

# 1 A Review of Rings

We do a quick, informal review of things that we should know already.

**Definition 1.1.** A **Ring**  $R$  is (put simply) a set with two binary operations.

**Definition 1.2.** An **ideal** is a subset of a ring which is closed under addition and has the absorption property of multiplication, namely

$$RI \subseteq I$$

**Definition 1.3.** A **quotient ring by an ideal**  $I$  denoted  $R/I$  is the ring whose elements are the cosets  $x + I$ , often denoted by  $\bar{x}$ . It is a frequent exercise to check that this notation is non-ambiguous.

**Proposition 1.4.** *There is a 1-1 correspondence between ideals in  $R/I$  and ideals in  $R$  containing  $I$ .*

*Proof. (Exercise) Hint: think about the projection map.* □

A number particular types of ideals will be of use to us over the course of this class, namely:

1. Principle: Generated by a single element
2. Maximal: Nontrivial ideal which isn't in any others
3. Prime Ideal: An ideal  $P$  such that  $x \notin P$  and  $y \notin P$  then  $xy \notin P$ . Moreover if  $xy \in P$ , then  $x \in P$  or  $y \in P$ . E.g. take any ideal in  $\mathbb{Z}$  generated by a prime number.

Notice that the structure of quotient rings by ideals classifies nicely:

**Theorem 1.5** (Quotient Ring Classification).

1.  $R/M$  is a field  $\longleftrightarrow M$  is maximal
2.  $R/P$  is an Integral domain  $\longleftrightarrow P$  is prime.

*Proof.* Solution given by Saskia:

(1.)

( $\Rightarrow$ )  $R/M$  is a field, so it therefore has no nontrivial ideals. Then also, by the correspondence theorem  $R$  has no ideals  $I$  with  $M \subset I \subset R$ .

( $\Leftarrow$ ) Works the same way as forward direction.  $\triangle$

(2.)

$\Rightarrow$  . We have that  $[x], [y] \in R/M$ , so then  $x_1 \in [x], y_1 \in [y]$ , but then  $[x_1 y_1] = [xy] = 0$  iff  $[x]$  or  $[y] = 0$ . Then, by definition of an ideal,  $x_1 y_1 \in M \Leftrightarrow x_1$  or  $y_1 \in M$ .

( $\Leftarrow$ ) Since  $M$  is a prime ideal,  $xy \in M$  means that one of  $x$  or  $y$  is in  $M$ , then  $[xy] = [x][y] = 0 \Leftrightarrow [x] = 0$  or  $[y] = 0$

$\square$

*Proof.* Alternative proof of the field condition given by Ishaan.

( $\Rightarrow$ ) Let  $I$  be an ideal such that  $M \subseteq I$ . Then let  $a \in I$  be given such that  $a \notin M$ . Then  $a + M \neq 0 + M$ , and also  $\exists b + M$  such that  $(a + M)(b + M) = 1 + M$ . Then  $ab + M = 1 + M$ , which indicates that  $ab - 1 \in M \subsetneq I$ . Thus  $ab \in I, 1 \in I$ , so  $1 \in I = R$ .

( $\Leftarrow$ ) Suppose that  $M$  is maximal, then let  $a + M \in R/M$  such such that  $a + M \neq 0 + M$ . Now notice that if we attempt to extend the ideal  $M$  via  $a$ , we will get the whole ring. So then since  $1 \in R, \exists r \in R, \exists m \in M$  such that  $1 = ra + m$ , so  $1 + M = ra + M$ , and  $(r + M)(a + M)$ .

$\square$

**Note.** Every maximal ideal is prime.

**Proposition 1.6.** *Every ring has a maximal ideal.*

*Proof.* (*Exercise*)  $\square$

**Corollary 1.7.** *Every ideal is contained in some maximal ideal*

**Corollary 1.8.** *Every non-unit is contained in a maximal ideal*

**Definition 1.9.** A **local ring** is a ring which has a single maximal ideal

**Proposition 1.10.** *Suppose  $m \subseteq R$  such that every  $\alpha \in m^c$  is a unit. Then  $R$  is local with maximal ideal  $m$ .*

**Example 1.11.** Let  $C^0(\mathbb{R}) = \{\text{continuous functions } j : \mathbb{R} \rightarrow \mathbb{R}\}$ , and take the ring identified by the collection of germs  $[f]_0$  which are centered at 0.

Notice that the collection of functions vanishing at 0 is a maximal ideal.

*Proof.* Define the evaluation map at zero  $\text{ev}_0 : C^0(R)_0 \rightarrow \mathbb{R}$   $\text{ev}_0(f) \mapsto f(0)$  .. Notice that for any continuous function  $f \neq 0$  at 0., the  $\frac{1}{f}$  is also continuous at 0, and  $f * \frac{1}{f} = 1$ .  $\square$

## 1.1 Exercises

1. Prove (1.4)
2. Prove (1.5)
3. Let  $K$  be a field show that  $(f(x)) = (g(x))$  iff  $f$  and  $g$  differ by a constant for  $f(x), g(x) \in K[x]$ ,

*Proof.* This proof was given by Grace.

