**Math 171** **Professor Sebastian Bozlee**
**Scribed by: Kyle Dituro** **April 14, 2023**

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**Definition 1.** A subset $I$ of $\mathbb{R}$ is said to be **convex** if for any $a, b \in I$ with $a < b$, we have $[a, b] \subseteq I$.

**Example 1.** $\emptyset$, $\{a\}$, $(a, b)$, $(a, b]$, $[a, b)$, $[a, b]$, $[a, \infty)$, $(a, \infty)$, $(-\infty, a)$, $(-\infty, a]$, $(\infty, \infty)$.

**Theorem 1.** *A subspace $I$ of $\mathbb{R}$ is connected if and only if it is convex.*

*Proof.* Suppose $I$ is connected. Suppose for the sake of contradiction that $I$ is not convex. Then $\exists a, b \in I$, $a < b$ such that $\exists z$ with $a < z < b$ and $z \notin I$.

Consider $U = (-\infty, z) \cap I$, $V = (z, \infty) \cap I$.

$U, V$ are clearly disjoint and nonempty (since $a \in U$ $b \in V$ by construction), open, and have union $I$.

This is therefore a separation, and a contradiction. So $I$ is convex.

Now suppose $I$ is convex. Suppose for contradiction that $U, V$ is a separation of $I$.

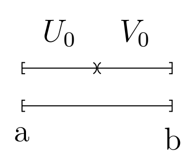
Let $a \in U, b \in V$. Now without loss of generality, assume that $a < b$. Consider the interval $[a, b]$. Let

$$U_0 = U \cap [a, b] \quad V_0 = V \cap [a, b].$$

This is a separation. $U_0, V_0$ are disjoint, open in $[a, b]$, nonempty since $a \in U$, $b \in V$, and their union is all of $[a, b]$.

Let $c = \sup\ U_0$

(Case 1):




Since $U_0$ is open in $[a, b]$ and $c < b$ and $c \in U_0$, there is some $d \in (a, b)$ such that $[c, d) \subseteq U_0$. But then $\frac{c+d}{2} \in U_0$ and $c < \frac{c+d}{2}$, so $c$ is not an upper bound on $U_0$, a contradiction.

(Case 2): ($c \in V_0$) Then $c = b$ or $a < c < b$.

Since $V_0$ is open in $[a, b]$ and $a < c$, $c \in V_0$, there is some $e \in (a, b)$ such that $(e, c] \subseteq V_0$. Then $e$ is an upper bound on $U_0$, but this contradicts that $c$ is the least upper bound on $U_x$.

Now $c \notin U_0$ and $c \notin V_0$, so $[a, b] \neq U_0 \cup V_0$, and thus $I$ is connected. $\square$

**Theorem 2** (Intermediate Value Theorem)**.** *Let $f : X \to \mathbb{R}$ be a continuous function where $X$ is a connected topological space. Then if $a, b \in X$ and $y$ is between $f(a)$ and $f(b)$, then there exists $c \in X$ such that $f(c) = y$.*

*Proof.* Since $X$ is connected and $f$ is continuous, $f(X)$ is connected. By the previous theorem, $f(X)$ is convex, so $y \in f(X)$. Then by the definition of the image, $\exists c \in X$ such that $f(c) = y$. $\square$

Now we will give yet another characterization of continuity. One that is perhaps a bit stronger than our prior ones, but should aide our intuition.

**Definition 2.** Let $X$ be a topological space, $p, q \in X$. A **path** from $p$ to $q$ in $X$ is a continuous function $f : [a, b] \to X$ such that $f(a) = p$, $f(b) = q$.

We say that $X$ is **path connected** if for any $p, q \in X$ there is a path in $X$ from $p$ to $q$.

**Proposition 1.** *If $X$ is path connected, then $X$ is connected.*

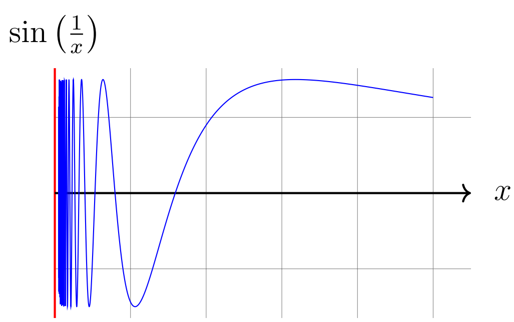
*Proof.* Suppose for the sake of contradiction that $U, V$ is a separation of $X$. Let $p \in U, q \in V$. Since $X$ is path connected, there is a continuous function $f : [a, b] \to X$. Then $f(a) = p$ and $q = f(b)$. Then $f([a, b])$ is connected since $[a, b]$ is connected.

Then $f([a, b]) \subseteq U$ or $f([a, b]) \subseteq V$, but this is impossible since $p \notin V$, $q \notin U$. $\square$

**Example 2.** The unit ball $B$ in $\mathbb{R}^n$ is connected, since it is path connected.

If $\mathbf{p}, \mathbf{q} \in B$, then $\begin{array}{c} f : [0,1] \to B \\ t \mapsto (1-t)\mathbf{p} + t\mathbf{q} \end{array}$ will be a path from $\mathbf{p}$ to $\mathbf{q}$.

**Example 3** (Connected $\not\Rightarrow$ Path connected)**.**




Let $S = \left\{ \left(t, \sin\left(\frac{1}{t}\right)\right) \;\middle|\; 0 < t \leq 1 \right\}$.

This is connected, since it is the image of a connected space. Take the closure:

$$\overline{S} = \{0\} \times [-1, 1] \cup S$$

**Fact.** *Closures of connected subspaces are also connected.*

Let $\mathbf{p} = (0,0)$, $q = (1, \sin(1))$.
Suppose $f : [0,1] \to \overline{S}$ is a path from $\mathbf{p}$ to $\mathbf{q}$.
notice $0 \times [-1,1]$ is a closed set, so its preimage under $f$ is a closed set, with some max.
Reparameterizing, we may assume this max is 0.
By the intermediate value theorem, for each integer $n > 0$, there is some $t_n$ such that

$$f(t_n) = \left(\frac{1}{2\pi n + \frac{\pi}{2}} . \sin\left(2\pi n + \frac{\pi}{2}\right)\right) = \left(\frac{1}{2\pi}, 1\right)$$

.

Then

$$\lim_{n\to\infty} f(t_n) = (0,1) \neq f(0) = (0,0)$$
$$\lim_{n\to\infty} t_n = 0$$

Moreover, $f\left(\lim_{n\to\infty} t_n\right) \neq (0,1)$ since the only value of $f$ with $x$-coordinate 0 is $(0,0)$. Therefore, $\lim_{n\to\infty}(t_n) \neq f(\lim_{n\to\infty} t_n)$, and therefore $f$ is not continuous.