

Herein we will outline some review content for the exam.

**Definition 1.** Let  $\{X_i\}_{i \in I}$  be a collection of sets indexed by  $I$ . A set  $P$  together with functions  $\pi_i : P \rightarrow X_i$  for each  $i \in I$  is said to have the **universal property of the product** if, for any set  $Z$  and functions  $f_i : Z \rightarrow X_i$  for each  $i \in I$ . Then there exists a unique function  $f : Z \rightarrow P$  such that

$$\begin{array}{ccc} Z & & \\ \downarrow f & \searrow f_i & \\ P & \xrightarrow{\pi_i} & X_i \end{array}$$

**Example 1.** A standard sort of product: Take  $X_1 = \mathbb{R}, X_2 = \mathbb{R}, P = \mathbb{R}^2$ , and  $\pi_1 : P \rightarrow X_1, (x, y) \mapsto x$ , and  $\pi_2 : P \rightarrow X_2$  taking  $(x, y) \mapsto y$ . then

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \vdots \exists! f & \searrow f_2 & \\ X_1 & \xleftarrow{\pi_1} & P & \xrightarrow{\pi_2} & X_2 \end{array}$$

And in fact we can see that such an  $f$  does exist, as  $z \mapsto (f_1(z), f_2(z))$ .

**Example 2.** Here's a bit of a weirder one.  $X_1 = \mathbb{R}, X_2 = \mathbb{R}, P' = \mathbb{R}^2$ , and the functions  $\pi'_1 : P' \rightarrow X_1$  a projection on the first coordinate, and  $\pi'_2 : P' \rightarrow X_2$  takes  $(x, y) \mapsto x + y$ .

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \vdots \exists! f & \searrow f_2 & \\ X_1 & \xleftarrow{\pi'_1} & P' & \xrightarrow{\pi'_2} & X_2 \end{array}$$

Notice that to get the  $f$  that we desire, we require that  $f(z) = (f_1(z), f_2(z) - f_1(z))$

**Theorem 1.** If  $(P, \pi_i)$  has the universal property of the product and  $(P', \pi'_i)$  has universal property of the product, then there exists a unique bijection  $f : P \rightarrow P'$  such that  $\pi_i = \pi'_i \circ f$  for all  $i \in I$

$$\begin{array}{ccc} P & & \\ \downarrow f & \searrow \pi_i & \\ P' & \xrightarrow{\pi'_i} & X_i \end{array}$$

This theorem was proved in class, so we omit the proof here.

**Definition 2.** Let  $X$  be a set,  $\sim$  be an equivalence relation on  $X$ . A function  $f : X \rightarrow Y$  is said to **respect the equivalence relation** if  $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$  for  $x_1, x_2 \in X$ . A set  $\overline{X}$  together with a function  $\pi : X \rightarrow \overline{X}$  is said to have the universal property of quotients if:

1.  $\pi : X \rightarrow \overline{X}$  respects the equivalence relation.
2. For any function  $f : X \rightarrow Y$  such that  $f$  respects the equivalence relation,  $\exists! \bar{f} : \overline{X} \rightarrow Y$  such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \pi & \nearrow \bar{f} & \\ \overline{X} & & \end{array}$$

**Example 3.** in general,  $\pi : X \rightarrow X/\sim$  taking  $x \mapsto [x]$  has the universal property.

**Example 4.**  $X = \mathbb{Z}$ ,  $a \sim b \Leftrightarrow a - b \equiv 0 \pmod{6}$ . Then  $X/\sim$  is called the integers modulo 6, written  $\mathbb{Z}/6$  (or more commonly  $\mathbb{Z}/6\mathbb{Z}$ ).

Let's define "multiplying by 2".

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z}/6 \\ \downarrow \pi & \nearrow & \\ \mathbb{Z}/6 & & \end{array}$$

well notice that what we really need here is

$$\begin{array}{ccc} a & \mapsto & [2a] \\ \downarrow & \nearrow & \\ [a] & & \end{array}$$

So the thing that we really want is  $\mathbb{Z} \rightarrow \mathbb{Z}/6$  that takes  $a \mapsto [2a]$ . Does this respect the equivalence relation?

If  $a \sim b$ , is it true that  $[2a] = [2b]$ ?

$$\begin{aligned} a \sim b &\Rightarrow a - b \equiv 0 \pmod{6} \\ &\Rightarrow a - b = 6n \quad (n \in \mathbb{Z}) \\ &\Rightarrow 2a - 2b = 6(2n) \\ &\Rightarrow 2a \sim 2b \\ &\Rightarrow [2a] = [2b]. \end{aligned}$$

So we have what we want.

**Definition 3.** Let  $Z \subseteq \mathbb{R}^n$ . A point  $\mathbf{x} \in \mathbb{R}^n$  is a **limit point of  $Z$**  if for all  $\varepsilon > 0$ ,

$$(B(\mathbf{x}, \varepsilon) - \{\mathbf{x}\}) \cap Z \neq \emptyset.$$

A point  $\mathbf{x} \in \mathbb{R}^n$  is said to be an **adherent point of  $Z$**  if for all  $\varepsilon > 0$ ,

$$B(\mathbf{x}, \varepsilon) \cap Z \neq \emptyset.$$

Equivalently,  $\mathbf{x}$  is adherent if  $\mathbf{x}$  is a limit point or  $\mathbf{x} \in Z$ .

**Example 5.** Let

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$x \mapsto \begin{cases} 0 & x < 0 \\ 20 & x \geq 0 \end{cases}.$$

Obviously, this should not be continuous. Recall the definition of continuity. Namely that  $\forall \varepsilon > 0, \forall x \exists \delta > 0 \forall y |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ . Furthermore, non-continuity entails that  $\exists \varepsilon > 0 \exists x \forall \delta > 0 \exists y |x - y| < \delta$  and  $|f(x) - f(y)| \geq \varepsilon$ .

So let  $x = 0, \varepsilon = 10$ . now let  $\delta > 0$ . Let  $y = \frac{\delta}{2}$ . Then  $|x - y| = \frac{\delta}{2} < \delta$ , but  $|f(x) - f(y)| = |20 - 0| = 20 > 10 = \varepsilon$

Now let's take  $g : \mathbb{R} \rightarrow \mathbb{R}$   
 $x \mapsto 3x$ . So we can get  $\forall \varepsilon > 0 \forall x, \exists \delta > 0, \forall y |x - y| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon$ . Now let  $\varepsilon > 0$ , and  $x \in \mathbb{R}$ . Take  $\delta = \frac{\varepsilon}{3}$ . Then for all  $y \in \mathbb{R}$  with  $|x - y| < \frac{\varepsilon}{3}$ ,

$$\begin{aligned} |g(x) - g(y)| &= |3x - 3y| \\ &= 3|x - y| \\ &< 3\left(\frac{\varepsilon}{3}\right) \\ &= \varepsilon. \end{aligned}$$

So  $g$  is continuous.