Math 171 **Professor Sebastian Bozlee**
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We will pick up with the tube lemma that we left unproven at the end of the last class.

Lemma 1 (Tube Lemma)**.** *If X is a topological space and Y is a compact space. Let $x_0 \in X$, and let $N \subseteq X \times Y$ be an open subset containing $x_0 \times Y$, then N contains some "tube" $W \times Y$, where W is an open neighborhood of x_0 in X.*

Proof. For each point $(x_0, y) \in x_0 \times Y \subseteq N$. choose a basic open subset $U_y \times V_y$ where $U_y \subseteq X$ is open, $V_y \subseteq Y$ open, and $(x_0, y) \in U_y \times V_y \subseteq N$. Then $\{U_y \times V_y\}_{y \in Y}$ is an open covering of $x_0 \times Y$. But $x_0 \times Y \cong Y$, so $x_0 \times Y$ is compact, so there exists a $y_1, \ldots, y_n \in Y$ such that $Y \subseteq (U_{y_1} \times V_{y_1}) \cup \ldots \cup (U_{y_n} \times V_{y_n})$. Let $W = U_{y_1} \cap \ldots \cap U_{y_n}$. Then this is an open neighborhood of $x_0 \in X$. Moreover,

$$\begin{aligned} W \times Y &= W \times \left(\bigcup_{i=1}^{n} V_{y_n} \right) \\ &= \bigcup_{i=1}^{n} (W \times V_{y_n}) \\ &\subseteq \bigcup_{i=1}^{n} (U_{y_i} \times V_{y_i}) \subseteq N \end{aligned}$$

□

Theorem 1. *If X and Y are compact topological spaces, then so is $X \times Y$.*

Proof. Let $\{A\}_{i \in I}$ be an open cover of $X \times Y$. Then, as in the proof of the tube lemma, $x_0 \times Y$ is compact, so finitely many of the A_is cover $x_0 \times Y$. Say $A_{i_1} \cup \ldots \cup A_{i_n} \supseteq x_0 \times Y$. Let $N = A_{i_1} \cup \ldots \cup A_{i_m}$.

Now by the tube lemma, there is an open neighborhood W_{x_0} of x_0 in X such that $W_{x_0} \times Y \subseteq N = A_{i_1} \cup \ldots \cup A_{i_n}$

Observe that $\{W_x\}_{x \in X}$ is an open cover of X. Since X is compact, there exists $x_1, \ldots, x_n$ such that $W_{x_1} \cup \ldots \cup W_{x_n} = X$ then $(W_{x_1} \times Y) \cup (W_{x_2} \times Y) \cup \ldots \cup (W_{x_n} \times Y) = X \times Y$.

Since there are finitely many W_{x_i}s, and each of the $W_{x_i} \times Y$ admits a cover by finitely many elements of $\{A_i\}_{i \in I}$, $X \times Y$ also admits a finite cover by A_is. Therefore $X \times Y$ is compact. □

Now moving on to a discussion of compact subspaces of $\mathbb{R}^n$.

We begin with the long-alluded to proof that closed intervals are compact.

Theorem 2. *A finite closed interval $[a, b] \subseteq \mathbb{R}$ is compact*

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover $[a, b]$ in the subspace topology.

Step 1: Our goal is to show that if x is a point of $[a, b)$ then there is $y > x$ in $[a, b]$ such that the interval $[x, y]$ is covered by just one element of $\mathcal{U}$.

Let $x \in [a, b)$. Since $\mathcal{U}$ is a cover, there is an element $U \in \mathcal{U}$ such that $x \in U$. Since U is open and $x \neq b$, there is some c in (x, b) such that $[x, c) \subseteq U$. Chose $y \in (x, c)$. Then $[x, y] \subseteq U$ $\triangle$

Step 2: Let C be the set of all points $y > a$ of $[a, b]$ such that the interval $[a, y]$ can be covered by finitely many elements of $\mathcal{U}$.

Applying Step 1 to $x = a$ we get that at least one $y \in C$, so C is nonempty. C is bounded above by b, so there exists a supremum c of C $\triangle$

Step 3: We now want to show $c \in C$.

Suppose not. Choose an element $U \in \mathcal{U}$ containing c. We know that $c > a$, so there exists $d \in [a, b]$ such that $(d, c] \subseteq U$. Choose an element z of $(d, c]$. We must have $z \in C$ since c is least upper bound. $\triangle$

Then since $z \in C$, the interval $[a, z]$ can be covered by finitely many elements of $\mathcal{U}$, say $U_1, \ldots, U_n$. But then $U_1 \cup \ldots \cup U_n \cup U \supseteq [a, z] \cup [z, c] = [a, c]$. Then $c \in C$ $-\!\!\times\!\!-$, so $c \in C$. $\triangle$

Step 4: Now we show that $c = b$.

Once again, suppose not. Then $c < b$, so by Step 1, there is a $y \in (c, b]$ and $U \in \mathcal{U}$ such that $[c, y] \subseteq U$. By definition of C, since $c \in C$ there is a finite collection of elements of $\mathcal{U}$, say $U_1, \ldots, U_n$ covering $[a, c]$. Then

$$U_1 \cup \ldots \cup U_n \cup U \supseteq [a, c] \cup [c, y] = [a, y].$$

But then $y > c$ and $y \in C$, which contradicts that c is an upper bound on C $-\!\!\times\!\!-$.
$\triangle$

Therefore $[a, b]$ is compact. $\square$

Definition 1. A subset $A \subseteq \mathbb{R}^n$ is said to be **bounded** if there is a radius r such that $A \subseteq B(\mathbf{0}, r)$. (where $B(\mathbf{x}, r)$ is a ball centered at $\mathbf{x}$ with radius r).

Fact. *A is bounded iff there exists $b \in \mathbb{R}$ such that $A \subseteq [-b, b]^n$.*

Theorem 3 (Extreme Value Theorem)**.** *Let X be a compact topological space and $f : X \to \mathbb{R}$ be a continuous function. Then there are elements $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in X$.*