

During the example last week, we found that it can be useful to consider the preimage of a function when looking at continuous functions. This leads us into the following major theorem.

**Theorem 1.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . Then  $f$  is continuous iff for each open subset  $U \subseteq \mathbb{R}^m$  the subset  $f^{-1}(U) \subseteq \mathbb{R}^n$  is open.*

*Proof.* ( $\Rightarrow$ ) Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuous. Let  $U \subseteq \mathbb{R}^m$  be an open set. Let  $\mathbf{x} \in f^{-1}(U)$ . Since  $U$  is open,  $f(\mathbf{x}) \in U$ , there  $\exists \varepsilon > 0$  such that  $B(f(\mathbf{x}), \varepsilon) \subseteq U$ .

Since  $f$  is continuous,  $\exists \delta > 0$  such that  $\forall \mathbf{y} \in \mathbb{R}^n$ , with  $|\mathbf{x} - \mathbf{y}| < \delta$ ,  $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$ . In other words, for all  $\mathbf{y} \in B(\mathbf{x}, \delta) \subseteq f^{-1}(U)$  therefore  $f^{-1}(U)$  is open.

( $\Leftarrow$ ) Assume that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  has the property that  $U \subseteq \mathbb{R}^m$  is open  $\Rightarrow f^{-1}(U) \subseteq \mathbb{R}^n$ . Now let  $\mathbf{x} \in \mathbb{R}^n$ ,  $\varepsilon > 0$ .

Now  $U = B(f(\mathbf{x}), \varepsilon) \subseteq \mathbb{R}^m$  is an open subset, so  $f^{-1}(U) \subseteq \mathbb{R}^n$  is open. So since  $f^{-1}(U)$  is open, and since  $\mathbf{x} \in f^{-1}(U)$ , so by the definition of openness, there exists a real number  $\delta > 0$  such that  $B(\mathbf{x}, \delta) \subseteq f^{-1}(U)$ . In other words, for all  $\mathbf{y} \in \mathbb{R}^n$  with  $|\mathbf{x} - \mathbf{y}| < \delta$ ,  $f(\mathbf{y}) \in U$ .

Therefore  $\forall \mathbf{y} \in \mathbb{R}^n$  with  $|\mathbf{x} - \mathbf{y}| < \delta$ ,  $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$ . Therefore  $f$  is continuous.  $\square$

We of course also know that  $f^{-1}$  preserves all the standard set-theoretic operations.

1.  $f^{-1}(\bigcup U_i) = \bigcup f^{-1}(U_i)$
2.  $f^{-1}(\bigcap U_i) = \bigcap f^{-1}(U_i)$
3.  $f^{-1}(U^c) = (f^{-1}(U))^c$

**Proposition 1.** 1.  $\emptyset, \mathbb{R}^n \subseteq \mathbb{R}^n$  are open in  $\mathbb{R}^n$

2. If  $U_{i \in I}$  is a collection of open subsets of  $\mathbb{R}^n$ , then their union is also open.

3. If  $U_1, \dots, U_n$  are open subsets of  $\mathbb{R}^n$  then  $\bigcap_{i=1}^n U_i$  is also an open subset of  $\mathbb{R}^n$ .

*Proof.* 1. The empty set is open vacuously.  $\forall \mathbf{x} \in \mathbb{R}^n$ ,  $B(\mathbf{x}, 1) \subseteq \mathbb{R}^n$ .

2. Unions come somewhat naturally. Namely let  $\mathbf{x} \in \bigcup_{i \in I} U_i$ , then there is an index  $j \in I$  such that  $\mathbf{x} \in U_j$ . Then since  $U_j$  is open, there is some  $\varepsilon > 0$  such that  $B(\mathbf{x}, \varepsilon) \subseteq U_j \subseteq \bigcup_{i \in I} U_i$ . Therefore  $\bigcup_{i \in I} U_i$  is open.

