

Assignment 1

1. \mathcal{C}^A : – The objects are things of the form $f : A \rightarrow Z$
 – Morphisms between objects are things like $\sigma \in \text{Hom}_{\mathcal{C}}(Z_1, Z_2)$ that make

$$\begin{array}{ccc} & A & \\ f_1 \swarrow & & \searrow f_2 \\ Z_1 & \xrightarrow{\sigma} & Z_2 \end{array}$$

commute.

- Let two morphisms $\varphi \in \text{Hom}(Z_1, Z_2)$ and $\psi \in \text{Hom}(Z_2, Z_3)$ be given. Then we have

$$\begin{array}{ccccc} & & A & & \\ & f_1 \swarrow & \downarrow f_2 & \searrow f_3 & \\ Z_1 & \xrightarrow{\varphi} & Z_2 & \xrightarrow{\psi} & Z_3 \end{array}$$

and we know that the left triangle and the right triangle commute. But then $f_3 = \psi \circ f_2$, and $f_2 = \varphi \circ f_1$, so $f_3 = \psi \circ \varphi \circ f_1$, which is precisely what we wanted.

- The identity is the identity on Z inherited from \mathcal{C} , namely the morphism $\text{id}_Z \in \text{Hom}_{\mathcal{C}}(Z, Z)$ such that

$$\begin{array}{ccc} & A & \\ f \swarrow & & \searrow f \\ Z & \xrightarrow{\text{id}_Z} & Z \end{array}$$

commutes. Notice that the identity here obeys the comp. rule on both sides. Namely, the diagrams

$$\begin{array}{ccccc} & & A & & \\ & z \swarrow & \downarrow z & \searrow g & \\ Z & \xrightarrow{\text{id}_Z} & Z & \xrightarrow{\psi} & G \end{array} \quad \begin{array}{ccccc} & & A & & \\ & f \swarrow & \downarrow z & \searrow z & \\ F & \xrightarrow{\varphi} & Z & \xrightarrow{\text{id}_Z} & Z \end{array}$$

Both commute and then also $\text{id}_Z \circ \varphi = \varphi$ and $\psi \circ \text{id}_Z = \psi$.

- $\mathcal{C}_{A,B}$: – The Objects are things that look like

$$\begin{array}{ccc} & & A \\ & f \nearrow & \\ Z & & \\ & g \searrow & \\ & & B \end{array}$$

- Morphisms between objects are maps $\sigma \in \text{Hom}_{\mathcal{C}}(Z_1, Z_2)$ such that the diagram

$$\begin{array}{ccccc} & & Z_1 & & \\ & \swarrow f_1 & \downarrow \sigma & \searrow g_1 & \\ A & & & & B \\ & \swarrow f_2 & \downarrow & \searrow g_2 & \\ & & Z_2 & & \end{array}$$

commutes. Notice that the terminal objects in this category are precisely products.

- We check now composition. Let $\varphi \in \text{Hom}(F, G), \psi \in \text{Hom}(G, H)$ be given. Then we have:

$$\begin{array}{ccccc} & & F & & \\ & \swarrow f_A & \downarrow \varphi & \searrow f_B & \\ A & & G & \xrightarrow{g_B} & B \\ & \swarrow g_A & \downarrow \psi & \searrow h_B & \\ & & H & & \end{array}$$

But we know that the upper part of the diagram commutes since φ is a morphism, and likewise for the bottom part of the diagram with ψ . We want to show that the diagram

$$\begin{array}{ccccc} & & F & & \\ & \swarrow f_A & \downarrow \psi \circ \varphi & \searrow f_B & \\ A & & H & & B \\ & \swarrow h_A & & \searrow h_B & \end{array}$$

commutes, but this is immediate since we know that $f_A = g_A \circ \varphi$, but also $g_A = h_A \circ \psi$, so $f_A = h_A \circ \psi \circ \varphi$, which is what we wanted to show. (Technically this is only half, but the B side follows identically).

- The identity morphism is obviously just the identity inherited from \mathcal{C} , namely, it is $\text{id}_Z \in \text{Hom}_{\mathcal{C}}(Z, Z)$, the morphism which causes the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow z_A & \downarrow \text{id}_Z & \searrow z_B & \\ A & & Z & & B \\ & \swarrow z_A & & \searrow z_B & \end{array}$$

to commute.

2. *Proof.* Suppose that we have $F, G \in \text{Obj}(\mathcal{C})$ that are both final objects in \mathcal{C} . Then, since F, G are final, then there exists unique morphisms $f \in \text{Hom}_{\mathcal{C}}(F, G)$, $g \in \text{Hom}_{\mathcal{C}}(G, F)$, and our goal is to show that the diagram

$$\begin{array}{ccc} & g & \\ F & \xleftarrow{\quad} & G \\ & f & \end{array}$$

commutes and namely that f undoes g , and visa-versa.

