

A GUIDed tour around Bayesian regression

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Abstract

This paper presents a graphical user interface (GUI) to perform Bayesian regression analysis in a very friendly environment without any programming skills. This paper is designed for teaching and applied purposes at introductory level; we present the basic theory underlying all regression models that we developed in our GUI, which in turn is based on an interactive web application using shiny, and libraries from R.

Keywords: Bayesian regression, Graphical user interface, Variable selection.

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1 Introduction

The main objective of this paper is to present an open source teaching graphical user interface (GUI) to implement Bayesian regression analysis using cross sectional and longitudinal data. We show the basic theoretical framework, and the tutorial for implementing these models. Therefore, practitioners and applied researchers can apply Bayesian regression analysis understanding its theoretical foundation without requiring programming skills. The latter seems to be one relevant impediment to increase the use of the Bayesian framework (Woodward, 2005; Karabatsos, 2016).

Table 1 shows available graphical user interfaces to perform Bayesian regression analysis. ShinyStan (Stan Development Team, 2017) is a very flexible open source, but users do require programming skills. BugsXLA (Woodward, 2005) is a less flexible open source; However, users do not require programming skills. Bayesian regression: Nonparametric and parametric models (Karabatsos, 2016) is a very flexible and friendly GUI that is based on MATLAB Compiler for 64-bit Windows computer. Its core is on Bayesian nonparametric regressions, and it can be though for users who have mastered basic parametric models as the onsw that we show in our GUI. On the other hand, MATLAB toolkit, Stata and BayES are not open sources.

We developed our GUI based on an interactive web application using shiny (Chang, 2018), and some libraries in R (R Core Team, 2018). In Table 2 can be seen specific libraries and commands that are used in our GUI. It has 9 univariate models, 4 multivariate, 3 hierarchical longitudinal, Bayesian bootstrap and 6 Bayesian model averaging frameworks. In addition, it gives basic summaries and diagnostics of the posterior chains, as well as the posterior chains, and different plots such as trace, autocorrelation and densities. Regarding flexibility and possibilities is between ShinyStan and BugsXLA, users do not require any programming skills, but it is as advanced as Karabatsos (2016)'s software. However, as our GUI is in RStudio cloud, it can be run in any operative system supported by this service. Our GUI, BEsmarter,¹ is freely available at <https://rstudio.cloud/project/211946>; so users have access to all developed code, and run our GUI typing `shiny::runApp()` on the console of our RStudio cloud project.

After this brief introduction, we present in Section 2 some general theoretical and computational aspects of the Bayesian framework. Section 3 shows specific technical details of the models implemented in our GUI. Section 4 introduces our GUI, and how to use it. Finally, Section 5 presents some concluding remarks and future developments.

¹Bayesian Econometrics: simulations, models and applications to research, teaching and encoding with responsibility.

2 Bayesian framework: A brief summary of theory and computation

The theoretical point of departure is the Bayes' rule, which formally establishes how prior beliefs are updated with new information (data),

$$\pi(\theta|y) = \frac{p(y|\theta)\pi(\theta)}{p(y)} \propto p(y|\theta)\pi(\theta),$$

where $\pi(\theta|y)$ is the posterior distribution, $\pi(\theta)$ is the prior, $p(y|\theta)$ is the likelihood, and $p(y) = \int_{\Theta} p(y|\theta)\pi(\theta)d\theta$ is the marginal likelihood.

The Bayesian framework allows to report the full posterior distribution. In case that just one value of the posterior distribution should be reported, decision theory offers an elegant framework, for instance, a quadratic loss function implies reporting posterior means. There are other appealing characteristics of this statistical framework, namely: There is a solid probability theory framework for hypothesis testing based on posterior odds (H_1 vs H_2), $PO_{12} = \frac{p(H_1|y)}{p(H_2|y)} = \frac{p(y|H_1)}{p(y|H_2)} \times \frac{\pi(H_1)}{\pi(H_2)}$, the first term is called the Bayes factor, and the second term is the prior odds, $\pi(H_j)$ is the prior probability of hypothesis H_j , and $p(y|H_j) = \int_{\Theta_j} p(y|\theta_j, H_j)\pi(\theta_j|H_j)d\theta_j$ (Kass and Raftery, 1995). Under this framework is also easy to account for model uncertainty based on posterior model probabilities, $p(M_j|y) = \frac{\pi(M_j)p(y|M_j)}{\sum_{m=1}^M \pi(M_m)p(y|M_m)}$, where $\pi(M_j)$ is the prior model probability, and $p(y|M_j) = \int_{\Theta_j} p(y|\theta_j, M_j)\pi(\theta_j|M_j)d\theta_j$ is the marginal likelihood under model j , $j = 1, 2, \dots, M$. In addition, it is also easy to perform inference of nonlinear functions of parameter estimates without requiring extra computational effort (re-sampling techniques) or asymptotic results (plug-in approach based on delta method). In a Bayesian setting, we can have parameters that are not identified, but perform inference of functions of these parameters that are well identified. Finally, predictive distributions acknowledge parameter uncertainty, $\pi(y^{New}|y) = \int_{\Theta} f(y^{New}|\theta)\pi(\theta|y)d\theta \approx \frac{1}{S} \sum_{s=1}^S f(y^{New}|\theta^{(s)})$, $\theta^{(s)} \sim \pi(\theta|y)$.

The main issue regarding estimation in the Bayesian approach is how to obtain draws from the posterior distribution when this distribution is not a standard one. The Metropolis-Hastings (M-H) algorithm is one of the most popular techniques to perform this task (Metropolis et al., 1953; Hastings, 1970). Given a target distribution $\pi(\theta|y)$, $\theta \in \Theta \subseteq \mathcal{R}^k$, such that $\pi(d\theta^c|y) = \int_{\Theta} p(\theta, d\theta^c)\pi(\theta|y)d\theta$ where $p(\theta, d\theta^c)$ is a conditional distribution function (transition kernel) that represents the probability of moving from θ to a point in $d\theta^c$, the M-H algorithm establishes

$$p(\theta, d\theta^c) = q(\theta, \theta^c)\alpha(\theta, \theta^c)d\theta^c + \left[1 - \int_{\Theta} q(\theta, \theta^c)\alpha(\theta, \theta^c)d\theta^c\right] \delta_{\theta}(d\theta^c),$$

where $q(\theta, \theta^c)$ is a proposal density function to draw candidates (θ^c), $\alpha(\theta, \theta^c) = \min \left\{ \frac{\pi(\theta^c|y)q(\theta, \theta^c)}{\pi(\theta|y)q(\theta^c, \theta)}, 1 \right\} = \min \left\{ \frac{p(y|\theta^c)\pi(\theta^c)q(\theta, \theta^c)}{p(y|\theta)\pi(\theta)q(\theta^c, \theta)}, 1 \right\}$ is the probability to move from θ to θ^c ,² $\delta_{\theta}(d\theta^c)$ is equal to 1 if

² $\pi(\theta|y)q(\theta, \theta^c) \neq 0$ is usually satisfied in practice.

$\theta \in d\theta^c$ and 0 otherwise, and $[1 - \int_{\Theta} q(\theta, \theta^c) \alpha(\theta, \theta^c) d\theta^c]$ is the probability that the chain remains at θ (Chib and Greenberg, 1995).³ The intuition behind the construction of this transition kernel is to transform a transition kernel that does not satisfy the reversibility condition into one that satisfies it.⁴ The algorithm is the following:

Algorithm A1 The Metropolis-Hastings algorithm

- 1: Initialized at an arbitrary value $\theta^{(0)}$.
 - 2: Draw θ^c from $q(\theta^{(s-1)}, \cdot)$, $s = 1, 2, \dots, S$, and u from $\mathcal{U}(0, 1)$.
 - 3: If $u \leq \alpha(\theta^{(s-1)}, \theta^c)$, set $\theta^{(s)} = \theta^c$, else $\theta^{(s)} = \theta^{(s-1)}$.
 - 4: Repeat this process S times, $s = 1, 2, \dots, S$ iterations.
 - 5: Return the values $\{\theta^{(b)}, \theta^{(b+d)}, \dots, \theta^{(S)}\}$ where b and d are burn-in and thinning parameters, respectively.
-

Observe that the M-H algorithm does not depend on the marginal likelihood, therefore it is not necessary to calculate the integral, which can be very computational demanding. This algorithm requires to define the proposal density, generates autocorrelated chains by construction, the chain remains at $\theta^{(s-1)}$ if θ^c is rejected, and it is necessary to discard some initial draws due to avoiding dependence to the initial value, and enough iterations to achieve convergence to the target distribution.

Regarding the proposal density, we implement in this GUI the random-walk and the tailored proposals. The former generates $\theta^c = \theta^{(s-1)} + \epsilon^{(s)}$, where the distribution of $\epsilon^{(s)}$ is specified. If this distribution is symmetric around zero, then $q(\theta, \theta^c) = q(\theta^c, \theta)$. The tailored proposal is based on a fat-tailed distribution, whose mean is the maximum of the logarithm of the posterior distribution, $\hat{\theta}$, and scale matrix equal to the negative of the inverse Hessian at the maximum, $\left(-\frac{\partial^2 \ln \pi(\theta|y)}{\partial \theta \partial \theta'} \Big|_{\hat{\theta}}\right)^{-1}$.

A crucial point associated with the proposal densities is the acceptance rate. A low or high acceptance rates are not ideal. A low rate implies poor mixing, that is, the chain does not move through the support of the posterior distribution. A high acceptance rate implies that chain will converge too slowly. A sensible value depends on the dimension of the parameter space. A rule of thumb in the case that the dimension is lower or equal to 2 is 0.50. If the dimension is greater than 2, the acceptance rate should be around 0.25 (Roberts et al., 1997).

Regarding convergence issues, we implement some diagnostics in order to check the adequacy of the posterior chains (Plummer et al., 2016). In particular, trace plots, which should look stable, and autocorrelation plots that should decrease very fast. In addition, Geweke's test (Geweke, 1992), which is a simple two-sample test of means. If the mean of the fraction in the first window (10%) is not significantly different from the mean of the fraction in the second window (50%), then we conclude the target distribution converged. Raftery and Lewis' test

³Observe that $\alpha(\theta, \theta^c)$ depends on y , and the proposal $q(\theta, \theta^c)$ can depend or cannot depend on y .

⁴Intuitively, the reversibility condition, $\pi(\theta|y)q(\theta, \theta^c) = \pi(\theta^c|y)q(\theta^c, \theta)$, establishes that the probability to move from θ to θ^c is equal that the probability to move from θ^c to θ .

(Raftery and Lewis, 1992) is designed to calculate the approximate number of iterations (S), burn-in (b) and thinning parameter (d) required to estimate $p[H(\theta) \leq h]$, $H(\theta) : \mathcal{R}^k \rightarrow \mathcal{R}$, given a specific quantile of interest (q), precision (r) and probability (p). Their diagnostic is based on the dependence factor, $I = \frac{S \times b}{S_{Min}}$, $S_{Min} = \Phi^{-1}(\frac{1}{2}(p+1))^2 q(1-q)/r^2$, $\Phi(\cdot)$ is the normal distribution function. Values of I much greater than 5 indicate a high level of dependence. Heidelberger and Welch's test (Heidelberger and Welch, 1983) is based on a Cramer-von-Mises statistic to test the null hypothesis that the sampled values, $\theta^{(s)}$, come from a stationary distribution, $CVM(B_S) = \int_0^1 B_S(t)^2 dt$, $B_S(t) = (S_{[St]} - [St] \bar{\theta}^S) / (Sp(0))^{1/2}$, $S_S = \sum_{s=1}^S \theta^{(s)}$, $\bar{\theta}^S = S_S / S$, $p(0)$ is the spectral density at 0, and $0 \leq t \leq 1$. $B_S(t)$ converges in distribution to the Brownian bridge under the null hypothesis. This test is recursively applied, until either the null hypothesis is not rejected, or $t = 50\%$ of the chain has been discarded. Then, the half-width test calculates a 95% confidence interval for the mean, using the portion of the chain which passed the stationarity test. If the ratio between the half-width of this interval and the mean is lower than 0.1, this test is passed.

Another popular MCMC algorithm that is extensively used in our models is the Gibbs sampler (Geman and Geman, 1984; Gelfand and Smith, 1990). This can be seen as a particular case of the M-H algorithm where the acceptance rate is equal to 1 (Gelman and Rubin, 1992; Robert and Casella, 2004). This is based on the fact that the full conditional distributions perfectly summarize the joint density (Hammersley–Clifford theorem),⁵ as a consequence, it is necessary to have the full set of conditional posterior distributions to implement it. The algorithm is the following:

Algorithm A2 The Gibbs sampler algorithm

- 1: Given $\theta = [\theta_1, \theta_2, \dots, \theta_k]$, initialized at an arbitrary value $\theta_{-1}^{(0)}$, $\theta_{-l}^{(0)} = [\theta_1, \theta_2, \dots, \theta_{l-1}, \theta_{l+1}, \dots, \theta_k]$.⁶
 - 2: Draw $\theta_1^{(s)}$ from $\pi(\theta_1 | \theta_{-1}^{(s-1)}; y)$.
 - 3: Draw $\theta_2^{(s)}$ from $\pi(\theta_2 | \theta_1^{(s)}, \theta_3^{(s-1)}, \dots, \theta_k^{(s-1)}; y)$.
 - 4: \vdots
 - 5: Draw $\theta_k^{(s)}$ from $\pi(\theta_k | \theta_{-k}^{(s)}; y)$.
 - 6: Repeat this process S times, $s = 1, 2, \dots, S$ iterations.
 - 7: Return the values $\{\theta^{(b)}, \theta^{(b+d)}, \dots, \theta^{(S)}\}$ where b and d are burn-in and thinning parameters, respectively.
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We should highlight that the Gibbs sampler can be applied in settings where the conditional posterior distributions are not standard. In this case, Metropolis-within-Gibbs algorithm can be adopted. Rather than simulating $\theta_l^{(s)} \sim \pi(\theta_l | \theta_1, \theta_2, \dots, \theta_{l-1}, \theta_{l+1}, \dots, \theta_k)$, we can implement a M-H step. The theoretical validity of this strategy is exactly the same as with the original Gibbs sampler (Robert and Casella, 2010).

