Math 245 **Genevieve Walsh**
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Assignment 1

1. $\mathcal{C}^A$:
   - The objects are things of the form $f : A \to Z$
   - Morphisms between objects are things like $\sigma \in \mathrm{Hom}_{\mathcal{C}}(Z_1, Z_2)$ that make

     $$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ Z_1 \arrow[rr, dashed, "\sigma"] & & Z_2 \end{tikzcd}$$

     commute.
   - Let two morphisms $\varphi \in \mathrm{Hom}(Z_1, Z_2)$ and $\psi \in \mathrm{Hom}(Z_2, Z_3)$ be given. Then we have

     $$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[d, "f_2"] \arrow[dr, "f_3"] & \\ Z_1 \arrow[r, "\varphi"'] & Z_2 \arrow[r, "\psi"'] & Z_3 \end{tikzcd}$$

     and we know that the left triangle and the right triangle commute. But then $f_3 = \psi \circ f_2$, and $f_2 = \varphi \circ f_1$, so $f_3 = \psi \circ \varphi \circ f_1$, which is precisely what we wanted.
   - The identity is the identity on Z inherited from $\mathcal{C}$, namely the morphism $\mathrm{id}_Z \in \mathrm{Hom}_{\mathcal{C}}(Z, Z)$ such that

     $$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "f"] & \\ Z \arrow[rr, "\mathrm{id}_Z"] & & Z \end{tikzcd}$$

     commutes. Notice that the identity here obeys the comp. rule on both sides. Namely, the diagrams

     $$\begin{tikzcd} & A \arrow[dl, "z"'] \arrow[d, "z"] \arrow[dr, "g"] & \\ Z \arrow[r, "\mathrm{id}_Z"'] & Z \arrow[r, "\psi"'] & G \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, "z"] \arrow[dr, "z"] & \\ F \arrow[r, "\varphi"'] & Z \arrow[r, "\mathrm{id}_Z"'] & Z \end{tikzcd}$$

     Both commute and then also $\mathrm{id}_Z \circ \varphi = \varphi$ and $\psi \circ \mathrm{id}_Z = \psi$.

   $\mathcal{C}_{A,B}$:
   - The Objects are things that look like

     $$\begin{tikzcd} & A \\ Z \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & B \end{tikzcd}$$

- Morphisms between objects are maps $\sigma \in \mathrm{Hom}_{\mathcal{C}}(Z_1, Z_2)$ such that the diagram

$$\begin{tikzcd}
 & Z_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[dd, dashed, "\sigma"] & \\
A & & B \\
 & Z_2 \arrow[ul, "f_2"] \arrow[ur, "g_2"'] &
\end{tikzcd}$$

  commutes. Notice that the terminal objects in this category are precisely products.
- We check now composition. Let $\varphi \in \mathrm{Hom}(F, G), \psi \in \mathrm{Hom}(G, H)$ be given. Then we have:

$$\begin{tikzcd}
 & F \arrow[dl, "f_A"'] \arrow[dr, "f_B"] \arrow[d, "\varphi"] & \\
A & G \arrow[l, "g_A"] \arrow[r, "g_B"] \arrow[d, "\psi"] & B \\
 & H \arrow[ul, "h_A"] \arrow[ur, "h_B"'] &
\end{tikzcd}$$

  But we know that the upper part of the diagram commutes since φ is a morphism, and likewise for the bottom part of the diagram with ψ. We want to show that the diagram

$$\begin{tikzcd}
 & F \arrow[dl, "f_A"'] \arrow[dr, "f_B"] \arrow[dd, "\psi\circ\varphi"] & \\
A & & B \\
 & H \arrow[ul, "h_A"] \arrow[ur, "h_B"'] &
\end{tikzcd}$$

  commutes, but this is immediate since we know that $f_A = g_A \circ \varphi$, but also $g_A = h_A \circ \psi$, so $f_A = h_A \circ \psi \circ \varphi$, which is what we wanted to show. (Technically this is only half, but the B side follows identically).
- The identity morphism is obviously just the identity inherited from $\mathcal{C}$, namely, it is $\mathrm{id}_Z \in \mathrm{Hom}_{\mathcal{C}}(Z, Z)$, the morphism which causes the diagram

$$\begin{tikzcd}
 & Z \arrow[dl, "z_A"'] \arrow[dr, "z_B"] \arrow[dd, "\mathrm{id}_Z"] & \\
A & & B \\
 & Z \arrow[ul, "z_A"] \arrow[ur, "z_B"'] &
\end{tikzcd}$$

  to commute.

2. *Proof.* Suppose that we have $F, G \in \mathrm{Obj}(\mathcal{C})$ that are both final objects in $\mathcal{C}$. Then, since F, G are final, then there exists unique morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(F, G)$, $g \in \mathrm{Hom}_{\mathcal{C}}(G, F)$, and our goal is to show that the diagram

$$\begin{tikzcd}
F \arrow[r, bend right, "f"'] & G \arrow[l, bend right, "g"']
\end{tikzcd}$$

commutes and namely that f undoes g, and visa-versa.

