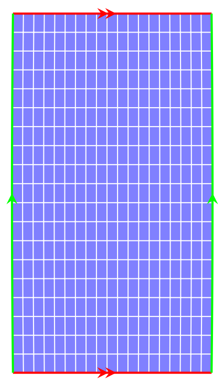
Math 171	Professor Sebastian Bozlee
Scribed by: Kyle Dituro	April 4, 2023

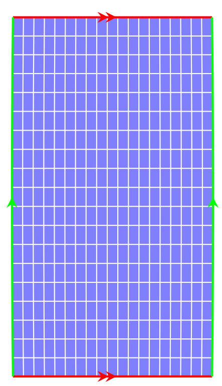
We begin thinking about the quotient topology, which will give us the power to "glue" topological spaces.

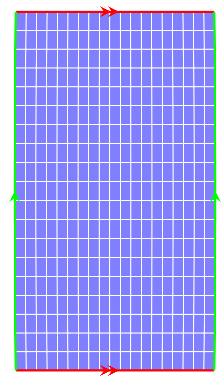
The most common example is the famous transformation of the unit square into a torus (doughnut) by associating opposite sides of squares as equal. Namely, we identify

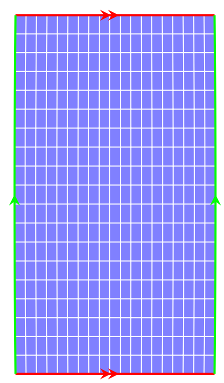
$$(0, y) \sim (1, y)$$
$$(x, 0) \sim (x, 1)$$

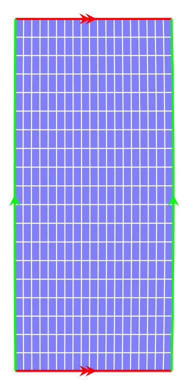
in the space $X = [0, 1] \times [0, 1]$.

