1 A Review of Rings

We do a quick, informal review of things that we should know already.

Definition 1. A **Ring** R is (put simply) a set with two binary operations.

Definition 2. An **ideal** is a subset of a ring which is closed under addition and has the absorption property of multiplication, namely

$$RI \subseteq I$$

Definition 3. A **quotient ring by an ideal** I denoted R/I is the ring whose elements are the cosets $x + I$, often denoted by $\overline{x}$. It is a frequent exercise to check that this notation is non-ambiguous.

Proposition 1.1. *There is a 1-1 correspondence between ideals in R/I and ideals in R containing I.*

Proof. *(Exercise) Hint: think about the projection map.* □

A number particular types of ideals will be of use to us over the course of this class, namely:

1. Principle: Generated by a single element
2. Maximal: Nontrivial ideal which isn't in any others
3. Prime Ideal: An ideal P such that $x \notin P$ and $y \notin P$ then $xy \notin P$. Moreover if $xy \in P$, then $x \in P$ or $y \in P$. E.g. take any ideal in $\mathbb{Z}$ generated by a prime number.

Notice that the structure of quotient rings by ideals classifies nicely:

Theorem 1.2 (Quotient Ring Classification)**.**

1. *R/M is a field $\longleftrightarrow$ M is maximal*
2. *R/P is an Integral domain $\longleftrightarrow$ P is prime.*

Proof. *(Exercise)* □

Note. Every maximal ideal is prime.

Proposition 1.3. *Every ring has a maximal ideal.*

Proof. (*Exercise*) ∎

Corollary 1.4. *Every ideal is contained in some maximal ideal*

Corollary 1.5. *Every non-unit is contained in a maximal ideal*

Definition 4. A **local ring** is a ring which has a single maximal ideal

Proposition 1.6. *Suppose $m \subseteq R$ such that every $\alpha \in m^c$ is a unit. Then R is local with maximal ideal m.*

Example 1. Let $C^0(\mathbb{R}) = \{\text{continuous functions} j : \mathbb{R} \to \mathbb{R}\}$, and take the ring identified by the collection of germs $[f]_0$ which are centered at 0.

Notice that the collection of functions vanishing at 0 is a maximal ideal.

Proof. Define the evaluation map at zero $\begin{array}{c} \mathrm{ev}_0 : C^0(R)_0 \to \mathbb{R} \\ \mathrm{ev}_0(f) \mapsto f(0) \end{array}$.. Notice that for any continuous function $f \neq 0$ at 0., the $\frac{1}{f}$ is also continuous at 0, and $f * \frac{1}{f} = 1$. ∎

1.1 Exercises

1. Prove (1.1)

2. Prove (1.2)

3. Let K be a field show that $(f(x)) = (g(x))$ iff f and g differ by a constant for $f(x), g(x) \in K[x]$,

4. Prove that if R is an integral domain, then $R[x]$ is too.

5. Define $\mathbb{C}[[x]] = \{\sum_{n=0}^{\infty} a_n x^n \mid a_n \in \mathbb{C}\}$. This is the ring of formal power series. Show that $\mathbb{C}[[x]]$ is a local ring.