

We begin with a reminder of what will be one of the most fundamental definitions for this course.

**Definition 1.** Let  $X$  be a set. A **topology on  $X$**  is a set  $\tau$  of subsets of  $X$  such that:

1.  $\emptyset, X \in \tau$
2. If  $U_i \in \tau$  of each  $i \in I$  then  $\bigcup_{i \in I} U_i \in \tau$
3. If  $U_1, \dots, U_n \in \tau$ , then  $\bigcap_{i=1}^n U_i \in \tau$ .

A **topological space** is a pair  $(X, \tau)$  where  $X$  is a set and  $\tau$  is a topology on  $X$ . Then the elements of  $\tau$  are called the **open subsets** of  $X$ .

And we remember the definition of continuity:

**Definition 2.** Let  $(X, \tau_X), (Y, \tau_Y)$  be topological spaces. A **continuous function**  $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$  is a function  $f : X \rightarrow Y$  such that whenever  $U \in \tau_Y$ , we have that  $f^{-1}(U) \in \tau_X$ .

And we'll proceed with a number of examples to guide our intuition of open-ness in a topological sense and continuity.

**Example 1.** Let  $X = \mathbb{R}$  be given. We know that the usual notion of open sets defines a topology on  $X$ :

$$\tau_1 = \{\text{open subsets of } \mathbb{R}\}$$

However, this is not the only topology that we can endow the real numbers with. In fact, I claim that:

$$\tau_2 = \tau_1 \cup \{\{0\} \cup U \mid U \in \tau_1\}$$

Let's check:

- $\emptyset, \mathbb{R} \in \tau_2$
- $\{U_i\}_{i \in I}$  with  $U_i \in \tau_1$ ,  $\{\{0\} \cup U_j\}_{j \in J}$  where  $U_j \in \tau_1$ . The union of these is either  $\bigcup_{i \in I} U_i$  if  $J$  is empty or  $\{0\} \cup (\bigcup_{i \in I \cup J} U_i)$
- $U_1, \dots, U_n, \{0\} \cup U_{n+1}, \dots, \{0\} \cup U_{n+m}$ , the intersection of these things is either  $\{0\} \cup (\bigcap U_i)$  if  $n = 0$  or  $\bigcap U_i$  if  $n \neq 0$ .

**Example 2.** Consider the function  $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$  Under the topology  $\tau_2$ , this function is actually continuous.

**Fact.** The continuous functions  $g : (\mathbb{R}, \tau_2) \rightarrow (\mathbb{R}, \tau_1)$  are the functions continuous at all  $x \in \mathbb{R} - \{0\}$  in the analysis sense of continuity. The continuous functions  $f : (\mathbb{R}, \tau_1) \rightarrow (\mathbb{R}, \tau_2)$  are the continuous functions with  $g(\mathbb{R}) \subseteq (-\infty, 0), (0, \infty)$ , or  $\{x\}$ .

**Idea.** Let  $f : X \rightarrow Y$ , the more open subsets  $Y$  has, the harder it is for  $f$  to be continuous, and likewise the more open subsets  $X$  has, the easier it is.

**Definition 3.** Let  $X$  be a set,  $\tau_1, \tau_2$  two topologies on  $X$ . We say that  $\tau_1$  is coarser than  $\tau_2$  and that  $\tau_2$  is finer than  $\tau_1$  if  $\tau_1 \subseteq \tau_2$ .

**Example 3.** Notice that the discrete topology is always the finest topology on a set  $X$ , and that the indiscrete topology is the finest.

**Proposition 1.** Let  $f : X \rightarrow Y$  be a function between topological spaces.

1. If  $X$  has the discrete topology then  $f$  is continuous.
2. If  $Y$  has the indiscrete topology then  $f$  is continuous.

*Proof.* 1. let  $f : X \rightarrow Y$  be any function. Let  $U \subseteq Y$  be an open subset. This is obvious because every subset of  $X$  is open, so of course  $f^{-1}(U)$  is open.

2. Let  $f : X \rightarrow Y$  be any function. Let  $U \subseteq Y$  be an open set. Now since  $Y$  has the indiscrete topology, we need only check  $f^{-1}(Y)$  and  $f^{-1}(\emptyset)$  but  $f^{-1}(Y) = X$  and  $f^{-1}(\emptyset) = \emptyset$ .

Thus, our function is continuous in either case. □

**Proposition 2.**  $\text{id}_X : (X, \tau_2) \rightarrow (X, \tau_1)$  is continuous iff  $\tau_2$  is finer than  $\tau_1$ . In particular  $\text{id}_X : (X, \tau) \rightarrow (X, \tau)$  is continuous.

*Proof.*  $\text{id}_X : (X, \tau_2) \rightarrow (X, \tau_1)$  is continuous iff  $\forall U \in \tau_1, \text{id}_X^{-1}(U) \in \tau_2$  iff  $\forall U \in \tau_1$ , have  $U \in \tau_2$  iff  $\tau_1 \subseteq \tau_2$  which is the definition of  $\tau_2$  is finer than  $\tau_1$ . □

**Lemma 1.** Let  $f : X \rightarrow Y, g : Y \rightarrow Z$  be functions,  $W \subseteq Z$ . Then  $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ .

*Proof.*

$$\begin{aligned}
 f^{-1}(g^{-1}(W)) &= \{x \in X \mid f(x) \in g^{-1}(W)\} \\
 &= \{x \in X \mid f(x) \in \{y \in Y \mid g(y) \in W\}\} \\
 &= \{x \in X \mid g(f(x)) \in W\} \\
 &= (g \circ f)^{-1}(W)
 \end{aligned}$$

□

**Proposition 3.** Let  $f : X \rightarrow Y, g : Y \rightarrow Z$  be continuous. Then  $(g \circ f) : X \rightarrow Z$  is as well.

*Proof.* Let  $W \subseteq Z$  be open. Then

$$(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$$

Notice here that we are taking the pre-image of an open set by continuous functions twice. Thus,  $g \circ f$  is continuous.  $\square$

And so it was that Kyle never wrote an  $\varepsilon$ - $\delta$  proof again. And we have officially moved out of the category of Sets, and have moved into the category of Topologies.