We start with a final theorem on Bases

Lemma 1. *Let X be a topological space. Suppose $\mathcal{C}$ is a collection of open subsets of X such that for each open subset U of X and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then C is a basis for the topology on X.*

Proof. We verify the basis axioms first:

1. **If $x \in X$, we want to show that then $\exists C \in \mathcal{C}$ such that $x \in C$.**

   By hypothesis: since $U = X$ is an open subset of X, for any $x \in X$, there is a $C \in \mathcal{C}$ such that $x \in C$, and the first axiom is verified.

2. **If $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, then $\exists C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$.**

   Let $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$. Then by definition, C_1, C_2 are open, so $C_1 \cap C_2$ is open, and then by assumption, for $U = C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$

△

So now we want to verify that the topology on X is that generated by $\mathcal{C}$, which we will do by double-containment on the topologies.

($\Rightarrow$) Suppose that U is open in X. For each $x \in U$, there is an element C_x of $\mathcal{C}$ such that $x \in C_x \subseteq U$. Then

$$U = \bigcup_{x \in U} C_x,$$

So U is open in the topology generated by $\mathcal{C}$.

($\Leftarrow$) Now suppose that $U \subseteq X$ that is open in the generated topology. Then $U = \bigcup_{i \in I} C_i$ where $C_i \in \mathcal{C}$ for each $i \in I$. This is a union of open subsets of X, so U is open in the original topology on X.

And thus we have double containment, and the lemma is proven. □

Now we will begin our discussion of homeomorphisms.

When we are working in the category of sets, the 'tool of choice' is functions between sets. Namely, an invertible function tells us a way in which two sets can be considered equivalent.

Likewise, we have found that in the category of topological spaces, continuous functions are our 'tool of choice'. Likewise, invertible continuous functions will be our form of equivalence of topological spaces.

Definition 1. Let X, Y be topological spaces. A **Homeomorphism from** $X \to Y$ is a continuous function $f : X \to Y$ with continuous inverse function $f^{-1} : Y \to X$.

Example 1. Notice that a continuous bijection need not always have a continuous inverse. Consider the identity function. $\mathrm{id}_{\{1,2\}} : (\{1,2\}, \tau_{\text{disc}}) \to (\{1,2\}, \tau_{\text{indisc}})$. As we have studied, any function from the discrete topology is continuous, and any function from the indiscrete topology is not, so obviously our function here is continuous without a continuous inverse.

Following this non-example, we give an actual example.

Example 2. Let $X = [0,1], Y = [0,2]$. Then take $\begin{array}{c} f : X \to Y \\ x \mapsto 2x \end{array}$, which is continuous and bijective, with inverse $\begin{array}{c} f^{-1} : Y \to X \\ y \mapsto \frac{y}{2} \end{array}$. This describes what we mean by the fact that topology has no sense of "distance".

Definition 2. If there exists a homeomorphism $f : X \to Y$, we say X is **homeomorphic** to Y, and we write $X \cong Y$.

Proposition 1. *Let $f : X \to Y$ be a homeomorphism, $g : Y \to X$ its continuous inverse. Then:*

1.

$$\begin{array}{ccc} \{U \in X \mid U \text{ is open}\} & \longleftrightarrow & \{V \subseteq Y \mid V \text{ is open}\} \\ U & \longmapsto & g^{-1}(U) \\ f^{-1}(v) & \longleftarrow & V \end{array}$$

2.

$$\begin{array}{ccc} \{W \subseteq X \mid W \text{ is closed}\} & \longleftrightarrow & \{Z \subseteq Y \mid Z \text{ is closed}\} \\ W & \longmapsto & g^{-1}(W) \\ f^{-1}(Z) & \longleftarrow & Z \end{array}$$

Proof.

1. The two maps are well-defined because f and g are continuous. To see that the maps are inverses, note that $\forall U \subseteq X$ open, we have

$$\begin{aligned} f^{-1}(g^{-1}(U)) &= (g \circ f)^{-1} \\ &= \mathrm{id}_X(U) \\ &= U. \end{aligned}$$

And we do similarly for all other cases.

□

Example 3. $(0,1) \cong \mathbb{R}$, with $f : (0,1) \to \mathbb{R}$ via $f(x) = \tan\left(\pi\left(x - \frac{1}{2}\right)\right)$

Lemma 2. *Rational functions, root functions, exponentials, logs, trig functions, and inverse trig functions are all continuous on their domains.*

Note. A rational function is a quotient of polynomial functions:

$$f(x_1, \dots, x_n) = \frac{P(x_1, \dots, x_n)}{Q(x_1 \dots, x_n)}$$

Lemma 3. *Let X, Y, Z be topological spaces. Then*

1. $\mathrm{id}_X : X \to X$ *is a homeomorphism*
2. *If $f : X \to Y$ is a homeomorphism, so is $f^{-1} : Y \to X$.*
3. *If $f : X \to Y$, $g : Y \to Z$, then $g \circ f : X \to Z$ is also a homeomorphism*

Proof. 1. Obvious (what's the inverse of id_X?) $\triangle$

2. Since f is a homeomorphism:
   - f is continuous
   - f has an inverse f^{-1}
   - f^{-1} is continuous/

   So...
   - f^{-1} is continuous
   - f^{-1} has an inverse f
   - and f^{-1} is a homeomorphism

   $\triangle$

3. $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, so f, g are continuous and have continuous inverses f^{-1}, g^{-1}. So:
   - $g \circ f$ is continuous as a composite of continuous functions
   - $(g \circ f)^{-1}$ is an inverse.
   - $f^{-1} \circ g^{-1}$ is continuous.

□

Lemma 4. *Let X, Y, Z be topological spaces. Then:*

1. $X \cong X$
2. *If $X \cong Y$, then $Y \cong X$,*
3. *If $X \cong Y$, and $Y \cong Z$, then $X \cong Z$.*

Proof. Admitted. **Hint:** Apply the definition of homeomorphic to the previous lemma. □