Corollary 1 ("Images are almost quotient spaces")**.** *Let $g : X \to Z$ be a surjective continuous map. Let $\sim$ be the equivalence relation on X given by*

$$x_1 \sim x_2 \iff g(x_1) = g(x_2).$$

Then g induces a continuous ***bijective*** *map $f : X/\sim \to Z$ given by $f([x]) = g(x)$, which is a homeomorphism if and only if g is a quotient.*

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ X/\sim \arrow[r, "f"'] & Z \end{tikzcd}$$

Proof. It is clear that $\sim$ is an equivalence relation.

We know f is well defined and continuous from the universal property of the quotient.

It is... uhhh... "clear" that f is bijective:

- (**injective**): Pretty clear
- (**surjective**): Let $z \in Z$. Since g is surjective, $\exists x \in X$ such that $g(x) = z$. Then $f([x]) = z$.

Suppose f is a homeomorphism. Then in particular f is a quotient map:

$$U \subseteq Z \text{ open} \iff f^{-1}(U) \subseteq X/\sim \text{ open}$$

Then $g = f \circ \pi$ is a composite of quotient maps, and is therefore a quotient map.

Conversely, if g is a quotient map, then it also has the universal property of the quotient:

By the universal property of g, $\exists! h : Z \to X/\sim$ such that

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ X/\sim & Z \arrow[l, "h"] \end{tikzcd}$$

Taking composites:

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\pi"] & \\ X/\sim \arrow[rr, "h \circ f"'] & & X/\sim \end{tikzcd}$$

So by the uniqueness in the universal property for π, we get that $h \circ f = \mathrm{id}_{X/\sim}$.

Symmetrically, $f \circ h = \mathrm{id}_Z$, s f is a homeomorphism, as desired. $\square$

Example 1. Recall $\mathbb{RP}^2 = \mathbb{R}^3 - \{0\} / \sim$.

This has a particularly nice open subset $D(z)$:

$$D(z) = \left\{ [x : y : z] \in \mathbb{RP}^2 \;\middle|\; z \neq 0 \right\}.$$

This is homeomorphic to $\mathbb{R}^2$ via

$$\begin{aligned} \mathbb{R}^2 &\to D(z) \\ (x, y) &\mapsto [x : y : 1] \\ \left(\frac{x}{z}, \frac{y}{z}\right) &\leftarrow\!\shortmid [x : y : z] \end{aligned}$$

Similarly, there are open subsets

$$\begin{aligned} D(x) &= \{[x : y : z] \mid x \neq 0\} \cong \mathbb{R}^2 \\ D(y) &= \{[x : y : z] \mid y \neq 0\} \cong \mathbb{R}^2 \end{aligned}$$

Notice that these open sets form an open cover of $\mathbb{RP}^2$: $D(x) \cup D(y) \cup D(z) = \mathbb{RP}^2$.

This is nice.

Definition 1. A topological space X is said to be a **Topological Manifold** if it has an open cover $\{U_i\}_{i \in I}$ such that:

1. each U_i is homeomorphic to an open subset of $\mathbb{R}^n$ for some n (n the dimension of X)

2. X is Hausdorff

3. ~~something technical that we don't care about~~

Example 2. $\mathbb{R}^n, \mathbb{RP}^n, \mathbb{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid |\mathbf{x}| = 1\}$, donuts.

Example 3.

$$\begin{aligned}
\mathbb{RP}^1 &= \{(x,y) \mid (x,y) \neq 0\} \,/\sim \text{ scaling} \\
D(x) &= \{[x,y] \mid x \neq 0\} \xrightarrow{\sim} \mathbb{R}^1 \\
[x:y] &\longmapsto \frac{y}{x} \\
D(y) &= \{[x:y] \mid y \neq 0\} \xrightarrow{\sim} \mathbb{R}^1 \\
[x:y] &\longmapsto \frac{x}{y}.
\end{aligned}$$

we now give the gluing construction:

Input: Two top spaces U_1, U_2
An open subset U_{12} of U_1
An open subset U_{21} of U_2
A homeomorphism $\varphi_{12} : U_{12} \to U_{21}$

Output: $X = U_1 \sqcup U_2 / \sim\, = \{1\} \times U_1 \cup \{2\} \times U_2 / \sim$ Where $(1, u) \sim (2, \varphi_{12}(u))$ when $u \in U_{12}$.

Theorem 1. *X has an open cover V_1, V_2 such that there exists homeomorphisms*

1. $\varphi_1 : U_1 \to V_1$
   $\varphi_2 : U_2 \to V_2$
2. $\varphi_1(U_{12}) = \varphi_2(U_{21})$
3. $\varphi_2^{-1} \circ \varphi_1 : U_{12} \to U_{21}$ *is equal to* φ_{12}

Example 4.

$$\begin{aligned}
U_1 = \mathbb{R} \quad U_{12} = \mathbb{R} - \{0\} \\
U_2 = \mathbb{R} \quad U_{21} = \mathbb{R} - \{0\}
\end{aligned}$$

$$\begin{aligned}
\varphi : \quad U_{12} &\to U_{21} \\
x &\mapsto \tfrac{1}{x}
\end{aligned}$$

This glues to $\mathbb{RP}^1$

Example 5. Now take the same U's but instead say $x \overset{\varphi}{\mapsto} x$. Then you get a cylinder