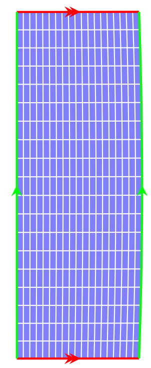


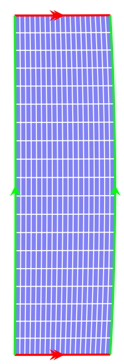


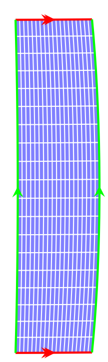


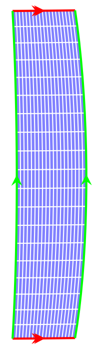


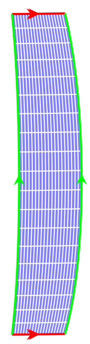


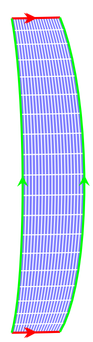


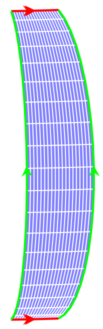


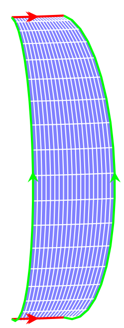


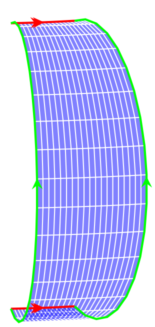


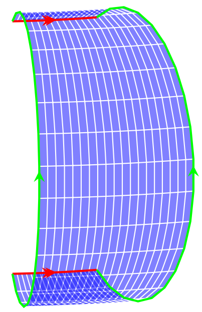


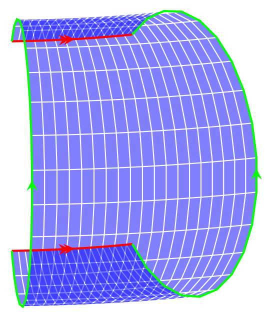


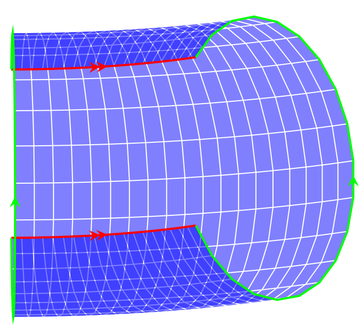


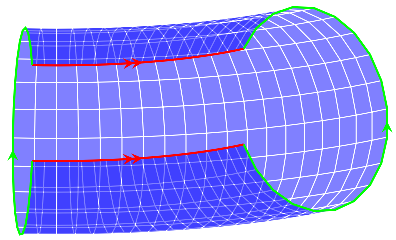


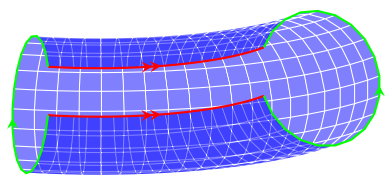


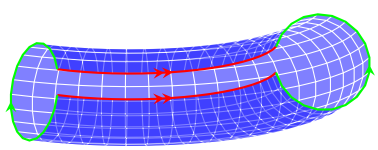


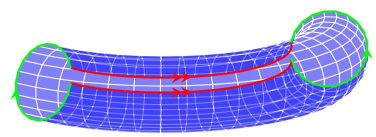


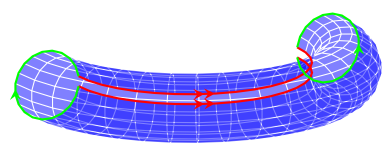


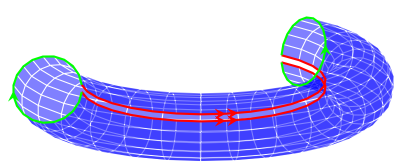


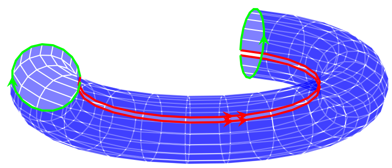


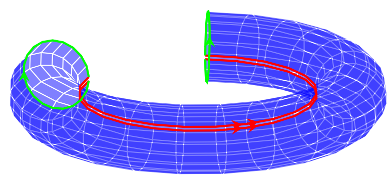


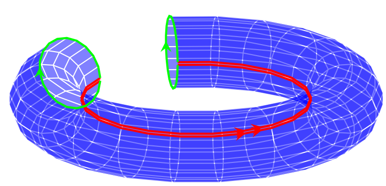


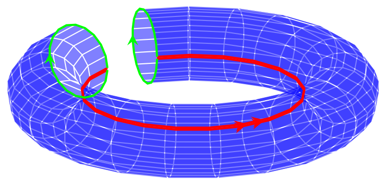


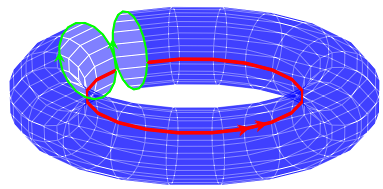


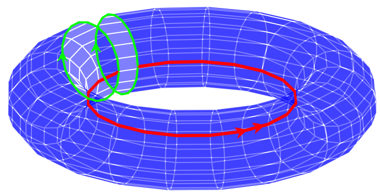


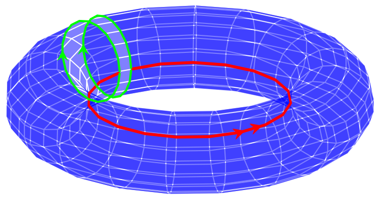


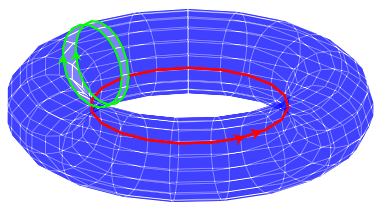


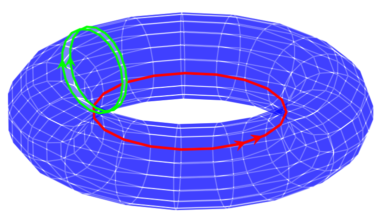


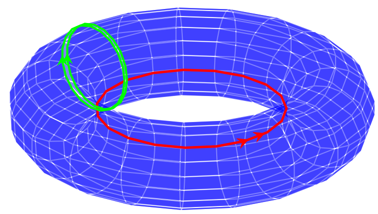


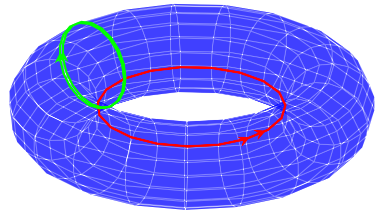












So really what we are doing is identifying equivalence classes, which immediately brings to mind quotients and their universal property:

Theorem 1. $\begin{aligned} p : X &\to X/\sim \\ x &\mapsto [x] \end{aligned}$ *a continuous function has the following property:*

1. *if $f : X/\sim \to Z$ is any continuous function, $f \circ p : X \to Z$ respects the equivalence relation.*

2. *If $g : X \varnothing Z$ is any continuous function that respects $\sim$, then there exists a unique continuous function $f : X/\sim \to Z$ such that*

$$\begin{tikzcd} X \arrow[d, "p"] \arrow[dr, "g"] & \\ X/\sim \arrow[r, dotted, "\exists! f"] & Z \end{tikzcd}$$

What should go in this topology?
Recall $S = \{1, 2\}$ with the topology $\tau_S = \{\emptyset, \{1\}, S\}$.

