We will begin thinking about the idea of closure and interior:

Definition 1. Let X be a topological space, let $A \subseteq X$. The **closure** $\overline{A}$ of A is the intersection of all closed subsets of X containing A.

In other words, it is the smallest closed set containing A

Definition 2. The **interior** of A, written Int A, (sometimes A°) is the union of all open subsets of X contained in A

Example 1. $A = (0, 1]$ in $X = \mathbb{R}$ then

$$\overline{A} = [0, 1] \qquad A^{\circ} = (0, 1)$$

Notice that the context of our closure is important. To see this, if $A = (0, 1], X = (0, 2]$, then $\overline{A} = (0, 1]$.

Lemma 1. *Let Y be a subspace of X, (a subset of X given the subspace topology). Then a subset $B \subseteq Y$ is closed iff there exists a closed set $Z \subseteq X$ such that $B = Z \cap Y$*

Proof. Observe that if $U \subseteq X$ then

$$Y - (U \cap Y) = (X - U) \cap Y$$

Then we have that $B \subseteq Y$ is closed in Y iff $Y - B$ is open in Y iff $\exists U \subseteq X$ open such that $Y - B = U \cap Y$ iff $\exists U \subseteq X$ open such that $B = Y - (U \cap Y)$ iff $\exists U \subseteq X$ open such that $B = (X - U) \cap Y$ iff $\exists Z \subseteq X$ closed such that $B = Z \cap Y$. $\square$

Theorem 1. *Let Y be a subspace of X, A be a subset of Y. Let $\overline{A}$ be the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$*

Proof. Let B be the closure of A in Y. Then $\overline{A}$ is closed in X, so by the lemma, $\overline{A} \cap Y$ is closed in Y. On the other hand, $\overline{A} \cap Y$ contains A.

Since B is in the intersection of the closed subsets of Y containing A, $B \subseteq \overline{A} \cap Y$. $\triangle$

On the other hand, since B is closed in Y, there exists a closed subset $Z \subseteq X$ such that $B = Z \cap Y$ by the lemma. This Z then is a closed subset of Z containing A. Since $\overline{A}$ is the intersection of all closed subsets of X containing A, $\overline{A} \subseteq Z$. Then

$$\overline{A} \cap Y \subseteq Z \cap Y = B$$

In such situations, $\overline{A}$ will be the closure in X. $\square$

We will now somehow demonstrate a relationship between the closure and the interior.

Lemma 2. *Let A be a subset of a topological space X. Then*

$$\overline{A} = ((A^c)^\circ)^c$$
$$A^\circ = (\overline{A^c})^c$$

Proof.

$$\begin{aligned}
\overline{A} &= \left(\bigcap_{\substack{Z\subseteq X \text{ closed} \\ A\subseteq Z}} Z \right) \\
A = \overline{A}^{cc} &= \left(\bigcap_{\substack{Z\subseteq X \text{ closed} \\ A\subseteq Z}} Z \right)^{cc} \\
&= \left(\bigcap_{\substack{Z\subseteq X \text{ closed} \\ A\subseteq Z}} Z^c \right)^{c} \\
&= \left(\bigcap_{\substack{Z^c\subseteq X \text{ open} \\ Z^c\subseteq A^c}} Z \right)^{c} \\
&= \left(\bigcap_{\substack{U\subseteq X \text{ open} \\ U\subseteq A^c}} U \right)^{c} \\
&= ((A^c)^\circ)^c
\end{aligned}$$

□

Theorem 2. *Let A be a subset of the topological space X. Then*

1. *$x \in \overline{A}$ iff every open neighborhood U of x intersects A. (where x is an adherent point).*
2. *If the topology on X is generated by a basis $\mathcal{B}$, then $x \in \overline{A}$ iff every basic open neighborhood B of x intersects A.*

Proof. 1. Let's prove by contrapositive:

$x \notin A$ iff There exists an open neighborhood of x such that U does not intersect A.

Now let's suppose $x \notin \overline{A}$ then $U = X - \overline{A}$ is an open subset of X containing x and $U \cap A = \emptyset$.

Conversely, suppose that U is an open neighborhood of x not containing A. then U^c is a closed subset of X containing A. Then $\overline{A} \subseteq U^c$). Thus $x \notin \overline{A}$. $\triangle$

2. Suppose now that all open neighborhoods U of x intersect A. Then in particular this holds of the basic open neighborhoods, so each basic open neighborhood of x intersects A.

   Conversely, suppose all basic open neighborhoods B of x intersect A.

   If U is an open neighborhood of x, then there exists a basis element B such that $x \in B \subseteq U$. Then $B \cap A \neq \emptyset$, so $U \cap A \neq \emptyset$ as well.

$\square$

Definition 3. A point $x \in X$ is said to be a **limit point** of a subset $A \subseteq X$ if each open neighborhood U of x intersects A in a point other than x (i.e. $(U - \{x\}) \cap A \neq \emptyset$).

Theorem 3. *Let A be a subset of a topological space X. Write A' for the set of limit points of A, then $\overline{A} = A \cup A'$.*

Proof. $(A \cup A') \subseteq \overline{A}$ Let $x \in A'$, then each open neighborhood of x intersects A in a point (other than x) so $X \in \overline{A}$.

If $x \in A$ then $x \in \overline{A}$ since $A \subseteq \overline{A}$ by definition.

So $A \cup A' \subseteq \overline{A}$.

For the reverse inclusion, let $x \in A$. Then $x \in A \cup A'$. If $x \notin A$ then any open neighborhood U of x intersects A. Since $x \notin A$ this must be a point other than x. So x is a limit point and $x \in A'$. So $x \in A \cup A'$. $\square$

Corollary 1. *A subset A of a topological space X is closed iff it contains all of its limit points.*