Math 171 **Professor Sebastian Bozlee**
Scribed by: Kyle Dituro **April 7, 2023**

Theorem 1 (Universal Property of the Quotient)**.** *Let $p : X\ toY$ be a quotient map. Then for any continuous function $g : X \to Z$ satisfying $p(x_1) = p(x_2) \Rightarrow g(x_1) = g(x_2)$, there exists a unique function $f : Y \to Z$ such that*

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "q"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Proof. We want to show that this f is well-defined, unique, makes the diagram ommute, and is continuous.

For well-definedness, we need that each element $y \in Y$ can be written as $p(x)$ for some $x \in x$ and if $p(x_1) = p(x_2)$ for some $x_1, x_2 \in X$ we have that $g(x_1) = g(x_2)$. The first part is true since p is surjective, and the second part is true by hypothesis.

in order for the diagram to commute, we need

$$f(p(x)) = g(x) \quad \text{for all } x \in X,$$

Which is true by construction, so $f(p(x)) = g(x)$ fully determines f, so we have uniqueness.

To check that f is continuous, suppose $U \subseteq Z$ is an open subset. Then since g is continuous, $g^{-1}(U) \subseteq X$ is open. Then $g^{-1}(U) = p^{-1}(f^{-1}(U))$ is open in X.

Then by definition of the quotient map, $f^{-1}(U) \subseteq Y$ is open, as desired. $\square$

Example 1 (Quotients need not behave well with respect to subspaces)**.** Write $\mathbb{R}^\delta$ for $\mathbb{R}$ with the discrete topology, and $\mathbb{R}^i$ for $\mathbb{R}$ with the indiscrete topology.

Say

$$X = \mathbb{R}^\delta \sqcup \mathbb{R}^i = \{(1, x) \mid x \in \mathbb{R}\} \cup \{(2, x) \mid x \in \mathbb{R}\}.$$

Which will gain the union topology: the open sets are those of the form $(1 \times U) \cup (2 \times V)$ where $U \subseteq \mathbb{R}^\delta$ ad $V \subseteq \mathbb{R}^i$ are open.

$\pi : X \to \mathbb{R}^i$
Consider p $(1, x) \mapsto x$., which we claim to be a quotient map.
$(2, x) \mapsto x$

Let $B \subseteq \mathbb{R}^i$, then $\pi^{-1}(B) = (1 \times B) \cup (2 \times B)$ is open if an only if $B = \emptyset, \mathbb{R}$

But! If we restrict π to $\mathbb{R}^\delta$, then we get $\begin{matrix} R^\delta \to \mathbb{R}^i \\ (1, x) \mapsto x \end{matrix}$ which is not a quotient, since too many pre-images are open.

Lemma 1. *Let $p : X \to Y$ be a quotient map, $B \subseteq Y$ is a subset, $A = p^{-1}(b)$ be its preimage. If B is open (or closed), then the restriction $q = p|_A : A \to B$ is also a quotient map. (e.g. $\widetilde{D}(z) \to D(z)$ from this weeks recitation is an example).*

Proof. Note that if $V \subseteq B$ then $q^{-1}(V) = p^{-1}(V)$. Assume that A is open, and let $V \subseteq B$ be a subset. We know that q is continuous and q is surjective by definition. All that remains to be shown is that $q^{-1}(V) \subseteq A$ is open, so $V \subseteq B$ is open.

Well $q^{-1}(V) = p^{-1}(V)$ since $q^{-1}(V) \subseteq A$ is open and $A \subseteq X$ is open, $q^{-1}(V) = p^{-1}(V)$ is open in X.

Since p is a quotient map, $V \subseteq Y$ is open. Since $V \subseteq B$ and $B \subseteq Y$ is open, $V \subseteq B$ is open.

For the case where B is closed, replace "open" with "closed" throughout. $\square$

Example 2 (Hausdorff not preserved by quotients)**.** Let $X = \mathbb{R} \sqcup \mathbb{R}$, and take the equivalence relation

$$(1, x) \sim (2, x) \quad \text{when } x \neq 0.$$

And consider $\pi : X \to X/\sim$. This construction is called the "bug-eyed line" or the "line with two origins"

Let $x = [(1, 0)], y = [(2, 0)]$. Suppose that U is an open neighborhood of x and V is an open neighborhood of Y.

Well since U is open, $\pi^{-1}(U)$ is open. Then for any arbitrary set, its pre-image will be the two copies of U which live in each copy of $\mathbb{R}$ everywhere but at 0.

Notice that U and V necisarilly intersect, so $X/\sim$ is not Hausdorff.

Question 1. What if $\sim$ was $(1, x) \sim \left(2, \frac{1}{x}\right)$ instead? What would $X/\sim$ be?

Lemma 2. *Let $p : X \to Y$ be a quotient map. $q : Y \to Z$ be a function. Then q is a quotient map if and only if $q \circ p$ is a quotient map.*

Proof. Suppose that q is a quotient map. then $q \circ p$ is continuous and surjective. Then

$$\begin{aligned} B \subseteq Z \text{ open} &\Leftrightarrow q^{-1}(B) \subseteq Y \text{ open} \\ &\Leftrightarrow p^{-1}(q^{-1}(B)) \subseteq X \text{ open} \\ &\Leftrightarrow (q \circ p)^{-1}(B) \subseteq X \text{ open.} \end{aligned}$$

$\triangle$

Conversely, suppose $q \circ p : X \to Z$ is a quotient map. Then since $q \circ p$ is surjective, q is surective. Suppose $B \subseteq Z$. Then

$$\begin{aligned} B \subseteq Z \text{ open} &\Leftrightarrow (q \circ p)^{-1}(B) \subseteq X \text{ open} \\ &\Leftrightarrow p^{-1}(q^{-1}(B)) \subseteq X \text{ open} \\ &\Leftrightarrow q^{-1}(B) \subseteq Y \text{ open.} \end{aligned}$$

Therefore q is a quotient map. $\square$

Example 3. Let $X = \mathbb{R}^3 - \{0\}$, $\sim$ be the relation

$$\mathbf{x} \sim \mathbf{y} \iff \exists c \in \mathbb{R} - \{0\} \text{ s.t. } c\mathbf{x} = \mathbf{y}.$$

Then $\pi : X \to X/\sim = \mathbb{RP}^2$.

Let

$$\begin{array}{c} p : X \to \mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \\ \mathbf{x} \mapsto \frac{1}{|\mathbf{x}|}\mathbf{x} \end{array}.$$

Believe that this is a quotient map.

Define $\begin{array}{c} q : \mathbb{S}^2 \to \mathbb{RP}^2 \\ \mathbf{x} \mapsto [x : y : z] \end{array}$.

The composite $q \circ p = \pi$.