In order to see this, recall that there must exist identity morphisms on F and G , so then the diagram

$$\begin{array}{ccccc} F & \xrightarrow{f} & G & & \\ & \searrow \text{id}_F & \downarrow g & \searrow \text{id}_G & \\ & & F & \xrightarrow{f} & G \end{array}$$

must commute with $g \circ f = \text{id}_F$ and $f \circ g = \text{id}_G$, so f and g are inverses, and so F and G are isomorphic. \square

3. The property ought to be that, for $A, B, C, Z \in \text{Obj}(\mathcal{C})$, $A \times B \times C$ is terminal, and has projections into A, B , and C such that for any $z_A : Z \rightarrow A$, $z_B : Z \rightarrow B$, $z_C : Z \rightarrow C$, there exists a unique φ such that the z s factor through φ . In other words, there exists a unique φ such that the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow z_A & \downarrow \varphi & \searrow z_B & \\ & & A \times B \times C & & B \\ & \swarrow \pi_A & \downarrow \pi_C & \searrow \pi_B & \\ & & A & & C \end{array}$$

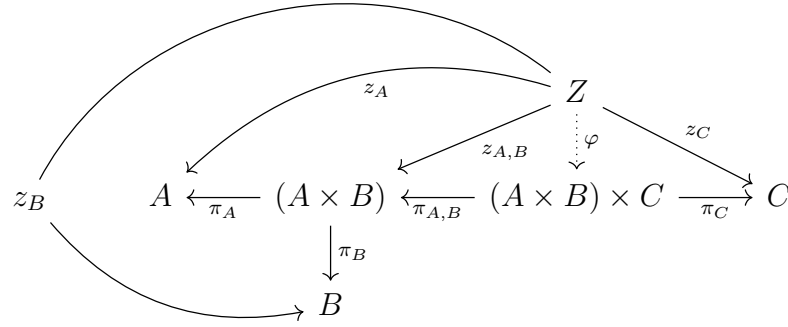
commutes.

Then we realize that $(A \times B) \times C$ has this property, and notice that the argument for $A \times (B \times C)$ will follow identically.

Proof. Notice that we have that

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow z_{A,B} & \downarrow \varphi & \searrow z_C & \\ (A \times B) & \xleftarrow{\pi_{A,B}} & (A \times B) \times C & \xrightarrow{\pi_C} & C \end{array}$$

From the universal property of the product on $(A \times B)$ and C . But then also we can extend this diagram by noticing that $(A \times B)$ can be further expanded on as



So notice that we get that the projections which allow us to factor through φ are $\pi_A \circ \pi_{A,B}$, $\pi_B \circ \pi_{A,B}$, and π_C . Then this diagram commutes and has the universal property of the triple product that I have asserted. \square

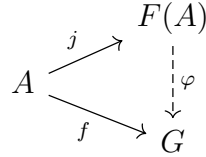
4. For each of $p = 5, 7, 17$:

$p = 5$: $(\mathbb{Z}/5\mathbb{Z})^* \cong \mathbb{Z}/4\mathbb{Z}$, so 2 generates the group

$p = 7$: $(\mathbb{Z}/7\mathbb{Z})^* \cong \mathbb{Z}/6\mathbb{Z}$, so 3 generates the group

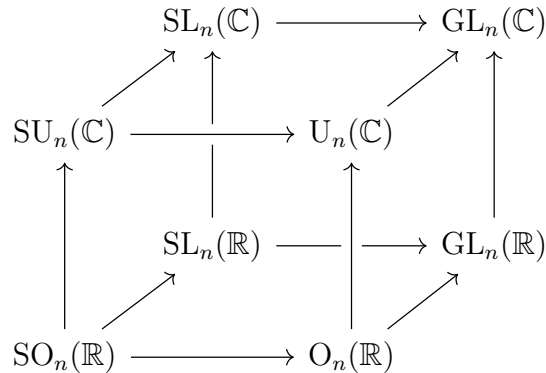
$p = 17$: $(\mathbb{Z}/17\mathbb{Z})^* \cong \mathbb{Z}/16\mathbb{Z}$, so 3 generates the group

5. Recall that the universal property of the free group tells us that for any group G and map $f : A \rightarrow G$, there exists a unique φ to make



So in particular, consider any group for which f can be made an injection. $\text{Aut}(A)$ will do¹. Then if f is an injection, then $\varphi \circ j$ must also be an injection, and thus j is an injection.

