We recall the definition of the product topology:

Definition 1. Let X, Y be topological spaces. Then the **product topology** is the topology on $X \times Y$ generated by the basis:

$$\mathcal{B} = \{U \times V \mid U \subseteq X \text{ open}, \ V \subseteq Y \text{ open}\}.$$

Example 1. $\mathbb{R}^2$ has the product topology induced by thinking of $\mathbb{R}^2$ as $\mathbb{R} \times \mathbb{R}$.

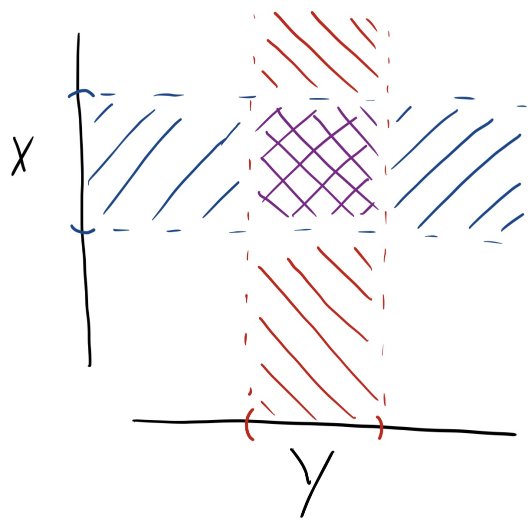

Fig. 1. Basis by open rectangles

Fact. *If $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X, Y, then*

$$\mathcal{B} = \{B_X \times B_Y \mid B_X \in \mathcal{B}_X, B_Y \in \mathcal{B}_Y\}$$

is also a basis for the product topology.

Proposition 1. *The set*

$$\{U \times V \mid U \subseteq X \text{ open}, V \subseteq Y \text{ open}\}$$

really is a basis.

Proof. 1. *(Each point belongs to a basis element)* Let $(x, y) \in X \times Y$. Then $(x, y) \in X \times Y$, and $X \subseteq X$ is open, $Y \subseteq Y$ is open, so we have a basis element containing (x, y).

2. Let $B_1 = U_1 \times V_1$, $B_2 = U_2 \times V_2$, where the U_i are open in X, and V_j are open in Y. Then

$$
\begin{aligned}
B_1 \cap B_2 &= (U_1 \times V_1) \cap (U_2 \times V_2) \\
&= \{(x,y) \in X \times Y \mid x \in U_1 \text{ and } y \in V_1\} \cap \{(x,y) \in X \times Y \mid x \in U_2 \text{ and } y \in V_2\} \\
&= \{(x,y) \in X \times Y \mid x \in U_1 \wedge x \in U_2 \wedge y \in V_1 \wedge Y \in V_2\} \\
&= \{(x,y) \in X \times Y \mid x \in U_1 \cap U_2 \wedge y \in V_1 \cap V_2\} \\
&= (U_1 \cap U_2) \times (V_1 \cap V_2)
\end{aligned}
$$

This is a basis element, so for any $(x, y) \in B_1 \cap B_2$ we can set $B_3 = B_1 \cap B_2$ to satisfy the axiom.

$\square$

Theorem 1. *Let X, Y be topological spaces.*

1. *The product topology on $X \times Y$ is the coarsest topology such that*

$$
\begin{aligned}
\pi_1 : X \times Y &\to X \qquad & \pi_2 : X \times Y &\to Y \\
(x,y) &\mapsto x & (x,y) &\mapsto y
\end{aligned}
$$

are continuous functions.

$X \times Y$ with the product topology and π_1, π_2 have the universal property of the product for topological spaces.

Proof. 1. Recall: π_1 is continuous iff for all $U \subseteq X$ open, we have that $\pi_1^{-1}(U = U \times Y)$ is open. Similarly for π_2, we need tat for all $V \subseteq Y$ open, we have that $\pi_1^{-1}(V) = X \times V$ is open. These are open in the product topology so π_1, π_2 are continuous. To complete the proof, suppose that τ is a topology on $X \times Y$ such that π_1, π_2 are continuous. Then for all $U \subseteq X$ open, $V \subseteq Y$ open, we have that $U \times Y$ and $X \times V$ are open. Then intersection $(U \times Y) \cap (X \times V) = (U \cap X) \times (Y \cap V) = U \times V$ is open as well.

Then since τ contains the basis for the product topology, we proved on the HW that τ contains the product topology too.

2. Now let's show that we have the universal property of the product. We need to show that for all diagrams of topological spaces

$$
\begin{tikzcd}
 & Z \arrow[dl, "f_1"'] \arrow[d, "\exists! f"] \arrow[dr, "f_2"] & \\
X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y
\end{tikzcd}
$$

there is a unique continuous function f such that the diagram commutes. By the universal property of products of sets, there is a unique function $f : Z \to X \times Y$

making the diagram commute, namely $z \mapsto (f_1(z), f_2(z))$. It suffices to show that this f is continuous.

By HW, it suffices to show $f^{-1}(B)$ is open for each basic open set in the product topology.

Let $B = U \times V$ be a basis element, where $U \subseteq X$ open $V \subseteq Y$ open. Then

$$\begin{aligned} f^{-1}(B) &= \{z \in Z \mid f(z) \in B\} \\ &= \{z \in Z \mid (f_1(z), f_2(z)) \in U \times V\} \\ &= \{z \in Z \mid f_1(z) \in U \text{ and } f_2(z) \in V\} \\ &= \{\in Z \mid f_1(z) \in U\} \cap \{z \in Z \mid f_1(z) \in V\} \\ &= \underbrace{\underbrace{f_1^{-1}(U)}_{\text{open}} \cap \underbrace{f_2^{-1}(V)}_{\text{open}}}_{\text{open}} \end{aligned}$$

$\square$

Corollary 1 (Universal property, short version). *If $f_1 : Z \to X$ and $f_2 : Z \to Y$ are continuous functions, then*

$$\begin{aligned} f_1 \times f_2 : &Z \to X \times Y \\ &z \mapsto (f_1(z), f_2(z)) \end{aligned}$$

is continuous too.

Example 2. $\mathbb{R}^{n_1+n_2}$ has the universal property of the product induced by thinking of $\mathbb{R}^{n_1+n_2}$ as $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Therefore $\mathbb{R}^{n_1+n_2}$ has the product topology.

Let's give one final lemma which should illustrate how we reason about the product topology.

Lemma 1. *Let X, Y be topological spaces. Let $Z \subseteq X$ and $W \subseteq Y$ be closed subsets. Then $Z \times W \subseteq X \times Y$ is closed.*

Proof. We want to show that $(Z \times W)^c$is open in the product topology. Notice that $(Z \times W)^c = \underbrace{(Z^c \times Y)}_{\text{open}} \cap \underbrace{(X \times W^c)}_{\text{open}}$. So $(Z \times W)^c$is open, as desired. $\square$

Question 1. What about more spaces? $X_1 \times \ldots \times X_n$ should have the topology generated by $U_1 \times \ldots U_n$.

$\prod_{i \in I} X_i$ should have the topology generated by $\prod_{i \in I} U_i$ where the U_i are open for each i, and $U_i = X_i$ *for all but finitely many* i.