⁵This statement requires the positivity condition, if $\pi(\theta_l | y) > 0$, then $\pi(\theta | y) > 0$, $l = 1, 2, \dots, k$.

⁶Note that θ_l , $l = 1, 2, \dots, k$ can be either scalars or vectors in Algorithm A2.

After this brief introduction to the general Bayesian framework, we present specific details of models in our GUI in the next section.

3 Models

3.1 Univariate models

Normal–Inverse Gamma: The Gaussian linear model specifies $\mathbf{y} = \mathbf{X}\beta + \mu$ such that $\mu \sim \mathcal{N}(0, \sigma^2 I_n)$ is an stochastic error, X is a $n \times k$ matrix of regressors, β is a k dimensional vector of coefficients, y is a n dimensional vector of a dependent variable, and n is the number of units.

The conjugate independent priors for the parameters are $\beta \sim \mathcal{N}(\beta_0, B_0)$ and $\sigma^2 \sim \mathcal{IG}(\alpha_0/2, \delta_0/2)$. Given the likelihood function, $p(\beta, \sigma^2 | y, X) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\}$, the conditional posterior distributions are

$$\beta | \sigma^2, y, X \sim \mathcal{N}(\beta^*, \sigma^2 B),^7$$

$$\sigma^2 | \beta, y, X \sim \mathcal{IG}(\alpha^*/2, \delta^*/2),^8$$

where $B = (B_0^{-1} + \sigma^{-2}X'X)^{-1}$, $\beta^* = B(B_0^{-1}\beta_0 + \sigma^{-2}X'y)$, $\alpha^* = \alpha_0 + n$ and $\delta^* = \delta_0 + (y - X\beta)'(y - X\beta)$.

We can apply Gibbs sampler in this model due to having standard conditional posterior distributions.

Logit: In the logit model the dependent variable is binary, then it follows a Bernoulli distribution, $y_i \stackrel{i.i.d.}{\sim} \mathcal{B}(\pi_i)$,⁹ that is $p(y_i = 1) = \pi_i$, such that $\pi_i = \frac{\exp\{x_i'\beta\}}{1 + \exp\{x_i'\beta\}}$. We specify a Gaussian distribution as prior $\beta \sim \mathcal{N}(\beta_0, B_0)$.

The logit model does not have a standard posterior distribution. Then, a random-walk Metropolis–Hastings algorithm is used to obtain draws from the posterior distribution (Martin et al., 2011, 2018). In particular, the proposal is multivariate Gaussian centered at the current value, and covariance matrix $S(B_0^{-1} + \hat{\Sigma}^{-1})^{-1}S$, where $S = sI_{\dim\beta}$, $s > 0$ is a tune parameter,¹⁰ and $\hat{\Sigma}$ is the sample covariance matrix from the maximum likelihood estimation.

Probit: The probit model also has as dependent variable a binary outcome. In this case, there is a latent variable (y_i^* , unobserved) that defines the structure of the estimation problem. In particular, $y_i = \begin{cases} 0, & y_i^* \leq 0 \\ 1, & y_i^* > 0 \end{cases}$ such that $y_i^* = x_i'\beta + \mu_i$, $\mu_i \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$.¹¹ This

⁷ \mathcal{N} denotes a normal density.

⁸ \mathcal{IG} denotes an inverse gamma density.

⁹ \mathcal{B} denotes a Bernoulli density.

¹⁰Tune parameters should be set such that achieve reasonable diagnostic criteria.

¹¹The variance in this model is set to 1 due to identification restrictions. Observe that multiplying y_i^* by a positive constant does not affect y_i .

implies $p(y_i = 1) = \pi_i = \Phi(x'_i \beta)$.

Albert and Chib (1993) implemented data augmentation (Tanner and Wong, 1987) to apply a Gibbs sampling algorithm in this model. Augmenting this model with y_i^* , we can have the likelihood contribution from observation i , $p(y_i|y_i^*) = 1_{y_i=0}1_{y_i^* \leq 0} + 1_{y_i=1}1_{y_i^* > 0}$, where 1_A is an indicator function that takes the value of 1 when condition A is satisfied.

Taking a Gaussian distribution as prior $\beta \sim \mathcal{N}(\beta_0, B_0)$, then $\pi(\beta, y^*|y, X) \propto \prod_{i=1}^n [1_{y_i=0}1_{y_i^* \leq 0} + 1_{y_i=1}1_{y_i^* > 0}] \mathcal{N}_n(y^*|X\beta, I_n) \times \mathcal{N}_n(\beta|\beta_0, B_0)$. This implies,

$$y_i^*|\beta, y, X \sim \begin{cases} \mathcal{TN}_{(-\infty, 0]}(x'_i \beta, 1) & , y_i = 0 \\ \mathcal{TN}_{(0, \infty)}(x'_i \beta, 1) & , y_i = 1 \end{cases},^{12}$$

$$\beta|y^*, X \sim \mathcal{N}(\beta^*, B),$$

where $B = (B_0^{-1} + X'X)^{-1}$, and $\beta^* = B(B_0^{-1}\beta_0 + X'y^*)$.

Multinomial probit: The multinomial probit model is used to model mutually exclusive discrete outcomes or qualitative response variables. We observe $y_{il} = \begin{cases} 1, & y_{il}^* \geq \max(y_i^*) \\ 0, & \text{otherwise} \end{cases}$

such that $y_i^* = X_i\beta + \mu_i$, $\mu_i \stackrel{i.i.d}{\sim} \mathcal{N}(0, \Sigma)$, y_i^* is an unobserved latent L dimensional vector, X_i is a $L \times k$ matrix of regressors for each alternative, $l = 1, 2, \dots, L$. We take into account simultaneously alternative-varying regressors (alternative attributes) and alternative-invariant regressors (individual characteristics).¹³ y_i^* can be stacked up into a multiple regression with correlated stochastic errors, $y^* = X\beta + \mu$, where $y^{*'} = [y_1^{*'}, y_2^{*'}, \dots, y_n^{*'}]$, $X' = [X_1', X_2', \dots, X_n']$, and $\mu' = [\mu_1', \mu_2', \dots, \mu_n']$.

Following the practice of expressing $y_{il}^{*'}$ relative to $y_{iL}^{*'}$ by letting $w_i' = [w_{i1}, w_{i2}, \dots, w_{iL-1}]$, $w_{il} = y_{il}^* - y_{iL}^*$, we can write $w_i = R_i\beta + \epsilon_i$, $\epsilon_i \sim \mathcal{N}(0, \Omega)$, where R_i denotes the $(L-1) \times k$ matrix of observation vectors obtained from X_i by subtracting the L -th row from the first $(L-1)$ rows.

The likelihood function in this model is $p(\beta, \Omega|y, R) = \prod_{i=1}^n \prod_{l=1}^L p_{il}^{y_{il}}$ where $p_{il} = p(y_{il}^* \geq \max(y_i^*))$. We assume independent priors, $\beta \sim \mathcal{N}(\beta_0, B_0)$ and $\Omega^{-1} \sim \mathcal{W}(\alpha_0, \Sigma_0)$.¹⁴ We can apply Gibbs sampling in this model because this is a standard Bayesian linear regression model when data augmentation in w is used. The posterior conditional distributions are

$$\beta|\Omega, w \sim \mathcal{N}(\beta^*, B),$$

$$\Omega^{-1}|\beta, w \sim \mathcal{W}(\alpha^*, \Sigma^*),$$

¹² \mathcal{TN} denotes a truncated normal density.

¹³Note that this model is not identified if Σ is unrestricted. The likelihood function is the same if a scalar random variable is added to each of the L latent regressions.

¹⁴ \mathcal{W} denotes the wishart density.

where $B = (B_0^{-1} + X^{*'}X^*)^{-1}$, $\beta^* = B(B_0^{-1}\beta_0 + X^{*'}w^*)$, $\Omega^{-1} = C'C$, $X_i^{*'} = C'R_i$, $w_i^* = C'w_i$,

$$X^* \begin{bmatrix} X_1^* \\ X_2^* \\ \vdots \\ X_n^* \end{bmatrix}, \alpha^* = \alpha_0 + n, \Sigma^* = (\Sigma_0 + \sum_{i=1}^n (w_i - R_i\beta)'(w_i - R_i\beta))^{-1}.$$

We can collapse the multinomial vector y_i into the indicator variable $d_i = \sum_{l=1}^{L-1} l \times I_{\max(w_l)=w_{il}}$,¹⁵ then the distribution of $w_i|\beta, \Omega^{-1}, d_i$ is a $L-1$ dimensional Gaussian distribution truncated over the appropriate cone in \mathcal{R}^{L-1} . McCulloch and Rossi (1994) propose to draw from the univariate conditional distributions $w_{il}|w_{i,-l}, \beta, \Omega^{-1}, d_i \sim \mathcal{TN}_{I_{il}}(m_{il}, \tau_{ll}^2)$, where $I_{il} = \begin{cases} w_{il} > \max(w_{i,-l}, 0), & d_i = l \\ w_{il} < \max(w_{i,-l}, 0), & d_i \neq l \end{cases}$, and permuting the columns and rows of Ω^{-1} such that the l -th column and row is the last,

$$\Omega^{-1} = \begin{bmatrix} \Omega_{-l,-l} & \omega_{-l,l} \\ \omega_{l,-1} & \omega_{l,l} \end{bmatrix}^{-1} = \begin{bmatrix} \Omega_{-l,-l}^{-1} + f'Ef & -f\tau_{ll}^{-2} \\ -\tau_{ll}^{-2}f' & \tau_{ll}^{-2} \end{bmatrix}$$

where $f = \Omega_{-l,-l}\omega_{-l,l}$, $\tau_{ll}^2 = \omega_{ll} - \omega_{l,-l}\Omega_{-l,-l}^{-1}\omega_{-l,l}$, $m_{il} = r'_{il}\beta + f'(w_{i,-l} - R_{i,-l}\beta)$, $w_{i,-l}$ is a $L-2$ dimensional vector of all components of w_i excluding w_{il} , r_{il} is the l -th row of R_i , $l = 1, 2, \dots, L-1$.

The identified parameters are obtained by normalizing with respect to one of the diagonal elements $\frac{1}{\omega_{1,1}^{0.5}}\beta$ and $\frac{1}{\omega_{1,1}}\Omega$.

Multinomial logit: The multinomial logit model is also used to model mutually exclusive discrete outcomes or qualitative response variables. We consider the multinomial mixed logit model (not to be confused with the random parameters logit model), that is, we take into account simultaneously alternative-varying regressors (conditional) and alternative-invariant regressors (multinomial).¹⁶

In this setting there are L mutually exclusive alternatives, and the dependent variable y_i is equal to l if the l th alternative is chosen by individual i , $l = \{1, 2, \dots, L\}$. The likelihood function is $p(\beta|y, X) = \prod_{i=1}^n p(y_i = l|\beta, X)$, where $p(y_i = l|\beta, X) = \frac{\exp\{x'_{il}\beta\}}{\sum_{j=1}^L \exp\{x'_{ij}\beta\}}$, X is a $nL \times k$ matrix, $k = k_1 + k_2$ is the total number of regressors, k_1 and k_2 are the number of alternative-varying and alternative-invariant regressors, respectively. In addition, we assume $\beta \sim \mathcal{N}(\beta_0, B_0)$ as prior distribution.

As the multinomial logit model does not have a standard posterior distribution, we propose a “tailored” Metropolis-Hastings algorithm as proposal distribution. In particular, a multivariate student-t distribution with v degrees of freedom (tune parameter), mean equal to the maximum likelihood estimator and scale equal to the inverse of the Hessian matrix.

¹⁵Observe that the identification issue in this model is due to scaling w_{il} by a positive constant does not change the value of d_i .

¹⁶The multinomial mixed logit model can be implemented as a conditional logit model.

Ordered probit: The ordered probit model is used when there is a natural order in the categorical response variable. In this case, there is a latent variable $y_i^* = x_i' \beta + \mu_i$, $\mu_i \stackrel{i.i.d}{\sim} \mathcal{N}(0, 1)$, such that $y_i = l$ if and only if $\alpha_{l-1} < y_i^* \leq \alpha_l$, $l = \{1, 2, \dots, L\}$, where $\alpha_0 = -\infty$, $\alpha_1 = 0$ and $\alpha_L = \infty$.¹⁷ Then, $p(y_i = l) = \Phi(\alpha_l - x_i' \beta) - \Phi(\alpha_{l-1} - x_i' \beta)$, and the likelihood function is $p(\beta, \alpha | y, X) = \prod_{i=1}^n p(y_i = l | \beta, \alpha, X)$. The independent priors of this model are $\beta \sim \mathcal{N}(\beta_0, B_0)$ and $\gamma \sim \mathcal{N}(\gamma_0, \Gamma_0)$, $\gamma = [\gamma_2, \gamma_3, \dots, \gamma_{L-1}]'$, such that $\alpha = \left[\exp\{\gamma_2\}, \sum_{l=2}^3 \exp\{\gamma_l\}, \dots, \sum_{l=2}^{L-1} \exp\{\gamma_l\} \right]'$.

