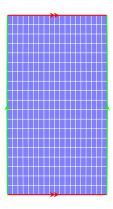
We begin thinking about the quotient topology, which will give us the power to "glue" topological spaces.

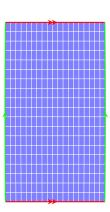
The most common example is the famous transformation of the unit square into a torus (doughnut) by associating opposite sides of squares as equal. Namely, we identify

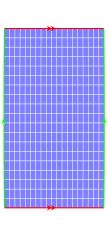
$$(0,y) \sim (1,y)$$

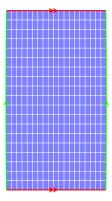
$$(x,0) \sim (x,1)$$

in the space $X = [0, 1] \times [0, 1]$.

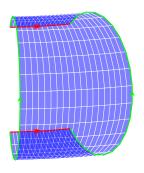


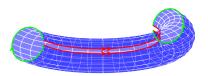


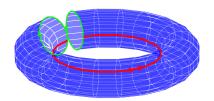


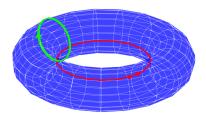












So really what we are doing is identifying equivalence classes, which immediately brings to mind quotients and their universal property:

Theorem 1. $p: X \to X/\sim a$ continuous function has the following property:

- 1. if $f: X/\sim \to Z$ is any continuous function, $f\circ p: X\to Z$ respects the equivalence relation.
- 2. If $g: X \otimes Z$ is any continuous function that respects \sim , then there exists a unique continuous function $f: X/\sim \to Z$ such that

$$X \downarrow p \qquad g \\ X/\sim \xrightarrow{\exists !f} Z$$

What should go in this topology?

Recall $S = \{1, 2\}$ with the topology $\tau_S = \{\emptyset, \{1\}, S\}$.

Then a continuous function $f: X \to S$ is "the same" as an open subset of X. In other words, given $f, f^{-1}(\{1\})$ is an open set.

Assume that $p: X \to X/\sim$ has the universal property. Then from part 2, for any function $g: X \to S$ respecting \sim , we should get a unique continuous $f: X/\sim S$ making the diagram commute. Since g respects \sim , if $x \in g^{-1}(\{1\})$, we better have for any $x_2 \sim x_1$ that $x_2 \in g^{-1}(\{1\})$ i.e. the possible open subsets we get from $g^{-1}(\{1\})$ are the open subsets with the property that $x_1 \in U$, $x_1 \sim x_2 \Rightarrow x_2 \in U$.

So using the universal property, we should have an $f: X/\sim \to S$ that makes $X/\sim \to S$ commute.

This function is defined by f([x]) = g(x).

Claim. The open subsets we get from g are the preimages of the open subsets we get from f. It follows that a subset $U \subseteq X/\sim$ should be open if and only if $p^{-1}(U)$ is open

Definition 1. A continuous function $p: X \to Y$ of topological spaces is a **quotient map** if $U \subset Y$ is open if and only iff $p^{-1}(U) \subset X$ is open.

This can be stated equivalently using complements as:

$$Z \subseteq Y$$
 is closed $\iff p^{-1}(Z) \subseteq X$ is closed.

Fact. If $p: X \to Y$ is surjective and $B \subseteq Y$ then

$$p(p^{-1}(B)) = B$$

Notice that this is not necessarily true, and in fact is often not, if p is not surjective.

Definition 2. A subset C of X is said to be **saturated** with respect to a function $p: X \to Y$ if there exists a subset B of Y such that $C = p^{-1}(B)$.

Lemma 1. $C \subseteq X$ is saturated with respect to $p: X \to Y$ if and only if C contains every fiber $p^{-1}(\{y\})$ that is intersects.

Proof. (\Rightarrow) Suppose that C is saturated, and let $B \subseteq Y$ be the subset such that $C = p^{-1}(B)$.

Then if $y \in y$, and $C \cap p^{-1}(\{y\})$, then $\exists c \in C$ such that p(c) = y.

Then
$$y \in B$$
 so $p^{-1}(\{y\}) \subseteq p^{-1}(B) = C$ by assumption \triangle

(\Leftarrow) Now suppose that C contains every fiber that it intersects. Consider B = p(C). Then since $B = \bigcup_{b \in B} \{b\}, \ p^{-1}B = \bigcup_{b \in B} p^{-1}(\{b\})$.

Since $p^{-1}(\{b\}) \cap C \neq \emptyset$ for all $b \in B$, $p^{-1}(\{b\}) \subseteq C$.

Then this implies that $p^{-1}(B) \subseteq C$.

On the other hand,
$$C \subseteq p^{-1}(p(C))$$
, so $C = p^{-1}(B)$.

Now with all of this done, we can finally re-write what a quotient map is.

$$p: X \to Y$$
 is a quorient map $\iff \begin{pmatrix} p \text{ is continuous, surjective, and} \\ \text{the image of each saturated} \\ \text{open set of } X \text{ open} \end{pmatrix}$

 $(\forall U \subseteq Y, U \text{ open}) \Leftrightarrow p(v) \text{ is open in } y \Leftrightarrow V \text{ is a saturated subset of } X \text{ open}$

Example 1. $\pi_1: X \times Y \to X$ is a quotient map.

Definition 3. If X is a topological space and A is a set, and $p: X \to A$ is a surjective function, there is a unique topology on A such that p becomes a quotient map. This is the **quotient topology** on X.

$$U \subseteq A \text{ open } \Leftrightarrow p^{-1}(U) \subseteq X \text{ open}$$

Let's now prove that this is a topology. This should be easy.

Proof. 1. $\emptyset \subseteq A$ is open since $p^{-1}(\emptyset) = \emptyset$ is open inp X

- 2. $A \subseteq A$ is open since $p^{-1}(A) = X$ is open in X.
- 3. If $\{U_i\}$ are open, then $p(\bigcup U_i) = \bigcup_{i \in I} \underbrace{p^{-1}(U_i)}_{\text{open in } X}$, so $\bigcup_{i \in I} U_i$ is open.