1 A Review of Rings

We do a quick, informal review of things that we should know already.

Definition 1. A **Ring** R is (put simply) a set with two binary operations.

Definition 2. An **ideal** is a subset of a ring which is closed under addition and has the absorption property of multiplication, namely

$$RI \subseteq I$$

Definition 3. A **quotient ring by an ideal** I denoted R/I is the ring whose elements are the cosets $x + I$, often denoted by $\overline{x}$. It is a frequent exercise to check that this notation is non-ambiguous.

Proposition 1.1. *There is a 1-1 correspondence between ideals in R/I and ideals in R containing I.*

Proof. *(Exercise) Hint: think about the projection map.* □

A number particular types of ideals will be of use to us over the course of this class, namely:

1. Principle: Generated by a single element
2. Maximal: Nontrivial ideal which isn't in any others
3. Prime Ideal: An ideal P such that $x \notin P$ and $y \notin P$ then $xy \notin P$. Moreover if $xy \in P$, then $x \in P$ or $y \in P$. E.g. take any ideal in $\mathbb{Z}$ generated by a prime number.

Notice that the structure of quotient rings by ideals classifies nicely:

Theorem 1.2 (Quotient Ring Classification)**.**

1. *R/M is a field $\longleftrightarrow$ M is maximal*
2. *R/P is an Integral domain $\longleftrightarrow$ P is prime.*

Proof. Solution given by Saskia:

(1.)

($\Rightarrow$) R/M is a field, so it therefore has no nontrivial ideals. Then also, by the correspondence theorem R has no ideals I with $M \subset I \subset R$.

($\Leftarrow$) Works the same way as forward direction. $\triangle$

(2.)

$\Rightarrow$. We have that $[x], [y] \in R/M$, so then $x_1 \in [x], y_1 \in [y]$, but then $[x_1 y_1] = [xy] = 0$ iff $[x]$ or $[y] = 0$. Then, by definition of an ideal, $x_1 y_1 \in M \Leftrightarrow x_1$ or $y_1 \in M$.

($\Leftarrow$) Since M is a prime ideal, $xy \in M$ means that one of x or y is in M, then $[xy] = [x][y] = 0 \Leftrightarrow [x] = 0$ or $[y] = 0$

$\square$

Proof. Alternative proof of the field condition given by Ishaan.

($\Rightarrow$) Let I be an ideal such that $M \subseteq I$. Then let $a \in I$ be given such that $a \notin M$. Then $a + M \neq 0 + M$, and also $\exists b + M$ such that $(a + M)(b + M) = 1 + M$. Then $ab + M = 1 + M$, which indicates that $ab - 1 \in M \subsetneq I$. Thus $ab \in I, 1 \in I$, so $1 \in I = R$.

($\Leftarrow$) Suppose that M is maximal, then let $a + M \in R/M$ such such that $a + M \neq 0 + M$. Now notice that if we attempt to extend the ideal M via a, we will get the whole ring. So then since $1 \in R$, $\exists r \in R$, $\exists m \in M$ such that $1 = ra + m$, so $1 + M = ra + M$, and $(r + M)(a + M)$.

$\square$

Note. Every maximal ideal is prime.

Proposition 1.3. *Every ring has a maximal ideal.*

Proof. *(Exercise)* $\square$

Corollary 1.4. *Every ideal is contained in some maximal ideal*

Corollary 1.5. *Every non-unit is contained in a maximal ideal*

Definition 4. A **local ring** is a ring which has a single maximal ideal

Proposition 1.6. *Suppose $m \subsetneq R$ such that every $\alpha \in m^c$ is a unit. Then R is local with maximal ideal m.*

Example 1. Let $C^0(\mathbb{R}) = \{\text{continuous functions} j : \mathbb{R} \to \mathbb{R}\}$, and take the ring identified by the collection of germs $[f]_0$ which are centered at 0.

Notice that the collection of functions vanishing at 0 is a maximal ideal.

Proof. Define the evaluation map at zero $\begin{matrix} \mathrm{ev}_0 : C^0(R)_0 \to \mathbb{R} \\ \mathrm{ev}_0(f) \mapsto f(0) \end{matrix}$.. Notice that for any continuous function $f \neq 0$ at 0., the $\frac{1}{f}$ is also continuous at 0, and $f * \frac{1}{f} = 1$. $\square$

1.1 Exercises

1. Prove (1.1)

2. Prove (1.2)

3. Let K be a field show that $(f(x)) = (g(x))$ iff f and g differ by a constant for $f(x), g(x) \in K[x]$,

4. Prove that if R is an integral domain, then $R[x]$ is too.

   Proof. ($\Rightarrow$) Suppose that $(f(x)) = (g(x))$, $f(x), g(x) \in K[x]$, then there exists some $k(x), k'(x) \in K[x]$ such that $f(x) = k(x)g(x)$ and $g(x) = k'(x)f(x)$. Then $\deg(f(x)) = \deg(k) + \deg(g) = \deg k + \deg k' + \deg f$, and so the degree of the ks must be zero.

   ($\Leftarrow$) Suppose that $\exists c, c' \in \mathbb{R}$ such that $f(x) = cg(x)$ and $g(x) = c'f(x)$

   $\square$

5. Define $\mathbb{C}[[x]] = \{\sum_{n=0}^{\infty} a_n x^n \mid a_n \in \mathbb{C}\}$. This is the ring of formal power series. Show that $\mathbb{C}[[x]]$ is a local ring.

   Proof. Consider the ideal of non-units $M = \langle x \rangle$, i.e. the ideal of all formal power series with constant term $= 0$.

   It is trivially obvious that this is an ideal.

   Now notice that $\mathbb{C}[[x]]/M \cong \mathbb{C}$ since $\forall a \in \mathbb{C}[[x]]$,

$$a = \sum_{i=0}^{\infty} a_i x^i = \underbrace{\left(\sum_{i=1}^{\infty} a_i x^i\right)}_{\in \langle x \rangle} + a_0,$$

   so in $\mathbb{C}[[x]]/M$, $\overline{a}$ is in bijection with a_0 in $\mathbb{C}$. Then, since $\mathbb{C}$ is a field, M must indeed be maximal. Now suppose that there was some other maximal ideal $M' \neq M$. But if this is the case, then there must be some unit $u \in M'$, which means that $1 = uu^{-1} \in M'$, so $M' = \mathbb{C}[[x]]$. And thus M is indeed a unique maximal ideal[1]. $\square$

[1] There is an assumption here that M really is the ideal containing all the non units, but this is easy to show