

Let's go over part (b) of the early problem...

**Proposition 1.** *Let  $X, Y$  be topological spaces, and let  $\mathcal{B}_X, \mathcal{B}_Y$  be bases for the resp. topologies of  $X, Y$ , then:*

1.  *$f : X \rightarrow Y$  is continuous iff for each  $B \in \mathcal{B}_Y$ ,  $f^{-1}(B)$  is open.*
2.  *$f : X \rightarrow Y$  is continuous iff for all  $x \in X$  and basic open neighborhoods  $B_{f(x)}$  of  $f(x)$ , there exists  $B_x \in \mathcal{B}_X$  such that  $x \in B_x$  and  $f(B_x) \subseteq B_{f(x)}$ .*

The proof of (a) is left to the early problem

*Proof of part b. ( $\Rightarrow$ )* Let  $f : X \rightarrow Y$  be continuous. Let  $x \in X$ , and  $B_{f(x)}$  be a basic open neighborhood of  $f(x)$ . Then, since  $f(x) \in B_{f(x)}$ ,  $x \in f^{-1}(B_{f(x)})$ . Since  $f$  is continuous,  $f^{-1}(B_{f(x)})$  is open. By the definition of open in a generated topology,  $\exists B_X \in \mathcal{B}_X$  such that  $x \in B_X$  and  $B_X \subseteq f^{-1}(B_{f(x)})$ . Then, using the Galois connection, we get that  $f(B_X) \subseteq B_{f(x)}$ .

( $\Leftarrow$ ) Assume that we have the property, now we want to show that  $f : X \rightarrow Y$  is continuous. So let  $U \subseteq Y$  be an open set. Now let  $x \in f^{-1}(U)$ . Then  $f(x) \in U$ . Since  $U$  is open, there is a basic open neighborhood  $B_{f(x)}$  such that  $f(x) \in B_{f(x)}$  and  $B_{f(x)} \subseteq U$ . By assumption, there is a basic open neighborhood  $B_x$  of  $x$  such that  $f(B_x) \subseteq B_{f(x)} \subseteq U$ . Then again by the Galois connection,  $B_x \subseteq f^{-1}(U)$ .

□

Now on to more stuff.

**Definition 1.** A set  $S$  of subsets of a set  $X$  whose union is all of  $X$  is a **subbasis for a topology on  $X$** . The **topology generated by  $S$**  is the set of unions of finite intersections of elements of  $S$ .

We claim that in order to show that this is a topology, it is enough to show that the set  $\mathcal{B}$  of finite intersections of elements of  $S$  is a basis. Now let's actually do the check:

*Proof.* 1. Let  $x \in X$ . Then since  $\bigcup_{T \in S} T = X$ , there is an element  $T \in S$  such that  $x \in T$ , and thus  $T \in \mathcal{B}$ .

2. Let  $B_1 = T_1 \cap \dots \cap T_n$ , where  $T_i \in S$ , and  $B_2 = T'_1 \cap \dots \cap T'_m$ , where  $T'_i \in S$ . Now let  $x \in B_1 \cap B_2$ . Since  $B_1 \cap B_2$  is a finite intersection of elements of  $S$ , it is a basis element, and so we can take  $B_3 = B_1 \cap B_2$ .

□

So where is this useful? We will now move into a discussion of product spaces. This is where all of the universal property juggling should pay off.

**Question 1.** Given two topological spaces  $X, Y$ , what topology “should” we put on  $X \times Y$ ?

More generally, given topological spaces  $X_i$  for  $i \in I$ , what topology should  $\prod_{i \in I} X_i$  have?

Let’s recall the universal property of the product for sets, and amend it for topological spaces.

**Theorem 1.** Given sets *topological spaces*  $X, Y$ , a set *topological space*  $P$  together with functions *continuous functions*  $\pi_1 : P \rightarrow X$  and  $\pi_2 : P \rightarrow Y$  is said to have the universal property if, for any set *topological space*  $Z$  and functions *continuous functions*  $f_1 : Z \rightarrow X$  and  $f_2 : Z \rightarrow Y$  there exists a unique function *continuous function*  $f : Z \rightarrow P$  such that.

