Math 171 **Professor Sebastian Bozlee**
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Definition 1. A subset I of $\mathbb{R}$ is said to be convex if for any $a, b \in I$ with $a < b$, we have $[a, b] \subseteq I$.

Example 1. $\emptyset$, $\{a\}$, (a, b), $(a, b]$, $[a, b)$, $[a, b]$, $[a, \infty)$, (a, ∞), $(-\infty, a)$, $(-\infty, a]$, (∞, ∞).

Theorem 1. *A subspace I of $\mathbb{R}$ is connected if and only if it is convex.*

Proof. Suppose I is connected. Suppose for the sake of contradiction that I is not convex. Then $\exists a, b \in I$, $a < b$ such that $\exists z$ with $a < z < b$ and $z \notin I$.

Consider $U = (-\infty, z) \cap I$, $V = (z, \infty) \cap I$.

U, V are clearly disjoint and nonempty (since $a \in U$ $b \in V$ by construction), open, and have union I.

This is therefore a separation, and a contradiction. So I is convex.

Now suppose I is convex. Suppose for contradiction that U, V is a separation of I.

Let $a \in U, b \in V$. Now without loss of generality, assume that $a < b$. Consider the interval $[a, b]$.

Let

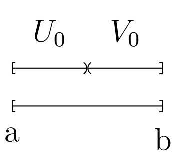
a

$$U_0 = U \cap [a, b] \quad V_0 = V \cap [a, b].$$

This is a separation. U_0, V_0 are disjoint, open in $[a, b]$, nonempty since $a \in U$, $b \in V$, and their union is all of $[a, b]$.

Let $c = \sup U_0$

(Case 1):



Since U_0 is open in $[a, b]$ and $c < b$ and $c \in U_0$, there is some $d \in (a, b)$ such that $[c, d) \subseteq U_0$. But then $\frac{c+d}{2} \in U_0$ and $c < \frac{c+d}{2}$, so c is not an upper bound on U_0, a contradiction.

(Case 2): ($c \in V_0$) Then $c = b$ or $a < c < b$.

Since V_0 is open in $[a, b]$ and $a < c$, $c \in V_0$, there is some $e \in (a, b)$ such that $(e, c] \subseteq V_0$. Then e is an upper bound on U_0, but this contradicts that c is the least upper bound on U_x.

Now $c \notin U_0$ and $c \notin V_0$, so $[a, b] \neq U_0 \cup V_0$, and thus I is connected. ∎

Theorem 2 (Intermediate Value Theorem)**.** *Let $f : X \to \mathbb{R}$ be a continuous function where X is a connected topological space. Then if $a, b \in X$ and y is between $f(a)$ and $f(b)$, then there exists $c \in X$ such that $f(c) = y$.*

Proof. Since X is connected and f is continuous, $f(X)$ is connected. By the previous theorem, $f(X)$ is convex, so $y \in f(X)$. Then by the definition of the image, $\exists c \in X$ such that $f(c) = y$. ∎

Now we will give yet another characterization of continuity. One that is perhaps a bit stronger than our prior ones, but should aide our intuition.

Definition 2. Let X be a topological space, $p, q \in X$. A **path** from p to q in X is a continuous function $f : [a, b] \to X$ such that $f(a) = p$, $f(b) = q$.

We say that X is **path connected** if for any $p, q \in X$ there is a path in X from p to q.

Proposition 1. *If X is path connected, then X is connected.*

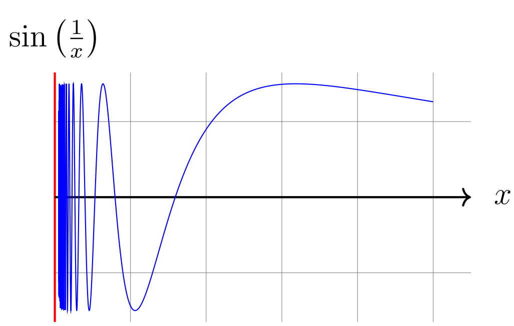
Proof. Suppose for the sake of contradiction that U, V is a separation of X. Let $p \in U, q \in V$. Since X is path connected, there is a continuous function $f : [a, b] \to X$. Then $f(a) = p$ and $q = f(b)$. Then $f([a, b])$ is connected since $[a, b]$ is connected.

Then $f([a, b]) \subseteq U$ or $f([a, b]) \subseteq V$, but this is impossible since $p \notin V$, $q \notin U$. ∎

Example 2. The unit ball B in $\mathbb{R}^n$ is connected, since it is path connected.

If $\mathbf{p}, \mathbf{q} \in B$, then $\begin{aligned} f : [0, 1] &\to B \\ t &\mapsto (1 - t)\mathbf{p} + t\mathbf{q} \end{aligned}$ will be a path from $\mathbf{p}$ to $\mathbf{q}$.

Example 3 (Connected $\not\Rightarrow$ Path connected)**.**



Let $S = \left\{(t, \sin\left(\frac{1}{t}\right) \mid 0 < t \leq 1)\right\}$.

This is connected, since it is the image of a connected space. Take the closure:

$$\overline{S} = \{0\} \times [-1, 1] \cup S$$

Fact. *Closures of connected subspaces are also connected.*

Let $\mathbf{p} = (0,0)$, $q = (1, \sin(1))$.

Suppose $f : [0,1] \to \overline{S}$ is a path from $\mathbf{p}$ to $\mathbf{q}$.

notice $0 \times [-1,1]$ is a closed set, so its preimage under f is a closed set, with some max.

Reparameterizing, we may assume this max is 0.

By the intermediate value theorem, for each integer $n > 0$, there is some t_n such that $f(t_n) = \left(\frac{1}{2\pi n + \frac{\pi}{2}}, \sin\left(2\pi n + \frac{\pi}{2}\right)\right) = \left(\frac{1}{2\pi}, 1\right)$.

Then

$$\lim_{n\to\infty} f(t_n) = (0,1) \neq f(0) = (0,0)$$
$$\lim_{n\to\infty} t_n = 0$$