1 C^∞ functions on $\mathbb{R}^n$

Definition 1. Let $U \subset \mathbb{R}^n$ be open, $p \in U$. $f : U \to \mathbb{R}$ is C^k at p if all partials of f of order $\leq k$ exist and are continuous at p. E.g. C^0 is continuous, C^1 is continuous w/ first partials also continuous.

For the sake of simplicity in this course, we care only about functions which are C^∞, in other words $f : U \to \mathbb{R}$ is C^∞ at $p \in U$ if f is C^k for all $k = 0, 1, 2, \ldots$.

We also care about another type of function, namely the analytic ones.

Definition 2. A function $f : U \to \mathbb{R}$ is **analytic** at p if $f(x)$ is equal to its Taylor's series at p in a neighborhood of p.

Importantly, an analytic function is infinitely differentiable within its radius of convergence, since the power series can be differentiated term by term. This suggests the following lemma:

Lemma 1. *Analytic* $\Rightarrow C^\infty$.

Notice that, notably, the converse is not true:

Example 1. Take $f(x) = \begin{cases} \frac{1}{e^{\frac{1}{x}}} & x > 0 \\ 0 & x \leq 0 \end{cases}$. It is evidently clear that this function is C^∞. Derivative can easily be computed, and verifying that they are continuous is easy. However, notice that $f^{(k)}(0) = 0$ $\forall k = 0, 1, 2, \ldots$. Actually demonstrating this is a simple matter of induction. Therefore, the Taylor's series of f at $x = 0$ is the zero function.

Thus $f(x)$ is not equal to its Taylor's series at 0 in any neighborhood of 0.

Therefore, f is not analytic at zero.

Luckily, for C^∞ functions there is an analogue that works.

The **Taylor's series with Remainder** of a function is equal to

$$f(x) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)(x^i - p^i) + \frac{1}{2!} \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}(p)(x^i - p^i)(x^j - p^j) + \ldots + k^{\text{th}} \text{ term } + \underbrace{k+1}_{\text{remainder}}$$

$p = (p^1, \ldots, p^n) \in \mathbb{R}^n$

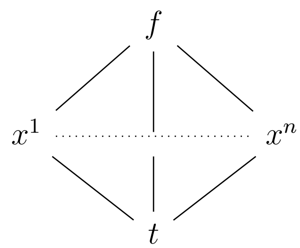
You know what a star-shaped polygon is. It's a polygon with a nonempty visibility kernel.

Theorem 1. *Taylor's Theorem with Remainder Let U be a star-shaped open set in $\mathbb{R}^n$ with respect to some point $p \in U$.*

If $f : U \to \mathbb{R}$ is C^∞, then $\exists C^\infty$ functions $g_1(x), \ldots, g_n(x)$ such that

$$f(x) = f(p) + \sum_{i=1}^{n} g_i(x)(x^i - p^i) \text{ and } g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

Now we remember calc III... uh oh.



$\frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} * \frac{dx^i}{dt}$. Yeah, it's the chain rule.

Proof. Let $y \in U$ The parametrization of $\overline{py}$ is $x(t) = p + t(y - p)$.

Note that $x(0) = p$, $x(1) = y$, $x^i(t) = p^i + t(y^i - p^i) \frac{dx^i}{dt} = y^i - p^i$.

And also see

$$\begin{aligned}
f(y) - f(p) &= f(x(1)) - f(x(0)) \\
&= \int_0^1 \frac{d}{dt} f(x(t)) dt \\
&= \int_0^1 \sum_i \frac{\partial f}{\partial x^i}(x(t)) \frac{dx^i}{dt} dt \\
&= \sum_i \underbrace{\int_0^1 \frac{\partial f}{\partial x^i}(p + t(y - p)) dt * (y^i - p^i)}_{g_i(y)}
\end{aligned}$$

And et cetera. □

2 Tangent Vectors

We have the issue where our current understanding of a tangent vector where our parameterization of the vectors are firmly rooted in $\mathbb{R}^3$. So we're going to re-define the tangent vector by what we want it to do: compute the **direcitonal derivative**.

Idea. *If $v_p \in T_p(U)$ and $f \in C^\infty(U)$, then*

$$\begin{aligned} D_{v_p} &= \textit{directional derivative of } f \textit{ in direction of} v_p \textit{ at } p \\ &= \left.\frac{d}{dt}\right|_{t=0} f(\underbrace{p + tv_p}_{\textit{line thru } p \textit{ with dir. } v_p}) \end{aligned}$$

Then, through application of the chain rule, we get

$$D_{v_p} = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p \tag{1}$$

Definition 3. Let (f, u) denote a C^∞ function $f : U \to \mathbb{R}$, $p \in U$. We say that $(f, U) \sim (g, V)$ if $\exists W \subset U \cap V$ such that $f = g$ on W.

Notation 1. $[(f, U)]$ = equiv class of (f, U) = germ of f at p.
$C_p^\infty = \{$germs of C^∞ func at $p\}$.

