Math 171 | **Professor Sebastian Bozlee**
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Definition 1. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called **continuous** (in an analysis sense) if $\forall \varepsilon > 0,\ \exists \delta$ s.t $\forall y \in \mathbb{R}^n, |\mathbf{x} - \mathbf{y}| < \delta \Rightarrow |f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$.

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$. We will now demonstrate that this is continuous by showing that given $x \in \mathbb{R}$, and we want to find some $\delta > 0$ such that $|x - y| < \delta$ implies $|x^2 - y^2| < \varepsilon$.

Proof. Say $h = x - y$. Then

$$\begin{aligned} x^2 - y^2 &= (x+y)(x-y) \\ &= h(2x - h) \\ |x^2 - y^2| &= |h(2x-h)| \\ &= |h||2x - h| \\ &\leq |h|(|2x| + |-h|) \\ &= |h|(2|x| + |h|) \\ &\leq |h|(2|x| + 1) \quad (\text{if } |h| < 1) \end{aligned}$$

So to make this less than ε, let's use $\delta = \min\left(\frac{\varepsilon}{2|x|+1}, 1\right)$. Then

$$\begin{aligned} |x^2 - y^2| &= |x+y||x-y| \\ &= \dots \\ &\leq |h|(2|x| + |h|) \end{aligned}$$

Then if $\frac{\varepsilon}{2|x|+1} < 1$, then

$$\begin{aligned} |h|(2|x|+1) &\leq |h|(2|x|+1) \\ &\leq \frac{\varepsilon}{2|x|+1}(2|x|+1) \\ &= \varepsilon \end{aligned}$$

And if $\frac{\varepsilon}{2|x|+1} \geq 1$, $|h|(2|x|+1) < 2|x|+1$ and since $\varepsilon \geq 2|x|+1$, we have what we need. □

Proposition 1. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ are continuous functions, then $g \circ f : \mathbb{R}^n \to \mathbb{R}^p$ is as well.*

Proof. let $\mathbf{x} \in \mathbb{R}^n$, $\varepsilon > 0$. Since g is continuous, there exists $\gamma > 0$ such that $\forall \mathbf{y} \in \mathbb{R}^m$,

$$f(\mathbf{x} - \mathbf{y}) < \gamma \Rightarrow |g(f(\mathbf{x})) - g(\mathbf{y})| < \varepsilon.$$

Furthermore, since f is continuous, there exists $\delta > 0$ such that for all $\mathbf{x}_2 \in \mathbb{R}^n$,

$$|\mathbf{x} - \mathbf{x}_2| < \delta \Rightarrow |f(\mathbf{x}) - f(\mathbf{x}_2)| < \delta.$$

And together $|\mathbf{x} - \mathbf{x}_2| < \delta \Rightarrow |f(\mathbf{x}) - f(\mathbf{x}_2)| < \gamma$ implies that our composition is bounded by epsilon. □

Proposition 2. *The following functions are continuous:*

1. $+ : \mathbb{R}^2 \to \mathbb{R}$
   $(x, y) \mapsto x + y$
2. $\cdot : \mathbb{R}^2 \to \mathbb{R}$
   $(x, y) \mapsto x \cdot y.$
3. *Constant functions.*

Proof. Let's just work with multiplication. We want to say that for each pair $(x_0, y_0) \in \mathbb{R}^2$, $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $(x_1, y_1) \in \mathbb{R}^2$ with $|(x_0, y_0) - (x_1, y_1)| < \delta$ we get that $|x_0 y_0 - x_1 y_1| < \varepsilon$. Let $\delta = \frac{\varepsilon}{|x_0|+|y_1|+1}$. This proof went slightly awry in class, but

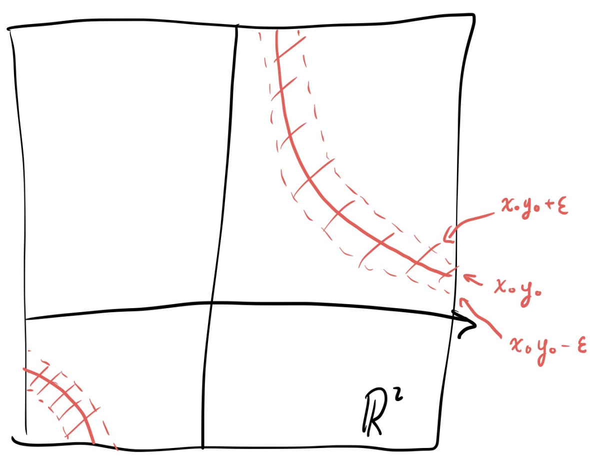

Fig. 1. Graph of a hyperbola and some tolerance ε.

was corrected in an email/canvas message later. □

Definition 2. Let $\mathbf{x} \in \mathbb{R}^n$ $r \in \mathbb{R}_{>0}$, the open ball of radius r centered at $\mathbf{x}$ is the set

$$B(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n \ : \ |\mathbf{x} - \mathbf{y}| < r\}.$$

A subset $U \subseteq \mathbb{R}^n$ is called **open** if for each $\mathbf{x} \in U$, there exists some $\varepsilon > 0$ such that $B(\mathbf{x}, \varepsilon) \subseteq U$.

Let $Z \subseteq \mathbb{R}^n$. A point $\mathbf{x} \in \mathbb{R}^n$ is said to be a **limit point** of Z if $B(\mathbf{x}, \varepsilon) \cap Z \neq \emptyset$ for all $\varepsilon > 0$.

A subset $Z \subseteq \mathbb{R}^n$ is said to be closed if Z contains all of its limit points.

Example 2.

1. An open interval $(a, b) \subseteq \mathbb{R}$ is open and not closed
2. A closed interval $[a, b] \subseteq \mathbb{R}$ is closed and not open
3. The half open interval $[a, b) \subseteq \mathbb{R}$ is neither open nor closed
4. $\mathbb{R} \subseteq \mathbb{R}$ is neither.

Theorem 1. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous iff for all open subsets $U \subseteq \mathbb{R}^m$, $f^{-1}(u) \subseteq \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$.*