$$

must commute with  $g \circ f = \mathrm{id}_F$  and  $f \circ g = \mathrm{id}_G$ , so f and g are inverses, and so F and G are isomorphic.

3. The property aught to be that, for  $A, B, C, Z \in \text{Obj}(\mathcal{C})$ ,  $A \times B \times C$  is terminal, and has projections into A, B, and C such that for any  $z_A : Z \to A$ ,  $z_B : Z \to B$ ,  $z_c : Z \to C$ , there exists a unique  $\varphi$  such that the zs factor through  $\varphi$ . In other words, there exists a unique  $\varphi$  such that the diagram



commutes.

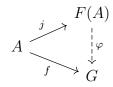
4. For each of p = 5, 7, 17:

 $p = 5 : (\mathbb{Z}/5\mathbb{Z})^* \cong \mathbb{Z}/4\mathbb{Z}$ , so 2 generates the group

 $p = 7 : (\mathbb{Z}/7\mathbb{Z})^* \cong \mathbb{Z}/6\mathbb{Z}$ , so 3 generates the group

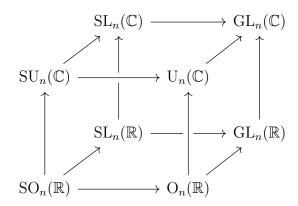
 $p=17\,:\,(\mathbb{Z}/17\mathbb{Z})^*\cong\mathbb{Z}/16\mathbb{Z},$  so 3 generates the group

5. Recall that the universal property of the free group tells us that for any group G and map  $f:A\to G$ , there exists a unique  $\varphi$  to make



So in particular, consider any group for which f can be made an injection.  $(\mathbb{R}, +)$  will do. Then if f is an injection, then  $\varphi \circ j$  must also be an injection, and thus j is an injection.

6. Let arrows denote a subset (subgroup) relation. Then



is a nice cube showing all inclusions. Now we move on to show that all these inclusions are subgroups... well, here we go I guess. We'll start at the bottom left corner, show that we have that  $\forall a, b \in H$ ,  $ab^{-1} \in H$ . Then, since  $GL_n(\mathbb{C})$  is a group, we'll get subgroup relations.

- $SO_n(\mathbb{R})$ :  $\forall Q, R \in SO_n(\mathbb{R})$ , notice that  $QR^{-1} = QR^T$ , but also  $QR^T*(QR^T)^T = QR^TRQ^T = I$ , and also  $det(QR^T) = 1$  by the multiplicative property of the determinant, so  $QR^T \in SO_n(\mathbb{R})$ .
  - $O_n(\mathbb{R})$ : Follows directly from above.
- $SU_n(\mathbb{C})$ : Replace the transposes above with the Hermitian adjoint and the proof is identical
  - $U_n(\mathbb{C})$ : Follows directly from above.
- $\mathrm{SL}_n(\mathbb{R})$ : Follows directly from the multiplicative property of the determinant, and the fact that  $\det(A^{-1}) = \det(A)^{-1}$ , so  $\forall Q, R \in \mathrm{SL}_n(\mathbb{R})$ ,  $QR^{-1}$  has determinant 1, since Q and  $R^{-1}$  each have determinant 1.
- $\mathrm{SL}_n(\mathbb{C})$ : Follows identically to above.
- $\mathrm{GL}_n(\mathbb{R})$ : replace "has determinant 1" in the  $\mathrm{SL}_n(\mathbb{R})$  argument with "has nonzero determinant" and you're done.
  - 7. Proof. Let  $h \in [G, G]$  be given. Then  $\forall g \in G, g^{-1}hg = h(h^{-1}g^{-1}hg) = h[h, g] \in [G, G]$ , so  $[G, G] \leq G$ .

Now we show that G/[G,G] is abelian. Let  $g[G,G]\in G/[G,G]$  and  $h[G,G]\in G/[G,G]$  be given. Then

$$gh[G,G] = gh[h,g][G,G] = ghh^{-1}g^{-1}hg[G,G] = hg[G,G]$$

so G/[G,G] is commutative.

- 8. Proof. I give no quarter to the abelian nature of the group, and appeal to Sylow's first theorem. Since n is odd, the multiplicity of 2 in the order of the group is 1, so there exists a Sylow 2-subgroup of G, and thus an element of order 2. Now suppose for the sake of contradiction that there exists more than 1 element of order 2. Then the subgroup generated by these two elements is the Vierergruppe, which has order 4, but  $4 \nmid 2n$ , since n is odd, so Lagrange tells us that there can't be more than one element of order 2.
- 9. We first show  $G^{\circ}$  is a group:
  - (Associative): Inherited from G
  - (Identity): Identities are two-sided anyway in groups, so the identity from G works.
  - (Inverses): Again, inverses are two-sided, so inverses in  $G^{\circ}$  are the same as inverses in G
  - *Proof.* First notice that id is bijective regardless of our assumption.
    - $(\Leftarrow)$  Suppose that G is commutative. Then

$$id(g \circ h) = g \circ h$$

$$= h \times g$$

$$= id(h) \times id(g)$$

$$= id(g) \times id(h).$$

 $\triangle$ 

so id is a homomorphism, and thus an isomorphism.

 $(\Rightarrow)$  Suppose now that id is an isomorphism. Then in particular

$$id(g \circ h) = g \times h$$

$$= h \circ g$$

$$= id(h \circ g)$$

$$= h \times g$$

so we have commutativity. If there's anything on this assignment that I don't believe, it's this.

• Proof. We give the isomorphism explicitly. Take  $\iota: G^{\circ} \to G$  be the map that takes  $g \longmapsto g^{-1}$ . Now in the language of group actions,  $\iota$  is a permutation, and thus a bijection.

Now we check that this is actually a homomorphism:

$$\iota(g \circ h) = (g \circ h)^{-1}$$

$$= h^{-1} \circ g^{-1}$$

$$= g^{-1} \times h^{-1}$$

$$= \iota(g) \times \iota(h)$$

So we have an isomorphism.

• If we have an action G acting on X on the right via  $(\cdot)$ , then define  $g \cdot x := x \cdot g^{-1}$ . Notice that this works because  $g \cdot (h \cdot x) = g \cdot (x \cdot h^{-1}) = (x \cdot h^{-1}) \cdot g^{-1}$ . But since we already know that the right action is compatible, we get that this equals  $(x \cdot h^{-1}g^{-1}) = (x \cdot (gh)^{-1}) = gh \cdot x$ .