

We begin with a quick reminder that, along  $\mathbb{S}^1$ , the open sets are open sets in  $\mathbb{R}^2$  which are intersected with the circle, and likewise that half-open intervals at the endpoints of  $[0, 1]$  are also open.

Now moving on to connectedness.

**Definition 1.** Let  $X$  be a topological space. A **separation** of  $X$  is a pair  $U, V$  of disjoint, nonempty, open subsets of  $X$  whose union is all of  $X$ . We say that  $X$  is **connected** if  $X$  does not have a separation.

**Remark.** • *The definition of connected is homeomorphism invariant*

- *If  $U, V$  is a separation of  $X$ , then  $U^c = V$  and  $V^c = U$  are open, so  $U$  and  $V$  are both open and closed.*
- *$X$  is connected iff the only subsets of  $X$  that are both open and closed are  $\emptyset$  and  $X$  itself. (To see this, simply take a complement of a nontrivial clopen set. What do you get?)*

**Example 1.** • Consider  $X = \{1, 2\}$  with the indiscrete topology. This is clearly connected.

- Let  $X = [-1, 0) \cup (0, 1]$  w/ subspace topology. This is disconnected, since both  $[-1, 0)$  and  $(0, 1]$  are disjoint open sets<sup>1</sup> which union to the entire space.
- Let  $X = [-1, 1]$ . This is connected, but that's kind of a truck to prove.
- Let  $X = \mathbb{Q}$ . The only subspaces of  $\mathbb{Q}$  which are connected are the singleton sets. If  $Y$  is any subspace of  $\mathbb{Q}$  containing  $p < q$ , then there is an irrational number  $a$  between  $p$  and  $q$ . Then  $U = (-\infty, a) \cap Y$ ,  $V = (a, \infty) \cap Y$  is a separation of  $Y$ .

**Theorem 1.** *Let  $X$  be a connected topological space,  $f : X \rightarrow Y$  a continuous function. Then  $f(X)$  is connected (w.r.t. the subspace topology)*

*Proof.* Let  $Z = f(X)$ , and notice that  $\begin{matrix} g : X \rightarrow Z \\ x \mapsto f(x) \end{matrix}$  is also continuous.

So suppose for the sake of contradiction that there exists a separation  $U, V$  of  $Z$ . Then we claim that  $U' = g^{-1}(U)$ ,  $V' = g^{-1}(V)$  is a separation of  $X$ .

- (*disjoint*):  $U' \cap V' = g^{-1}(U) \cap g^{-1}(V) = g^{-1}(U \cap V) = g^{-1}(\emptyset) = \emptyset$
- (*nonempty*):  $U', V' \neq \emptyset$  since  $g$  is surjective.

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<sup>1</sup>Remember, this is in the subspace topology on  $\mathbb{R}$

- (*open*): Follows trivially since  $g$  is continuous.
- (*Union is  $X$* ):  $U' \cup V' = g^{-1}(U) \cup g^{-1}(V) = g^{-1}(U \cup V) = g^{-1}(Z) = X$ .

But then we've given a separation of  $X$ , which is connected. A contradiction! So we must have that  $f(x) = Z$  is connected.  $\square$

Now we'll prove a sequence of lemmas, with the goal of proving that product spaces are connected.

**Lemma 1.** *Suppose that  $X$  is a topological space, and further suppose that  $A$  is a connected subspace. If  $U, V$  is a separation of  $X$ , then  $A \subseteq U$  or  $A \subseteq V$ .*

*Proof.* Consider  $U' = U \cap A$  and  $V' = V \cap A$ . Then  $U'$  and  $V'$  are disjoint, open in  $A$  and  $U' \cup V' = A$ .

Since  $A$  is connected, it must be so that  $U'$  or  $V'$  is empty, otherwise we would have a separation.

If  $U' = \emptyset$ ,  $A \subseteq V$ , and if  $V' = \emptyset$ , then  $A \subseteq U$ .  $\square$

**Theorem 2.** *If  $\{A_i\}_{i \in I}$  is a collection of connected subspaces of a topological space  $X$  and  $\bigcap_{i \in I} A_i \neq \emptyset$ , then  $\bigcup_{i \in I} A_i$  (with the subspace topology) is connected.*

*Proof.* Suppose – again for the sake of contradiction – that  $U, V$  is a separation of  $\bigcup_{i \in I} A_i$ .

Let  $p \in \bigcup_{i \in I} A_i$ . Suppose without loss of generality that  $p \in U$ .

Consider  $A_i$  for some  $i \in I$ . Then by the lemma,  $A_i \subseteq U$  or  $A_i \subseteq V$ . Since  $p \in A_i$  and  $p \in U$ , it must follow that  $A_i \subseteq U$ . Since this is true for all  $i$ ,  $\bigcup_{i \in I} A_i \subseteq U$ . But then  $V = \bigcup_{i \in I} A_i - U = \emptyset$ , again a contradiction.

So  $\bigcup_{i \in I} A_i$  must be connected.  $\square$

**Theorem 3.** *Let  $X, Y$  be connected topological spaces. Then  $X \times Y$  is connected as well.*

*Proof.* First, notice that if  $X, Y = \emptyset$ , then  $X \times Y = \emptyset$  is connected.

So let's suppose that  $X$  and  $Y$  are not empty. Then  $\exists(a, b) \in X \times Y$ .

Notice:  $X \times \{b\} \cong X$  is connected. Likewise, for any  $x \in X$ ,  $\{x\} \times Y \cong Y$  is connected.

Thus by the lemma,  $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$  is connected.

Then  $\bigcup_{x \in X} T_x = X \times Y$  and  $(a, b) \in \bigcup_{x \in X} T_x$ . So again by the lemma,  $X \times Y$  is connected.  $\square$