Then a continuous function $f : X \to S$ is "the same" as an open subset of X. In other words, given f, $f^{-1}(\{1\})$ is an open set.

Assume that $p : X \to X/\sim$ has the universal property. Then from part 2, for any function $g : X \to S$ respecting $\sim$, we should get a unique continuous $f : X/\sim \to S$ making the diagram commute. Since g respects $\sim$, if $x \in g^{-1}(\{1\})$, we better have for any $x_2 \sim x_1$ that $x_2 \in g^{-1}(\{1\})$. i.e. the possible open subsets we get from $g^{-1}(\{1\})$ are the open subsets with the property that $x_1 \in U,\ x_1 \sim x_2 \Rightarrow x_2 \in U$.

So using the universal property, we should have an $f : X/\sim \to S$ that makes
$$\begin{tikzcd} X \arrow[d] \arrow[dr] & \\ X/\sim \arrow[r] & S \end{tikzcd}$$
commute.

This function is defined by $f([x]) = g(x)$.

Claim. *The open subsets we get from g are the preimages of the open subsets we get from f. It follows that a subset $U \subseteq X/\sim$ should be open if and only if $p^{-1}(U)$ is open*

Definition 1. A continuous function $p : X \to Y$ of topological spaces is a **quotient map** if $U \subseteq Y$ is open if and only iff $p^{-1}(U) \subseteq X$ is open.

This can be stated equivalently using complements as:

$$Z \subseteq Y \text{ is closed} \iff p^{-1}(Z) \subseteq X \text{ is closed.}$$

Fact. *If $p : X \to Y$ is surjective and $B \subseteq Y$ then*

$$p(p^{-1}(B)) = B$$

Notice that this is not necessarily true, and in fact is often not, if p is not surjective.

Definition 2. A subset C of X is said to be **saturated** with respect to a function $p : X \to Y$ if there exists a subset B of Y such that $C = p^{-1}(B)$.

Lemma 1. *$C \subseteq X$ is saturated with respect to $p : X \to Y$ if and only if C contains every fiber $p^{-1}(\{y\})$ that is intersects.*

Proof. ($\Rightarrow$) Suppose that C is saturated, and let $B \subseteq Y$ be the subset such that $C = p^{-1}(B)$.

Then if $y \in y$, and $C \cap p^{-1}(\{y\})$, then $\exists c \in C$ such that $p(c) = y$.

Then $y \in B$ so $p^{-1}(\{y\}) \subseteq p^{-1}(B) = C$ by assumption $\triangle$

($\Leftarrow$) Now suppose that C contains every fiber that it intersects. Consider $B = p(C)$. Then since $B = \bigcup_{b \in B} \{b\}$, $p^{-1}B = \bigcup_{b \in B} p^{-1}(\{b\})$.

Since $p^{-1}(\{b\}) \cap C \neq \emptyset$ for all $b \in B$, $p^{-1}(\{b\}) \subseteq C$.

Then this implies that $p^{-1}(B) \subseteq C$.

On the other hand, $C \subseteq p^{-1}(p(C))$, so $C = p^{-1}(B)$. $\square$

Now with all of this done, we can finally re-write what a quotient map is.

$$p : X \to Y \text{is a quorient map} \Longleftrightarrow \begin{pmatrix} p \text{ is continuous, surjective, and} \\ \text{the image of each saturated} \\ \text{open set of } X \text{ open} \end{pmatrix}$$

$$(\forall U \subseteq Y, U \text{ open}) \Leftrightarrow p(v) \text{ is open in } y \Leftrightarrow V \text{ is a saturated subset of } X \text{ open}$$

Example 1. $\pi_1 : X \times Y \to X$ is a quotient map.

Definition 3. If X is a topological space and A is a set, and $p : X \to A$ is a surjective function, there is a unique topology on A such that p becomes a quotient map. This is the **quotient topology** on X.

$$U \subseteq A \text{ open} \Leftrightarrow p^{-1}(U) \subseteq X \text{ open}$$

Let's now prove that this is a topology. This should be easy.

Proof.

1. $\emptyset \subseteq A$ is open since $p^{-1}(\emptyset) = \emptyset$ is open inp X
2. $A \subseteq A$ is open since $p^{-1}(A) = X$ is open in X.
3. If $\{U_i\}$ are open, then $p\left(\bigcup U_i\right) = \underbrace{\bigcup_{i \in I} \underbrace{p^{-1}(U_i)}_{\text{open in } X}}_{\text{open in } X}$, so $\bigcup_{i \in I} U_i$ is open.

□