The final will be take home, posted on Gradescope Thursday, May 11th.

There will be a 24 hour window. The exam will be open notes, open book, closed other people, closed internet.

$\frac{1}{3}$ should be pre-midterm content, the remainder being new.

Definition 1. A topological space X is **compact** if each open cover of X admits a finite subcover.

Theorem 1. *A closed subspace of a compact space is compact.*

Proof. Let X be a compact topological space, and let $Z \subseteq X$ be a closed subspace. Recall that Z is compact iff every collection of open sets $\{U_i\}$ in X with union containing Z has a finite subcollection with the same property.

Let $\{U_i\}_{i\in I}$ be a collection of open subsets of X such that $\bigcup_{i\in I} U_i \supseteq Z$. Notice now that since Z is closed, $\{U_i\}_{i\in I} \cup \{X - Z\}$ is an open cover of X. Since X is compact, there exists $i_1, \ldots, i_n$ such that $\bigcup_{j=1}^n U_{i_j} \cup (X - Z) = X$.

Then $\bigcup_{j=1}^n U_{i_j} \supseteq Z$, and thus Z is compact. $\square$

Theorem 2. *Let X be a Hausdorff space, and let $Z \subseteq X$ be a compact subspace. Then Z is closed inside of X.*

Proof. Let X be a Hausdorff space, Z a compact subspace. We want to show that $Z \supseteq \overline{Z}$. (Then $Z = \overline{Z}$, so Z is closed).

We'll show if $y \in X - Z$, then $y \notin \overline{Z}$.

Let $y \in X - Z$. Then for each point $x \in Z$, we can find disjoint open neighborhoods U_x of x and V_x of y. Observe $\{U_x\}_{x\in X}$ are a collection of open sets whose union contains Z. Since Z is compact, there exists some finite subcover: there are $x_1, \ldots, x_n$ such that $U_{x_1} \cup \ldots \cup U_{x_n} \supseteq Z$. Then observe that $V_{x_1} \cap \ldots \cap V_{x_n}$ is an open neighborhood of y disjoint from $U_{x_1} \cup \ldots \cup U_{x_n}$.

This implies $(V_{x_1} \cap \ldots \cap V_{x_n}) \cap Z = \emptyset$. Thus $y \notin \overline{Z}$. Then Z is closed, as desired. $\square$

Example 1. $(0, 1]$ is not compact. A quick way to show this is to note $(0, 1]$ is not a closed subspace of $\mathbb{R}$.

Theorem 3. *Let $f : X \to Y$ be a continuous function. If Z is a compact subspace of X, then $f(Z)$ is a compact subspace of Y.*

Proof. We can assume $X = Z$.

Then we want to show $f(X)$ is compact. Let $\{U_i\}_{i\in I}$ be a collection of open subsets of Y with $\bigcup_{i\in I} U_i \supseteq f(X)$.

Taking pre-images, $\{f^{-1}(U_i)\}_{i\in I}$ is a collection of open subsets of X with union.

$$\bigcup_{i\in I} f^{-1}(U_i) = f^{-1}\left(\bigcup_{i\in I} U_i\right) \supseteq f^{-1}(f(X)) = X.$$

Since X is compact, there are indices $i_1, \ldots, i_n$ such that $\bigcup_{j=1}^{n} f^{-1}(U_{i_j}) = X$. Then we claim $\bigcup_{j=1}^{n} U_{i_j} \supseteq f(X)$.

Suppose for the sake of contradiction that there's some $x \in X$ such that $f(x) \notin U_{i_j}$ for all $j = 1, \ldots, n$. But this implies $x \notin f^{-1}(U_{i_j})$ for all $j = 1, \ldots, n$ $-\!\!\!\times\!\!\!-$ $\square$

Corollary 1. *If $f : X \to Y$ is a continuous function from a compact space to a Hausdorff space, then f is a closed map.*

Proof. Suppose $Z \subseteq X$ is closed. Then Z is compact. Then $f(Z)$ is compact. Then $f(Z)$ is closed. $\square$

Corollary 2. *If $f : X \to Y$ is bijective continuous function from a compact space to a Hausdorff space, then f is a homeomorphism.*

Example 2. Let $X = S^1$. consider $\sim$, $\mathbf{x} \sim \pm\mathbf{x}$

$$\begin{tikzcd}
S^1 \arrow[d, "p"'] \arrow[dr, "g"] & \\
S^1/\{\pm\} \arrow[r, dashed] & S^1
\end{tikzcd}$$

$S^1/\{\pm\}$ is compact, and S^1 is Hausdorff, so by corollary $S^1/\{\pm\} \cong S^1$.

> "Compact + Hausdorff = Super closed".
>
> If $i : X \hookrightarrow X$ is the inclusion of a compact subspace to be a Hausdorff space, then $Z' \subseteq Z$ is closed iff $i(Z')$ in X is closed.

Theorem 4. *If X and Y are compact subspaces, then $X \times Y$ is also compact.*

Lemma 1 (Tube Lemma)**.** *If X is a topological space and Y is a compact space. Let $x_0 \in X$, and let $N \subseteq X \times Y$ be an open subset containing $x_0 \times Y$, then N contains some "tube" $W \times Y$, where W is an open neighborhood of x_0 in X.*