This model does not have a standard conditional posterior distribution for γ (α), but it does have a standard conditional distribution for β once data augmentation is used. Then, we use a Metropolis-within-Gibbs sampling algorithm. In particular, we use Gibbs sampling algorithms to draw β and y^* ,

$$\beta | y^*, \alpha, X \sim \mathcal{N}(\beta^*, B),$$

where $B = (B_0^{-1} + X'X)^{-1}$, $\beta^* = B(B_0^{-1}\beta_0 + X'y^*)$, and $y_i^* | \beta, \alpha, y, X \sim \mathcal{TN}_{(\alpha_{y_i-1}, \alpha_{y_i})}(x_i' \beta, 1)$.

We use a random-walk Metropolis-Hastings algorithm for γ that has as proposal a Gaussian distribution with mean equal to the current value, and covariance matrix $s^2(B_0^{-1} + \hat{\Sigma}^{-1})^{-1}$, where $s > 0$ is a tune parameter, and $\hat{\Sigma}$ is the sample covariance matrix from the maximum likelihood estimation.

Negative binomial: The dependent variable in the negative binomial model is a non-negative integer or count. In contrast to the poisson model, the negative binomial model takes into account over-dispersion. The Poisson model has mean and variance equal.

We assume that $y_i \stackrel{i.n.d}{\sim} \mathcal{NB}(\gamma, \theta_i)$, that is, the density function for individual i is $\frac{\Gamma(y_i + \gamma)}{\Gamma(\gamma) y_i!} \theta_i^{y_i} (1 - \theta_i)^\gamma$, the success probability is $\theta_i = \frac{\lambda_i}{\lambda_i + \gamma}$, $\lambda_i = \exp\{x_i' \beta\}$ and $\gamma = \exp\{\alpha\}$. The independent priors for this model are $\beta \sim \mathcal{N}(\beta_0, B_0)$ and $\alpha \sim \mathcal{G}(\alpha_0, \delta_0)$.¹⁸

This model does not have standard conditional posterior distributions, then we use a random-walk Metropolis-Hastings algorithm where the proposal distribution for β is Gaussian centered at the current stage and covariance matrix $s_\beta^2 \hat{\Sigma}_\beta$ where s_β is a tune parameter and $\hat{\Sigma}_\beta$ is the maximum likelihood covariance estimator. In addition, the proposal for α is normal centered at the current value, and variance $s_\alpha^2 \hat{\sigma}_\alpha^2$ where s_α is a tune parameter and $\hat{\sigma}_\alpha^2$ is the maximum likelihood variance estimator.

Tobit: The dependent variable is partially observed in Tobit models due to sampling schemes, whereas regressors are completely observed. In particular,

$$y_i = \begin{cases} L, & y_i^* < L \\ y_i^*, & L \leq y_i^* < U \\ U, & y_i^* \geq U \end{cases},$$

¹⁷Identification issues imply setting the variance in this model equal to 1 and $\alpha_1 = 0$. Observe that multiplying y_i^* by a positive constant or adding a constant to all of the cut-offs and subtracting the same constant from the intercept does not affect y_i .

¹⁸ \mathcal{G} denotes a gamma density.

where $y_i^* \stackrel{i.n.d}{\sim} \mathcal{N}(x'_i \beta, \sigma^2)$.¹⁹

We use conjugate independent priors $\beta \sim \mathcal{N}(\beta_0, B_0)$ and $\sigma^2 \sim \mathcal{IG}(\alpha_0/2, \delta_0/2)$, and data augmentation using y_C^* such that $y_{C_i}^* \stackrel{i.n.d}{\sim} \mathcal{N}(x'_i \beta, \sigma^2)$, $y_{C_i} = \{y_{C_i^L}^* \cup y_{C_i^U}^*\}$ are lower and upper censored data. This allows to implement the Gibbs sampling algorithm (Chib, 1992). Then, $\pi(\beta, \sigma^2, y^* | y, X) \propto \prod_{i=1}^n \left[1_{y_i=L} 1_{y_{C_i^L}^* < L} + 1_{L \leq y_i < U} + 1_{y_i=U} 1_{y_{C_i^U}^* \geq U} \right] \mathcal{N}(y_i^* | x'_i \beta, \sigma^2) \times \mathcal{N}_n(\beta | \beta_0, B_0) \times \mathcal{IG}(\sigma^2 | \alpha_0/2, \delta_0/2)$

The posteriors distributions are

$$y_{C_i}^* | \beta, \sigma^2, y, X \sim \begin{cases} \mathcal{TN}_{(-\infty, L)}(x'_i \beta, \sigma^2), & y_i = L \\ \mathcal{TN}_{[U, \infty)}(x'_i \beta, \sigma^2), & y_i = U \end{cases},$$

$$\beta | \sigma^2, y, X \sim \mathcal{N}(\beta^*, \sigma^2 B),$$

$$\sigma^2 | \beta, y, X \sim \mathcal{IG}(\alpha^*/2, \delta^*/2),$$

where $B = (B_0^{-1} + \sigma^{-2} X' X)^{-1}$, $\beta^* = B(B_0^{-1} \beta_0 + \sigma^{-2} X' y)$, $\alpha^* = \alpha_0 + n$ and $\delta^* = \delta_0 + (y^* - X\beta)'(y^* - X\beta)$.

Quantile: In quantile regression the location parameters vary according to the quantile of the dependent variable. Let $q_\tau(x_i) = x'_i \beta_\tau$ denote the τ -th ($0 < \tau < 1$) quantile regression function of y_i given x_i such that $y_i = x'_i \beta_\tau + \mu_i$ where $\int_{-\infty}^0 f_\tau(\mu_i) d\mu_i = \tau$. In particular, $f_\tau(\mu_i) = \tau(1-\tau) \exp\{\mu_i(\tau - I_{\mu_i < 0})\}$ (asymmetric Laplace distribution). Kozumi and Kobayashi (2011) propose the location-scale mixture of normals representation given by $\mu_i = \theta e_i + \psi \sqrt{e_i} z_i$ where $\theta = \frac{1-2\tau}{\tau(1-\tau)}$, $\psi^2 = \frac{2}{\tau(1-\tau)}$, $e_i \sim \mathcal{E}(1)$ and $z_i \sim \mathcal{N}(0, 1)$, $e_i \perp z_i$.²⁰ As a consequence of this representation and *i.i.d* sample, $p(y | \beta_\tau, e, X) \propto \left(\prod_{i=1}^n e_i^{-1/2} \right) \exp \left\{ - \sum_{i=1}^n \frac{(y_i - x'_i \beta_\tau - \theta e_i)^2}{2\psi^2 e_i} \right\}$.

Taking as prior a normal distribution for β_τ , that is, $\beta_\tau \sim \mathcal{N}(\beta_{\tau 0}, B_{\tau 0})$, and using data augmentation for e , we can implement a Gibbs sampling algorithm in this model. The posterior distributions are

$$\beta_\tau | e, y, X \sim \mathcal{N}(\beta_\tau^*, B_\tau),$$

$$e_i | \beta_\tau, y, X \sim \mathcal{GIG}(1/2, \alpha_i^*, \delta_i^*),$$
²¹

where $B_\tau = \left(B_{\tau 0}^{-1} + \sum_{i=1}^n \frac{x_i x'_i}{\psi^2 e_i} \right)^{-1}$, $\beta_\tau^* = B_\tau \left(B_{\tau 0}^{-1} \beta_{\tau 0} + \sum_{i=1}^n \frac{x_i (y_i - \theta e_i)}{\psi^2 e_i} \right)$, $\alpha_i^* = ((y_i - x'_i \beta_\tau)^2 / \psi^2)^{0.5}$ and $\delta_i^* = (2 + \theta^2 / \psi^2)^{0.5}$.

¹⁹We can set L and U equal to $-\infty$ and ∞ to model censored data in just one side.

²⁰ \mathcal{E} denotes an exponential density.

²¹ \mathcal{GIG} denotes a generalized inverse Gaussian density.

3.2 Multivariate models

Multivariate regression: This model is used when there are m multiple dependent variables which share the same set of regressors, and their stochastic errors are contemporaneously correlated. In particular, $Y = [\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_m]$ is a $n \times m$ matrix that is generated as $Y = XB + U$ where X is a $n \times k$ matrix, $B = [\beta_1, \beta_2, \dots, \beta_m]$ is a $k \times m$ matrix of parameters, and $U = [u_1, u_2, \dots, u_m]$ is a matrix of stochastic random errors such that $u_i \sim \mathcal{N}(0, \Sigma)$, $i = 1, 2, \dots, n$ is each row of U . Then, $p(B, \Sigma | Y, X) \propto |\Sigma|^{-n/2} \exp \left[-\frac{1}{2} \text{tr}(Y - XB)'(Y - XB)\Sigma^{-1} \right]$, tr denotes the trace operator.

Rossi et al. (2005) propose the natural conjugate priors $\pi(\text{vec}(B) | \Sigma) \sim \mathcal{N}(\text{vec}(B_0), \Sigma \otimes \Delta_0)$ and $\pi(\Sigma) \sim \mathcal{IW}(\alpha_0, \Sigma_0)$ where vec is the vectorization operator and \otimes is the Kronecker operator. Therefore, the conditional posterior distributions are

$$\beta | \Sigma, Y, X \sim \mathcal{N}(\beta^*, \Sigma \otimes \Delta^*),$$

$$\Sigma | Y, X \sim \mathcal{IW}(\alpha^*, \Sigma^*),^{22}$$

where $\Delta^* = (X'X + \Delta_0^{-1})^{-1}$, $\beta^* = \text{vec}(B^*)$, $B^* = \Delta^*(\Delta_0^{-1}B_0 + X'X\hat{B})$, $\hat{B} = (X'X)^{-1}X'Y$, $\alpha^* = \alpha_0 + n$ and $\Sigma^* = \Sigma_0 + (Y - X\hat{B})'(Y - X\hat{B})$.

We can apply a Gibbs sampling algorithm in this model due to having standard conditional posterior distributions.

Seemingly unrelated regression: In this model there are m dependent variables with potentially different regressors and stochastic errors contemporaneously correlated. In particular, $y_j = X_j\beta_j + \mu_j$, where β_j is a k_j vector, $j = 1, 2, \dots, m$. Setting $\mu_i = [\mu_{1i}, \mu_{2i}, \dots, \mu_{mi}]'$ such that $\mu_i \sim \mathcal{N}(0, \Sigma)$, and stacking the m equations, we can write $y = X\beta + \mu$ where $y = [y_1', y_2', \dots, y_m']'$, $\beta = [\beta_1', \beta_2', \dots, \beta_m']'$ is a K dimensional vector, $K = \sum_{j=1}^m k_j$, X is a $mn \times K$ block diagonal matrix composed by X_j and $\mu = [\mu_1', \mu_2', \dots, \mu_m']'$ such that $\mu \sim \mathcal{N}(0, \Sigma \otimes I_n)$. Then, $p(\beta, \Sigma | y, X) \propto |\Sigma|^{-n/2} \exp \left[-\frac{1}{2} (y - X\beta)'(\Sigma^{-1} \otimes I_n)(y - X\beta) \right]$.

Using independent priors $\pi(\beta) \sim \mathcal{N}(\beta_0, B_0)$ and $\pi(\Sigma^{-1}) \sim \mathcal{W}(\alpha_0, \Sigma_0)$, the posterior distributions are

$$\beta | \Sigma, y, X \sim \mathcal{N}(\beta^*, B^*),$$

$$\Sigma^{-1} | \beta, y, X \sim \mathcal{W}(\alpha^*, \Sigma^*),$$

where $B^* = (X'(\Sigma^{-1} \otimes I_n)X + B_0^{-1})^{-1}$, $\beta^* = B^*(B_0^{-1}\beta_0 + X'(\Sigma^{-1} \otimes I_n)y)$, $\alpha^* = \alpha_0 + n$ and $\Sigma^* = (\Sigma_0^{-1} + U'U)^{-1}$, where U is a $n \times m$ matrix whose columns are $y_j - X_j\beta_j$.

We can apply a Gibbs sampling algorithm in this model due to having standard conditional posterior distributions.

Instrumental variables: This model is used when there are endogeneity problems caused by feedback, omitted relevant variables or measurement error in regressors. So, we

²² \mathcal{IW} denotes an inverse wishart density.

specify the dependent variable as a linear function of one endogenous regressors and exogenous regressors. That is $y_i = x'_{ei}\beta_1 + \beta_s x_{si} + \mu_i$ where $x_{si} = x'_{ei}\gamma_1 + z'_i\gamma_2 + v_i$, x_s is the variable which generates endogeneity issues, such that x_e are k_1 exogenous regressors and z are k_2 instruments. Assuming $(u_i, v_i)' \stackrel{i.i.d}{\sim} \mathcal{N}(0, \Sigma)$, $\Sigma = [\sigma_{lm}]$, $l, m = 1, 2$, the likelihood function is $p(\beta, \gamma, \Sigma | y, X, Z) = \frac{1}{(2\pi)^{\frac{n}{2}|\Sigma|^{\frac{n}{2}}}} \exp \left\{ -\frac{1}{2} \sum_{i=1}^n (y_i - x'_i\beta, x_{si} - w'_i\gamma) \Sigma^{-1} \begin{pmatrix} y_i - x'_i\beta \\ x_{si} - w'_i\gamma \end{pmatrix} \right\}$ where $\beta = [\beta'_1, \beta'_s]'$, $\gamma = [\gamma'_1, \gamma'_2]'$, $x_i = [x'_{ei}, x_{si}]'$ and $w_i = [x'_{ei}, z'_i]'$.

