

**Theorem 1** (Universal Property of the Quotient). *Let  $p : X \rightarrow Y$  be a quotient map. Then for any continuous function  $g : X \rightarrow Z$  satisfying  $p(x_1) = p(x_2) \Rightarrow g(x_1) = g(x_2)$ , there exists a unique function  $f : Y \rightarrow Z$  such that the following diagram commutes:*

$$\begin{array}{ccc} X & & \\ p \downarrow & \searrow g & \\ Y & \xrightarrow{f} & Z \end{array}$$

*Proof.* We want to show that this  $f$  is well-defined, unique, makes the diagram commute, and is continuous.

For well-definedness, we need that each element  $y \in Y$  can be written as  $p(x)$  for some  $x \in X$  and if  $p(x_1) = p(x_2)$  for some  $x_1, x_2 \in X$  we have that  $g(x_1) = g(x_2)$ . The first part is true since  $p$  is surjective, and the second part is true by hypothesis.

in order for the diagram to commute, we need

$$f(p(x)) = g(x) \quad \text{for all } x \in X,$$

Which is true by construction, so  $f(p(x)) = g(x)$  fully determines  $f$ , so we have uniqueness.

To check that  $f$  is continuous, suppose  $U \subseteq Z$  is an open subset. Then since  $g$  is continuous,  $g^{-1}(U) \subseteq X$  is open. Then  $g^{-1}(U) = p^{-1}(f^{-1}(U))$  is open in  $X$ .

Then by definition of the quotient map,  $f^{-1}(U) \subseteq Y$  is open, as desired.  $\square$

**Example 1** (Quotients need not behave well with respect to subspaces). Write  $\mathbb{R}^\delta$  for  $\mathbb{R}$  with the discrete topology, and  $\mathbb{R}^i$  for  $\mathbb{R}$  with the indiscrete topology.

Say

$$X = \mathbb{R}^\delta \sqcup \mathbb{R}^i = \{(1, x) \mid x \in \mathbb{R}\} \cup \{(2, x) \mid x \in \mathbb{R}\}.$$

Which will gain the union topology: the open sets are those of the form  $(1 \times U) \cup (2 \times V)$  where  $U \subseteq \mathbb{R}^\delta$  and  $V \subseteq \mathbb{R}^i$  are open.

$$\pi : X \rightarrow \mathbb{R}^i$$

Consider  $p(1, x) \mapsto x$ , which we claim to be a quotient map.

$$(2, x) \mapsto x$$

Let  $B \subseteq \mathbb{R}^i$ , then  $\pi^{-1}(B) = (1 \times B) \cup (2 \times B)$  is open if and only if  $B = \emptyset, \mathbb{R}$

But! If we restrict  $\pi$  to  $\mathbb{R}^\delta$ , then we get  $\begin{matrix} \mathbb{R}^\delta \rightarrow \mathbb{R}^i \\ (1, x) \mapsto x \end{matrix}$  which is not a quotient, since too many pre-images are open.

**Lemma 1.** *Let  $p : X \rightarrow Y$  be a quotient map,  $B \subseteq Y$  is a subset,  $A = p^{-1}(b)$  be its preimage. If  $B$  is open (or closed), then the restriction  $q = p|_A : A \rightarrow B$  is also a quotient map. (e.g.  $\tilde{D}(z) \rightarrow D(z)$  from this weeks recitation is an example).*

*Proof.* Note that if  $V \subseteq B$  then  $q^{-1}(V) = p^{-1}(V)$ . Assume that  $A$  is open, and let  $V \subseteq B$  be a subset. We know that  $q$  is continuous and  $q$  is surjective by definition. All that remains to be shown is that  $q^{-1}(V) \subseteq A$  is open, so  $V \subseteq B$  is open.

Well  $q^{-1}(V) = p^{-1}(V)$  since  $q^{-1}(V) \subseteq A$  is open and  $A \subseteq X$  is open,  $q^{-1}(V) = p^{-1}(V)$  is open in  $X$ .

Since  $p$  is a quotient map,  $V \subseteq Y$  is open. Since  $V \subseteq B$  and  $B \subseteq Y$  is open,  $V \subseteq B$  is open.

For the case where  $B$  is closed, replace “open” with “closed” throughout.  $\square$

**Example 2** (Hausdorff not preserved by quotients). Let  $X = \mathbb{R} \sqcup \mathbb{R}$ , and take the equivalence relation

$$(1, x) \sim (2, x) \quad \text{when } x \neq 0.$$

And consider  $\pi : X \rightarrow X/\sim$ . This construction is called the “bug-eyed line” or the “line with two origins”

Let  $x = [(1, 0)], y = [(2, 0)]$ . Suppose that  $U$  is an open neighborhood of  $x$  and  $V$  is an open neighborhood of  $Y$ .

Well since  $U$  is open,  $\pi^{-1}(U)$  is open. Then for any arbitrary set, its pre-image will be the two copies of  $U$  which live in each copy of  $\mathbb{R}$  everywhere but at 0.

Notice that  $U$  and  $V$  necessarily intersect, so  $X/\sim$  is not Hausdorff.

**Question 1.** What if  $\sim$  was  $(1, x) \sim (2, \frac{1}{x})$  instead? What would  $X/\sim$  be?

**Lemma 2.** Let  $p : X \rightarrow Y$  be a quotient map.  $q : Y \rightarrow Z$  be a function. Then  $q$  is a quotient map if and only if  $q \circ p$  is a quotient map.

*Proof.* Suppose that  $q$  is a quotient map. then  $q \circ p$  is continuous and surjective. Then

$$\begin{aligned} B \subseteq Z \text{ open} &\Leftrightarrow q^{-1}(B) \subseteq Y \text{ open} \\ &\Leftrightarrow p^{-1}(q^{-1}(B)) \subseteq X \text{ open} \\ &\Leftrightarrow (q \circ p)^{-1}(B) \subseteq X \text{ open.} \end{aligned}$$

$\triangle$

Conversely, suppose  $q \circ p : X \rightarrow Z$  is a quotient map. Then since  $q \circ p$  is surjective,  $q$  is surjective. Suppose  $B \subseteq Z$ . Then

$$\begin{aligned} B \subseteq Z \text{ open} &\Leftrightarrow (q \circ p)^{-1}(B) \subseteq X \text{ open} \\ &\Leftrightarrow p^{-1}(q^{-1}(B)) \subseteq X \text{ open} \\ &\Leftrightarrow q^{-1}(B) \subseteq Y \text{ open.} \end{aligned}$$

Therefore  $q$  is a quotient map.  $\square$

**Example 3.** Let  $X = \mathbb{R}^3 - \{0\}$ ,  $\sim$  be the relation

$$\mathbf{x} \sim \mathbf{y} \iff \exists c \in \mathbb{R} - \{0\} \text{ s.t. } c\mathbf{x} = \mathbf{y}.$$

Then  $\pi : X \rightarrow X/\sim = \mathbb{RP}^2$ .

Let

$$p : X \rightarrow \mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

$$\mathbf{x} \mapsto \frac{1}{|\mathbf{x}|}\mathbf{x}.$$

Believe that this is a quotient map.

Define  $q : \mathbb{S}^2 \rightarrow \mathbb{RP}^2$

$\mathbf{x} \mapsto [x : y : z]$

The composite  $q \circ p = \pi$ .