

Definition 1. let X be a set. We say that a collection \mathcal{B} of subsets of X is a **basis** for a topology on X if:

1. For each point $x \in X$, there is a **basis element** (or **basic open subset**) $B \in \mathcal{B}$ such that $x \in B$
2. For each pair of basis elements $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there exists a basis element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

The **topology** τ **generated by the basis** \mathcal{B} is the topology on X where:

1. a subset $U \subseteq X$ is said to be open if for each $x \in U$ there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.

Example 1. In the usual topology on \mathbb{R}^n , the open balls $B(\mathbf{x}, \varepsilon)$ form a basis for the usual topology. I.E. the set

$$\mathcal{B} = \{B(\mathbf{x}, \varepsilon) | \mathbf{x} \in \mathbb{R}^n, \varepsilon > 0\}$$

Is a basis for the usual topology on \mathbb{R}^n .

Proof. Let's check the axioms:

1. For any $\mathbf{x} \in \mathbb{R}^n$, we can use $B(\mathbf{x}, \varepsilon)$ as a basis element containing \mathbf{x} .
2. Now for any $B_1 = B(\mathbf{x}_1, \varepsilon_1), B_2 = B(\mathbf{x}_2, \varepsilon_2)$, and $\mathbf{x} \in B_1 \cap B_2$. then there exists $B_3 = B(\mathbf{x}, \min\{|\varepsilon_1 - \mathbf{x}_1|, |\varepsilon_2 - \mathbf{x}_2|\})$

□

Recall that this is almost exactly how we usually define the usual topology on \mathbb{R}^n , barring minor differences.

This next proposition will hi-light and explain a commonly used hand-wave. Namely, handling finite unions by claiming that it 'follows by induction'.

Proposition 1. *Let X be any set, and let \mathcal{B} be a basis for a topology on X . Then we claim that the "topology" τ generated by \mathcal{B} truly is a topology.*

Proof. 1. \emptyset , vacuously open, since there are no points in the empty set. X is open, since for every point \mathbf{x} , the first basis axiom guarantees that there exists a $B \in \mathcal{B}$ such that $\mathbf{x} \in B$.

2. Suppose $U_i \in \tau$ for all $i \in I$. So let some point $x \in \bigcup_{i \in I} U_i$. Then there is some $j \in I$ such that $x \in U_j$. Since U_j is open, there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U_j$. So then $x \in B \subseteq \bigcup_{i \in I} U_i$, so $\bigcup_{i \in I} U_i$ is open

3. Let's induct for the final part.

Suppose $U, V \in \tau$, and consider $U \cap V$. Now Let $x \in U \cap V$. Since U is open, there is a basis element B_1 such that $x \in B_1$, and $B_1 \subseteq U$. Then also since V is open, $\exists B_2$ such that $x \in B_2$ and $B_2 \subseteq V$. Thus by basis axiom two, $\exists B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$. So $U \cap V$ is open. This completes the next part of our induction.

Suppose that there exists a $k \in \mathbb{Z}_{>0}$ such that if $U_1, \dots, U_k \in \tau$, then $\bigcup_{i=1}^k U_i \in \tau$ for the sake of induction. now suppose that we have some collection U_1, \dots, U_{n+1} . Then the intersection of the first n sets will be open, and so we return to just taking an intersection of two open sets. Therefore, we have shown by induction that we can take finite intersections.

□

Example 2. If $X = \mathbb{R}^2$ take

$$\mathcal{B} = \left\{ (a, b) \times (c, d) \mid \begin{array}{l} a, b, c, d \in \mathbb{R} \\ a < b, c < d \end{array} \right\}$$

This is also a basis for a topology. How does it compare to the usual topology, I wonder...

Example 3. If X is any set

$$\mathcal{B} = \{\{x\} \mid x \in X\}$$

is also a basis for a topology. This is (pretty clearly) a basis for the discrete topology.

Lemma 1. Let X be a set, \mathcal{B} be a basis for a topology on X , τ the topology generated by \mathcal{B} . Then τ is the set of all possible unions of the basis elements:

$$\tau = \left\{ \bigcup_{i \in I} B_i \mid \begin{array}{l} I \text{ is any indexing set} \\ B_i \in \mathcal{B} \text{ for all } i \in I \end{array} \right\}.$$

Proof. Suppose $U = \bigcup_{i \in I} B_i$, where $B_i \in \mathcal{B}$. Then if $x \in U$ there exists $j \in I$ such that $x \in B_j$. Then $x \in B_j \subseteq U$, so U is open in the generated topology. Conversely, suppose $U \subseteq X$ is open in the generated topology.

Then since U is open for each $x \in U$ there is a basis element $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Then

$$\begin{aligned} U &= \bigcup_{x \in U} \{x\} \\ &\subseteq \bigcup_{x \in U} B_x \\ &\subseteq \bigcup_{x \in U} U = U. \end{aligned}$$

So $U = \bigcup_{x \in U} B_x$

□

Proposition 2. *Let X be a set. Let $\mathcal{B}', \mathcal{B}$ be bases for topologies on X . Let τ', τ be the respective generated topologies. Then the following are equivalent:*

1. τ' is finer than τ
2. For each $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Proof.

- (1 \Rightarrow 2) Assume that τ' is finer than τ , and let $B \in \mathcal{B}$ and $x \in B$. But then $B \in \tau$, so $B \in \tau'$. By definition of τ' , since B is open, and $x \in B$, $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$.
- (2 \Rightarrow 1) Assume $\forall B \in \mathcal{B}, x \in B, \exists B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$.

Now let $U \in \tau$ be given. Then by definition, for any $x \in U$, there exists a basis element $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then we apply our hypothesis. for such an x , we have by assumption a $B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$.

Then, for all x , $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq U$. So U is open in τ' .

□