( $\Rightarrow$ ) Suppose that  $(f(x)) = (g(x))$ ,  $f(x), g(x) \in K[x]$ , then there exists some  $k(x), k'(x) \in K[x]$  such that  $f(x) = k(x)g(x)$  and  $g(x) = k'(x)f(x)$ . Then  $\deg(f(x)) = \deg(k) + \deg(g) = \deg k + \deg k' + \deg f$ , and so the degree of the  $k$ s must be zero.

( $\Leftarrow$ ) Suppose that  $\exists c, c' \in \mathbb{R}$  such that  $f(x) = cg(x)$  and  $g(x) = c'f(x)$

□

4. Prove that if  $R$  is an integral domain, then  $R[x]$  is too.
5. Define  $\mathbb{C}[[x]] = \{\sum_{n=0}^{\infty} a_n x^n \mid a_n \in \mathbb{C}\}$ . This is the ring of formal power series. Show that  $\mathbb{C}[[x]]$  is a local ring.

*Proof.* This proof was given by Me.

Consider the ideal of non-units  $M = \langle x \rangle$ , i.e. the ideal of all formal power series with constant term = 0.

It is trivially obvious that this is an ideal.

Now notice that  $\mathbb{C}[[x]]/M \cong \mathbb{C}$  since  $\forall a \in \mathbb{C}[[x]]$ ,

$$a = \sum_{i=0}^{\infty} a_i x^i = \underbrace{\left( \sum_{i=1}^{\infty} a_i x^i \right)}_{\in \langle x \rangle} + a_0,$$

so in  $\mathbb{C}[[x]]/M$ ,  $\bar{a}$  is in bijection with  $a_0$  in  $\mathbb{C}$ . Then, since  $\mathbb{C}$  is a field,  $M$  must indeed be maximal. Now suppose that there was some other maximal ideal  $M' \neq M$ . But if this is the case, then there must be some unit  $u \in M'$ , which means that  $1 = uu^{-1} \in M'$ , so  $M' = \mathbb{C}[[x]]$ . And thus  $M$  is indeed a unique maximal ideal<sup>1</sup>. □

---

<sup>1</sup>There is an assumption here that  $M$  really is the ideal containing all the non units, but this is easy to show

## 2 Ideal Quotient and Radicals

**Definition 2.1** (Ideal Quotient). The **ideal quotient** is defined as

$$\begin{aligned}(i : j) &= \{x \in R \mid xJ \subseteq I\} \\ &= \{x \in R \mid xj \in I \ \forall j \in J\}\end{aligned}$$

**Example 2.2.**  $(6 : 18) = \mathbb{R}$   
 $(18 : 6) = (3)$

**Definition 2.3.**  $r(I)$ , called the radical of an ideal  $I$  is defined as

$$\{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$$

**Example 2.4.**  $r(8) = (2)$ .

**Proposition 2.5.** For  $m = p_1^{r_1} \dots p_n^{r_n}$ ,  $r(m) = (p_1 \cdot p_2 \cdot \dots \cdot p_n)$

*Proof.* *Exercise.* □

### 2.1 Algebraic Geometry

Let  $K$  be an algebraically closed field, and let  $R = k[x_1, \dots, x_n]$ .

**Definition 2.6.** The **zero locus** of a polynomial  $f \in R$  is defined as

$$Z(f) = \{p \in K^n \mid f(p) = 0\}$$

**Example 2.7.**  $y - x^2 \in \mathbb{R}[x, y]$  will have a zero locus shaped like a parabola in the  $x - y$  plane.

**Definition 2.8.** For  $A \subseteq K^n$ , define

$$I(A) = \{f \in R \mid f(p) = 0 \ \forall p \in A\}$$

**Example 2.9.** Taking  $A = \{-2, 2\}$ , and we get  $(x - 2) \cap (x + 2) = (x^2 - 4)$ .

Ideally<sup>2</sup> we would like for there to be a 1-1 correspondence between the zero loci of polynomials and ideals... Is this so?

**Example 2.10.**  $Z(x^2) = 0 = Z(0)$ .

Well dang. There goes that idea. Here's a better one:

**Theorem 2.11** (Hilbert's Nullstellensatz (I spelt this wrong)). *Let  $I \subseteq R$  and suppose that  $f$  vanishes on all of  $Z(I)$ , then  $f^r \in I$  for some  $r$ .*

---

<sup>2</sup>pun intended

## 2.2 Exercises

1. For  $\mathbb{Z}$ , show:

(a)  $(m) + (n) = (\gcd(m, n))$

(b)  $(m) \cap (n) = (\text{lcm}(m, n))$

(c)  $(m)(n) = (mn)$

2. For  $\mathbb{Z}$   $(m : n) = \left( \frac{m}{\gcd(m, n)} \right)$

3. Prove Proposition 2.5

4. Problem 15 in Atiyah.

5. Problem 17 in Atiyah.

6. Problem 19 in Atiyah.