Math 171 **Professor Sebastian Bozlee**
Scribed by: Kyle Dituro **February 17, 2023**

We begin with a reminder of what will be one of the most fundimental definitions for this course.

Definition 1. Let X be a set. A **topology on** X is a set τ of subsets of X such that:

1. $\emptyset, X \in \tau$
2. If $U_i \in \tau$ of each $i \in I$ then $\bigcup_{i\in I} U_i \in \tau$
3. If $U_1, \ldots, U_n \in \tau$, then $\bigcap_{i=1}^n U_i \in \tau$.

A **topological space** is a pair (X, τ) where X is a set and τ is a topology on X. Then the elements of τ are called the **open subsets** of X.

And we remember the definition of continuity:

Definition 2. Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces. A **continuous function** $f : (X, \tau_X) \to (Y, \tau_Y)$ is a function $f : X \to Y$ such that whenever $U \in \tau_Y$, we have that $f^{-1}(U) \in \tau_X$.

And we'll proceed with a number of examples to guide our intuition of open-ness in a topological sense and continuity.

Example 1. Let $X = \mathbb{R}$ be given. We know that the usual notion of open sets defines a topology on X:

$$\tau_1 = \{\text{open subsets of } \mathbb{R}\}$$

However, this is not the only topology that we can endow the real numbers with. In fact, I claim that:

$$\tau_2 = \tau_1 \cup \{\{0\} \cup U \ | U \in \tau_1\}$$

Let's check:

- $\emptyset, \mathbb{R} \in \tau_2$
- $\{U_i\}_{i\in I}$with $U_i \in \tau_1$, $\{\{0\} \cup U_j\}_{j\in J}$ where $U)j \in \tau_1$. The union of these is either $\bigcup_{i\in I} U_i$ if J is empty or $\{0\} \cup \left(\bigcup_{i\in I\cup J} U_i\right)$
- $U_1, \ldots, U_n, \{0\} \cup U_{n+1}, \ldots, \{0\} \cup U_{n+m}$, the intersection of these tings is either $\{0\} \cup (\bigcap U_i)$if $n = 0$ or $\bigcap U_i$ if $n \neq 0$.

Example 2. Consider the function $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$ Under the topology τ_2, this function is actually continuous.

Fact. *The continuous funcions $g : (\mathbb{R}, \tau_2) \to (\mathbb{R}, \tau_1)$ are the funtions continuous at all $x \in \mathbb{R} - \{0\}$ in the analysis sense of continuity. The continuous functions $f : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_2)$ are the continuous function with $g(\mathbb{R}) \subseteq (-\infty, 0), (0, \infty),$ or $\{0\}$.*

Idea. *Let $f : X \to Y$, the more open subsets Y has, the harder it is for f to be continuous, and likewise the more open susbsets X has, the easier it s.*

Definition 3. Let X be a set, τ_1, τ_2 two topologies on X. We say that τ_1 is coarser than τ_2 and that τ_2 is finer than τ_1 if $\tau_1 \subseteq \tau_2$.

Example 3. Notice that the discrete topology is always the finest topology on a set X, and that the indiscrete topology is the finest.

Proposition 1. *Let $f : X \to Y$ be a function between topological spaces.*

1. *If X has the discrete topology then f is continuous.*
2. *If Y has the indiscrete topology then f is continuous.*

Proof.

1. let $f : X \to Y$ be any function. Let $U \subseteq Y$ be an open subset. This is obvious because every subset of X is open, so of course $f^{-1}(U)$ is open.
2. Let $f : X \to Y$ be any function. Let $U \subseteq Y$ be an open set. Now since Y has the indiscrete topology, we need only check $f^{-1}(Y)$ and $f^{-1}(\emptyset)$ but $f^{-1}(Y) = X$ and $f^{-1}(\emptyset) = \emptyset$.

Thus, our function is continuous in either case. □

Proposition 2. $\mathrm{id}_X : (X, \tau_2)) \to (X, \tau_1)$*is continuous iff τ_2 is finer than τ_1. In particular* $\mathrm{id}_X : (X, \tau) \to (X, \tau)$ *is continuous.*

Proof. $\mathrm{id}_X : (X, \tau_2) \to (X, \tau_1)$ is continuous iff $\forall U \in \tau_1$, $\mathrm{id}_X^{-1}(U) \in \tau_2$ iff $\forall U \in \tau_1$, have $U \in tau_2$ iff $\tau_1 \subseteq \tau_2$ which is the definition of τ_2 is finer than τ_1. □

Lemma 1. *Let $f : X \to Y$, $g : Y \to Z$ be functions, $W \subseteq Z$. Then $(g \circ f)^{-1}(W) = f^{-1}\left(g^{-1}\left(W\right)\right)$.*

Proof.

$$\begin{aligned} f^{-1}(g^{-1}(W)) &= \left\{x \in X \middle| f(x) \in g^{-1}(W)\right\} \\ &= \{x \in X | f(x) \in \{y \in Y | g(y) \in W\}\} \\ &= \{x \in X | g(f(x)) \in W\} \\ &= (g \circ f)^{-1}(W) \end{aligned}$$

□

Proposition 3. *Let $f : X \to Y$, $g : Y \to Z$ be continuous. Then $(g \circ f) : X \to Z$ is as well.*

Proof. Let $W \subseteq Z$ be open. Then

$$(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$$

Notice here that we are taking the pre-image of an open set by continuous functions twice. Thus, $g \circ f$ is continuous. $\square$

And so it was that Kyle never wrote an ε-δ proof again. And we have officielly moved out of the category of Sets, and have moved into the category of Topologies.