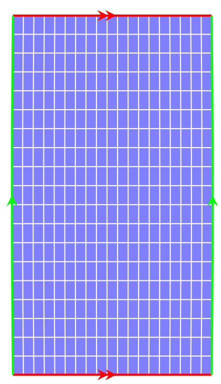
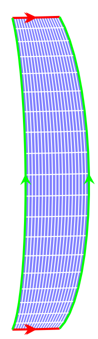
We begin thinking about the quotient topology, which will give us the power to "glue" topological spaces.

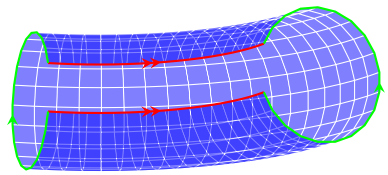
The most common example is the famous transformation of the unit square into a torus (doughnut) by associating opposite sides of squares as equal. Namely, we identify

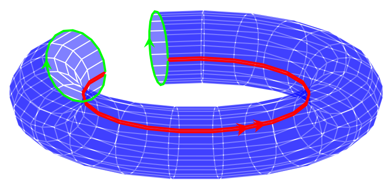
$$(0, y) \sim (1, y)$$
$$(x, 0) \sim (x, 1)$$

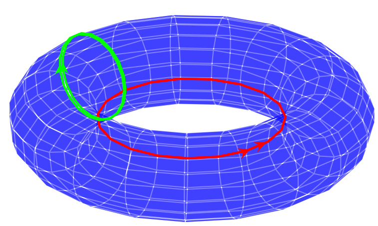
in the space $X = [0, 1] \times [0, 1]$.











So really what we are doing is identifying equivalence classes, which immediately brings to mind quotients and their universal property:

Theorem 1. $\begin{array}{c} p : X \to X/\sim \\ x \mapsto [x] \end{array}$ *a continuous function has the following property:*

1. *if* $f : X/\sim \to Z$ *is any continuous function,* $f \circ p : X \to Z$ *respects the equivalence relation.*

2. *If $g : X \to Z$ is any continuous function that respects $\sim$, then there exists a unique continuous function $f : X/\sim \to Z$ such that*

$$\begin{tikzcd} X \arrow[d, "p"] \arrow[dr, "g"] & \\ X/\sim \arrow[r, dotted, "\exists! f"] & Z \end{tikzcd}$$

What should go in this topology?

Recall $S = \{1, 2\}$ with the topology $\tau_S = \{\emptyset, \{1\}, S\}$.

Then a continuous function $f : X \to S$ is "the same" as an open subset of X. In other words, given f, $f^{-1}(\{1\})$ is an open set.

Assume that $p : X \to X/\sim$ has the universal property. Then from part 2, for any function $g : X \to S$ respecting $\sim$, we should get a unique continuous $f : X/\sim \to S$ making the diagram commute. Since g respects $\sim$, if $x \in g^{-1}(\{1\})$, we better have for any $x_2 \sim x_1$ that $x_2 \in g^{-1}(\{1\})$. i.e. the possible open subsets we get from $g^{-1}(\{1\})$ are the open subsets with the property that $x_1 \in U,\ x_1 \sim x_2 \Rightarrow x_2 \in U$.

So using the universal property, we should have an $f : X/\sim \to S$ that makes $\begin{tikzcd} X \arrow[d] \arrow[dr] & \\ X/\sim \arrow[r] & S \end{tikzcd}$ commute.

This function is defined by $f([x]) = g(x)$.

Claim. *The open subsets we get from g are the preimages of the open subsets we get from f. It follows that a subset $U \subseteq X/\sim$ should be open if and only if $p^{-1}(U)$ is open*

Definition 1. A continuous function $p : X \to Y$ of topological spaces is a **quotient map** if $U \subseteq Y$ is open if and only iff $p^{-1}(U) \subseteq X$ is open.

This can be stated equivalently using complements as:

$$Z \subseteq Y \text{ is closed} \iff p^{-1}(Z) \subseteq X \text{ is closed.}$$

Fact. *If $p : X \to Y$ is surjective and $B \subseteq Y$ then*

$$p(p^{-1}(B)) = B$$

Notice that this is not necessarily true, and in fact is often not, if p is not surjective.

Definition 2. A subset C of X is said to be **saturated** with respect to a function $p : X \to Y$ if there exists a subset B of Y such that $C = p^{-1}(B)$.

Lemma 1. *$C \subseteq X$ is saturated with respect to $p : X \to Y$ if and only if C contains every fiber $p^{-1}(\{y\})$ that is intersects.*

Proof. ($\Rightarrow$) Suppose that C is saturated, and let $B \subseteq Y$ be the subset such that $C = p^{-1}(B)$.

Then if $y \in y$, and $C \cap p^{-1}(\{y\})$, then $\exists c \in C$ such that $p(c) = y$.

Then $y \in B$ so $p^{-1}(\{y\}) \subseteq p^{-1}(B) = C$ by assumption $\triangle$

($\Leftarrow$) Now suppose that C contains every fiber that it intersects. Consider $B = p(C)$. Then since $B = \bigcup_{b \in B} \{b\}$, $p^{-1}B = \bigcup_{b \in B} p^{-1}(\{b\})$.
Since $p^{-1}(\{b\}) \cap C \neq \emptyset$ for all $b \in B$, $p^{-1}(\{b\}) \subseteq C$.
Then this implies that $p^{-1}(B) \subseteq C$.
On the other hand, $C \subseteq p^{-1}(p(C))$, so $C = p^{-1}(B)$. $\square$

Now with all of this done, we can finally re-write what a quotient map is.

$$p : X \to Y \text{is a quorient map} \Longleftrightarrow \begin{pmatrix} p \text{ is continuous, surjective, and} \\ \text{the image of each saturated} \\ \text{open set of } X \text{ open} \end{pmatrix}$$

$$(\forall U \subseteq Y, U \text{ open}) \Leftrightarrow p(v) \text{ is open in } y \Leftrightarrow V \text{ is a saturated subset of } X \text{ open}$$

Example 1. $\pi_1 : X \times Y \to X$ is a quotient map.

Definition 3. If X is a topological space and A is a set, and $p : X \to A$ is a surjective function, there is a unique topology on A such that p becomes a quotient map. This is the **quotient topology** on X.

$$U \subseteq A \text{ open} \Leftrightarrow p^{-1}(U) \subseteq X \text{ open}$$

Let's now prove that this is a topology. This should be easy.

Proof.

1. $\emptyset \subseteq A$ is open since $p^{-1}(\emptyset) = \emptyset$ is open inp X
2. $A \subseteq A$ is open since $p^{-1}(A) = X$ is open in X.
3. If $\{U_i\}$ are open, then $p\left(\bigcup U_i\right) = \underbrace{\bigcup_{i \in I} \underbrace{p^{-1}(U_i)}_{\text{open in } X}}_{\text{open in } X}$, so $\bigcup_{i \in I} U_i$ is open.

$\square$