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Math 171 **Professor Sebastian Bozlee**
Scribed by: Kyle Dituro **February 17, 2023**

We begin with a reminder of what will be one of the most fundimental definitions for this course.

Definition 1. Let X be a set. A **topology on** X is a set τ of subsets of X such that:

1. $\emptyset, X \in \tau$
2. If $U_i \in \tau$ of each $i \in I$ then $\bigcup_{i\in I} U_i \in \tau$
3. If $U_1, \ldots, U_n \in \tau$, then $\bigcap_{i=1}^{n} U_i \in \tau$.

A **topological space** is a pair (X, τ) where X is a set and τ is a topology on X. Then the elements of τ are called the **open subsets** of X.

And we remember the definition of continuity:

Definition 2. Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces. A **continuous function** $f : (X, \tau_X) \to (Y, \tau_Y)$ is a function $f : X \to Y$ such that whenever $U \in \tau_Y$, we have that $f^{-1}(U) \in \tau_X$.

And we'll proceed with a number of examples to guide our intuition of open-ness in a topological sense and continuity.

Example 1. Let $X = \mathbb{R}$ be given. We know that the usual notion of open sets defines a topology on X:

$$\tau_1 = \{\text{open subsets of } \mathbb{R}\}$$

However, this is not the only topology that we can endow the real numbers with. In fact, I claim that:

$$\tau_2 = \tau_1 \cup \{\{0\} \cup U \ |U \in \tau_1\}$$

Let's check:

- $\emptyset, \mathbb{R} \in \tau_2$
- $\{U_i\}_{i\in I}$with $U_i \in \tau_1$, $\{\{0\} \cup U_j\}_{j\in J}$ where $U)j \in \tau_1$. The union of these is either $\bigcup_{i\in I} U_i$ if J is empty or $\{0\} \cup \left(\bigcup_{i\in I\cup J} U_i\right)$
- $U_1, \ldots, U_n, \{0\} \cup U_{n+1}, \ldots, \{0\} \cup U_{n+m}$, the intersection of these tings is either $\{0\} \cup (\bigcap U_i)$if $n = 0$ or $\bigcap U_i$ if $n \neq 0$.

Example 2. Consider the function $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$ Under the topology τ_2, this function is actually continuous.

Fact 1. *The continuous funcions* $g : (\mathbb{R}, \tau_2) \to (\mathbb{R}, \tau_1$ *are the funtions continuous at all* $x \in \mathbb{R} - \{0\}$ *in the analysis sense of continuity. The continuous functions* $f : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_2)$ *are the continuous function with* $g(\mathbb{R}) \subseteq (-\infty, 0), (0, \infty),$ *or* $\{x\}$.

Idea 1. *Let* $f : X \to Y$*, the more open subsets* Y *has, the harder it is for* f *to be continuous, and likewise the more open susbsets* X *has, the easier it s.*