In order to see this, recall that there must exist identity morphisms on F and G, so then the diagram

$$\begin{tikzcd}
F \arrow[r, "f"] \arrow[dr, "\mathrm{id}_F"'] & G \arrow[d, "g"] \arrow[dr, "\mathrm{id}_G"] & \\
& F \arrow[r, "f"'] & G
\end{tikzcd}$$

must commute with $g \circ f = \mathrm{id}_F$ and $f \circ g = \mathrm{id}_G$, so f and g are inverses, and so F and G are isomorphic. $\square$

3. The property aught to be that, for $A, B, C, Z \in \mathrm{Obj}(\mathcal{C})$, $A \times B \times C$ is terminal, and has projecitons into A, B, and C such that for any $z_A : Z \to A$, $z_B : Z \to B$, $z_c : Z \to C$, there exists a unique φ such that the zs factor through φ. In other words, there exists a unique φ such that the diagram

$$\begin{tikzcd}
& Z \arrow[ddl, "z_A"'] \arrow[dd, dashed, "\varphi"] \arrow[dr, "z_B"] \arrow[dddr, "z_C"] & \\
& & B \arrow[dd, dotted, no head] \\
A \arrow[urr, dotted, no head] \arrow[drr, dotted, no head] & A \times B \times C \arrow[l, "\pi_A"] \arrow[ur, "\pi_B"] \arrow[dr, "\pi_C"] & \\
& & C
\end{tikzcd}$$

commutes.

Then we realize that $(A \times B) \times C$ has this property, and notice that the argument for $A \times (B \times C)$ will follow identically.

Proof. Notice that we have that

$$\begin{tikzcd}
& Z \arrow[dl, "z_{A,B}"'] \arrow[d, dotted, "\varphi"] \arrow[dr, "z_C"] & \\
(A \times B) & (A \times B) \times C \arrow[l, "\pi_{A,B}"] \arrow[r, "\pi_C"'] & C
\end{tikzcd}$$

From the universal property of the product on $(A \times B)$ and C. But then also we can extend this diagram by noticing that $(A \times B)$ can be further expanded on as

$$
\begin{tikzcd}
 & & Z \arrow[dll, bend right, "z_A"'] \arrow[dl, "z_{A,B}"'] \arrow[d, dotted, "\varphi"] \arrow[dr, "z_C"] \arrow[ddl, bend right=100, "z_B"'] & \\
A & (A \times B) \arrow[l, "\pi_A"] \arrow[d, "\pi_B"] & (A \times B) \times C \arrow[l, "\pi_{A,B}"] \arrow[r, "\pi_C"] & C \\
 & B & &
\end{tikzcd}
$$

So notice that we get that the projections which allow us to factor through φ are $\pi_A \circ \pi_{A,B}$, $\pi_B \circ \pi_{A,B}$, and π_C. Then this diagram commutes and haas the universal property of the triple product that I have asserted. $\square$

4. For each of $p = 5, 7, 17$:

$$
\begin{aligned}
\text{p} = 5 &: (\mathbb{Z}/5\mathbb{Z})^* \cong \mathbb{Z}/4\mathbb{Z}, \text{ so 2 generates the group} \\
\text{p} = 7 &: (\mathbb{Z}/7\mathbb{Z})^* \cong \mathbb{Z}/6\mathbb{Z}, \text{ so 3 generates the group} \\
\text{p} = 17 &: (\mathbb{Z}/17\mathbb{Z})^* \cong \mathbb{Z}/16\mathbb{Z}, \text{ so 3 generates the group}
\end{aligned}
$$

5. Recall that the universal property of the free group tells us that for any group G and map $f : A \to G$, there exists a unique φ to make

$$
\begin{tikzcd}
 & F(A) \arrow[d, dashed, "\varphi"] \\
A \arrow[ur, "j"] \arrow[r, "f"'] & G
\end{tikzcd}
$$

So in particular, consider any group for which f can be made an injection. $\mathrm{Aut}(A)$ will do[1]. Then if f is an injection, then $\varphi \circ j$ must also be an injection, and thus j is an injection.

6. Let arrows denote a subset (subgroup) relation. Then

$$
\begin{tikzcd}
 & \mathrm{SL}_n(\mathbb{C}) \arrow[rr] & & \mathrm{GL}_n(\mathbb{C}) \\
\mathrm{SU}_n(\mathbb{C}) \arrow[ur] \arrow[rr] & & \mathrm{U}_n(\mathbb{C}) \arrow[ur] & \\
 & \mathrm{SL}_n(\mathbb{R}) \arrow[uu] \arrow[rr] & & \mathrm{GL}_n(\mathbb{R}) \arrow[uu] \\
\mathrm{SO}_n(\mathbb{R}) \arrow[uu] \arrow[ur] \arrow[rr] & & \mathrm{O}_n(\mathbb{R}) \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

[1]A note to Flame: we initially posited that $\mathbb{R}$ would work

o is a nice cube showing all inclusions. Now we move on to show that all these inclusions are subgroups... well, here we go I guess. We'll start at the bottom left corner, show that we have that $\forall a, b \in H,\ ab^{-1} \in H$. Then, since $\mathrm{GL}_n(\mathbb{C})$ is a group, we'll get subgroup relations.