We get standard conditional posterior densities using the following independent priors $\gamma \sim \mathcal{N}(\gamma_0, G_0)$, $\beta \sim \mathcal{N}(\beta_0, B_0)$ and $\Sigma^{-1} \sim \mathcal{W}(\alpha_0, \Sigma_0)$. In particular,

$$\beta | \gamma, \Sigma, y, X, Z \sim \mathcal{N}(\beta^*, B^*)$$

$$\gamma | \beta, \Sigma, y, X, Z \sim \mathcal{N}(\gamma^*, G^*)$$

$$\Sigma^{-1} | \beta, \gamma, y, X, Z \sim \mathcal{W}(\alpha^*, \Sigma^*)$$

where $B^* = (\omega_1^{-1} \sum_{i=1}^n x_i x'_i + B_0^{-1})^{-1}$, $\beta^* = B^* \left(B_0^{-1} \beta_0 + \omega_1^{-1} \sum_{i=1}^n \left[x_i \left(y_i - \frac{\sigma_{12}(x_{si} - w'_i\gamma)}{\sigma_{22}} \right) \right] \right)$, $\omega_1 = \sigma_{11} - \sigma_{12}^2 / \sigma_{22}$, $G^* = (\omega_2^{-1} \sum_{i=1}^n w_i w'_i + G_0^{-1})^{-1}$, $\gamma^* = G^* \left(G_0^{-1} \gamma_0 + \omega_2^{-1} \sum_{i=1}^n \left[w_i \left(x_{si} - \frac{\sigma_{12}(y_i - x'_i\beta)}{\sigma_{11}} \right) \right] \right)$, $\omega_2 = \sigma_{22} - \sigma_{12}^2 / \sigma_{11}$, $\alpha^* = \alpha_0 + n$ and $\Sigma^* = \left[\Sigma_0^{-1} + \sum_{i=1}^n \begin{pmatrix} y_i - x'_i\beta \\ x_{si} - w'_i\gamma \end{pmatrix} (y_i - x'_i\beta, x_{si} - w'_i\gamma) \right]^{-1}$.

We also use a Gibbs sampling algorithm in this model given that we obtain standard conditional posterior distributions.

Multivariate probit: In the multivariate probit model (Edwards and Allenby, 2003), the response variable $y_{il} = \{0, 1\}$ indicates that individual (unit) i makes binary choices regarding alternative l , $i = 1, 2, \dots, n$, $l = 1, 2, \dots, L$.²³ In particular, $y_{il} = \begin{cases} 0, & y_{il}^* \leq 0 \\ 1, & y_{il}^* > 0 \end{cases}$ such that $y_i^* = X_i \beta + \mu_i \stackrel{i.i.d}{\sim} \mathcal{N}(0, \Sigma)$, y_i^* is an unobserved latent L dimensional vector, X_i is a $L \times K$ design matrix of regressors, $K = L \times k$, k is the number of regressors, and $\beta = [\beta'_1, \beta'_2, \dots, \beta'_k]'$, β_j are a L dimensional vector of coefficients, $j = 1, 2, \dots, k$. We take into account simultaneously alternative-varying regressors (alternative attributes) and alternative-invariant regressors (individual characteristics).

The likelihood function in this model is $p(\beta, \Sigma | y, X) = \prod_{i=1}^n \prod_{l=1}^L p_{il}^{y_{il}}$ where $p_{il} = p(y_{il}^* \geq 0)$. Observe that $p(y_i^* \geq 0) = p(\Lambda y_i^* \geq 0)$, where $\Lambda = \text{diag}\{\lambda_{il}\}$, $\lambda_{il} > 0$. This generates identification issues. We follow the post processing strategy proposed by Edwards and Allenby (2003) to get identified parameters, that is, $\tilde{\beta} = \text{vec}\{D\Gamma\}$ and the correlation matrix $R = D\Sigma D$, where $D = \text{diag}\{\sigma_{il}\}^{-1/2}$ and $\Gamma = [\beta_1, \beta_2, \dots, \beta_k]$.²⁴

²³These alternatives are not mutually exclusive.

²⁴In a Bayesian setting, we can have a no identified model; however, the posterior of the model parameters exists given a proper prior distribution (Edwards and Allenby, 2003).

We assume independent priors, $\beta \sim \mathcal{N}(\beta_0, B_0)$ and $\Sigma^{-1} \sim \mathcal{W}(\alpha_0, \Sigma_0)$. We can apply Gibbs sampling in this model because this is a standard Bayesian linear regression model when data augmentation in y^* is used. The posterior conditional distributions are

$$\beta|\Sigma, w \sim \mathcal{N}(\beta^*, B),$$

$$\Sigma^{-1}|\beta, w \sim \mathcal{W}(\alpha^*, \Sigma^*),$$

$$y_{il}^*|y_{i,-l}^*, \beta, \Sigma^{-1}, y_i \sim \mathcal{TN}_{I_{il}}(m_{il}, \tau_{ll}^2)$$

where $B = (B_0^{-1} + X^{*'}X^*)^{-1}$, $\beta^* = B(B_0^{-1}\beta_0 + X^{*'}y^{**})$, $\Sigma^{-1} = C'C$, $X_i^{*'} = C'X_i$, $y_i^{**} = C'y_i^*$,

$$X^* \begin{bmatrix} X_1^* \\ X_2^* \\ \vdots \\ X_n^* \end{bmatrix}, \alpha^* = \alpha_0 + n, \Sigma^* = (\Sigma_0 + \sum_{i=1}^n (y_i^* - R_i\beta)'(y_i^* - X_i\beta))^{-1}, I_{il} = \begin{cases} y_{il}^* > 0, & y_{il} = 1 \\ y_{il}^* \leq 0, & y_{il} = 0 \end{cases},$$

$m_{il} = x'_{il}\beta + f'(y_{i,-l}^* - X_{i,-l}\beta)$, $y_{i,-l}^*$ is a $L - 1$ dimensional vector of all components of y_i^* excluding y_{il}^* , x_{il} is the l -th row of X_i , $X_{i,-l}$ is X_i deleting l -th row, $\tau_{ll}^2 = 1/\sigma^{ll}$, and

$$f = -\sigma^{ll}\omega_{l,-l}, \sigma^{jl} \text{ is the } jl\text{-th element of } \Sigma^{-1}, \Sigma^{-1} = \begin{bmatrix} \omega'_1 \\ \omega_2 \\ \vdots \\ \omega'_L \end{bmatrix}, \omega'_{l,-l} \text{ is the } l\text{-th row of } \Sigma^{-1}$$

extracting the l -th element.

Bivariate probit: In the bivariate probit model (Edwards and Allenby, 2003), the response variable $y_{ij} = \{0, 1\}$ indicates that individual (unit) i makes binary choices regarding alternative j , $i = 1, 2, \dots, n$, $j = 1, 2$. In particular, $y_{ij} = \begin{cases} 0, & y_{ij}^* \leq 0 \\ 1, & y_{ij}^* > 0 \end{cases}$ such that $y_{ij}^* = x'_{ij}\beta_j + \mu_{ij}$.

Setting $y_i^* = X_i\beta + \mu_i$ where $y_i^* = \begin{bmatrix} y_{i1}^* \\ y_{i2}^* \end{bmatrix}$, $X_i = \begin{bmatrix} x'_{i1} & 0' \\ 0' & x'_{i2} \end{bmatrix}$, $\beta = [\beta'_{i1} \ \beta'_{i2}]'$, and $\mu_i = \begin{bmatrix} \mu_{i1} \\ \mu_{i2} \end{bmatrix} \sim \mathcal{N}(0, \Sigma)$,²⁵ and priors $\pi(\beta) \propto \text{Exp}\left\{-\frac{1}{2}(\beta - \beta_0)'B_0^{-1}(\beta - \beta_0)\right\}$ and $\pi(\rho) \propto 1_{\rho \in (-1, 1)} \text{Exp}\left\{-\frac{1}{2\sigma_0^2}(\rho - \rho_0)^2\right\}$ then

²⁵The covariance matrix in this model is not identified due to identifying just the signs of the latent variables through the likelihood functions. So, the main diagonal of the covariance matrix is set equal to 1, and the off diagonal element is the coefficient of correlation ρ .

$$\begin{aligned}
\pi(\beta, \rho, y^* | y, X) &\propto |\Sigma|^{-n/2} \text{Exp} \left\{ -\frac{1}{2} \sum_{i=1}^n (y_i^* - X_i \beta)' \Sigma^{-1} (y_i^* - X_i \beta) \right\} \\
&\times \prod_{i,j=1}^n \left[1_{y_{ij}=0} 1_{y_{ij}^* \leq 0} + 1_{y_{ij}=1} 1_{y_{ij}^* > 0} \right] \\
&\times \text{Exp} \left\{ -\frac{1}{2} (\beta - \beta_0)' B_0^{-1} (\beta - \beta_0) \right\} \\
&\times 1_{\rho \in (-1,1)} \text{Exp} \left\{ -\frac{1}{2\sigma_0^2} (\rho - \rho_0)^2 \right\}.
\end{aligned}$$

This implies,

$$\begin{aligned}
y_{i1}^* | \beta, \rho, y_{i2}^*, y, X &\sim \begin{cases} \mathcal{TN}_{(-\infty, 0]}(x'_{i1} \beta_1 + \rho(y_{i2}^* - x'_{i2} \beta_2), \sqrt{1 - \rho^2}), & y_{i1} = 0 \\ \mathcal{TN}_{(0, \infty)}(x'_{i1} \beta_1 + \rho(y_{i2}^* - x'_{i2} \beta_2), \sqrt{1 - \rho^2}) & y_{i1} = 1 \end{cases}, \\
y_{i2}^* | \beta, \rho, y_{i1}^*, y, X &\sim \begin{cases} \mathcal{TN}_{(-\infty, 0]}(x'_{i2} \beta_2 + \rho(y_{i1}^* - x'_{i1} \beta_1), \sqrt{1 - \rho^2}), & y_{i2} = 0 \\ \mathcal{TN}_{(0, \infty)}(x'_{i2} \beta_2 + \rho(y_{i1}^* - x'_{i1} \beta_1), \sqrt{1 - \rho^2}) & y_{i2} = 1 \end{cases}, \\
\beta | y^*, X &\sim \mathcal{N}(\beta^*, B),
\end{aligned}$$

where $B = (B_0^{-1} + \sum_{i=1}^n X_i' \Sigma^{-1} X_i)^{-1}$, $\beta^* = B(B_0^{-1} \beta_0 + \sum_{i=1}^n X_i' \Sigma^{-1} y_i^*)$.

We use a Gibbs sampling algorithm for the previous part of the model. However, the conditional posterior distribution of ρ is not standard, then we use a random-walk Metropolis-Hastings algorithm where the proposal distribution is $\mathcal{TN}_{(-1,1)}(\rho, s_\rho^2)$ where s_ρ is a tune parameter.

3.3 Hierarchical longitudinal models

Normal: The hierarchical longitudinal normal model establishes $y_i = X_i \beta + W_i b_i + \mu_i$ where y_i are n_i vectors corresponding to unit $i = 1, 2, \dots, m$, X_i are $n_i \times k$ matrices, β is a k dimensional vector of “fixed” effects, W_i are $n_i \times q$ matrices associated with random effects (q typically less than k), b_i is a q dimensional vector of unit-specific random effects such that $b_i \sim \mathcal{N}(0, D)$, and $\mu_i \sim \mathcal{N}(0, \sigma^2 I_{n_i})$ are stochastic errors.

We use standard conjugate prior distributions following [Chib and Carlin \(1999\)](#). In particular, $\beta \sim \mathcal{N}(\beta_0, B_0)$, $\sigma^2 \sim \mathcal{IG}(\alpha_0, 1/\delta_0)$ and $D \sim \mathcal{IW}(d_0, d_0 D_0)$, and taking into account that $y_i | \beta, D, \sigma^2 \sim \mathcal{N}(X_i \beta, V_i)$ where $V_i = \sigma^2 I + W_i D W_i'$, then

$$\begin{aligned}
\beta | \sigma^2, D, y, X, W &\sim \mathcal{N}(\beta^*, B), \\
b_i | \beta, \sigma^2, D, y, X, W &\sim \mathcal{N}(b_i^*, B_i), \\
D | b &\sim \mathcal{IW}(d^*, D^*),
\end{aligned}$$

$$\sigma^2|\beta, D, b, y, X, W \sim \mathcal{IG}(\alpha^*, \delta^*),$$

where $B = (B_0^{-1} + \sigma^{-2} \sum_{i=1}^n X_i' V_i^{-1} X_i)^{-1}$, $\beta^* = B(B_0^{-1} \beta_0 + \sigma^{-2} \sum_{i=1}^n X_i' V_i^{-1} y_i)$, $B_i = (D^{-1} + \sigma^{-2} W_i' W_i)^{-1}$, $b_i^* = B_i(\sigma^{-2} W_i'(y_i - X_i \beta))$, $d^* = d_0 + m$ and $D^* = d_0 D_0 + \sum_{i=1}^m b_i b_i'$, $\alpha^* = \alpha_0 + \frac{1}{2} \sum_{i=1}^m n_i$ and $\delta^* = 1/\delta_0 + \frac{1}{2} \sum_{i=1}^m (y_i - X_i \beta - W_i b_i)'(y_i - X_i \beta - W_i b_i)$.

