Math 285 — Professor Loring Tu

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1 C^∞ functions on $\mathbb{R}^n$

Definition 1. Let $U \subset \mathbb{R}^n$ be open, $p \in U$. $f : U \to \mathbb{R}$ is C^k at p if all partials of f of order $\leq k$ exist and are continuous at p. E.g. C^0 is continuous, C^1 is continuous w/ first partials also continuous.

For the sake of simplicity in this course, we care only about functions which are C^∞, in other words $f : U \to \mathbb{R}$ is C^∞ at $p \in U$ if f is C^k for all $k = 0, 1, 2, \ldots$.

We also care about another type of function, namely the analytic ones.

Definition 2. A function $f : U \to \mathbb{R}$ is **analytic** at p if $f(x)$ is equal to its Taylor's series at p in a neighborhood of p.

Importantly, an analytic function is infinitely differentiable within its radius of convergence, since the power series can be differentiated term by term. This suggests the following lemma:

Lemma 1. *Analytic* $\Rightarrow C^\infty$.

Notice that, notably, the converse is not true:

Example 1. Take $f(x) = \begin{cases} \frac{1}{e^{\frac{1}{x}}} & x > 0 \\ 0 & x \leq 0 \end{cases}$. It is evidently clear that this function is C^∞. Derivative can easily be computed, and verifying that they are continuous is easy. However, notice that $f^{(k)}(0) = 0\ \forall k = 0, 1, 2, \ldots$. Actually demonstrating this is a simple matter of induction. Therefore, the Taylor's series of f at $x = 0$ is the zero function.

Thus $f(x)$ is not equal to its Taylor's series at 0 in any neighborhood of 0.

Therefore, f is not analytic at zero.

Luckily, for C^∞ functions there is an analogue that works.

The **Taylor's series with Remainder** of a function is equal to

$$f(x) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)(x^i - p^i) + \frac{1}{2!} \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}(p)(x^i - p^i)(x^j - p^j) + \ldots + k^{\text{th}} \text{ term } + \underbrace{k+1}_{\text{remainder}}$$

$p = (p^1, \ldots, p^n) \in \mathbb{R}^n$

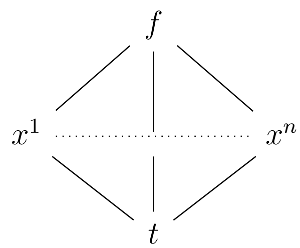
You know what a star-shaped polygon is. It's a polygon with a nonempty visibility kernel.

Theorem 1. *Taylor's Theorem with Remainder Let U be a star-shaped open set in $\mathbb{R}^n$ with respect to some point $p \in U$.*

If $f : U \to \mathbb{R}$ is C^∞, then $\exists C^\infty$ functions $g_1(x), \dots, g_n(x)$ such that

$$f(x) = f(p) + \sum_{i=1}^{n} g_i(x)(^i - p^i) \text{ and } g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

Now we remember calc III... uh oh.



$\frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} * \frac{dx^i}{dt}$. Yeah, it's the chain rule.

Proof. Let $y \in U$ The parametrization of $\overline{py}$ is $x(t) = p + t(y - p)$.

Note that $x(0) = p$, $x(1) = y$, $x^i(t) = p^i + t(y^i - p^i)\frac{dx^i}{dt} = y^i - p^i$.

And also see

$$\begin{aligned} f(y) - f(p) &= f(x(1)) - f(x(0)) \\ &= \int_0^1 \frac{d}{dt} f(x(t))dt \\ &= \int_0^1 \sum_i \frac{\partial f}{\partial x^i}(x(t))\frac{dx^i}{dt}dt \\ &= \sum_i \underbrace{\int_0^1 \frac{\partial f}{\partial x^i}(p + t(y-p))dt * (y^i - p^i)}_{g_i(y)} \end{aligned}$$

And et cetera. $\square$