3. Intersections are a little bit more of a trick. Namely suppose that  $x \in \cap_{i=1}^n U_i$ . Then  $x \in U_i$  for each  $i$ , so there exists a positive real number  $\varepsilon_i > 0$  for each  $i = 1, \dots, n$  such that  $B(\mathbf{x}, \varepsilon_i) \subseteq U_i$  for each  $i = 1, \dots, n$ . Let  $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$ . Then  $B(\mathbf{x}, \varepsilon) \subseteq B(\mathbf{x}, \varepsilon_i) \subseteq U_i$  for all  $i$ , so  $B(\mathbf{x}, \varepsilon) \subseteq \cap_{i=1}^n U_i$ .

□

It aught to be suspicious that we can only do finite intersections... why is that?

**Example 1.** Set  $U_n = B(\mathbf{x}, \frac{1}{n})$  for each  $n \in \mathbb{Z}_{>0}$ . Then  $\cap_{n=1}^{\infty} U_n = \{\mathbf{0}\}$ , which is not open.

**Fig. 1.** We really can only do finite intersections.

**Proposition 2.** Let  $Z \subseteq \mathbb{R}^n$  then  $Z$  is a closed subset of  $\mathbb{R}^n$  iff  $Z^c$  is an open subset of  $\mathbb{R}^n$ .

*Proof.* ( $\Rightarrow$ ) . Suppose that  $Z$  is closed. Then let  $\mathbf{x} \in Z^c$ . Then  $\mathbf{x}$  is not a limit point of  $Z$  so  $\exists \varepsilon > 0$  such that  $B(\mathbf{x}, \varepsilon) \cap Z = \emptyset$ . Then  $B(\mathbf{x}, \varepsilon) \subseteq Z^c$  so  $Z^c$  is open.

( $\Leftarrow$ ) Suppose that  $Z^c$  is open. Then let  $\mathbf{x} \in \mathbb{R}^n$  be a limit point of  $Z$ . Then for all  $\varepsilon > 0$ ,  $B(\mathbf{x}, \varepsilon) \cap Z \neq \emptyset$ . So  $\forall \varepsilon > 0$ ,  $B(\mathbf{x}, \varepsilon) \not\subseteq Z^c$ . This implies that  $\mathbf{x} \notin Z^c$  since  $Z^c$  is open. Thus  $\mathbf{x} \in Z$ , so  $Z$  is closed.

□

**Definition 1.** Let  $X$  be a set. A **topology on  $X$**  is a set  $\tau$  of subsets of  $X$  such that:

1.  $\emptyset, X \in \tau$
2. If  $U_i \in \tau$  of each  $i \in I$  then  $\cup_{i \in I} U_i \in \tau$
3. If  $U_1, \dots, U_n \in \tau$ , then  $\cap_{i=1}^n U_i \in \tau$ .

A **topological space** is a pair  $(X, \tau)$  where  $X$  is a set and  $\tau$  is a topology on  $X$ . Then the elements of  $\tau$  are called the **open subsets** of  $X$ .

**Example 2.**  $\mathbb{R}^n$  with the usual definition of open sets is a topological space.

**Example 3.** We can define a topological space with only finitely many points. For example

$$\begin{aligned}
 X &= \{1, 2\} \\
 \tau &= \{\emptyset, \{1, 2\}\} && \text{(indiscrete topology)} \\
 \tau_1 &= \{\emptyset, \{1\}, \{1, 2\}\} && \text{(Sierpinski space)} \\
 \tau_2 &= \mathcal{P}(X) && \text{(The discrete topology)}
 \end{aligned}$$

**Definition 2.** If  $(X, \tau_X), (Y, \tau_Y)$  are topological spaces, then a function  $f : X \rightarrow Y$  is said to be **continuous** if  $f^{-1}(U)$  is open in  $X$  whenever  $U$  is open in  $Y$ .

**Example 4.** What are the continuous functions from  $S$  (the Sierpinski space) to  $S$ .  
 There are four functions:

$x$	1	2	cont?
$f_1(x)$	1	2	Yes
$f_2(x)$	1	1	Yes
$f_3(x)$	2	1	$f_3^{-1}(1) = 2$
$f_4(x)$	2	2	Yes