Logit: The hierarchical longitudinal logit model establishes $y_{ij} \sim \mathcal{B}(\pi_{ij})$ where $\text{logit}(\pi_{ij}) = \log\left(\frac{\pi_{ij}}{1-\pi_{ij}}\right) \equiv y_{ij}^* = x'_{ij}\beta + w'_{ij}b_i + \mu_{ij}$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n_i$ where x_{ij} are k dimensional vectors of regressors, β is a k dimensional vector of “fixed” effects, w_{ij} are q dimensional vectors of regressors associated with random effects (q typically less than k), b_i is a q dimensional vector of unit-specific random effects such that $b_i \sim \mathcal{N}(0, D)$, and $\mu_{ij} \sim \mathcal{N}(0, \sigma^2)$ are stochastic errors.

We use standard conjugate prior distributions following Chib and Carlin (1999). In particular, $\beta \sim \mathcal{N}(\beta_0, B_0)$, $\sigma^2 \sim \mathcal{IG}(\alpha_0, 1/\delta_0)$ and $D \sim \mathcal{IW}(d_0, d_0 D_0)$, and taking into account that $y_i^*|\beta, D, \sigma^2 \sim \mathcal{N}(X_i \beta, V_i)$ where $V_i = \sigma^2 I + W_i D W_i'$, then

$$y_{ij}^*|\beta, \sigma^2, b, y, X, W \propto \pi_{ij}^{y_{ij}} (1 - \pi_{ij})^{1-y_{ij}} \times \mathcal{N}(x'_{ij}\beta + w'_{ij}b_i, \sigma^2),$$

$$\beta|\sigma^2, D, y^*, X, W \sim \mathcal{N}(\beta^*, B),$$

$$b_i|\beta, \sigma^2, D, y^*, X, W \sim \mathcal{N}(b_i^*, B_i),$$

$$D|b \sim \mathcal{IW}(d^*, D^*),$$

$$\sigma^2|\beta, D, b, y^*, X, W \sim \mathcal{IG}(\alpha^*, \delta^*),$$

where $B = (B_0^{-1} + \sigma^{-2} \sum_{i=1}^n X_i' V_i^{-1} X_i)^{-1}$, $\beta^* = B(B_0^{-1} \beta_0 + \sigma^{-2} \sum_{i=1}^n X_i' V_i^{-1} y_i^*)$, $B_i = (D^{-1} + \sigma^{-2} W_i' W_i)^{-1}$, $b_i^* = B_i(\sigma^{-2} W_i'(y_i^* - X_i \beta))$, $d^* = d_0 + m$ and $D^* = d_0 D_0 + \sum_{i=1}^m b_i b_i'$, $\alpha^* = \alpha_0 + \frac{1}{2} \sum_{i=1}^m n_i$ and $\delta^* = 1/\delta_0 + \frac{1}{2} \sum_{i=1}^m (y_i^* - X_i \beta - W_i b_i)'(y_i^* - X_i \beta - W_i b_i)$.

We can implement a Gibbs sampling algorithm for the latter four standard conditional posterior distributions. However, this model has been augmented with the latent variable $y^* = [y_{ij}^*]$ which should be drawn using a random-walk Metropolis-Hastings algorithm such that the proposal distribution is Gaussian with mean y_{ij}^* and variance equal to 1, that is, $y_{ij}^{*c} = y_{ij}^* + \epsilon_{ij}$ where $\epsilon_{ij} \sim \mathcal{N}(0, 1)$ which implies $\pi_{ij} = \frac{1}{1+e^{-y_{ij}^*}}$ and $\pi_{ij}^c = \frac{1}{1+e^{-y_{ij}^{*c}}}$.

Poisson: The hierarchical longitudinal poisson model establishes $y_{ij} \sim \mathcal{P}(\lambda_{ij})$,²⁶ where $\log(\lambda_{ij}) \equiv y_{ij}^* = x'_{ij}\beta + w'_{ij}b_i + \mu_{ij}$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n_i$ where x_{ij} are k dimensional vectors of regressors, β is a k dimensional vector of “fixed” effects, w_{ij} are q dimensional vectors of regressors associated with random effects (q typically less than k), b_i is a q dimensional vector of unit-specific random effects such that $b_i \sim \mathcal{N}(0, D)$, and $\mu_{ij} \sim \mathcal{N}(0, \sigma^2)$ are stochastic errors.

We use standard conjugate prior distributions following Chib and Carlin (1999). In particular, $\beta \sim \mathcal{N}(\beta_0, B_0)$, $\sigma^2 \sim \mathcal{IG}(\alpha_0, 1/\delta_0)$ and $D \sim \mathcal{IW}(d_0, d_0 D_0)$, and taking into account that $y_i^*|\beta, D, \sigma^2 \sim \mathcal{N}(X_i \beta, V_i)$ where $V_i = \sigma^2 I + W_i D W_i'$, then

²⁶ \mathcal{P} denotes a poisson density.

$$\begin{aligned}
y_{ij}^* | \beta, \sigma^2, b, y, X, W &\propto \lambda_{ij}^{y_{ij}} e^{-\lambda_{ij}} \times \mathcal{N}(x'_{ij}\beta + w'_{ij}b_i, \sigma^2), \\
\beta | \sigma^2, D, y^*, X, W &\sim \mathcal{N}(\beta^*, B), \\
b_i | \beta, \sigma^2, D, y^*, X, W &\sim \mathcal{N}(b_i^*, B_i), \\
D | b &\sim \mathcal{IW}(d^*, D^*), \\
\sigma^2 | \beta, D, b, y^*, X, W &\sim \mathcal{IG}(\alpha^*, \delta^*),
\end{aligned}$$

where $B = (B_0^{-1} + \sigma^{-2} \sum_{i=1}^n X'_i V_i^{-1} X_i)^{-1}$, $\beta^* = B(B_0^{-1}\beta_0 + \sigma^{-2} \sum_{i=1}^n X'_i V_i^{-1} y_i^*)$, $B_i = (D^{-1} + \sigma^{-2} W'_i W_i)^{-1}$, $b_i^* = B_i(\sigma^{-2} W'_i (y_i^* - X_i \beta))$, $d^* = d_0 + m$ and $D^* = d_0 D_0 + \sum_{i=1}^m b_i b'_i$, $\alpha^* = \alpha_0 + \frac{1}{2} \sum_{i=1}^m n_i$ and $\delta^* = 1/\delta_0 + \frac{1}{2} \sum_{i=1}^m (y_i^* - X_i \beta - W_i b_i)'(y_i^* - X_i \beta - W_i b_i)$.

We can implement a Gibbs sampling algorithm for the latter four standard conditional posterior distributions. However, this model has been augmented with the latent variable $y^* = [y_{ij}^*]$ which should be drawn using a random-walk Metropolis-Hastings algorithm such that the proposal distribution is Gaussian with mean y_{ij}^* and variance equal to 1, that is, $y_{ij}^{*c} = y_{ij}^* + \epsilon_{ij}$ where $\epsilon_{ij} \sim \mathcal{N}(0, 1)$ which implies $\lambda_{ij} = e^{y_{ij}^*}$ and $\lambda_{ij}^c = e^{y_{ij}^{*c}}$.

3.4 Bayesian bootstrap

We implement the Bayesian bootstrap (Rubin, 1981) to linear regression models. In particular, the Bayesian bootstrap simulates posterior distributions assuming that the sample cumulative distribution function (cdf) is the population cdf (this assumption is also implicit in the frequentist bootstrap (Efron, 1979)).

Given $y_i \stackrel{i.i.d}{\sim} \mathcal{F}$ where \mathcal{F} does not define a particular parametric family distribution, $i = 1, 2, \dots, n$, but sets $E(Y_i | x_i) = x'_i \beta$, such that x_i is a k dimensional vector of regressors and β is a k dimensional vector of parameters, the Bayesian bootstrap generates posterior probabilities for each y_i where values of Y that are not observed have zero posterior probability.

The algorithm to implement the Bayesian bootstrap is the following:

Algorithm A3 Bayesian bootstrap in linear regression

- 1: Draw $g \sim \text{Dir}(\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $\alpha_i = 1 \ \forall i$.
- 2: $g = (g_1, g_2, \dots, g_n)$ is the vector of probabilities to attach to $(y_1, x'_1), (y_2, x'_2), \dots, (y_n, x'_n)$ in each Bayesian bootstrap replication.
- 3: Sample S_1 times (y_i, x'_i) , $i = 1, 2, \dots, n$ with replacement and probabilities g_i .
- 4: Estimate β using ordinary least squares in the model $E(Y|X) = X\beta$, y is a S_1 dimensional vector of realizations of Y , and X is a $S_1 \times k$ matrix from the previous stage.*
- 5: Repeat this process S_2 times.
- 6: The distribution of $\beta^{(s_2)}$ is the Bayesian distribution of β .

*Ordinary least squares is the posterior mean of β using Jeffrey's prior in a linear regression.

3.5 Bayesian model averaging

Bayesian model averaging (BMA) is an approach which accounts for model uncertainty. In particular, we consider regressors uncertainty (variable selection) in a regression framework where there are k possible explanatory variables. This implies $\mathcal{M} = \{M_1, M_2, \dots, M_{2^k}\}$ potential models indexed by parameters θ_m , $m = 1, 2, \dots, 2^k$. Following [Simmons et al. \(2010\)](#), the posterior model probability is

$$\pi(M_j|y) = \frac{p(y|M_j)\pi(M_j)}{\sum_{m=1}^{2^k} p(y|M_m)\pi(M_m)},$$

where $\pi(M_j)$ is the prior model probability,²⁷ $p(y|M_j) = \int_{\Theta_j} p(y|\theta_j, M_j)\pi(\theta_j|M_j)d\theta_j$ is the marginal likelihood, and $\pi(\theta_j|M_j)$ is the prior distribution of θ_j .

Following [Raftery \(1993\)](#), the posterior distribution of θ is

$$\pi(\theta|y) = \sum_{m=1}^{2^k} \pi(\theta_m|y, M_m)\pi(M_m|y)$$

where $\pi(\theta_m|y, M_m)$ is the posterior distribution of θ under model m , $E(\theta|y) = \sum_{m=1}^{2^k} \hat{\theta}_m \pi(M_m|y)$, $Var(\theta|y) = \sum_{m=1}^{2^k} \pi(M_m|y) \widehat{Var}(\theta|y, M_m) + \sum_{m=1}^{2^k} \pi(M_m|y) (\hat{\theta}_m - E(\theta|y))^2$, $\hat{\theta}_m$ and $\widehat{Var}(\theta|y, M_m)$ are the posterior mean and variance under model m , respectively.

The posterior variance highlights how BMA methodology accounts for model uncertainty. The first term is the weighted variance for each model, averaged over all potential models, and the second term indicates how stable the estimates are across models. The more the estimates differ between models, the greater is the posterior variance.

The posterior predictive distribution is

$$\pi(y^{New}|y) = \sum_{m=1}^{2^k} p_m(y^{New}|y, M_m)\pi(M_m|y)$$

where $p_m(y^{New}|y, M_m) = \int_{\Theta_m} p(y^{New}|y, \theta_m, M_m)\pi(\theta_m|y, M_m)d\theta_m$ is the posterior predictive distribution under model m .

Another important statistic in BMA is the posterior inclusion probability associated with variable x_l , $l = 1, 2, \dots, k$ is

$$PIP(x_l) = \sum_{m=1}^{2^k} \pi(M_m|y) \times I_{l,m},$$

²⁷We set equal prior probability to each model. However, this choice gives more prior probability to the set of models of medium size (think about the k -th row of Pascal's triangle). An interesting alternative is to use the Beta-Binomial prior proposed by [Ley and Steel \(2009\)](#).

where $I_{l,m} = \begin{cases} 1 & \text{if } x_l \in M_m \\ 0 & \text{if } x_l \notin M_m \end{cases}$.

BMA allows to incorporate model uncertainty in a regression framework, but sometimes it is desirable to select just one model. Two compelling alternatives are the model with the highest posterior model probability, and the median probability model. The latter is the model which includes every predictor that has posterior inclusion probability higher than 0.5. The first model is the best alternative for prediction in the case of a 0-1 loss function (Clyde and George, 2004), whereas the second is the best alternative when there is a quadratic loss function (Barbieri and Berger, 2004).

There are two main computational issue implementing BMA. First, the marginal likelihood $p(y|M_j) = \int_{\Theta_j} p(y|\theta_j, M_j)\pi(\theta_j|M_j)d\theta_j$, which most of the cases does not have an analytic solution, and second, the amount of models in the model space, 2^k , this some times can be enormous.

The Bayesian information criterion is a possible solution for the first issue. Defining $h(\theta|M_j) = -\frac{\log(p(y|\theta_j, M_j)\pi(\theta_j|M_j))}{n}$, then $p(y|M_j) = \int_{\Theta_j} \exp\{-nh(\theta|M_j)\}d\theta_j$. If n is sufficiently large ($n \rightarrow \infty$), we can make the following assumptions (Hoeting et al., 1999):

- We can use the Laplace method for approximating integrals (Tierney and Kadane, 1986).
- The posterior mode is reached at the same point as the maximum likelihood estimator (MLE) denoted as $\hat{\theta}_{MLE}$.

