Math 245 **Genevieve Walsh**
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We examine the free group via the set that it's defined over. In the language of categories $\mathscr{F}^A$ is the free categoy of A, where the objects are like : $A \to G$, and the morphisms are group homomorphisms σ like

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[dr, "j_2"] & G \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

Now we define

Definition 0.1. $F(A)$ is an initial object in $\mathscr{F}^A$.

Claim. *The maps $\{a\} \to \langle a \rangle$ are initial in the category.*

This defined $F(A)$ up to isomorphism, but why do they exist? In particular, define the resolution of the first cancelation relation among the words by $r : W(A) \to W(A)$, and define moreover $R : W(A) \to W(A)$ by $w \longmapsto Rr^{\lfloor n/2 \rfloor}(w)$. Then $F(A) = \left(R\left(W\left(A\right)\right), \cdot\right)$, where $\cdot$ denotes concatonation.

Fact. *This is a group, and this is easy to check for yourself.*

Then define the map $j : A \to F(A)$ as the map that takes a letter as a set object to a letter as a word in the group $F(A)$. Then the homomorphisms σ are defined letterwise in order to force the homomorphism condition.

In other words, just take $abc \overset{\sigma}{\mapsto} j(a)j(b)j(c)$.

1 Subgroups

Definition 1.1. A *Subgroup* H of a group G (denoted by $H \leq G$) is a subset $H \subseteq G$ such that H is a group inheriting the group operation of G.

Lemma 1.2. *$H \subseteq G$ is a subgroup iff $\forall a, b \in H,\ ab^{-1} \in H$.*

Proof. Trivially verified. □

Example 1.3. The image of a group homomorphism φ is a subgroup of the domain.

Now we can define a slightly more interesting object.

Definition 1.4. A subgroup $H \leq G$ is called normal if $g \in G$ $gHg^{-1} \in H$

Example 1.5. ker φ is a normal subgroup of the domain of φ since $\forall h \in H, g \in G$, $\varphi(g)\varphi(h)\varphi(g)^{-1} = \varphi(g)\varphi(g)^{-1} = e$

Claim. *All normal subgroups are the kernel of some homomorphism.*

Proof. First notice that the left cosets of some normal subgroup $K \trianglelefteq G$ partition G. Now we claim that this gives us a well-defined group operation. This can be seen by noticing that

$$\begin{aligned} gK \cdot hK &= ghK \\ gk_gK \cdot hk_hK &= gk_ghk_hK \\ &= ghk'_gk_hK \\ &= ghK. \end{aligned}$$

when K is normal, (where $kh = hk'$ for $k' \in K$).

Now we denote this group G/K, with $g \overset{\pi}{\mapsto} gK$. $\square$

Then as a corollary of this, we get the following:

Corollary 1.6. *If $\varphi : G \to G'$ is onto, then*

$$G/\text{ker } \varphi \cong G'$$

And moreover,

Theorem 1.7 (LaGrange's theorem)**.** *The order of any subgroup divides $|G|$. in other words: $|G| = [G : H]|H|$.*

Quotient groups allow one to say that if $\varphi : G \to G'$:

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[rr, bend right, "\varphi"'] & G/\text{ker } \varphi \cong \text{Im}\varphi \arrow[r, hook] & G' \end{tikzcd}$$

2 Group Actions

The *action* of a group on a set is a homomorphism

$$\sigma : G \to \text{Aut}(A)$$

. Namely, a left action $\rho : G \times A \to A$ is defined such that $\rho(gh, a) = \rho(g, \rho(h, a))$.

Fact. *Every group acts faithfully on some set. Therefore it is a subgroup of a permutation group. Yes, we are just stating Cayley's theorem as a fact, eat shit idiot.*