6. Let arrows denote a subset (subgroup) relation. Then



¹A note to Flame: we initially posited that \mathbb{R} would work

o is a nice cube showing all inclusions. Now we move on to show that all these inclusions are subgroups... well, here we go I guess. We'll start at the bottom left corner, show that we have that $\forall a, b \in H, ab^{-1} \in H$. Then, since $GL_n(\mathbb{C})$ is a group, we'll get subgroup relations.

$SO_n(\mathbb{R})$: $\forall Q, R \in SO_n(\mathbb{R})$, notice that $QR^{-1} = QR^T$, but also $QR^T * (QR^T)^T = QR^T RQ^T = I$, and also $\det(QR^T) = 1$ by the multiplicative property of the determinant, so $QR^T \in SO_n(\mathbb{R})$.

$O_n(\mathbb{R})$: Follows directly from above.

$SU_n(\mathbb{C})$: Replace the transposes above with the Hermitian adjoint and the proof is identical

$U_n(\mathbb{C})$: Follows directly from above.

$SL_n(\mathbb{R})$: Follows directly from the multiplicative property of the determinant, and the fact that $\det(A^{-1}) = \det(A)^{-1}$, so $\forall Q, R \in SL_n(\mathbb{R})$, QR^{-1} has determinant 1, since Q and R^{-1} each have determinant 1.

$SL_n(\mathbb{C})$: Follows identically to above.

$GL_n(\mathbb{R})$: replace "has determinant 1" in the $SL_n(\mathbb{R})$ argument with "has nonzero determinant" and you're done.

7. *Proof.* Let $h \in [G, G]$ be given. Then $\forall g \in G, g^{-1}hg = h(h^{-1}g^{-1}hg) = h[h, g] \in [G, G]$, so $[G, G] \trianglelefteq G$. \triangle

Now we show that $G/[G, G]$ is abelian. Let $g[G, G] \in G/[G, G]$ and $h[G, G] \in G/[G, G]$ be given. Then

$$gh[G, G] = gh[h, g][G, G] = gh h^{-1} g^{-1} h g [G, G] = hg[G, G]$$

so $G/[G, G]$ is commutative. \square

8. *Proof.* I appeal to Sylow's first theorem. Since n is odd, the multiplicity of 2 in the order of the group is 1, so there exists a Sylow 2-subgroup of G , and thus an element of order 2. Now suppose for the sake of contradiction that there exists more than 1 element of order 2. Then the subgroup generated by these two elements is the Vierergruppe, which has order 4, but $4 \nmid 2n$, since n is odd, so Lagrange tells us that there can't be more than one element of order 2. \square

Notice also that D_6 is a group with multiplicity $2 * 3$ without

9. • We first show G° is a group:

- **(Associative)**: Inherited from G
- **(Identity)**: Identities are two-sided anyway in groups, so the identity from G works.
- **(Inverses)**: Again, inverses are two-sided, so inverses in G° are the same as inverses in G

• *Proof.* First notice that id is bijective regardless of our assumption.

(\Leftarrow) Suppose that G is commutative. Then

$$\begin{aligned}\text{id}(g \circ h) &= g \circ h \\ &= h \times g \\ &= \text{id}(h) \times \text{id}(g) \\ &= \text{id}(g) \times \text{id}(h).\end{aligned}$$

so id is a homomorphism, and thus an isomorphism. \triangle

(\Rightarrow) Suppose now that id is an isomorphism. Then in particular

$$\begin{aligned}\text{id}(g \circ h) &= \text{id}(g) \times \text{id}(h) \\ &= g \times h \\ &= \text{id}(h \circ g) \\ &= h \times g\end{aligned}$$

so we have commutativity. \square

- *Proof.* We give the isomorphism explicitly. Take $\iota : G^\circ \rightarrow G$ be the map that takes $g \mapsto g^{-1}$. Now in the language of group actions, ι is a permutation, and thus a bijection. \triangle

Now we check that this is actually a homomorphism:

$$\begin{aligned}\iota(g \circ h) &= (g \circ h)^{-1} \\ &= h^{-1} \circ g^{-1} \\ &= g^{-1} \times h^{-1} \\ &= \iota(g) \times \iota(h)\end{aligned}$$

So we have an isomorphism. \square

- If we have an action G acting on X on the right via (\cdot) , then define $g \cdot x := x \cdot g^{-1}$. Notice that this works because $g \cdot (h \cdot x) = g \cdot (x \cdot h^{-1}) = (x \cdot h^{-1}) \cdot g^{-1}$. But since we already know that the right action is compatible, we get that this equals $(x \cdot h^{-1} g^{-1}) = (x \cdot (gh)^{-1}) = gh \cdot x$.