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \vdots & \searrow & \\ & f_1 & \exists! f & f_2 & \\ X & \xleftarrow{\pi_1} & P & \xrightarrow{\pi_2} & Y \end{array}$$

**Proposition 2.** “Products are unique up to unique homeomorphism.” i.e. Given  $(P, \pi_1, \pi_2)$  and  $(P', \pi'_1, \pi'_2)$  with the universal property, there exists a unique homeomorphism  $\varphi : P \rightarrow P'$  such that  $\pi_1 = \pi'_1 \circ \varphi$  and  $\pi_2 = \pi'_2 \circ \varphi$ .

*Proof.* Let’s do some good old-fashioned diagram chasing.

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \vdots & \searrow & \\ & \pi_1 & \exists! \varphi & \pi_2 & \\ X & \xleftarrow{\pi'_1} & P' & \xrightarrow{\pi'_2} & Y \end{array} \quad \begin{array}{ccccc} & & Z & & \\ & \swarrow & \vdots & \searrow & \\ & & \psi & & \\ X & \xleftarrow{\quad} & P & \xrightarrow{\quad} & Y \end{array}$$

Combining these two diagrams, we get:

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \downarrow & \searrow & \\ & \pi_1 & \psi \circ \varphi & \pi_2 & \\ X & \xleftarrow{\pi_1} & P & \xrightarrow{\pi_2} & Y \end{array}$$

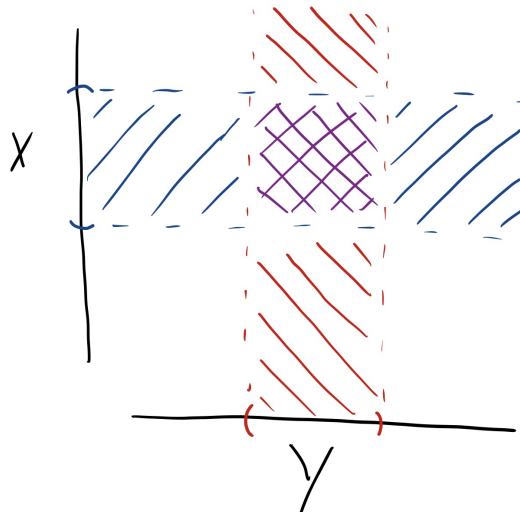
So then  $\psi \circ \varphi$  is the identity map, and we have what we need.  $\square$

So we can revise our goal: We now want to find a topology on  $X \times Y$  so that  $(X \times Y, \pi_1, \pi_2)$  has the universal property.

In other words, we want to find a topology on  $X \times Y$  such that  $\pi_1 : X \times Y \rightarrow X$  and  $\pi_2 : X \times Y \rightarrow Y$  are continuous.

$(x, y) \mapsto x$   
So we need

1. For each  $U \subseteq X$  open,  $\pi_1^{-1}(U) = U \times Y$  is open
2. For each  $V \subseteq Y$  open,  $\pi_2^{-1}(V) = X \times V$  is open.



**Fig. 1.** A diagram outlining what our “Product space” should look like

This is not yet a topology, we still need finite intersections. So, let's add in finite intersections:  $\mathcal{B} = \{U \times V \mid U \subseteq X \text{ open}, V \subseteq Y \text{ open}\}$ . This in and of itself is still not a topology, but it is a basis.

**Proposition 3.** *Let  $X, Y$  be topological spaces, then the set:*

$$\mathcal{B} = \{U \times V \mid U \subseteq X \text{ open}, V \subseteq Y \text{ open}\}$$

**Definition 2.** The **product topology on  $X \times Y$**  is the topology generated by the basis  $\mathcal{B}$  above.

**Example 1.**  $\mathbb{R}^2$  with the usual topology has the product topology from  $\mathbb{R} \times \mathbb{R}$ . With the usual topology, we were taking a basis of open balls, but in this construction, we are doing it with open rectangles, which we previously showed was equivalent.