

Let's go over part (b) of the early problem...

Proposition 1. *Let X, Y be topological spaces, and let $\mathcal{B}_X, \mathcal{B}_Y$ be bases for the resp. topologies of X, Y , then:*

1. $f : X \rightarrow Y$ is continuous iff for each $B \in \mathcal{B}_Y$, $f^{-1}(B)$ is open.
2. $f : X \rightarrow Y$ is continuous iff for all $x \in X$ and basic open neighborhoods $B_{f(x)}$ of $f(x)$, there exists $B_x \in \mathcal{B}_X$ such that $x \in B_x$ and $f(B_x) \subseteq B_{f(x)}$.

The proof of (a) is left to the early problem

Proof of part b. (\Rightarrow) Let $f : X \rightarrow Y$ be continuous. Let $x \in X$, and $B_{f(x)}$ be a basic open neighborhood of $f(x)$. Then, since $f(x) \in B_{f(x)}$, $x \in f^{-1}(B_{f(x)})$. Since f is continuous, $f^{-1}(B_{f(x)})$ is open. By the definition of open in a generated topology, $\exists B_X \in \mathcal{B}_X$ such that $x \in B_X$ and $B_X \subseteq f^{-1}(B_{f(x)})$. Then, using the Galois connection, we get that $f(B_X) \subseteq B_{f(x)}$.

(\Leftarrow) Assume that we have the property, now we want to show that $f : X \rightarrow Y$ is continuous. So let $U \subseteq Y$ be an open set. Now let $x \in f^{-1}(U)$. Then $f(x) \in U$. Since U is open, there is a basic open neighborhood $B_{f(x)}$ such that $f(x) \in B_{f(x)}$ and $B_{f(x)} \subseteq U$. By assumption, there is a basic open neighborhood B_x of x such that $f(B_x) \subseteq B_{f(x)} \subseteq U$. Then again by the Galois connection, $B_x \subseteq f^{-1}(U)$.

□

Now on to more stuff.

Definition 1. A set S of subsets of a set X whose union is all of X is a **subbasis for a topology on X** . The **topology generated by S** is the set of unions of finite intersections of elements of S .

We claim that in order to show that this is a topology, it is enough to show that the set \mathcal{B} of finite intersections of elements of S is a basis. Now let's actually do the check:

Proof. 1. Let $x \in X$. Then since $\bigcup_{T \in S} T = X$, there is an element $T \in S$ such that $x \in T$, and thus $T \in \mathcal{B}$.

2. Let $B_1 = T_1 \cap \dots \cap T_n$, where $T_i \in S$, and $B_2 = T'_1 \cap \dots \cap T'_m$, where $T'_i \in S$. Now let $x \in B_1 \cap B_2$. Since $B_1 \cap B_2$ is a finite intersection of elements of S , it is a basis element, and so we can take $B_3 = B_1 \cap B_2$.

□

So where is this useful? We will now move into a discussion of product spaces. This is where all of the universal property juggling should pay off.

Question 1. Given two topological spaces X, Y , what topology “should” we put on $X \times Y$?

More generally, given topological spaces X_i for $i \in I$, what topology should $\prod_{i \in I} X_i$ have?

Let’s recall the universal property of the product for sets, and amend it for topological spaces.

Theorem 1. Given sets *topological spaces* X, Y , a set *topological space* P together with functions *continuous functions* $\pi_1 : P \rightarrow X$ and $\pi_2 : P \rightarrow Y$ is said to have the universal property if, for any set *topological space* Z and functions *continuous functions* $f_1 : Z \rightarrow X$ and $f_2 : Z \rightarrow Y$ there exists a unique function *continuous function* $f : Z \rightarrow P$ such that.

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \vdots & \searrow & \\ & f_1 & \exists! f & f_2 & \\ X & \xleftarrow{\pi_1} & P & \xrightarrow{\pi_2} & Y \end{array}$$

Proposition 2. “Products are unique up to unique homeomorphism.” i.e. Given (P, π_1, π_2) and (P', π'_1, π'_2) with the universal property, there exists a unique homeomorphism $\varphi : P \rightarrow P'$ such that $\pi_1 = \pi'_1 \circ \varphi$ and $\pi_2 = \pi'_2 \circ \varphi$.

Proof. Let’s do some good old-fashioned diagram chasing.

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \vdots & \searrow & \\ & \pi_1 & \exists! \varphi & \pi_2 & \\ X & \xleftarrow{\pi'_1} & P' & \xrightarrow{\pi'_2} & Y \end{array} \quad \begin{array}{ccccc} & & Z & & \\ & \swarrow & \vdots & \searrow & \\ & & \psi & & \\ X & \xleftarrow{\quad} & P & \xrightarrow{\quad} & Y \end{array}$$

Combining these two diagrams, we get:

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \downarrow & \searrow & \\ & \pi_1 & \psi \circ \varphi & \pi_2 & \\ X & \xleftarrow{\pi_1} & P & \xrightarrow{\pi_2} & Y \end{array}$$

So then $\psi \circ \varphi$ is the identity map, and we have what we need. \square

So we can revise our goal: We now want to find a topology on $X \times Y$ so that $(X \times Y, \pi_1, \pi_2)$ has the universal property.

In other words, we want to find a topology on $X \times Y$ such that $\pi_1 : X \times Y \rightarrow X$ and $(x, y) \mapsto x$

$\pi_2 : X \times Y \rightarrow Y$ are continuous.
 $(x, y) \mapsto y$

So we need

1. For each $U \subseteq X$ open, $\pi_1^{-1}(U) = U \times Y$ is open
2. For each $V \subseteq Y$ open, $\pi_2^{-1}(V) = X \times V$ is open.

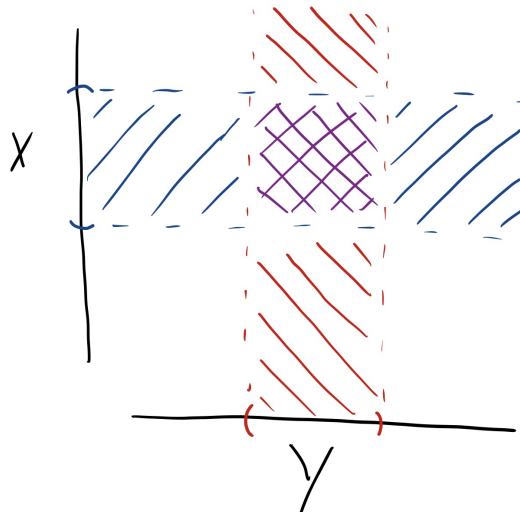


Fig. 1. A diagram outlining what our “Product space” should look like

This is not yet a topology, we still need finite intersections. So, let's add in finite intersections: $\mathcal{B} = \{U \times V \mid U \subseteq X \text{ open}, V \subseteq Y \text{ open}\}$. This in and of itself is still not a topology, but it is a basis.

Proposition 3. *Let X, Y be topological spaces, then the set:*

$$\mathcal{B} = \{U \times V \mid U \subseteq X \text{ open}, V \subseteq Y \text{ open}\}$$

Definition 2. The **product topology on $X \times Y$** is the topology generated by the basis \mathcal{B} above.

Example 1. \mathbb{R}^2 with the usual topology has the product topology from $\mathbb{R} \times \mathbb{R}$. With the usual topology, we were taking a basis of open balls, but in this construction, we are doing it with open rectangles, which we previously showed was equivalent.