Math 171 **Professor Sebastian Bozlee**
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Recall the definition of the subspace topology:

Definition 1. If (X, τ_X) is a topological space, and $Y \subseteq X$, then the **subspace topology on** Y is the topology

$$\tau_{\text{sub}} = \{U \cap Y | U \in \tau_X\}$$

.

We now give a proposition which somehow justifies that this is not just any topology, but is somehow a good choice.

Proposition 1. *With notation as above:*

1. $i : (Y, \tau_{\text{sub}}) \to (X, \tau_X)$ *taking* $y \mapsto y$ *(the inclusion function) is continuous.*
2. τ_{sub} *is the coarsest topology on* Y *such that* i *is continuous.*
3. *If* $f : (X, \tau_X) \to (Z, \tau_Z)$ $x \mapsto f(x)$ *is continuous, then* $f|_y : (Y, \tau_{\text{sub}}) \to (Z, \tau_Z)$ $y \mapsto f(y)$ *is alco continuous.*
4. *If* (W, τ_W) *is a topological space and* $g : W \to Y$ *is a function, then*

$$\begin{array}{ccc} g : (W, \tau_W) \to (Y, \tau_{\text{sub}}) \text{ is continuous} & \Leftrightarrow & i \circ g : (W, \tau_W) \to (X, \tau_X) \text{ is continuous} \\ w \mapsto g(w) & & w \mapsto g(w) \end{array}$$

Proof.

1. This part is mosly trivial (try writing out what i^{-1} of an open set must be! The proof follows directly).
2. Let τ_2 be a topology on Y such that $i : (Y, \tau_2) \to (X, \tau_X)$ is continuous. Since i is continuous, for all $U \subseteq X$ an open subset of X, we have that $i^{-1}(U) : U \cap Y$ is open in Y with topology τ_2. This shows that for any $U \subseteq X$ an open subset of X, $U \cap Y \in \tau_2$ i.e. an arbitrary element of τ_{sub} also belongs to τ_2. So $\tau_2 \supseteq \tau_{\text{sub}}$, and, by definition τ_{sub} is coarser than τ_2.
3. $$(Y, \tau_{\text{sub}}) \xrightarrow[\text{cont}]{i} (X, \tau_X) \xrightarrow[\text{cont}]{f} (Z, \tau_Z)$$

   is a composition of continuous functions which is continuous, so $f|_y = f \circ i$ is continuous.
4. ($\Rightarrow$) if $g : (W, \tau_W) \to (Y, \tau_{\text{sub}})$ is continuous, then again $i \circ g$ is a ocmposite of continuous functions, so $i \circ g$ is continuous.

($\Leftarrow$) Assume $i \circ g$ is continuous. Let some $U \subseteq Y$ is open. By definition $U = V \cap Y$ for some $V \subseteq X$ an open subset of X. So we know $(i \circ g)^{-1}(V)$ is open in W.

$$\begin{aligned}(i \circ g)^{-1}(V) &= g^{-1}(i^{-1}(V)) \\ &= g^{-1}(V \cap Y) \\ &= g^{-1}(U).\end{aligned}$$

So $g^{-1}(U)$ is open, and thus g is continuous.

$\square$

Notice that if U is open in Y, we may not have that U is open in X. $\{0, 1\} \subseteq \mathbb{R}$, then $\{0, 1\}$ is an open subset of Y, but not an open subset of X.

Proposition 2. *If Y is an open subset of a topological space X and $U \subseteq Y$, then $U \subseteq Y$ is and open subset of Y in the subspace topology iff U is open in X.*

Proof. ($\Rightarrow$) Let U be an open subset of Y in the subspace topology. Then there exists some open set $V \subseteq X$ such that $V \cap Y = U$. But we know that both V and Y are open in X, their intersection must also be open in X.

($\Leftarrow$) Let $U \subseteq Y$, and we know that U is open in X. But since $U \subseteq Y$, $U = U \cap Y$, but then since U is open in X by assumption, U is open in Y.

$\square$

Remember that in the early problem, we (should have) realized that

$$\{(a, b) \mid a < b\} \cup \{\emptyset, \mathbb{R}\}$$

is not a topology.

Exercise 1. Why?

Notice that if we allow also for unions of these open intervals, we realize that we get the usual topology on $\mathbb{R}$, which better be a topology (otherwise our definition would be quite bad).

Definition 2. let X be a set. We say that a collection $\mathcal{B}$ of subsets of X is a **basis** for a topology on X if:

1. For each point $x \in X$, there is a **basis element** (or **basic open subset**) $B \in \mathcal{B}$ such that $x \in B$

2. For each pair of basis elements $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there exists a basis element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

The **topology τ generated by the basis $\mathcal{B}$** is the topology on X where:

1. a subset $U \subseteq X$ is said to be open if for each $x \in U$ there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.

Example 1. In the usual topology on $\mathbb{R}^n$, the open balls $B(\mathbf{x}, \varepsilon)$ form a basis for the usual topology.