

We examine the free group via the set that it's defined over. In the language of categories  $\mathcal{F}^A$  is the free category of  $A$ , where the objects are like  $A \rightarrow G$ , and the morphisms are group homomorphisms  $\sigma$  like

$$\begin{array}{ccc} A & \xrightarrow{j_1} & G \\ & \searrow j_2 & \downarrow \sigma \\ & & H \end{array}$$

Now we define

**Definition 0.1.**  $F(A)$  is an initial object in  $\mathcal{F}^A$ .

**Claim.** The maps  $\{a\} \rightarrow \langle a \rangle$  are initial in the category.

This defined  $F(A)$  up to isomorphism, but why do they exist? In particular, define the resolution of the first cancelation relation among the words by  $r : W(A) \rightarrow W(A)$ , and define moreover  $R : W(A) \rightarrow W(A)$  by  $w \mapsto Rr^{\lfloor n/2 \rfloor}(w)$ . Then  $F(A) = (R(W(A)), \cdot)$ , where  $\cdot$  denotes concatenation.

**Fact.** This is a group, and this is easy to check for yourself.

Then define the map  $j : A \rightarrow F(A)$  as the map that takes a letter as a set object to a letter as a word in the group  $F(A)$ . Then the homomorphisms  $\sigma$  are defined letterwise in order to force the homomorphism condition.

In other words, just take  $abc \xrightarrow{\sigma} j(a)j(b)j(c)$ .

## 1 Subgroups

**Definition 1.1.** A Subgroup  $H$  of a group  $G$  (denoted by  $H \leq G$ ) is a subset  $H \subseteq G$  such that  $H$  is a group inheriting the group operation of  $G$ .

**Lemma 1.2.**  $H \subseteq G$  is a subgroup iff  $\forall a, b \in H, ab^{-1} \in H$ .

*Proof.* Trivially verified. □

**Example 1.3.** The image of a group homomorphism  $\varphi$  is a subgroup of the domain.

Now we can define a slightly more interesting object.

**Definition 1.4.** A subgroup  $H \leq G$  is called normal if  $g \in G, gHg^{-1} \in H$

**Example 1.5.**  $\ker \varphi$  is a normal subgroup of the domain of  $\varphi$  since  $\forall h \in H, g \in G, \varphi(g)\varphi(h)\varphi(g)^{-1} = \varphi(g)\varphi(h) = e$

**Claim.** All normal subgroups are the kernel of some homomorphism.

*Proof.* First notice that the left cosets of some normal subgroup  $K \trianglelefteq G$  partition  $G$ . Now we claim that this gives us a well-defined group operation. This can be seen by noticing that

$$\begin{aligned} gK \cdot hK &= ghK \\ gk_gK \cdot hk_hK &= gk_g h k_h K \\ &= ghk'_g k_h K \\ &= ghK. \end{aligned}$$

when  $K$  is normal, (where  $kh = hk'$  for  $k' \in K$ ).

Now we denote this group  $G/K$ , with  $g \mapsto gK$ . □

Then as a corollary of this, we get the following:

**Corollary 1.6.** *If  $\varphi : G \rightarrow G'$  is onto, then*

$$G/\ker \varphi \cong G'$$

And moreover,

**Theorem 1.7** (LaGrange's theorem). *The order of any subgroup divides  $|G|$ . in other words:  $|G| = [G : H]|H|$ .*

Quotient groups allow one to say that if  $\varphi : G \rightarrow G'$ :

$$\begin{array}{ccccc} G & \xrightarrow{\pi} & G/\ker \varphi \cong \operatorname{Im} \varphi & \hookrightarrow & G' \\ & & \searrow \varphi & & \uparrow \end{array}$$

## 2 Group Actions

The *action* of a group on a set is a homomorphism

$$\sigma : G \rightarrow \operatorname{Aut}(A).$$

Namely, a left action  $\rho : G \times A \rightarrow A$  is defined such that  $\rho(gh, a) = \rho(g, \rho(h, a))$ .

**Fact.** *Every group acts faithfully on some set. Therefore it is a subgroup of a permutation group. Yes, we are just stating Cayley's theorem as a fact, eat shit idiot.*