We get the following results under these assumptions:

$$p(y|M_j) \approx \left(\frac{2\pi}{n}\right)^{k_j/2} |\Sigma|^{-1/2} \exp\left\{-nh(\hat{\theta}_j^{MLE}|M_j)\right\}, \quad n \rightarrow \infty,$$

where Σ is the Hessian matrix of $h(\hat{\theta}_j^{MLE}|M_j)$, and $k_j = \dim\{\theta_j\}$.

This implies

$$\log(p(y|M_j)) \approx \frac{k_j}{2}\log(2\pi) - \frac{k_j}{2}\log(n) - \frac{1}{2}\log(|\Sigma|) + \log(p(y|\hat{\theta}_j^{MLE}, M_j)) + \log(\pi(\hat{\theta}_j^{MLE}|M_j)), \quad n \rightarrow \infty.$$

Taking into account that $\frac{k_j}{2}\log(2\pi)$ and $\log(\pi(\hat{\theta}_j^{MLE}|M_j))$ are constants as functions of y , and $|\Sigma|$ is bounded by a finite constant, then

$$\log(p(y|M_j)) \approx \frac{k_j}{2}\log(n) + \log(p(y|\hat{\theta}_j^{MLE}, M_j)) = -\frac{BIC}{2}, \quad n \rightarrow \infty.$$

The second computational issue, which is related to the size of the model space (2^k), is basically a problem of ranking models. This can be tackled from different approaches such as Occam's window criterion (Madigan and Raftery, 1994; Raftery et al., 1997), reversible jump

Markov chain Monte Carlo computation (Green, 1995), Markov chain Monte Carlo model composition (Madigan et al., 1995b), and multiple testing using intrinsic priors Casella and Moreno (2006) or nonlocal prior densities (Jhonson and Rossell, 2012). In this GUI we focus on Occam’s window and Markov chain Monte Carlo model composition.²⁸

In Occam’s window a model is discarded if its predictive performance is far less well than the best model (Madigan and Raftery, 1994; Raftery et al., 1997). Thus models not belonging to $\mathcal{M}' = \left\{ M_j : \frac{Max_r \pi(M_r|y)}{\pi(M_j|y)} \leq c \right\}$ should be discarded, where c is chosen by the user (Madigan and Raftery (1994) propose $c = 20$). In addition, complicated models than are less supported by the data than simpler models are also discarded, that is, $\mathcal{M}'' = \left\{ M_j : \exists M_m \in \mathcal{M}', M_m \subset M_j, \frac{\pi(M_m|y)}{\pi(M_j|y)} > 1 \right\}$. Then, the set of models used in BMA is $\mathcal{M}^* = \mathcal{M}' \cap \mathcal{M}''^c \in \mathcal{M}$. Raftery et al. (1997) find that the number of models in \mathcal{M}^* is normally fewer than 25.

However, the previous theory framework requires calculating $Max_r \pi(M_r|y)$, which implies to calculate all possible models in \mathcal{M} . This is computationally burdensome. Then, a heuristic approach is proposed by Raftery et al. (2012) based on ideas of Madigan and Raftery (1994). The search strategy is based on a series of nested comparisons of ratios of posterior model probabilities. Denote M_0 a model with one regressor less than model M_1 . If $\log(\pi(M_0|y)/\pi(M_1|y)) > \log(O_R)$ then M_1 is rejected and M_0 is considered, if $\log(\pi(M_0|y)/\pi(M_1|y)) \leq -\log(O_L)$ then M_0 is rejected, and M_1 is considered, and if $\log(O_L) < \log(\pi(M_0|y)/\pi(M_1|y)) \leq \log(O_R)$, M_0 and M_1 are considered. The search strategy can be “up”, adding one regressor, or “down”, dropping one regressor (see Madigan and Raftery (1994), down and up algorithms for details). The leaps and bounds algorithm (Furnival and Wilson, 2000) is implemented to improve computational efficiency of this search strategy (Raftery et al., 2012). Once the set of potentially acceptable models is defined; we discard all the models that are not in \mathcal{M}' , and the models that are in \mathcal{M}'' where 1 is replaced by $\exp\{O_R\}$ due to the leaps and bounds algorithm giving an approximation to BIC, so to ensure that no good models are discarded.²⁹

The second approach that we consider in this GUI to tackle the model space size issue is Markov chain Monte Carlo model composition (Madigan et al., 1995a). In particular, given the space of models \mathcal{M} , we simulate a chain of M_s models, $s = 1, 2, \dots, S \ll 2^k$, where the algorithm randomly extracts a candidate model M_c from a neighborhood of models ($nbd(M)$) that consists of the actual model itself, and the set of models with either one variable more or one variable fewer (Raftery et al., 1997). Therefore, there is a transition kernel in the space of models $q(M \rightarrow M_c)$, such that $q(M \rightarrow M_c) = 0 \ \forall M_c \notin nbd(M)$ and $q(M \rightarrow M_c) = \frac{1}{|nbd(M)|} \ \forall M \in nbd(M)$, $|nbd(M)|$ is the number of neighbors of M . This candidate model is accepted with probability

²⁸Variable selection (model selection) is a related topic to model uncertainty. Approaches such as stochastic search variable selection (spike and slab) (George and McCulloch, 1993, 1997; Ishwaran and Rao, 2005), and Bayesian Lasso (Park and Casella, 2008) are good examples to tackle this issue. The former allows to calculate an estimate similar to *PIP*. However, these frameworks do not offer a clear way to introduce posterior model probabilities.

²⁹ $O_L = 1/O_R^2$ is defined by default in Raftery et al. (2012).

$$\alpha(M_{s-1}, M_c) = \text{Min} \left\{ \frac{|nbd(M)|p(y|M_c)\pi(M_c)}{|nbd(M_c)|p(y|M_{s-1})\pi(M_{s-1})}, 1 \right\}.$$

Observe that by construction $|nbd(M)| = |nbd(M_c)| = \frac{1}{k}$, except in pathological cases where a model just has one regressor or all regressors.

Normal: The Gaussian linear model specifies $\mathbf{y} = \mathbf{X}\beta + \mu$ such that $\mu \sim \mathcal{N}(0, \sigma^2 I_n)$, and the conjugate priors for the parameters are $\beta|\sigma^2 \sim \mathcal{N}(\beta_0, \sigma^2 B_0)$ and $\sigma^2 \sim \mathcal{IG}(\alpha_0/2, \delta_0/2)$. Given the likelihood function, $p(\beta, \sigma^2|y, X) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right\}$, the marginal distribution $p(y)$ is $\mathcal{T}\left(X\beta_0, \frac{\alpha_0(I + XB_0X')}{\delta_0}, \delta_0\right)$.

- Bayesian information criterion: We implement the BIC approximation in the Gaussian linear model using the Occam's window approach.
- Markov chain Monte Carlo model composition (MC³): We implement the Gaussian linear model using MC³ using the marginal multivariate Student's t distribution.
- Instrumental variables: Let's consider the previous instrumental variable framework (Multivariate models section) assuming $\gamma \sim \mathcal{N}(0, I)$, $\beta \sim \mathcal{N}(0, I)$ and $\Sigma^{-1} \sim \mathcal{W}(3, I)$ (Karl and Lenkoski, 2012). Then, the posterior distributions are

$$\beta|\gamma, \Sigma, y, X, Z \sim \mathcal{N}(\beta^*, B^*)$$

$$\gamma|\beta, \Sigma, y, X, Z \sim \mathcal{N}(\gamma^*, G^*)$$

$$\Sigma^{-1}|\beta, \gamma, y, X, Z \sim \mathcal{W}(\alpha^*, \Sigma^*)$$

where $B^* = (\omega_1^{-1} \sum_{i=1}^n x_i x_i' + I)^{-1}$, $\beta^* = B^* \left(\omega_1^{-1} \sum_{i=1}^n \left[x_i \left(y_i - \frac{\sigma_{12}(x_{si} - w_i' \gamma)}{\sigma_{22}} \right) \right] \right)$, $\omega_1 = \sigma_{11} - \sigma_{12}^2 / \sigma_{22}$, $G^* = (\omega_2^{-1} \sum_{i=1}^n w_i w_i' + I)^{-1}$, $\gamma^* = G^* \left(\omega_2^{-1} \sum_{i=1}^n \left[w_i \left(x_{si} - \frac{\sigma_{12}(y_i - x_i' \beta)}{\sigma_{11}} \right) \right] \right)$, $\omega_2 = \sigma_{22} - \sigma_{12}^2 / \sigma_{11}$, $\alpha^* = 3 + n$ and $\Sigma^* = \left[I + \sum_{i=1}^n \begin{pmatrix} y_i - x_i' \beta \\ x_{si} - w_i' \gamma \end{pmatrix} \begin{pmatrix} y_i - x_i' \beta & x_{si} - w_i' \gamma \end{pmatrix} \right]^{-1}$.

Lenkoski et al. (2013) propose an algorithm based on conditional Bayes factors (Dickey and Gunel, 1978) that allows embedding MC³ within a Gibbs sampling algorithm. Given the candidate (M_c^{2nd}) and actual (M_{s-1}^{2nd}) models for the iteration s in the second stage, the conditional Bayes factor is $CBF^{2nd} = \frac{p(y|M_c^{2nd}, \gamma, \Sigma)}{p(y|M_{s-1}^{2nd}, \gamma, \Sigma)}$, where $p(y|M_c^{2nd}, \gamma, \Sigma) =$

$\int_{\mathcal{M}^{2nd}} p(y|\beta, \gamma, \Sigma) \pi(\beta|M_c^{2nd}) d\beta \propto |B^*|^{1/2} \text{Exp} \left\{ \frac{1}{2} \beta^{*'} B^{*-1} \beta^* \right\}$. For the first stage, $CBF^{1st} = \frac{p(y|M_c^{1st}, \beta, \Sigma)}{p(y|M_{s-1}^{1st}, \beta, \Sigma)}$, where $p(y|M_c^{1st}, \beta, \Sigma) = \int_{\mathcal{M}^{1st}} p(y|\gamma, \beta, \Sigma) \pi(\gamma|M_c^{1st}) d\gamma \propto |G^*|^{1/2} \text{Exp} \left\{ \frac{1}{2} \gamma^{*'} G^{*-1} \gamma^* \right\}$.³⁰

These conditional Bayes factors assume $\pi(M^{1st}, M^{2sd}) \propto 1$. See Lenkoski et al. (2013) for more details of the instrumental variable BMA algorithm.³¹

³⁰In the case that $\beta_s = 0$, the update is based on seemingly unrelated regressions framework.

³¹Koop et al. (2012) and Lenkoski et al. (2014) propose other frameworks for BMA taking into account endogeneity.

The Gaussian linear model is an example of a generalized linear model. A GLM is characterized by a distribution function that is in the exponential family, that is, $p_i(y_i|\theta_i, \phi) = h(y_i, \phi) \text{Exp}\{(\theta_i y_i - b(\theta_i))/a(\phi)\}$ (canonical representation), $y_i \stackrel{i.i.d.}{\sim} p_i$, $i = 1, 2, \dots, n$. It also has a linear predictor $\theta_i = x_i' \beta$, and a link function g such that $E(Y_i|x_i) \equiv \mu_i = b'(\theta_i) = g^{-1}(x_i' \beta)$ (g is monotonic and differentiable), and $V(Y_i) = b''(\theta_i)a(\phi)$ (McCullagh and Nelder, 1989). The identity function $\mu_i = x_i' \beta$ is the canonical link function in the case of the Gaussian model.³² This statistical framework can help us to characterize:

Logit: The logit model is also a GLM where the link function is $x_i' \beta = \log\left(\frac{\mu_i}{1-\mu_i}\right)$. We perform BMA using the BIC approximation and the Occam's window approach in the logit model.

Gamma: The gamma model is also a GLM where the link function is $x_i' \beta = \mu_i^{-1}$. We perform BMA using the BIC approximation and the Occam's window approach in the logit model.

Poisson: The logit model is also a GLM where the link function is $x_i' \beta = \log(\mu_i)$. We perform BMA using the BIC approximation and the Occam's window approach in the logit model.

4 Using BEsmarter

Users should have a RStudio cloud account, such that after clicking our RStudio cloud project <https://rstudio.cloud/project/211946>, user should log in her/his account. See Figure 1.

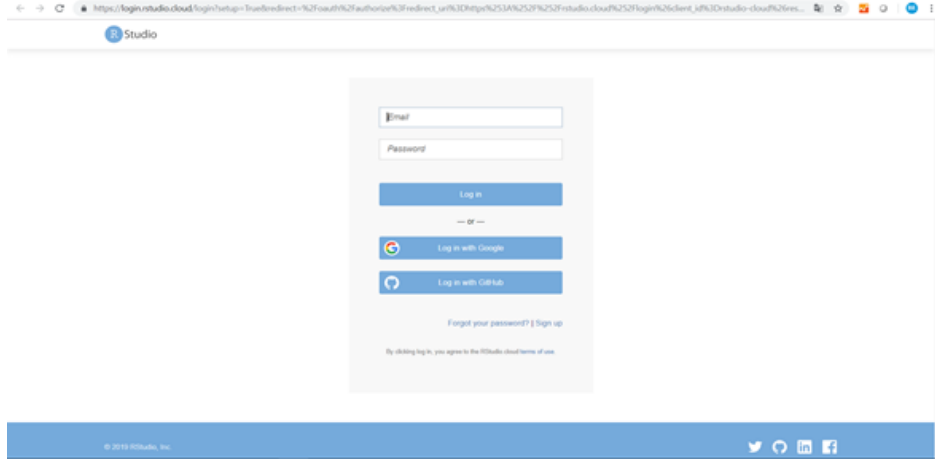


Figure 1: Log in RStudio cloud account

After our project is deployed and open, users have access to all our codes, and all our toy examples data sets in folder *DataSim* (see Table 3 for details). Use these data sets templates as a guide for structuring your own data sets. User should type `shiny::runApp()` in console to run our GUI. See Figure 2.

