We begin with a quick reminder that, along $\mathbb{S}^1$, the open sets are open sets in $\mathbb{R}^2$ which are intersected with the circle, and likewise that half-open intervals at the endpoints of $[0,1]$ are also open.

Now moving on to connectedness.

Definition 1. Let X be a topological space. A **separation** of X is a pair U, V of disjoint, nonempty, open subsets of X whose union is all of X. We say that X is **connected** if X does not have a separation.

Remark.

- *The definition of connected is homeomorphism invariant*
- *If U, V is a separation of X, then $U^c = V$ and $V^c = U$ are open, so U and V are both open and closed.*
- *X is connected iff the only subsets of X that are both open and closed are $\emptyset$ and X itself. (To see this, simply take a complement of a nontrivial clopen set. What do you get?)*

Example 1.

- Consider $X = \{1, 2\}$ with the indiscrete topology. This is clearly connected.
- Let $X = [-1, 0) \cup (0, 1]$ w/ subspace topology. This is disconnected, since both $[-1, 0)$ and $(0, 1]$ are disjoint open sets[1] which union to the entire space.
- Let $X = [-1, 1]$. This is connected, but that's kind of a truck to prove.
- Let $X = \mathbb{Q}$. The only subspaces of $\mathbb{Q}$ which are connected are the singleton sets. If Y is any subspace of $\mathbb{Q}$ containing $p < q$, then there is an irrational number a between p and q. Then $U = (-\infty, a) \cap Y$, $V = (a, \infty) \cap Y$ is a separation of Y.

Theorem 1. *Let X be a connected topological space, $f : X \to Y$ a continuous function. Then $f(X)$ is connected (w.r.t. the subspace topology)*

Proof. Let $Z = f(X)$, and notice that $\begin{matrix} g : X \to Z \\ x \mapsto f(x) \end{matrix}$ is also continuous.

So suppose for the sake of contradiction that there exists a separation U, V of Z. Then we claim that $U' = g^{-1}(U)$, $V' = g^{-1}(V)$ is a separation of X.

- *(disjoint):* $U' \cap V' = g^{-1}(U) \cap g^{-1}(V) = g^{-1}(U \cap V) = g^{-1}(\emptyset) = \emptyset$
- *(nonempty):* $U', V' \neq \emptyset$ since g is surjective.

[1]Remember, this is in the subspace topology on $\mathbb{R}$

- *(open):* Follows trivially since g is continuous.
- *(Union is X):* $U' \cup V' = g^{-1}(U) \cup g^{-1}(V) = g^{-1}(U \cup V) = g^{-1}(Z) = X$.

But then we've given a separation of X, which is connected. A contradiction! So we must have that $f(x) = Z$ is connected. $\square$

Now we'll prove a sequence of lemmas, with the goal of proving that product spaces are connected.

Lemma 1. *Suppose that X is a topological space, and further suppose that A is a connected subspace. If U, V is a separation of X, then $A \subseteq U$ or $A \subseteq V$.*

Proof. Consider $U' = U \cap A$ and $V' = V \cap A$. Then U' and V' are disjoint, open in A and $U' \cup V' = A$.

Since A is connected, it must be so that U' or V' is empty, otherwise we would have a separation.

If $U' = \emptyset$, $A \subseteq V$, and if $V' = \emptyset$, then $A \subseteq U$. $\square$

Theorem 2. *If $\{A_i\}_{i \in I}$ is a collection of connected subspaces of a topological space X and $\bigcap_{i \in I} A_i \neq \emptyset$, then $\bigcup_{i \in I} A_i$ (with the subspace topology) is connected.*

Proof. Suppose – again for the sake of contradiction – that U, V is a separation of $\bigcup_{i \in I} A_i$.

Let $p \in \bigcup_{i \in I} A_i$. Suppose without loss of generality that $p \in U$.

Consider A_i for some $i \in I$. Then by the lemma, $A_i \subseteq U$ or $A_i \subseteq V$. Since $p \in A_i$ and $p \in U$, it must follow that $A_i \subseteq U$. Since this is true for all i, $\bigcup_{i \in I} A_i \subseteq U$. But then $V = \bigcup_{i \in I} A_i - U = \emptyset$, again a contradiction.

So $\bigcup_{i \in I} A_i$ must be connected. $\square$

Theorem 3. *Let X, Y be connected topological spaces. Then $X \times Y$ is connected as well.*

Proof. First, notice that if $X, Y = \emptyset$, then $X \times Y = \emptyset$ is connected.

So let's suppose that X and Y are not empty. Then $\exists (a, b) \in X \times Y$.

Notice: $X \times \{b\} \cong X$ is connected. Likewise, for any $x \in X$, $\{x\} \times Y \cong Y$ is connected.

Thus by the lemma, $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$ is connected.

Then $\bigcup_{x \in X} T_x = X \times Y$ and $(a, b) \in \bigcup_{x \in X} T_x$. So again by the lemma, $X \times Y$ is connected. $\square$