$\mathrm{SO}_n(\mathbb{R})$: $\forall Q, R \in \mathrm{SO}_n(\mathbb{R})$, notice that $QR^{-1} = QR^T$, but also $QR^T * (QR^T)^T = QR^T RQ^T = I$, and also $\det(QR^T) = 1$ by the multiplicative property of the determinant, so $QR^T \in \mathrm{SO}_n(\mathbb{R})$.

$\mathrm{O}_n(\mathbb{R})$: Follows directly from above.

$\mathrm{SU}_n(\mathbb{C})$: Replace the transposes above with the Hermitian adjoint and the proof is identical

$\mathrm{U}_n(\mathbb{C})$: Follows directly from above.

$\mathrm{SL}_n(\mathbb{R})$: Follows directly from the multiplicative property of the determinant, and the fact that $\det(A^{-1}) = \det(A)^{-1}$, so $\forall Q, R \in \mathrm{SL}_n(\mathbb{R})$, QR^{-1} has determinant 1, since Q and R^{-1} each have determinant 1.

$\mathrm{SL}_n(\mathbb{C})$: Follows identically to above.

$\mathrm{GL}_n(\mathbb{R})$: replace "has determinant 1" in the $\mathrm{SL}_n(\mathbb{R})$ argument with "has nonzero determinant" and you're done.

7. *Proof.* Let $h \in [G, G]$ be given. Then $\forall g \in G$, $g^{-1}hg = h(h^{-1}g^{-1}hg) = h[h, g] \in [G, G]$, so $[G, G] \trianglelefteq G$. $\triangle$

   Now we show that $G/[G, G]$ is abelian. Let $g[G, G] \in G/[G, G]$ and $h[G, G] \in G/[G, G]$ be given. Then

$$gh[G, G] = gh[h, g][G, G] = ghh^{-1}g^{-1}hg[G, G] = hg[G, G]$$

   so $G/[G, G]$ is commutative. □

8. *Proof.* I appeal to Sylow's first theorem. Since n is odd, the multiplicity of 2 in the order of the group is 1, so there exists a Sylow 2-subgroup of G, and thus an element of order 2. Now suppose for the sake of contradiction that there exists more than 1 element of order 2. Then the subgroup generated by these two elements is the Vierergruppe, which has order 4, but $4 \nmid 2n$, since n is odd, so Lagrange tells us that there can't be more than one element of order 2. □

   Notice also that D_6 is a group with multiplicity $2 * 3$ without

9.
   - We first show G° is a group:
     - **(Associative)**: Inherited from G
     - **(Identity)**: Identities are two-sided anyway in groups, so the identity from G works.
     - **(Inverses)**: Again, inverses are two-sided, so inverses in G° are the same as inverses in G
   - *Proof.* First notice that id is bijective regardless of our assumption.

($\Leftarrow$) Suppose that G is commutative. Then

$$\begin{aligned}\mathrm{id}(g \circ h) &= g \circ h \\ &= h \times g \\ &= \mathrm{id}(h) \times \mathrm{id}(g) \\ &= \mathrm{id}(g) \times \mathrm{id}(h).\end{aligned}$$

so id is a homomorphism, and thus an isomorphism. $\triangle$

($\Rightarrow$) Suppose now that id is an isomorphism. Then in particular

$$\begin{aligned}\mathrm{id}(g \circ h) &= \mathrm{id}(g) \times \mathrm{id}(h) \\ &= g \times h \\ &= \mathrm{id}(h \circ g) \\ &= h \times g\end{aligned}$$

so we have commutativity.

$\square$

- *Proof.* We give the isomorphism explicitly. Take $\iota : G^\circ \to G$ be the map that takes $g \longmapsto g^{-1}$. Now in the language of group actions, ι is a permutation, and thus a bijection. $\triangle$

  Now we check that this is actually a homomorphism:

  $$\begin{aligned}\iota(g \circ h) &= (g \circ h)^{-1} \\ &= h^{-1} \circ g^{-1} \\ &= g^{-1} \times h^{-1} \\ &= \iota(g) \times \iota(h)\end{aligned}$$

  So we have an isomorphism. $\square$

- If we have an action G acting on X on the right via $(\cdot)$, then define $g \cdot x := x \cdot g^{-1}$. Notice that this works because $g \cdot (h \cdot x) = g \cdot (x \cdot h^{-1}) = (x \cdot h^{-1}) \cdot g^{-1}$. But since we already know that the right action is compatible, we get that this equals $(x \cdot h^{-1}g^{-1}) = (x \cdot (gh)^{-1}) = gh \cdot x$.