Math 171 **Professor Sebastian Bozlee**
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During the example last week, we found that it can be useful to consider the preimage of a function when looking at continuous functions. This leads us into the following major theorem.

Theorem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Then f is continuous iff for each open subset $U \subseteq \mathbb{R}^m$ the subset $f^{-1}(U) \subseteq \mathbb{R}^n$ is open.*

Proof. ($\Rightarrow$) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous. Let $U \subseteq \mathbb{R}^m$ be an open set. Let $\mathbf{x} \in f^{-1}(U)$. Since U is open, $f(\mathbf{x}) \in U$, there $\exists \varepsilon > 0$ such that $B(f(\mathbf{x}), \varepsilon) \subseteq U$.

Since f is continuous, $\exists \delta > 0$ such that $\forall \mathbf{y} \in \mathbb{R}^n$, with $|\mathbf{x} - \mathbf{y}| < \delta$, $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$. In other words, for all $\mathbf{y} \in B(\mathbf{x}, \delta) \subseteq f^{-1}(U)$ therefore $f^{-1}(U)$ is open.

($\Leftarrow$) Assume that $f : \mathbb{R}^n \to \mathbb{R}^m$ has the property that $U \subseteq \mathbb{R}^m$ is open $\Rightarrow f^{-1}(U) \subseteq \mathbb{R}^n$. Now let $\mathbf{x} \in \mathbb{R}^n$, $\varepsilon > 0$.

Now $U = B(f(\mathbf{x}), \varepsilon) \subseteq \mathbb{R}^m$ is an open subset, so $f^{-1}(U) \subseteq \mathbb{R}^n$) is open. So since $f^{-1}(U)$ is open, and since $\mathbf{x} \in f^{-1}(U)$, so by the definition of openness, there exists a real number $\delta > 0$ such that $B(\mathbf{x}, \delta) \subseteq f^{-1}(U)$. In other words, for all $\mathbf{y} \in \mathbb{R}^n$ with $|\mathbf{x} - \mathbf{y}| < \delta$, $f(\mathbf{y}) \in U$.

Therefore $\forall \mathbf{y} \in \mathbb{R}^n$ with $|\mathbf{x} - \mathbf{y}| < \delta$, $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$. Therefore f is continuous.

□

We of course also know that f^{-1} preserves all the standard set-theoretic operations.

1. $f^{-1}(\bigcup U_i) = \bigcup f^{-1}(U_i)$
2. $f^{-1}(\bigcap U_i) = \bigcap f^{-1}(U_1)$
3. $f^{-1}(U^c) = (f^{-1}(U))^c$

Proposition 1. *1. $\emptyset, \mathbb{R}^n \subseteq \mathbb{R}^n$ are open in $\mathbb{R}^n$*

2. If $U_{i \in I}$ is a collection of open subsets of $\mathbb{R}^n$, then their union is also open.

3. If $U_1, \ldots, U_n$ are open subsets of $\mathbb{R}^n$ then $\cap_{i=1}^n U_i$is also an open subset of $\mathbb{R}^n$.

Proof. 1. The empty set is open vacuously. $\forall \mathbf{x} \in \mathbb{R}^n$, $B(\mathbf{x}, 1) \subseteq \mathbb{R}^n$.

2. Unions come somewhat naturally. Namely let $\mathbf{x} \in \cup_{i \in I} U_i$, then there is an index $j \in I$ such that $\mathbf{x} \in U_j$. Then since U_j is open, there is some $\varepsilon > 0$ such that $B(\mathbf{x}, \varepsilon) \subseteq U_j \subseteq \cup_{i \in I} U_i$. Therefore $\cup_{i \in I} U_i$ is open.

3. Intersections are a little bit more of a trick. Namely suppose that $x \in \cap_{i=1}^{n} U_i$. Then $x \in U_i$for each i, so there exists a positive real number $\varepsilon_i > 0$ for each $i = 1, \dots, n$ such that $B(\mathbf{x}, \varepsilon_i) \subseteq U_i$ for each $i = 1, \dots, n$. Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$. Then $B(\mathbf{x}, \varepsilon) \subseteq B(\mathbf{x}, \varepsilon_i) \subseteq U_i$ for all i, so $B(\mathbf{x}, \varepsilon) \subseteq \cap_{i=1}^{n} U_i$.

$\square$

It aught to be suspicious that we can only do finite intersections... why is that?

Example 1. Set $U_n = B(\mathbf{x}, \frac{1}{n})$ for each $n \in \mathbb{Z}_{>0}$. Then $\cap_{n=1}^{\infty} U_n = \{\mathbf{0}\}$, which is not open.

Fig. 1. We really can only do finite intersections.

Proposition 2. *Let $Z \subseteq \mathbb{R}^n$ then Z is a closed subset of $\mathbb{R}^n$ iff Z^c is an open subset of $\mathbb{R}^n$.*

Proof. ($\Rightarrow$) . Suppose that Z is closed. Then let $\mathbf{x} \in Z^c$. Then $\mathbf{x}$ is not a limit point of Z so $\exists \varepsilon > 0$ such that $B(\mathbf{x}, \varepsilon) \cap Z) = \emptyset$. Then $B(\mathbf{x}, \varepsilon) \subseteq Z^c$ so Z^c is open.

($\Leftarrow$) Suppose that Z^c is open. Then let $\mathbf{x} \in \mathbb{R}^n$ be a limit point of Z. Then for all $\varepsilon > 0$, $B(\mathbf{x}, \varepsilon) \cap Z \neq \emptyset$. So $\forall \varepsilon > 0$, $B(\mathbf{x}, \varepsilon) \not\subseteq Z^c$. This implies that $\mathbf{x} \notin Z^c$ since Z^c is open. Thus $\mathbf{x} \in Z$, so Z is closed.

$\square$

Definition 1. Let X be a set. A **topology on** X is a set τ of subsets of X such that:

1. $\emptyset, X \in \tau$
2. If $U_i \in \tau$ of each $i \in I$ then $\cup_{i \in I} U_i \in \tau$
3. If $U_1, \dots, U_n \in \tau$, then $\cap_{i=1}^{n} U_i \in \tau$.

A **topological space** is a pair (X, τ) where X is a set and τ is a topology on X., Then the elements of τ are called the **open subsets** of X.

Example 2. $\mathbb{R}^n$ with the usual definition of open sets is a topological space.

Example 3. We can define a topological space with only finitely many points. For example

$$
\begin{aligned}
X &= \{1, 2\} \\
\tau &= \{\emptyset, \{1, 2\}\} && \text{(indiscreet topology)} \\
\tau_1 &= \{\emptyset, \{1\}, \{1, 2\}\} && \text{(Sierpinski space)} \\
\tau_2 &= \mathcal{P}(X) && \text{(The discrete topology)}
\end{aligned}
$$

Definition 2. If $(X, \tau_X), (Y, \tau_Y)$ are topological spaces, then a function $f : X \to Y$ is said to be **continuous** if $f^{-1}(U)$ is open in Y whenever U is open in X.

Example 4. What are the continuous functions from S (the Sierpinski space) to S.

There are four functions:

x	1	2	cont?
$f_1(x)$	1	2	Yes
$f_2(x)$	1	1	Yes
$f_3(x)$	2	1	$f_3^{-1}(1) = 2$
$f_4(x)$	2	2	Yes