

Herein we will outline some review content for the exam.

Definition 1. Let $\{X_i\}_{i \in I}$ be a collection of sets indexed by I . A set P together with functions $\pi_i : P \rightarrow X_i$ for each $i \in I$ is said to have the **universal property of the product** if, for any set Z and functions $f_i : Z \rightarrow X_i$ for each $i \in I$. Then there exists a unique function $f : Z \rightarrow P$ such that

$$\begin{array}{ccc} Z & & \\ \downarrow f & \searrow f_i & \\ P & \xrightarrow{\pi_i} & X_i \end{array}$$

Example 1. A standard sort of product: Take $X_1 = \mathbb{R}, X_2 = \mathbb{R}, P = \mathbb{R}^2$, and $\pi_1 : P \rightarrow X_1, (x, y) \mapsto x$, and $\pi_2 : P \rightarrow X_2$ taking $(x, y) \mapsto y$. then

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \vdots \exists! f & \searrow f_2 & \\ & f_1 & & & \\ X_1 & \xleftarrow{\pi_1} & P & \xrightarrow{\pi_2} & X_2 \end{array}$$

And in fact we can see that such an f does exist, as $z \mapsto (f_1(z), f_2(z))$.

Example 2. Here's a bit of a weirder one. $X_1 = \mathbb{R}, X_2 = \mathbb{R}, P' = \mathbb{R}^2$, and the functions $\pi'_1 : P' \rightarrow X_1$ a projection on the first coordinate, and $\pi'_2 : P' \rightarrow X_2$ takes $(x, y) \mapsto x + y$.

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \vdots \exists! f & \searrow f_2 & \\ & f_1 & & & \\ X_1 & \xleftarrow{\pi'_1} & P' & \xrightarrow{\pi'_2} & X_2 \end{array}$$

Notice that to get the f that we desire, we require that $f(z) = (f_1(z), f_2(z) - f_1(z))$

Theorem 1. If (P, π_i) has the universal property of the product and (P', π'_i) has universal property of the product, then there exists a unique bijection $f : P \rightarrow P'$ such that $\pi_i = \pi'_i \circ f$ for all $i \in I$

$$\begin{array}{ccc} P & & \\ \downarrow f & \searrow \pi_i & \\ P' & \xrightarrow{\pi'_i} & X_i \end{array}$$

This theorem was proved in class, so we omit the proof here.

Definition 2. Let X be a set, \sim be an equivalence relation on X . A function $f : X \rightarrow Y$ is said to **respect the equivalence relation** if $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$ for $x_1, x_2 \in X$. A set \overline{X} together with a function $\pi : X \rightarrow \overline{X}$ is said to have the universal property of quotients if:

1. $\pi : X \rightarrow \overline{X}$ respects the equivalence relation.
2. For any function $f : X \rightarrow Y$ such that f respects the equivalence relation, $\exists! \bar{f} : \overline{X} \rightarrow Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \pi & \nearrow \bar{f} & \\ \overline{X} & & \end{array}$$

Example 3. in general, $\pi : X \rightarrow X/\sim$ taking $x \mapsto [x]$ has the universal property.

Example 4. $X = \mathbb{Z}$, $a \sim b \Leftrightarrow a - b \equiv 0 \pmod{6}$. Then X/\sim is called the integers modulo 6, written $\mathbb{Z}/6$ (or more commonly $\mathbb{Z}/6\mathbb{Z}$).

Let's define "multiplying by 2".

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z}/6 \\ \downarrow \pi & \nearrow & \\ \mathbb{Z}/6 & & \end{array}$$

well notice that what we really need here is

$$\begin{array}{ccc} a & \mapsto & [2a] \\ \downarrow & \nearrow & \\ [a] & & \end{array}$$

So the thing that we really want is $\mathbb{Z} \rightarrow \mathbb{Z}/6$ that takes $a \mapsto [2a]$. Does this respect the equivalence relation?

If $a \sim b$, is it true that $[2a] = [2b]$?

$$\begin{aligned} a \sim b &\Rightarrow a - b \equiv 0 \pmod{6} \\ &\Rightarrow a - b = 6n \quad (n \in \mathbb{Z}) \\ &\Rightarrow 2a - 2b = 6(2n) \\ &\Rightarrow 2a \sim 2b \\ &\Rightarrow [2a] = [2b]. \end{aligned}$$

So we have what we want.

Definition 3. Let $Z \subseteq \mathbb{R}^n$. A point $\mathbf{x} \in \mathbb{R}^n$ is a **limit point of Z** if for all $\varepsilon > 0$,

$$(B(\mathbf{x}, \varepsilon) - \{\mathbf{x}\}) \cap Z \neq \emptyset.$$

A point $\mathbf{x} \in \mathbb{R}^n$ is said to be an **adherent point of Z** if for all $\varepsilon > 0$,

$$B(\mathbf{x}, \varepsilon) \cap Z \neq \emptyset.$$

Equivalently, \mathbf{x} is adherent if \mathbf{x} is a limit point or $\mathbf{x} \in Z$.

Example 5. Let

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$x \mapsto \begin{cases} 0 & x < 0 \\ 20 & x \geq 0 \end{cases}.$$

Obviously, this should not be continuous. Recall that the definition of continuity. Namely that $\forall \varepsilon > 0, \forall x \exists \delta > 0 \forall y |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Furthermore, non-continuity entails that $\exists x \forall \delta > 0 \exists y |x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

So let $x = 0, \varepsilon = 10$. now let $\delta > 0$. Let $y = \frac{\delta}{2}$. Then $|x - y| = \frac{\delta}{2} < \delta$, but $|f(x) - f(y)| = |20 - 0| = 20 > 10 = \varepsilon$

Now let's take $g : \mathbb{R} \rightarrow \mathbb{R}$
 $x \mapsto 3x$. So we can get $\forall \varepsilon > 0 \forall x > 0, \exists \delta > 0, \forall y |x - y| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon$. Now let $\varepsilon > 0$, and $x \in \mathbb{R}$. Take $\delta = \frac{\varepsilon}{3}$. Then for all $y \in \mathbb{R}$ with $|x - y| < \frac{\varepsilon}{3}$,

$$\begin{aligned} |g(x) - g(y)| &= |3x - 3y| \\ &= 3|x - y| \\ &< 3\left(\frac{\varepsilon}{3}\right) \\ &= \varepsilon. \end{aligned}$$

So g is continuous.