³²A canonical link functions is characterized by the existence of a sufficient statistic $(X'y)$ equal in dimension to β .

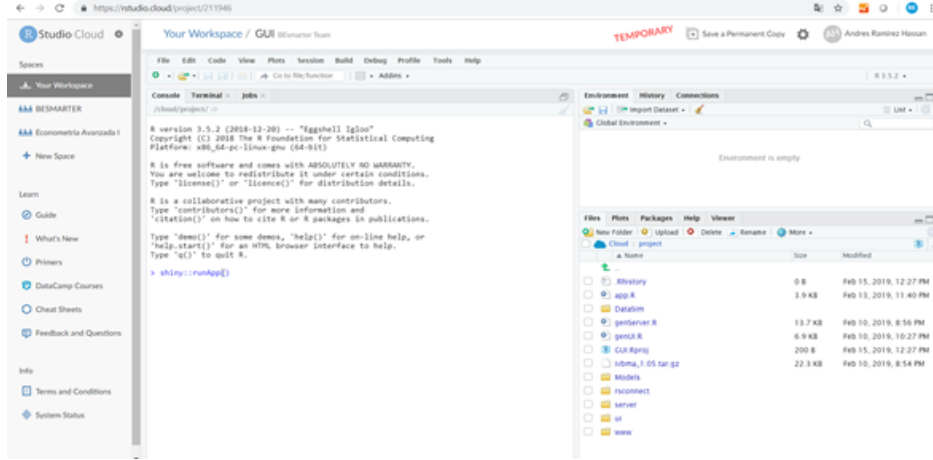


Figure 2: Run BEsmarter application in RStudio cloud

After this, users can see a new window where the presentation of our research team is displayed. In addition, the top panel in Figure 3 shows the class of models that can be estimated in our GUI.

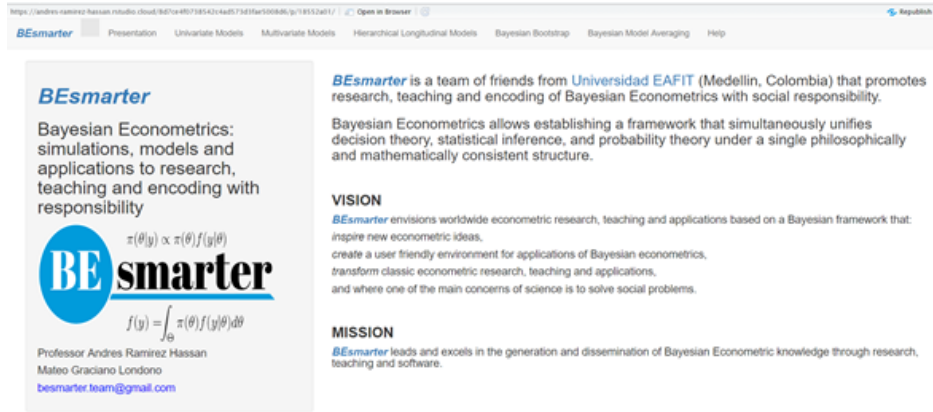


Figure 3: BEsmarter GUI

We select univariate models such that the radio button on the left hand side shows the specific models inside this generic class. In particular, users can see that the normal model is selected inside univariate class models. See Figure 4.

Then, the right hand side panel displays a widget to upload input data set, which should be a csv file with headers in the first row, this file has to be closed, we also should select what kind of separator is used in input file: comma, semicolon or tab.³³ Range sliders help to set MCMC iterations and burn-in size, and thinning parameter can be selected as well.

³³The folder called *DataSim* has examples of input data sets for each model that can be estimated using our GUI.

BEsmarter Presentation Univariate Models Multivariate Models Hierarchical Longitudinal Models Bayesian Bootstrap Bayesian Model Averaging Help

Models

- ☐ No Selection
- ☒ Normal
- ☐ Logit
- ☐ Probit
- ☐ Multinomial Probit
- ☐ Multinomial Logit
- ☐ Ordered Probit
- ☐ Negative Binomial(Poisson)
- ☐ Tobit
- ☐ Quantile

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$$\pi(\theta|y) \propto \pi(\theta)f(y|\theta)$$

$$f(y) = \int_{\theta} \pi(\theta)f(y|\theta)d\theta$$

Choose File 115cmNormalModel.csv

MCMC iterations: 10,000 20,000 30,000 40,000 50,000 60,000 70,000 80,000 90,000 100,000

Burn-in Size: 1,000 2,000 3,000 4,000 5,000 6,000 7,000 8,000 9,000 10,000

Thinning parameter: 5

Main Equation Introduce Formula. Example: y=x1*x2 where y is the dependent variable, and x1 and x2 are independent variables. Model specification includes the constant term by default.

Introduce prior mean vector location parameters

	cte	x1	x2	x3
cte	0			
x1	0	0		
x2	0	0	0	
x3	0	0	0	0

Introduce prior covariances location parameters by row. It has to be symmetric

	cte	x1	x2	x3
cte	1	0	0	0
x1	0	1	0	0
x2	0	0	1	0
x3	0	0	0	1

Prior Shape Parameter: Variance Parameter

Prior Scale Parameter: Variance Parameter

Figure 4: Univariate models: Specification

After this, users should specify the equation, this is set in R software form (see Figure 4, $y \sim x1 + x2 + x3$). Take into account that univariate models include intercept by default, except ordered probit, where specification has to do this explicit, for instance $y \sim x1 + x2 + x3 - 1$. Finally, we should define prior hyperparameters, for instance mean, covariance, shape and scale in the normal-inverse gamma model (see Figure 4).

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$$\pi(\theta|y) \propto \pi(\theta)f(y|\theta)$$

$$f(y) = \int_{\theta} \pi(\theta)f(y|\theta)d\theta$$

Introduce prior covariances location parameters by row. It has to be symmetric

	cte	x1	x2	x3
cte	1	0	0	0
x1	0	1	0	0
x2	0	0	1	0
x3	0	0	0	1

Prior Shape Parameter: Variance Parameter

Prior Scale Parameter: Variance Parameter

Introduce Prior shape Parameter
Example: 0.001

Introduce prior scale parameter
Example: 0.001

Click the button (Go!) after importing the dataset and selecting the model to update the value displayed in the main panel.

Warning: Be patient! This may take several minutes!!!

Summary

Iterations = 5000: 10000
Thinning interval = 5
Number of chains = 1
Sample size per chain = 6000

1. Empirical mean and standard deviation for each variable,
plus standard error of the mean:

	Mean	SD	Naive SE	Time-series SE	SE
(Intercept)	0.9522	0.04050	0.0000269	0.0000143	
x1	1.3529	0.05474	0.0007067	0.0007067	
x2	-0.9624	0.05422	0.0007000	0.0007000	
x3	1.5738	0.05341	0.0006895	0.0006895	

Figure 5: Univariate models: Results

After this specification process, users should click the Go! button to initiate the estimation process. We display summary statistics and convergence diagnostics after this process is finished (see Figure 5). We also have widgets to download posterior chains (csv file) and graphs (pdf and eps files). Take into account that order of coefficients in results (summary, posterior chains and graphs) is first for location parameters, and then, scale parameters.

Multinomial models (probit and logit) require a data set file to have in the first column the dependent variable, then alternative specific regressors (for instance alternatives' prices), and

finally, non-alternative regressors (for instance income). Specification also requires to define the base category, number of alternatives (this is also required in ordered probit), number of alternative specific regressors, and number of non-alternative regressors (see Figure 6). Multinomial logit also allows to define the tune parameter, degrees of freedom in this case, for the Metropolis-Hastings algorithm. This is a feature in our GUI when estimation of models is based on Metropolis-Hastings algorithm. The order of the coefficients in results of these models is first intercepts, and then, non-alternative specific regressors, these by the order of categories, and finally, coefficients for alternative specific regressors. Take into account that non-alternative specific regressors associated with base category are equal to zero (they do not appear in results). In addition, some coefficients of the main diagonal of the covariance matrix are constant by identification issues in multinomial and multivariate probit models.

The screenshot shows the BEsmarter software interface. The sidebar on the left lists various models under 'Univariate Models', with 'Multinomial Logit' selected. The main panel is titled 'Choose File' and includes a 'Browse' button, a file path '155intFullLogitmodel.csv', and a 'Load complete' button. Below this, there are sliders for 'MCMC iterations' (set to 10,000) and 'Burn-in Size' (set to 1,000). A 'Thinning parameter' dropdown is set to 5. The 'Main Equation' section contains a text box with the formula $y = x11 + x12 + x13 + x21 + x22 + x23 + x31 + x32 + x33 + x4$. Below the formula, there are three input fields: 'Base Alternative' (set to 3), 'Number of choice categorical alternatives' (set to 3), and 'Number of alternative specific variables' (set to 3). The 'Number of Non-Alternative specific variables' is set to 1. At the bottom, there is a table for 'Introduce prior mean vector location parameters' with columns for 'mean' and values for categories 1 through 5.

Figure 6: Univariate models: Multinomial

In the case of the negative binomial model, users should define dispersion parameter (α , see negative binomial model). User should define censoring points and quantile in Tobit and quantile models, respectively.

Figure 7 displays the multivariate regression setting. In this case, the input file should have first the dependent variables, and then regressors, if there are intercepts in each equation, there should be a column of 1's after the dependent variables in the input file. Users also has to define the number of dependent variables and the number of regressors, if necessary include the intercept (see Figure 7).

The input file in seemingly unrelated regressions should have first the dependent variables, and then regressors by equation, include intercept in each equation if necessary (column of 1's). Users should define the number of dependent variables (equations), the number of total regressors, that is, the sum of all regressors associated with equation (if necessary include intercepts, each intercept is an additional regressor), and the number of regressors by equation (if necessary include intercept).

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$$\pi(\theta|y) \propto \pi(\theta)f(y|\theta)$$

$$f(y) = \int_{\theta} \pi(\theta)f(y|\theta)d\theta$$

Choose File: 21SimMultivariate.csv
MCMC Iterations: 100,000
Burn-in Size: 1,000
Thinning parameter: 10

Number Endogenous Variables: m: 2
Number Exogenous Variables: k: 3

y1	y2
0.000	0.000
0.000	0.000
0.000	0.000

Figure 7: Multivariate models: Simple multivariate

Results of the simple multivariate and seemingly unrelated regressions show first posterior location parameters by equation, and then, the posterior covariance matrix.

In the instrumental variable setting, users should specify the main equation, and the auxiliary equation. This setting includes intercept by default. The first variable on the right hand side in the main equation has to be the variable with endogeneity issues. Users can find first the results associated with the endogeneous regressor, then location parameters of the auxiliary regression, and location parameters of the exogenous regressors. Finally, the posterior covariance matrix.

BEsmarter
Bayesian Econometrics:
simulations, models and
applications to research,
teaching and encoding with
responsibility

$$\pi(\theta|y) \propto \pi(\theta)f(y|\theta)$$

$$f(y) = \int_{\theta} \pi(\theta)f(y|\theta)d\theta$$

Choose File: 24SimMultiProbit.csv
MCMC Iterations: 100,000
Burn-in Size: 1,000
Thinning parameter: 10

Number of individuals: n: 5000
Number of exogenous Variables: k: 4
Number of choices: l: 2

Beta11	Beta12	Beta13	Beta14	Beta21	Beta22	Beta23	Beta24
100	0	0	0	0	0	0	0

Figure 8: Multivariate models: Multivariate probit

Multivariate probit model requires an input data set ordered by unit, for instance 3 choices implies repeat 3 times each unit, the first column has to be identification of each unit, users should use ordered integers, then the dependent variable, just one vector, composed by 0's and 1's, then regressors, which should include a column of 1's for intercepts. Users should

define number of units, number of regressors, and number of choices (see Figure 8). Results display first posterior location parameters by equation, and then the posterior covariance matrix.

Input files of hierarchical longitudinal models should have first the dependent variable, then regressors and a cross sectional identifier ($i = 1, 2, \dots, m$). It is not a requirement to have a balanced data set, n_i can be different for each i . Users should specify the fixed part equation, and random part equation. Both in R software form. In case of just requiring random intercepts, do not introduce anything in this part (see Figure 9). Users should also type the name of the cross sectional identifier variable. Results include fixed and random effects posteriors, but do not include convergence diagnostics tests.

The screenshot shows the 'Hierarchical Longitudinal Models' tab in the BEsmarter application. On the left, there's a sidebar with the BEsmarter logo and a list of models: No Selection, Normal (selected), Logit, and Poisson. Below this is a brief description of Bayesian Econometrics and a mathematical formula. The main panel is titled 'Choose File' and shows a file named '31SimLogitudinalNormal.csv' has been uploaded. It includes sliders for 'MCMC Iterations' (set to 10,000) and 'Burn-in Size' (set to 1,000), and a 'Thinning parameter' dropdown set to 5. Below these are input fields for 'Main Equation: Fixed Effects' (y~x1+x2), 'Main Equation: Random Effects' (empty), and 'Name of grouping variable' (species). There are also sections for 'Prior mean' and 'Prior covariance matrix' of fixed effects, each with a table of values. The 'Prior mean' table has columns 'cte', 'x1', and 'x2' with values 0, 0, and 0 respectively. The 'Prior covariance matrix' table has columns 'cte', 'x1', and 'x2' with values 1, 0, and 0 respectively.

