Let's go over part (b) of the early problem...

Proposition 1. *Let X, Y be topological spaces, and let $\mathcal{B}_X, \mathcal{B}_Y$ be bases for the resp. topologies of X, Y, then:*

1. *$f : X \to Y$ is continuous iff for each $B \in \mathcal{B}_Y$, $f^{-1}(B)$ is open.*
2. *$f : X \to Y$ is continuous iff for all $x \in X$ and basic open neighborhoods $B_{f(x)}$ of $f(x)$, there exists $B_x \in \mathcal{B}_x$ such that $x \in B_x$ and $f(B_x) \subseteq B_{f(x)}$.*

The proof of (a) is left to the early problem

Proof of part b. ($\Rightarrow$) Let $f : X \to Y$ be continuous. Let $x \in X$, and $B_{f(x)}$ be a basic open neighborhood of $f(x)$. Then, since $f(x) \in B_{f(x)}$, $x \in f^{-1}(B_{f(x)})$. Since f is continuous, $f^{-1}(B_{f(x)})$ is open. By the definition of open in a generated topology, $\exists B_X \in \mathcal{B}_X$ such that $x \in B_X$ and $B_x \subseteq f^{-1}(B_{f(x)})$. Then, using the Galois connection, we get that $f(B_X) \subseteq B_{f(x)}$.

($\Leftarrow$) Assume that we have the property, now we want to show that $f : X \to Y$ is continuous. So let $U \subseteq Y$ be an open set. Now let $x \in f^{-1}(U)$ Then $f(x) \in U$. Since U is open, there is a basic open neighborhood $B_{f(x)}$ such that $f(x) \in B_{f(x)}$ and $B_{f(x)} \subseteq U$. By assumption, there is a basic open neighborhood B_x of x such that $f(B_x) \subseteq B_{f(x)} \subseteq U$. Then again by the Galois connection, $B_x \subseteq f^{-1}(U)$.

□

Now on to more stuff.

Definition 1. A set S of subsets of a set X whose union is all of X is a **subbasis for a topology on** X. The **topology generated by** S is the set of unions of finite intersections of elements of S.

We claim that in order to show that this is a topology, it is enough to show that the set $\mathcal{B}$ of finite intersections of elements of S is a basis. Now let's actually do the check:

Proof.

1. Let $x \in X$. Then since $\bigcup_{T \in S} T = X$, there is an element $T \in S$ such that $x \in T$, and thus $T \in \mathcal{B}$.
2. Let $B_1 = T_1 \cap \ldots \cap T_n$, where $T_i \in S$, and $B_2 = T_1' \cap \ldots \cap T_m'$, where $T_i' \in S$. Now let $x \in B_1 \cap B_2$. Since $B_1 \cap B_2$ is a finite intersection of elements of S, it is a basis element, and so we can take $B_3 = B_1 \cap B_2$.

□

So where is this useful? We will now move into a discussion of product spaces. This is where all of the universal property juggling should pay off.

Question 1. Given two topological spaces X, Y, what topology "should" we put on $X \times Y$?

More generally, given topological spaces X_i for $i \in I$, what topology should $\prod_{i \in I} X_i$ have?

Let's recall the universal property of the product for sets, and amend it for topological spaces.

Theorem 1. *Given ~~sets~~ topological spaces X, Y, a ~~set~~ topological spaces P together with ~~functions~~ continuous functions $\pi_1 : O \to X$ and $\pi_2 : P \to Y$ is said to have the universal property if, for any ~~set~~ topological spaces Z and ~~functions~~ continuous functions $f_1 : Z \to X$ and $f_2 : Z \to Y$ there exists a unique ~~function~~ continuous function $f : Z \to P$ such that.*

$$\begin{tikzcd} & Z \arrow[dl, "f_1"'] \arrow[d, dotted, "\exists! f"] \arrow[dr, "f_2"] & \\ X & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

Proposition 2. *"Products are unique up to unique homeomorphism." i.e. Given (P, π_1, π_2) and (P', π_1', π_2') with the universal property, there exists a unique homeomorphism $\varphi : P \to P'$ such that $\pi_1 = \pi_1' \circ \varphi$ and $\pi_2 = \pi_2' \circ \varphi$.*

Proof. Let's do some good old-fashioned diagram chasing.

$$\begin{tikzcd} & Z \arrow[dl, "\pi_1"'] \arrow[d, dotted, "\exists! \varphi"] \arrow[dr, "\pi_2"] & \\ X & P' \arrow[l, "\pi_1'"] \arrow[r, "\pi_2'"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & Z \arrow[dl] \arrow[d, dotted, "\psi"] \arrow[dr] & \\ X & P \arrow[l] \arrow[r] & Y \end{tikzcd}$$

Combining these two diagrams, we get:

$$\begin{tikzcd} & Z \arrow[dl, "\pi_1"'] \arrow[d, "\psi \circ \varphi"] \arrow[dr, "\pi_2"] & \\ X & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

So then $\psi \circ \varphi$ is the identity map, and we have what we need. $\square$

So we can revise our goal: We now want to find a topology on $X \times Y$ so that $(X \times Y, \pi_1, \pi_2)$ has the universal property.

In other words, we want to find a topology on $X \times Y$ such that $\begin{array}{c} \pi_1 : X \times Y \to X \\ (x, y) \mapsto x \end{array}$ and $\begin{array}{c} \pi_2 : X \times Y \to X \\ (x, y) \mapsto y \end{array}$ are continuous.

So we need

1. For each $U \subseteq X$ open, $\pi_1^{-1}(U) = U \times Y$ is open
2. For each $V \subseteq Y$ open, $\pi_2^{-1}(V) = X \times Y$ is open.

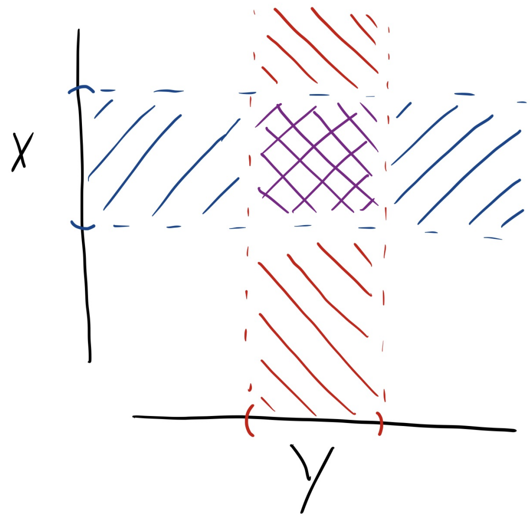

Fig. 1. A diagram outlining what our "Product space" should look like

This is not yet a topology, we still need finite intersections. So, lets add in finite intersections: $\mathcal{B} = \{U \times V \mid U \subseteq X \text{ open}, V \subseteq Y \text{ open}\}$. This in and of itself is still not a topology, but it is a basis.

Proposition 3. *Let X, Y be topological spaces, then the set:*

$$\mathcal{B} = \{U \times V \mid U \subseteq X \text{ open}, V \subseteq Y \text{ open}\}$$

Definition 2. The **product topology on** $X \times Y$ is the topology generated by the basis $\mathcal{B}$ above.

Example 1. $\mathbb{R}^2$ with the usual topology has the product topology from $\mathbb{R} \times \mathbb{R}$. With the usual topology, we were taking a basis of open balls, but in this construction, we are doing it with open rectangles, which we previously showed was equivalent.