Lemma 2. *Germs are **module-like**, and we will call them a C_p^∞ algebra over $\mathbb{R}$.*

Proposition 1. $D_{v_p} : C_p^\infty \to \mathbb{R}$.

(i) D_{v_p} *is $\mathbb{R}$-linear.*

$$\begin{aligned} D_{v_p}(f + g) = D_{v_p}f + D_{v_p}g \\ D_{v_p}(\lambda f) = \lambda D_{v_p} f. \end{aligned}$$

(ii) $D_{v_p}(fg) = (D_{v_p}f)g(p) + f(p)D_{v_p}g$ *(Liebniz rule)*

Definition 4. A function $D = C_p^\infty \to \mathbb{R}$ satisfying conditions (i) and (ii) prior is called a **derivation at** p or a **point derivation** of C_p^∞.

Definition 5. $\mathcal{D}_p(\mathbb{R}^n) = \left\{\text{all point-derivations of } C_p^\infty\right\}$

Notice that the structure of $\mathcal{D}_p(\mathbb{R}^n)$ has an addition and scalar multiplication structure, but notably not a vector multiplication structure. So this is a vector space.

Theorem 2. *The map* $\begin{aligned} \varphi : \quad T_p(\mathbb{R}^n) &\to \mathcal{D}_p(\mathbb{R}^n) \\ v_p &\mapsto D_{v_p} \end{aligned}$ *is a linear isomorphism of vector spaces.*

Lemma 3. *If $D \in \mathcal{D}_p(\mathbb{R}^n)$ then $Dc = 0$ $\forall c \in \mathbb{R}$.*

Proof. $D1 = D(1 \cdot 1) = (D1) \cdot 1 + 1 \cdot D1$, so $0 = D1$. Thus $Dc = D(c \cdot 1) = cD1 = c \cdot 0 = 0$.

Thus

$$\begin{aligned}
Df &= D(f(p) + \sum g_i(x)(x^i - p^i)) \\
&= D(f(p)) + \sum (Dg_i)(p^i - p^i) + g_i(pD(x^i - p^i)) \\
&= \sum (Dx^i) \left. \frac{\partial}{\partial x^i} \right|_p f \\
&= \sum (Dx^i) e_{ip}
\end{aligned}$$

Thus we have established the one-to-one correspondane

$$v_p = \sum v^i e_{i,p} \longleftrightarrow D_{v_p} = \sum v^i \left. \frac{\partial}{\partial x^i} \right|_p \tag{2}$$

□

Of theorem. We do the usual thing.

Inj. Suppose $\varphi(v_p) = D = \sum v^i \left. \frac{\partial}{\partial x^i} \right|_p = 0$. Now apply to x^j: $\sum v^i \left. \frac{\partial}{\partial x^i} \right|_p x^j = 0$. And

$$\begin{aligned}
\frac{\partial x^j}{\partial x^i} &= \begin{cases} 1 & \text{for } j = i, \\ 0 & \text{for } j \neq i \end{cases}. \\
&= \delta_i^\gamma.
\end{aligned}$$

$$\begin{aligned}
\sum_i v^i \frac{\partial x^j}{\partial x^i}(p) &= \sum_i v^i \delta_i^j \\
&= v^j \forall j.
\end{aligned}$$

So $v_p = 0$.

Surj. Let $D \in \mathcal{D}_p(\mathbb{R}^n)$. Suppose $D = D_{v_p}$ for $v_p \in T_p(\mathbb{R}^n)$. So then

$$D = \sum_i v^i \left. \frac{\partial}{\partial x^i} \right|_p .$$

Apply both sides to x^j:

$$Dx^j = \sum v^i \left. \frac{\partial}{\partial x^i} \right|_p x^j = \sum_i v^i \delta_i^j = v^j.$$

So $v_p = \sum (Dx^i) \left. \frac{\partial}{\partial x^i} \right|_p$. then by Taylor's Theorem with Remainder , $\exists g_i \in C^\infty(U)$ such that $f(x) = f(p) + \sum g_i(x)(x^i - p^i)$, and $g_i(p) = \frac{\partial f}{\partial x^i}(p)$.

$\square$

Definition 6. $T_p(U) = \left\{\text{point-derivations of } C_p^\infty\right\}$

Definition 7. A **vector field** on open $U \subset \mathbb{R}^n$ $X : U \to \coprod_{p \in U} T_p(U)$ (where $\coprod$ denotes the disjoint union) such that $X_p \in T_p(U)$.

Note. If $X_p \in T_p(U)$, then $X_p = \sum a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$.

Lemma 4. *$X = \sum a6i \frac{\partial}{\partial x_i}$, where $a^i : U \to \mathbb{R}$ are functions.*

Definition 8. A vector field $X = \sum a^i \frac{\partial}{\partial x^i}$ on U is C^∞ if all $a^i : U \to \mathbb{R}$ are C^∞ functions.

Notation 2. $\mathcal{X}(U) = \{C^\infty \text{ v.f. on U }\}$ is a vector space on $\mathbb{R}$.