**Definition 1.** let $X$ be a set. We say that a collection $\mathcal{B}$ of subsets of $X$ is a **basis** for a topology on $X$ if:

1. For each point $x \in X$, there is a **basis element** (or **basic open subset**) $B \in \mathcal{B}$ such that $x \in B$
2. For each pair of basis elements $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there exists a basis element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

The **topology $\tau$ generated by the basis $\mathcal{B}$** is the topology on $X$ where:

1. a subset $U \subseteq X$ is said to be open if for each $x \in U$ there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.

**Example 1.** In the usual topology on $\mathbb{R}^n$, the open balls $B(\mathbf{x}, \varepsilon)$ form a basis for the usual topology. I.E. the set

$$\mathcal{B} = \{B(\mathbf{x}, \varepsilon) | \mathbf{x} \in \mathbb{R}^n, \varepsilon > 0\}$$

Is a basis for the usual topology on $\mathbb{R}^n$.

*Proof.* Let's check the axioms:

1. For any $\mathbf{x} \in \mathbb{R}^n$, we can use $B(\mathbf{x}, \varepsilon)$ as a basis element containing $\mathbf{x}$.
2. Now for any $B_1 = B(\mathbf{x}_1, \varepsilon_1), B_2 = B(\mathbf{x}_2, \varepsilon_2,$ and $\mathbf{x} \in B_1 \cap B_2$. then there exists $B_3 = B(\mathbf{x}, \min\{|\varepsilon_1 - \mathbf{x_1}|, |\varepsilon_2 - \mathbf{x_2}|\}$

$\square$

Recall that this is almost exactly how we usually define the usual topology on $\mathbb{R}^n$, baring minor differences.

This next proposition will hi-light and explain a commonly used hand-wave. Namely, handling finite unions by claiming that it 'follows by induction'.

**Proposition 1.** *Let $X$ be any set, and let $\mathcal{B}$ be a basis for a topology on $X$. Then we claim that the "topology" $\tau$ generated by $\mathcal{B}$ truly is a topology.*

*Proof.*
1. $\emptyset$, vacuously open, since there are no points in the empty set. $X$ is open, since for every point $\mathbf{x}$, the first basis axiom guarantees that there exists a $B \in \mathcal{B}$ such that $\mathbf{x} \in B$.
2. Suppose $U_i \in \tau$ for all $i \in I$. So let some point $x \in \bigcup_{i \in I} U_i$. Then there is some $j \in I$ such that $x \in U_j$. Since $U_j$ is open, there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U_j$. So then $x \in B \subseteq \bigcup_{i \in I} U_i$, so $U_{i \in I} U_i$ is open

3. Let's induct for the final part.

   Suppose $U, V \in \tau$, and consider $U \cap V$. Now Let $x \in U \cap V$. Since $U$ is open, there is a basis element $B_1$ such that $x \in B_1$, and $B_1 \subseteq U$. Then also since $V$ is open, $\exists B_2$ such that $x \in B_2$ and $B_2 \subseteq V$. Thus by basis axiom two, $\exists B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$. So $U \cap V$ is open. This completes the next part of our induction.

   Suppose that there exists a $k \in \mathbb{Z}_{>0}$ such that if $U_1, \ldots U_k \in \tau$, then $\bigcup_{i=1}^{k} U_i \in tau$ for the sake of induction. now suppose that we have some collection $U_1, \ldots, U_{n+1}$. Then the intersection of the first $n$ sets will be open, and so we return to just taking an intersection of two open sets. Therefore, we have shown by induction that we can take finite intersections.

$\square$

**Example 2.** If $X = \mathbb{R}^2$ take

$$\mathcal{B} = \left\{ (a,b) \times (c,d) \;\middle|\; \begin{matrix} a,b,c,d \in \mathbb{R} \\ a < b, \;\; c < d \end{matrix} \right\}$$

This is also a basis for a topology. How does it compare to the usual topology, I wonder...

**Example 3.** If $X$ is any set

$$\mathcal{B} = \{\{x\} \mid x \in X\}$$

is also a basis for a topology. This is (pretty clearly) a basis for the discrete topology.

**Lemma 1.** *Let $X$ be a set, $\mathcal{B}$ be a basis for a topology on $X$, $\tau$ the topology generated by $\mathcal{B}$. Then $\tau$ is the set of all possible unions of the basis elements:*

$$\tau = \left\{ \bigcup_{i \in I} B_i \;\middle|\; \begin{matrix} I \text{ is any indexing set} \\ B_i \in \mathcal{B} \text{ for all } i \in I \end{matrix} \right\}.$$

*Proof.* Suppose $U = \bigcup_{i \in I} B_i$, where $B_i \in \mathcal{B}$. Then if $x \in U$ there exists $j \in I$ such that $x \in B_j$. Then $x \in B_j \subseteq U$, so $U$ is open in the generated topology. Conversely, suppose $U \subseteq X$ is open in the generated topology.

Then since $U$ is open for each $x \in U$ there is a basis element $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Then

$$\begin{aligned} U &= \bigcup_{x \in U} \{x\} \\ &\subseteq \bigcup_{x \in U} B_x \\ &\subseteq \bigcup_{x \in U} U = U. \end{aligned}$$

So $U = \bigcup_{x \in U} B_x$ $\square$

**Proposition 2.** *Let $X$ be a set. Let $\mathcal{B}', \mathcal{B}$ be bases for topologies on $X$. Let $\tau', \tau$ be the respective generated topologies. Then the following are equivalent:*

1. *$\tau'$ is finer than $\tau$*
2. *For each $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

*Proof.*

$(1 \Rightarrow 2)$ Assume that $\tau'$ is finer than $\tau$, and let $B \in \mathcal{B}$ and $x \in B$. But then $B \in \tau$, so $B \in \tau'$. By definition of $\tau'$, since $B$ is open, and $x \in B$, $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$.

$(2 \Rightarrow 1)$ Assume $\forall B \in \mathcal{B}$, $x \in B$, $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$.

Now let $U \in \tau$ be given. Then by definition, for any $x \in U$, there exists a basis element $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then we apply our hypothesis. for such an $x$, we have by assumption a $B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$.

Then, for all $x$, $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq U$. So $U$ is open in $\tau'$.

$\square$