Figure 9: Hierarchical longitudinal models: Specification

Bayesian bootstrap just requires upload a data set, specify MCMC iterations, resampling size and equation. Users can see the progress bar after clicking the Go! button (see Figure 10).

The screenshot shows the 'Bayesian Bootstrap' tab in the BEsmarter application. It features a similar layout to Figure 9, with a sidebar on the left. The main panel shows a file named '41SimBootstrapmodel.csv' uploaded. It includes sliders for 'MCMC Iterations' (set to 10,000) and 'Resampling Size' (set to 1,000). The 'Main Equation' field contains 'y~x1+x2+x3'. A 'Go!' button is visible. Below the equation, there's a 'plyr progress' bar showing the progress of the computation. At the bottom, there are links to 'Download Posterior Chains' and 'Download Posterior Graphs'. A warning message states: 'Warning: Be patient this may take several minutes!!!'.

Figure 10: Bayesian bootstrap: Specification

Bayesian model averaging based on a Gaussian distribution can be performed using Bayesian information criterion (BIC) approximation, Markov chain Monte Carlo model composition (MC3) or instrumental variables (see Figure 11). The former two approaches require an input data set where the first column is the dependent variable, and then, the potentially important regressors. Users should define the band width model selection parameter (O_R) and number of iterations for BIC and MC3, respectively. Results include posterior inclusion probability ($p^!$), expected value (EV), and standard deviation (SD) of coefficients associated with each regressors. The BIC framework also displays the most relevant models including number of regressors, R^2 , BIC, and posterior model probability. Users can download two csv files: *Best models* and *Descriptive statistics coefficients*. The former is a 0-1 matrix such that columns are regressors and rows are models. 1 indicates presence of regressor l in model M_m , 0 otherwise. Take into account that the last column of this file is the posterior model probability. The latter file shows posterior inclusion probabilities, expected values and standard deviations associated with each regressor taking into account the BMA procedure based on the best models.

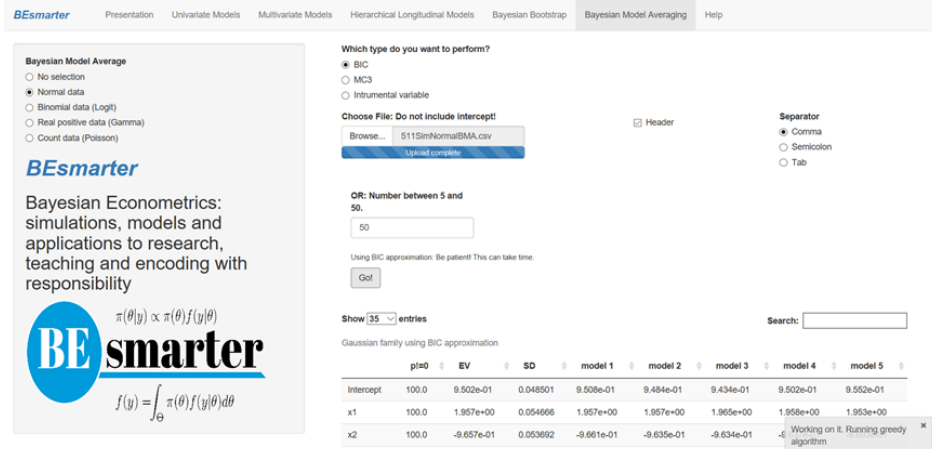


Figure 11: Bayesian model averaging: Specification and results

Bayesian model averaging with endogeneity issues requires two input files. The first one has the dependent variable in the first column, the next columns are regressors with endogeneity issues, and then, exogenous regressors, users should include a column of 1's if an intercept is required. The second input file has all instruments. Users should also introduce number of regressors with endogeneity issues (see Figure 12).

Results include posterior inclusion probabilities and expected values for each regressor. Users can find results of the main equation, and then, auxiliary equations. Users can download csv files of BMA results for the second stage (main equation), and first stage (auxiliary equations). In addition, users can download posterior chains of location parameters of the main equation, β_l , $l = 1, 2, \dots, \dim\{\beta\}$, location parameters of auxiliary equations, $\gamma_{j,i}$, $j = 1, 2, \dots, \dim\{\beta_s\}$ where $\dim\{\beta_s\}$ is the number of regressors with endogeneous issues, $i = 1, 2, \dots, \dim\{\gamma\}$, where $\dim\{\gamma\}$ is the number of regressors in auxiliary regressors (exo-

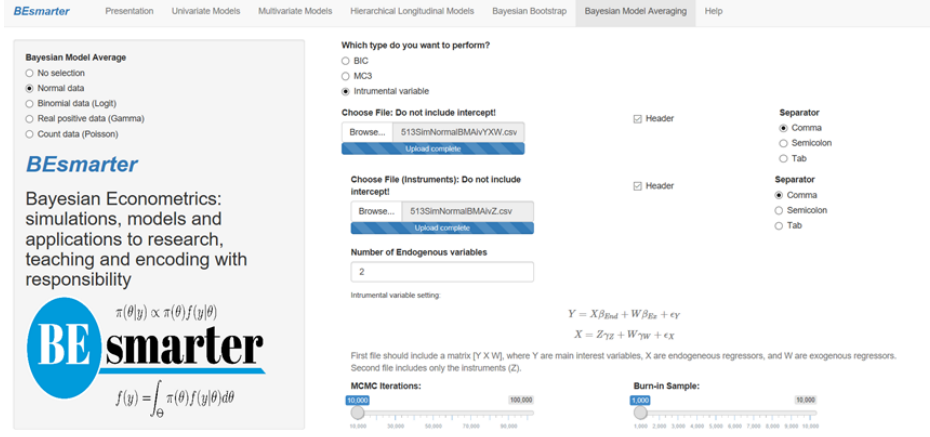


Figure 12: Bayesian model averaging: Instrumental variable specification

geneous regressors + instruments), and elements of covariance matrix. $\sigma_{j,k}$.

Bayesian model averaging based on BIC approximation for non-linear models, logit, gamma, and poisson, requires an input data set where the first column is the dependent variable, and other columns are potentially relevant regressors. Users should specify band width model selection parameters (O_R and O_L). Our GUI displays PIP ($p! = 0$), expected value of posterior coefficients (EV), and standard deviation (SD). In addition, users can see results associated with models with the highest posterior model probabilities, and download csv files with results of specifications of the best models, and descriptive statistics of posterior coefficients from the BMA procedure. These files are similar to the results of the BIC approximation of the Gaussian model.

5 Concluding remarks

The Bayesian statistical framework has become very popular among scientists since the computational revolution in 1990's. In particular, computational burdensome procedures such as MCMC algorithms can be easily applied nowadays. However, most of the open source software to apply these procedures require programming skills. This may be a cause why the Bayesian framework is not very popular among applied researchers and practitioners. We introduce in this paper a graphical user interface to implement Bayesian regression analysis under different frameworks, explaining the basic theory, such that users can understand the basic principles of Bayesian statistics, and apply them easily. Our objective is to increase popularity of the Bayesian statistical framework among applied researchers and practitioners.

Table 1: Graphical user interfaces to perform Bayesian regression analysis.

Name	Language	Models	Open source
ShinyStan	R+Stan	MCMC Implementation*	Yes
Bayesian regression: NP&P	MATLAB Compiler	Bayesian infinite-mixture regression Bayesian normal regression Hierarchical linear regression Binary regression Ordered regression Censoring regression Quantile regression Survival regression Density estimation Variable selection (spike-and-slab)	Yes
BugsXLA	OpenBUGS + Excel	Normal linear models GLM: Binomial GLM: Poisson GLM: Survival GLM: Multivariate categorical data Normal linear mixed Generalized linear mixed Bayesian variable selection Robust models	Yes
MATLAB toolkit: E&E ⁺	MATLAB	Linear Regression Regression with non-spherical errors Regime switch regression Regression with restricted parameters Seemingly unrelated regression (SUR) Vector AutoRegression (VAR) Instrumental variable Probit and logit Tobit Model Panel Data Analysis Stochastic search variable selection Highest posterior density (HPD) region Marginal likelihood of linear regression	No
Stata	Stata	MCMC implementation*	No
BayES	C++	Simple linear model Random-effects Random-coefficients Stochastic frontiers Inefficiency-effects Random-effects stochastic frontiers Dynamic stochastic frontier Probit and logit Random-effects probit and logit Multinomial probit and logit Ordered probit and logit Poisson and negative-binomial Type I Tobit Type II Tobit Seemingly unrelated regressions (SUR) Vector Autoregressive (VAR)	No

*User should define prior and likelihood.

⁺Toolkit on econometrics and economics teaching.

Table 2: Libraries and commands in BESmarter GUI.

Univariate models			
Model	Library	Command	Reference
Normal	MCMCpack	MCMCregress	Martin et al. (2018)
Logit	MCMCpack	MCMClogit	Martin et al. (2018)
Probit	bayesm	rbprobitGibbs	Rossi (2017)
Multinomial(Mixed) Probit	bayesm	rmnpGibbs	Rossi (2017)
Multinomial(Mixed) Logit	bayesm	rmnlIndepMetrop	Rossi (2017)
Ordered Probit	bayesm	rordprobitGibbs	Rossi (2017)
Negative Binomial(Poisson)	bayesm	rnegbinRw	Rossi (2017)
Tobit	MCMCpack	MCMCtobit	Martin et al. (2018)
Quantile	MCMCpack	MCMCquantreg	Martin et al. (2018)
Multivariate models			
Model	Library	Command	Reference
Multivariate	bayesm	rmultireg	Rossi (2017)
Seemingly Unrelated Regression	bayesm	rsurGibbs	Rossi (2017)
Instrumental Variable	bayesm	rivGibbs	Rossi (2017)
Bivariate Probit	bayesm	rmvpGibbs	Rossi (2017)
Hierarchical longitudinal models			
Model	Library	Command	Reference
Normal	MCMCpack	MCMChregress	Martin et al. (2018)
Logit	MCMCpack	MCMChlogit	Martin et al. (2018)
Poisson	MCMCpack	MCMChpoisson	Martin et al. (2018)
Bayesian Bootstrap			
Model	Library	Command	Reference
Bayesian bootstrap	bayesboot	bayesboot	Baath (2018)
Bayesian model averaging			
Model	Library	Command	Reference
Normal (BIC)	BMA	bic.glm	Raftery et al. (2012)
Normal (MC ³)	BMA	MC3.REG	Raftery et al. (2012)
Normal (instrumental variables)	ivbma	ivbma	Lenkoski et al. (2013)
Logit (BIC)	BMA	bic.glm	Raftery et al. (2012)
Gamma (BIC)	BMA	bic.glm	Raftery et al. (2012)
Poisson (BIC)	BMA	bic.glm	Raftery et al. (2012)
Diagnostics			
Diagnostic	Library	Command	Reference
Trace plot	coda	traceplot	Plummer et al. (2016)
Autocorrelation plot	coda	autocorr.plot	Plummer et al. (2016)
Geweke test	coda	geweke.diag	Plummer et al. (2016)
Raftery & Lewis test	coda	raftery.diag	Plummer et al. (2016)
Heidelberger & Welch test	coda	heidel.diag	Plummer et al. (2016)

Table 3: Data sets templates in folder *DataSim*.

Univariate models		
Model	Data set file	Data set simulation
Normal	11SimNormalmodel.csv	11SimNormal.R
Logit	12SimLogitmodel.csv	12SimLogit
Probit	13SimProbitmodel.csv	13SimProbit.R
Multinomial(Mixed) Probit	14SimMultProbmodel.csv	14SimMultinomialProbit.R
Multinomial(Mixed) Logit	15SimMultLogitmodel.csv	15SimMultinomialLogit.R
Ordered Probit	16SimOrderedProbitmodel.csv	16SimOrderedProbit.R
Negative Binomial(Poisson)	17SimNegBinmodel.csv	17SimNegBin.R
Tobit	18SimTobitmodel.csv	18SimTobit.R
Quantile	19SimQuantilemodel.csv	19SimQuantile.R
Multivariate models		
Model	Library	Command
Multivariate	21SimMultivariate.csv	21SimMultReg.R
Seemingly Unrelated Regression	22SimSUR.csv	22SimSUR.R
Instrumental Variable	23SimIV.csv	23SimIV.R
Bivariate Probit	24SimMultProbit.csv	24SimMultProbit.R
Hierarchical longitudinal models		
Model	Library	Command
Normal	31SimLogitudinalNormal.csv	31SimLogitudinalNormal.R
Logit	32SimLogitudinalLogit.csv	32SimLogitudinalLogit.R
Poisson	33SimLogitudinalPoisson.csv	33SimLogitudinalPoisson.R
Bayesian Bootstrap		
Model	Library	Command
Bayesian bootstrap	41SimBootstrapmodel.csv	41SimBootstrapmodel.R
Bayesian model averaging		
Model	Library	Command
Normal (BIC)	511SimNormalBMA.csv	511SimNormalBMA.R
Normal (MC ³)	512SimNormalBMA.csv	512SimNormalBMA.R
Normal (instrumental variables)	513SimNormalBMAivYXW.csv	513SimNormalBMAiv.R
	513SimNormalBMAivZ.csv	
Logit (BIC)	52SimLogitBMA.csv	52SimLogitBMA.R
Gamma (BIC)	53SimGammaBMA.csv	53SimGammaBMA.R
Poisson (BIC)	53SimPoissonBMA.csv	53SimPoissonBMA.R

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