Math 171 **Professor Sebastian Bozlee**
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We begin with a definition:

Definition 1. Let X be a topological space. We say that $Z \subseteq X$ is **closed** if $Z^c \subseteq X$ is open in X.

Proposition 1. *Suppose that X is a topological space. Then:*

1. *$X, \emptyset$ are closed subsets of X.*
2. *If $Z_i \subset X$ for each $i \in I$ is a closed subset of X, then so is $\bigcap_{i\in I} Z_i \subseteq X$ is closed is X as well.*
3. *If $Z_1, \ldots, Z_n$ are closed subsets of X, then their union $\bigcup_{i=1}^n Z_i \subseteq X$ is as well.*

Proof. We will show that this follows directly from the three tenants of what it means to be a topology.

1. Notice that $\emptyset = X^c$ and likewise $X = \emptyset^c$, so X and $\emptyset$ are trivially clopen.
2. For this and the following part, we will leverage De Morgan's laws. Let $Z_i \subseteq X$ be a closed subset for each $i \in I$ then

$$\begin{aligned}\bigcap_{i\in I} Z_i &= \left(\bigcap_{i\in I} Z_i^c\right)^c \\ &= \left(\bigcup_{i\in I} Z_i^c\right)^c\end{aligned}$$

   But we know that Z_i^c is open, so this union is an open set, so its complement is closed.

3. Let Z_i, $i = 1, \ldots, n$ be a closed collection of subsets of X. Then again:

$$\begin{aligned}\bigcup_{i=1}^n Z_i &= \left(\bigcup_{i=1}^n Z_i^c\right)^c \\ &= \left(\bigcap_{i=1}^n Z_i^c\right)^c\end{aligned}$$

   So we have the result again in the same way.

$\square$

Idea. *Any statement made in terms of open sets can be rephrased in terms of an equivalent statement about closed sets.*

Fact. *A topological space can be defined by specifying a collection of closed sets and showing that they satisfy the equivalent closed set definition of a topology.*

Proposition 2. *If X, Y are topological spaces, then a function $f : X \to Y$ s continuous iff whenever $Z \subseteq Y$ is closed in Y, $f^{-1}(Z)$ is a closed subset of X.*

Proof. ($\Rightarrow$) Assume $f : X \to Y$ is continuous. Let $Z \subseteq Y$ be a closed subset. Then $f^{-1}(Z^c)^c$ is also closed. But Z^c is open, so $f^{-1}(Z^c)$ is open, and thus $f^{-1}(Z^c)^c$ is closed.

($\Leftarrow$) Assume that pre-images of closed subsets are closed. So let $U \subseteq Y$ be an open subset. Then $f^{-1}(U) = f^{-1}(U^c)^c$, and the same thing happens again. $f^{-1}(U^c)$ is the pre-image of a closed set, which is closed, and so its complement is open. And the result falls out.

□

Definition 2. A polynomial in n variable is a function $f : \mathbb{R}^n \to \mathbb{R}$.

$$f(x_1 \dots x_n) = \sum c_I x^I$$

The **Zero Locus** of a set of polynomials $\{f_i\}_{i \in I}$is the subset

$$Z(\{f_i\}) = \{(x_1, \dots, x_n) \in \mathbb{R}^n | f_i(x_1 \dots x_n) = 0\}.$$

For example: $f(x, y) = y - x^2$: $Z(f)$ should be a parabola.

Definition 3. The zero locus of a set of polynomials is called an **algebraic variety**. The **Zariski** topology on $\mathbb{R}^n$ is the topology on $\mathbb{R}^n$ in which the algebraic varieties are the closed subsets.

Definition 4. Given a topological space (X, τ_X) and a subset $Y \subseteq X$, the **subspace topology on** Y is the topology

$$\tau_Y = \{U \cap Y | U \in \tau_X\}$$

This is possibly the most important example that we will give.

Example 1. $\{0, 1\} \subseteq \mathbb{R}$. Then $\tau_Y = \{\emptyset, \{1\}, \{0\}, \{0, 1\}\}$ where τ_Y is the subspace topology of the usual topology on $\mathbb{R}$. This gives us a reason to call the discrete topology what it is...

Proposition 3. *Let (X, τ_X) a topological space, (Y, τ_Y) a subset of X with the subspace "topology". Then τ_Y is a topology.*

Proof. 1. $\emptyset_{\tau_y} = \emptyset_{\tau_x} capY$, and furthermore $Y = Y \cap X$.

2. Suppose that $U_i \in \tau_y$ for each $i \in I$. By the definition of τ_y, for each U_i, $\exists V_i \in \tau_x$ such that $V_i \cap Y = U_i$. Then

$$\bigcup_{i \in I} U_i = \bigcup_{i \in I} (V_i \cap Y) = \left(\bigcup_{i \in I} V_i \right) \cap Y \in \tau_Y.$$

   And the same idea for the intersections.

3. Let $U_1 \dots U_n \in \tau_Y$. Then we can find $V_1, \dots, V_n \in \tau_X$ such that $U_i = V_i \cap Y$ for each $i = 1 \dots n$. Then

$$\begin{aligned} \bigcap_{i=1}^{n} U_i &= \bigcap_{i=1}^{n} (V_i \cap Y) \\ &= \left(\bigcap_{i=1}^{n} V_i \right) \cap Y \in \tau_Y \end{aligned}$$

So we really do have a topology. $\square$

Note that it is actually important to do these intersections, since open subsets of a subset might **NOT** be open in the larger space.