

Corollary 1 (“Images are almost quotient spaces”). *Let $g : X \rightarrow Z$ be a surjective continuous map. Let \sim be the equivalence relation on X given by*

$$x_1 \sim x_2 \iff g(x_1) = g(x_2).$$

*Then g induces a continuous **bijective** map $f : X/\sim \rightarrow Z$ given by $f([x]) = g(x)$, which is a homeomorphism if and only if g is a quotient map.*

$$\begin{array}{ccc} X & & \\ \pi \downarrow & \searrow g & \\ X/\sim & \xrightarrow{f} & Z \end{array}$$

Proof. It is clear that \sim is an equivalence relation.

We know f is well defined and continuous from the universal property of the quotient. It is... uhhh... “clear” that f is bijective:

- **(injective)**: Pretty clear
- **(surjective)**: Let $z \in Z$. Since g is surjective, $\exists x \in X$ such that $g(x) = z$. Then $f([x]) = z$.

Suppose f is a homeomorphism. Then in particular f is a quotient map:

$$U \subseteq Z \text{ open} \iff f^{-1}(U) \subseteq X/\sim \text{ open}$$

Then $g = f \circ \pi$ is a composite of quotient maps, and is therefore a quotient map. Conversely, if g is a quotient map, then it also has the universal property of the quotient: By the universal property of g , $\exists! h : Z \rightarrow X/\sim$ such that

$$\begin{array}{ccc} X & & \\ \pi \downarrow & \searrow g & \\ X/\sim & \xleftarrow{h} & Z \end{array}$$

Taking composites:

$$\begin{array}{ccc} & X & \\ \pi \swarrow & & \searrow \pi \\ X/\sim & \xrightarrow{h \circ f} & X/\sim \end{array}$$

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So by the uniqueness in the universal property for π , we get that $h \circ f = \text{id}_{X/\sim}$.

Symmetrically, $f \circ h = \text{id}_Z$, so f is a homeomorphism, as desired. □

Example 1. Recall $\mathbb{RP}^2 = \mathbb{R}^3 - \{0\} / \sim$.

This has a particularly nice open subset $D(z)$:

$$D(z) = \{[x : y : z] \in \mathbb{RP}^2 \mid z \neq 0\}.$$

This is homeomorphic to \mathbb{R}^2 via

$$\begin{aligned} \mathbb{R}^2 &\rightarrow D(z) \\ (x, y) &\mapsto [x : y : 1] \\ \left(\frac{x}{z}, \frac{y}{z}\right) &\mapsto [x : y : z] \end{aligned}$$

Similarly, there are open subsets

$$\begin{aligned} D(x) &= \{[x : y : z] \mid x \neq 0\} \cong \mathbb{R}^2 \\ D(y) &= \{[x : y : z] \mid y \neq 0\} \cong \mathbb{R}^2 \end{aligned}$$

Notice that these open sets form an open cover of \mathbb{RP}^2 : $D(x) \cup D(y) \cup D(z) = \mathbb{RP}^2$.

This is nice.

Definition 1. A topological space X is said to be a **Topological Manifold** if it has an open cover $\{U_i\}_{i \in I}$ such that:

1. each U_i is homeomorphic to an open subset of \mathbb{R}^n for some n (n the dimension of X)
2. X is Hausdorff
3. ~~something technical that we don't care about~~

Example 2. $\mathbb{R}^n, \mathbb{RP}^n, \mathbb{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid |\mathbf{x}| = 1\}$, donuts.

Example 3.

$$\begin{aligned}
\mathbb{RP}^1 &= \{(x, y) \mid (x, y) \neq 0\} / \sim \text{ scaling} \\
D(x) &= \{[x, y] \mid x \neq 0\} \xrightarrow{\sim} \mathbb{R}^1 \\
[x : y] &\longmapsto \frac{y}{x} \\
D(y) &= \{[x : y] \mid y \neq 0\} \xrightarrow{\sim} \mathbb{R}^1 \\
[x : y] &\longmapsto \frac{x}{y}.
\end{aligned}$$

we now give the gluing construction:

Input: Two top spaces U_1, U_2

An open subset U_{12} of U_1

An open subset U_{21} of U_2

A homeomorphism $\varphi_{12} : U_{12} \rightarrow U_{21}$

Output: $X = U_1 \sqcup U_2 / \sim = \{1\} \times U_1 \cup \{2\} \times U_2 / \sim$ Where $(1, u) \sim (2, \varphi_{12}(u))$ when $u \in U_{12}$.

Theorem 1. X has an open cover V_1, V_2 such that there exists homeomorphisms

1. $\varphi_1 : U_1 \rightarrow V_1$
 $\varphi_2 : U_2 \rightarrow V_2$
2. $\varphi_1(U_{12}) = \varphi_2(U_{21})$
3. $\varphi_2^{-1} \circ \varphi_1 : U_{12} \rightarrow U_{21}$ is equal to φ_{12}

Example 4.

$$\begin{aligned}
U_1 &= \mathbb{R} & U_{12} &= \mathbb{R} - \{0\} \\
U_2 &= \mathbb{R} & U_{21} &= \mathbb{R} - \{0\}
\end{aligned}$$

$$\begin{aligned}
\varphi : U_{12} &\rightarrow U_{21} \\
x &\mapsto \frac{1}{x}
\end{aligned}$$

This glues to \mathbb{RP}^1

Example 5. Now take the same U 's but instead say $x \xrightarrow{\varphi} x$. Then you get a cylinder