Math 171 **Professor Sebastian Bozlee**
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Definition 1. let X be a set. We say that a collection $\mathcal{B}$ of subsets of X is a **basis** for a topology on X if:

1. For each point $x \in X$, there is a **basis element** (or **basic open subset**) $B \in \mathcal{B}$ such that $x \in B$

2. For each pair of basis elements $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there exists a basis element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

The **topology τ generated by the basis $\mathcal{B}$** is the topology on X where:

1. a subset $U \subseteq X$ is said to be open if for each $x \in U$ there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.

Example 1. In the usual topology on $\mathbb{R}^n$, the open balls $B(\mathbf{x}, \varepsilon)$ form a basis for the usual topology. I.E. the set

$$\mathcal{B} = \{B(\mathbf{x}, \varepsilon) | \mathbf{x} \in \mathbb{R}^n, \varepsilon > 0\}$$

Is a basis for the usual topology on $\mathbb{R}^n$.

Proof. Let's check the axioms:

1. For any $\mathbf{x} \in \mathbb{R}^n$, we can use $B(\mathbf{x}, \varepsilon)$ as a basis element containing $\mathbf{x}$.

2. Now for any $B_1 = B(\mathbf{x}_1, \varepsilon_1), B_2 = B(\mathbf{x}_2, \varepsilon_2,$ and $\mathbf{x} \in B_1 \cap B_2$. then there exists $B_3 = B(\mathbf{x}, \min\{|\varepsilon_1 - \mathbf{x_1}|, |\varepsilon_2 - \mathbf{x_2}|\}$

$\square$