Some review:

Definition 1. Let $\{X_i\}_{i \in I}$ be a collection of sets indexed by I. A set P together with functions $\pi_i : P \to X_i$ for each $i \in I$ is said to have the **universal property of the product** if, for any set Z and functions $f_i : Z \to X_i$ for each $i \in I$. Then there exists a unique function $f : Z \to P$ such that

$$\begin{tikzcd} Z \arrow[d, "f"] \arrow[dr, "f_i"] & \\ P \arrow[r, "\pi_i"] & X_i \end{tikzcd}$$

Example 1. A standard sort of product: Take $X_1 = \mathbb{R}, X_2 = \mathbb{R}, P = \mathbb{R}^2$, and $\pi_1 : P \to X_1$, $(x, y) \mapsto x$, and $\pi_2 : P \to X_2$ taking $(x, y) \mapsto y$. then

$$\begin{tikzcd} & Z \arrow[dl, "f_1"'] \arrow[d, dotted, "\exists! f"] \arrow[dr, "f_2"] & \\ X_1 & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"] & X_2 \end{tikzcd}$$

And in fact we can see that such an f does exist, as $z \mapsto (f_1(z), f_2(z))$.

Example 2. Here's a bit of a weirder one. $X_1 = \mathbb{R}$, $X_2 = \mathbb{R}$, $P' = \mathbb{R}^2$, and the functions $\pi_1' : P' \to X$ a projection on the first coordinate, and $\pi_2' : P' \to X_2$ takes $(x, y) \mapsto x + y$.

$$\begin{tikzcd} & Z \arrow[dl, "f_1"'] \arrow[d, dotted, "\exists! f"] \arrow[dr, "f_2"] & \\ X_1 & P' \arrow[l, "\pi_1'"] \arrow[r, "\pi_2'"] & X_2 \end{tikzcd}$$

Notice that to get the f that we desire, we require that $f(z) = (f_1(z), f_2(z) - f_1(z))$

Theorem 1. *If (P, π_i) has the universal property of the product and (P', π_i') has universal property of the product, then there exists a unique bijuection $f : P \to P'$ such that $\pi_i = \pi_i'$ for all $i \in I$*

$$\begin{tikzcd} P \arrow[d, "f"] \arrow[dr, "\pi_i"] & \\ P' \arrow[r, "\pi_i'"] & X_i \end{tikzcd}$$

This theorem was proved in calss, so we omit the proof here.

Definition 2. Let X be a set, $\sim$ be an equivalence relation on X. A function $f : X \to Y$ is said to **respect the equvalence relation** if $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$ for $x_1, x_2 \in X$. A set $\overline{X}$together with a function $\pi : X \to \overline{X}$ is said to have the universal property of quotients if:

1. $\pi : X \to \overline{X}$ resepects the equivalence relation.
2. For any function $f : X \to Y$ such that f respects the equivalence relation, $\exists! \overline{f} : \overline{X} \to Y$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & Y \\ \overline{X} \arrow[ur, "\overline{f}"] & \end{tikzcd}$$

Example 3. in general, $\pi : X \to X/\sim$ taking $x \mapsto [x]$ has the universal property.

Example 4. $X = \mathbb{Z}$, $a \sim b \Leftrightarrow a - b \equiv 0 \pmod 6$. Then $X/\sim$ is called the integers modulo 6, written $\mathbb{Z}/6$ (or more commonly $\mathbb{Z}/6\mathbb{Z}$).

Let's define "multiplying by 2".

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "\pi"] & \mathbb{Z}/6 \\ \mathbb{Z}/6 \arrow[ur, dotted] & \end{tikzcd}$$

well notice that ehat we really need here is

$$\begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto] & {[2a]} \\ {[a]} \arrow[ur, mapsto] & \end{tikzcd}$$

So the thing that we really want is $\mathbb{Z} \to \mathbb{Z}/6$ that takes $a \mapsto [2a]$. Does this respect the equivalance relation?

If $a \sim b$, is it true that $[2a] = [2b]$?

$$\begin{aligned} a \sim b &\Rightarrow a - b \equiv 0 \pmod 6 \\ &\Rightarrow a - b = 6n \ (n \in \mathbb{Z}) \\ &\Rightarrow 2a - 2b = 6(2n) \\ &\Rightarrow 2a \sim 2b \\ &\Rightarrow [2a] = [2b]. \end{aligned}$$

So we have what we want.

Definition 3. Let $Z \subseteq \mathbb{R}^n$. A point $\mathbf{x} \in \mathbb{R}6n$ is a **limit point of** Z if for all $\varepsilon > 0$,

$$(B(\mathbf{x}, \varepsilon) - \{\mathbf{x}\}) \cap Z \neq \emptyset.$$

A point $\mathbf{x} \in \mathbb{R}^n$ is said to be an **adherent point of** Z if for all $\varepsilon > 0$,

$$B(\mathbf{x}, \varepsilon) \cap Z \neq \emptyset.$$

Equivalently, $\mathbf{x}$ is adherent if $\mathbf{x}$ is a limit point or $\mathbf{x} \in Z$.

Example 5. Let

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ x &\mapsto \begin{cases} 0 & x < 0 \\ 20 & x \geq 0 \end{cases} . \end{aligned}$$

Obviously, this should not be continuous. Recall that the definition of continuity. Namely that $\forall \varepsilon > 0,\ \forall x\ \exists \delta > 0\ \forall y |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Furthermore, non-continuity entails that $\exists > 0\ \exists x\ \forall \delta > 0\ \exists y |x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

So let $x = 0, \varepsilon = 0$. now let $\delta > 0$. Let $y = \frac{-\delta}{2}$. Then $|x - y| = \frac{\delta}{2} < \delta$, but $|f(x) - f(y)| = |20 - 0| = 20 > 10 = \varepsilon$

Now let's take $\begin{aligned} g &: \mathbb{R} \to \mathbb{R} \\ x &\mapsto 3x \end{aligned}$. So we can get $\forall \varepsilon > 0\ \forall x > 0,\ \exists \delta > 0,\ \forall y |x - y| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon$. Now let $\varepsilon > 0$, and $x \in \mathbb{R}$. Take $\delta = \frac{\varepsilon}{3}$. Then for all $y \in \mathbb{R}$ with $|x - y| < \frac{\varepsilon}{3}$,

$$\begin{aligned} |g(x) - g(y)| &= |3x - 3y| \\ &= 3|x - y| \\ &< 3\left(\frac{\varepsilon}{3}\right) \\ &= \varepsilon. \end{aligned}$$

So g is continuous.