Math 245 **Geniveve Walsh**
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We discuss a little bit more before moving on to "abstract nonsense".

Recall that we have the universal property of the product, where something is a product if maps to projections factor through the product. Diagramatically, create a category whose morphisms are the σ such that the below diagram commutes,

$$\begin{tikzcd} & Z_1 \arrow[dl, "g_1"] \arrow[d, "\sigma"] \arrow[dr, "f_1"] & \\ B & Z_2 \arrow[l, "g_2"] \arrow[r, "f_2"] & A \end{tikzcd}$$

and the objects are things of the type

$$\begin{tikzcd} & Z \arrow[dl, "g"] \arrow[dr, "f"] & \\ B & & A \end{tikzcd}$$

Then notice that, in this cattegory, the product

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_B"] \arrow[dr, "\pi_A"] & \\ B & & A \end{tikzcd}$$

Is final, with morphisms

$$\begin{tikzcd} & Z_1 \arrow[dl, "g"] \arrow[d, "\sigma"] \arrow[dr, "f"] & \\ B & A \times B \arrow[l, "\pi_B"] \arrow[r, "\pi_A"] & A \end{tikzcd}$$

Where $\sigma(z) = (f(z), g(z))$. Here it is clear that the product map is unique, so this truly is final.

Definition 0.1. We say that a category $\mathcal{C}$ *has products* if $\forall A, B \in \mathrm{Obj}(\mathcal{C})$, then $\mathcal{C}_{A,B}$ has a final object. in SET, final objects are disjoint unions.

1 Groups

Definition 1.1 (Group)**.** G is a set with a closed binary operation

$$(\cdot) : G \times G \to G \quad \cdot (g, h) \mapsto g \times h$$

such that

1. $\cdot$ is associative

2. $\exists e_G \in G$ such that $\forall g \in G$ $g \cdot e = e \cdot g = g$

3. We get inverses: $\forall g \in G$, there exists $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e_G$.

Example 1.2. The trivial group. Yep.

Example 1.3. $(\mathbb{Z}, +)$, $(\mathbb{C}, +)$, $(\pm 1, \cdot)$. Notice that all of these groups are abelian.

Example 1.4. Matrix groups: $(\mathrm{GL}_n(F),\ \cdot)$, the multiplicative group of invertible $n \times n$ matrices over a field. Notice that matrix multiplication doesn't commute: a trivial fact.

Proposition 1.5. *Both the identity and g^{-1} are unique.*

Proof. Trivial. Suppose $\exists e_1, e_2$ both identities for the sake of contradiction. But then their product blah blah blah. And likewise for inverses. □

Notation. By associativity, we can justify that

$$g^n = \underbrace{g \cdot g \cdot \ldots \cdot g}_{n \text{times}}$$

As always, a group is called *Abelian* if the binary operation is commutative.

Example 1.6. The Commutator group for a non-abelian group is such that the binary op. is

$$[a, b] = aba^{-1}b^{-1}$$

Likewise the commutator subgroup is a subgroup generated by all commutators of elements from G.

Definition 1.7. The order of an element $g \in G$ is finite if $\exists n \in \mathbb{N}$ such that $g^n = \mathrm{Id}$. The order of the element is the least such n. An element has infinite order otherwise.

Lemma 1.8. *If $g^n = e$ for some $n > 0$, then $|g|$ divides n.*

Proof. Trivial exercise in number theory. □