Math 171 **Professor Sebastian Bozlee**
Scribed by: Kyle Dituro **March 31, 2023**

Let's cover a bit of the theorem that we stated during class yesterday before we get to the recitation.

Theorem 1. *Let X, Y be topological spaces, let $\{U_i\}_{i \in I}$ be an open cover of X.*

1. *If $f : X \to Y$, $g : X \to Y$ then $f = g$ iff $f|_{U_i} = g|_{U_i}$ for all $i \in I$.*
2. *If $f_i : U_i \to Y$ is a continuous function for each $i \in I$ and $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$ then there exists a unique continuous function $f : X \to Y$ such that $f|_{U_i} = f_i$ for all $i \in I$.*

Remark. *These two properties together are known as the "sheaf property".*

Let's see if we can give this a proof

Proof.

1. Suppose $f = g$. Then clearly, $f|_{U_i} = g|_{U_i}$ for all $i \in I$.

   Conversely suppose $f|_{U_i} = g|_{U_i}$ for all $i \in I$. We want to show that $f = g$. Let $x \in X$. Since $X = \bigcup_{i \in I} U_i$, there exists an i such that $x \in U_i$. Then $f(x) = f|_{U_i}(x) = g|_{U_i}(x) = g(x)$, and thus since X and x are arbitrary, $f = g$. $\triangle$

2. Let $f : X \to Y$ be defined by $f(x) = f_i(x)$ when $x \in U_i$. Notice that there is a question of well-definedness here. To see if this f is well-defined, we need to check that $f_i(x) = f_j(x)$ when $x \in U_i$ and $x \in U_j$ (for some $i, j \in I$). This follows directly from the second part of our assumption. Namely, $x \in U_i \cap U_j$, and we know that $f_i|_{U_i \cap U_j}(x) = f_j|_{U_i \cap U_j}(x)$, so $f_i(x) = f_j(x)$ so f is well-defined.

   Note $f|_{U_i} = f_i$ for each $i \in I$ by definition. By the local characterization of continuity, we know that an everywhere locally continuous function is continuous, so we get that f is continuous.

   Now we need uniqueness. Suppose $g : X \to Y$ is a second continuous function such that $g|_{U_i} = f_i$ for all $i \in I$. Then, by part (1), we get that $f = g$

$\square$

We better get to the actual recitation now...