

The final will be take home, posted on Gradescope Thursday, May 11th.

There will be a 24 hour window. The exam will be open notes, open book, closed other people, closed internet.

$\frac{1}{3}$  should be pre-midterm content, the remainder being new.

**Definition 1.** A topological space  $X$  is **compact** if each open cover of  $X$  admits a finite subcover.

**Theorem 1.** *A closed subspace of a compact space is compact.*

*Proof.* Let  $X$  be a compact topological space, and let  $Z \subseteq X$  be a closed subspace. Recall that  $Z$  is compact iff every collection of open sets  $\{U_i\}$  in  $X$  with union containing  $Z$  has a finite subcollection with the same property.

Let  $\{U_i\}_{i \in I}$  be a collection of open subsets of  $X$  such that  $\bigcup_{i \in I} U_i \supseteq Z$ . Notice now that since  $Z$  is closed,  $\{U_i\}_{i \in I} \cup \{X - Z\}$  is an open cover of  $X$ . Since  $X$  is compact, there exists  $i_1, \dots, i_n$  such that  $\bigcup_{j=1}^n U_{i_j} \cup (X - Z) = X$ .

Then  $\bigcup_{j=1}^n U_{i_j} \supseteq Z$ , and thus  $Z$  is compact.  $\square$

**Theorem 2.** *Let  $X$  be a Hausdorff space, and let  $Z \subseteq X$  be a compact subspace. Then  $Z$  is closed inside of  $X$ .*

*Proof.* Let  $X$  be a Hausdorff space,  $Z$  a compact subspace. We want to show that  $Z \supseteq \overline{Z}$ . (Then  $Z = \overline{Z}$ , so  $Z$  is closed).

We'll show if  $y \in X - Z$ , then  $y \notin \overline{Z}$ .

Let  $y \in X - Z$ . Then for each point  $x \in Z$ , we can find disjoint open neighborhoods  $U_x$  of  $x$  and  $V_x$  of  $y$ . Observe  $\{U_x\}_{x \in X}$  are a collection of open sets whose union contains  $Z$ . Since  $Z$  is compact, there exists some finite subcover: there are  $x_1, \dots, x_n$  such that  $U_{x_1} \cup \dots \cup U_{x_n} \supseteq Z$ . Then observe that  $V_{x_1} \cap \dots \cap V_{x_n}$  is an open neighborhood of  $y$  disjoint from  $U_{x_1} \cup \dots \cup U_{x_n}$ .

This implies  $(V_{x_1} \cap \dots \cap V_{x_n}) \cap Z = \emptyset$ . Thus  $y \notin \overline{Z}$ . Then  $Z$  is closed, as desired.  $\square$

**Example 1.**  $(0, 1]$  is not compact. A quick way to show this is to note  $(0, 1]$  is not a closed subspace of  $\mathbb{R}$ .

**Theorem 3.** *Let  $f : X \rightarrow Y$  be a continuous function. If  $Z$  is a compact subspace of  $X$ , then  $f(Z)$  is a compact subspace of  $Y$ .*

*Proof.* We can assume  $X = Z$ .

Then we want to show  $f(X)$  is compact. Let  $\{U_i\}_{i \in I}$  be a collection of open subsets of  $Y$  with  $\bigcup_{i \in I} U_i \supseteq f(X)$ .

Taking pre-images,  $\{f^{-1}(U_i)\}_{i \in I}$  is a collection of open subsets of  $X$  with union.

$$\bigcup_{i \in I} f^{-1}(U_i) = f^{-1} \left( \bigcup_{i \in I} U_i \right) \supseteq f^{-1}(f(X)) = X.$$

Since  $X$  is compact, there are indices  $i_1, \dots, i_n$  such that  $\bigcup_{j=1}^n f^{-1}(U_{i_j}) = X$ . Then we claim  $\bigcup_{j=1}^n U_{i_j} \supseteq f(X)$ .

Suppose for the sake of contradiction that there's some  $x \in X$  such that  $f(x) \notin U_{i_j}$  for all  $j = 1, \dots, n$ . But this implies  $x \notin f^{-1}(U_{i_j})$  for all  $j = 1, \dots, n$  — ✖ —  $\square$

**Corollary 1.** *If  $f : X \rightarrow Y$  is a continuous function from a compact space to a Hausdorff space, then  $f$  is a closed map.*

*Proof.* Suppose  $Z \subseteq X$  is closed. Then  $Z$  is compact. Then  $f(Z)$  is compact. Then  $f(Z)$  is closed.  $\square$

**Corollary 2.** *If  $f : X \rightarrow Y$  is bijective continuous function from a compact space to a Hausdorff space, then  $f$  is a homeomorphism.*

**Example 2.** Let  $X = S^1$ . consider  $\sim$ ,  $\mathbf{x} \sim \pm \mathbf{x}$

$$\begin{array}{ccc} S^1 & & \\ p \downarrow & \searrow g & \\ S^1/\{\pm\} & \dashrightarrow & S^1 \end{array}$$

$S^1/\{\pm\}$  is compact, and  $S^1$  is Hausdorff, so by corollary  $S^1/\{\pm\} \cong S^1$ .

“Compact + Hausdorff = Super closed”.

If  $i : X \hookrightarrow X$  is the inclusion of a compact subspace to be a Hausdorff space, then  $Z' \subseteq Z$  is closed iff  $i(Z')$  in  $X$  is closed.

**Theorem 4.** *If  $X$  and  $Y$  are compact subspaces, then  $X \times Y$  is also compact.*

**Lemma 1** (Tube Lemma). *If  $X$  is a topological space and  $Y$  is a compact space and  $Y$  is a compact space.  $x_0 \in X$  and  $Y \subseteq X \times Y$  is an open subset containing  $X_0 \times Y$ , then there is an open neighborhood  $W$  of  $x_0$  such that  $W \times Y \subseteq N$ .*