Bijective enumeration of general stacks

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Supplement material: bijective enumeration of the number of m-regular 3-contact stacks

The generating function of m-regular 3-contact stacks satisfies the equation,

$$\sum_{k=0}^{17} c_{m,3,k}(x) S_{m,3}^k(x) = 0,$$

where $C_{m,3}(x) = \frac{x^{3(m-1)}-1}{x^3-1}$, C is short for $C_{m,3}(x)$, and the coefficients $c_{m,3,k}(x)$ are detailed as follows.

- $c_{m,3,0} = -1 - 9C^{18}x^{18} + \left(-102x^{17} + 99x^{16} \right)C^{17} + \left(-1087x^{16} + 363x^{15} - 27x^{14} \right)C^{16} + \left(-3200x^{15} + 1838x^{14} - 104x^{13} + x^{12} \right)C^{15} + \left(-8196x^{14} + 3978x^{13} - 554x^{12} + 3x^{11} \right)C^{14} + \left(-12059x^{13} + 7729x^{12} - 1014x^{11} + 40x^{10} \right)C^{13} + \left(-16655x^{12} + 9448x^{11} - 1758x^{10} + 43x^9 - x^8 \right)C^{12} + \left(-15901x^{11} + 10961x^{10} - 1792x^9 + 54x^8 \right)C^{11} + \left(-15037x^{10} + 8990x^9 - 1752x^8 + 25x^7 \right)C^{10} + \left(-10197x^9 + 7269x^8 - 1196x^7 + 18x^6 \right)C^9 + \left(-7214x^8 + 4186x^7 - 761x^6 + 3x^5 \right)C^8 + \left(-3546x^7 + 2445x^6 - 330x^5 + x^4 \right)C^7 + \left(-1988x^6 + 973x^5 - 137x^4 \right)C^6 + \left(-701x^5 + 412x^4 - 30x^3 \right)C^5 + \left(-322x^4 + 101x^3 - 7x^2 \right)C^4 + \left(-75x^3 + 32x^2 \right)C^3 + \left(-28x^2 + 3x \right)C^2 + \left(-3x + 1 \right)C$,

- $c_{m,3,1} = 9C^{19}x^{20} + \left(93x^{19} - 90x^{18} \right)C^{18} + \left(1012x^{18} - 162x^{17} - 72x^{16} \right)C^{17} + \left(3040x^{17} - 1594x^{16} - 286x^{15} + 26x^{14} \right)C^{16} + \left(22487x^{16} - 1856x^{15} - 1995x^{14} + 102x^{13} - x^{12} \right)C^{15} + \left(46952x^{15} - 16338x^{14} - 4774x^{13} + 850x^{12} - 3x^{11} \right)C^{14} +$

$$\left(121549x^{14} - 32860x^{13} - 5626x^{12} + 1670x^{11} - 75x^{10}\right)C^{13} + \left(150784x^{13} - 72539x^{12} - 4254x^{11} + 2638x^{10} - 82x^9 + 2x^8\right)C^{12} + \left(208249x^{12} - 83478x^{11} + 1386x^{10} + 2640x^9 - 93x^8\right)C^{11} + \left(171424x^{11} - 98777x^{10} + 4253x^9 + 2441x^8 - 43x^7\right)C^{10} + \left(159294x^{10} - 74498x^9 + 6016x^8 + 1704x^7 - 32x^6\right)C^9 + \left(91256x^9 - 58571x^8 + 4990x^7 + 1041x^6 - 3x^5\right)C^8 + \left(62340x^8 - 29847x^7 + 3189x^6 + 436x^5 - x^4\right)C^7 + \left(24503x^7 - 16177x^6 + 1381x^5 + 181x^4\right)C^6 + \left(13051x^6 - 5283x^5 + 493x^4 + 30x^3\right)C^5 + \left(3276x^5 - 1950x^4 + 81x^3 + 7x^2\right)C^4 + \left(1426x^4 - 317x^3 + 2x^2\right)C^3 + \left(176x^3 - 80x^2 - 3x\right)C^2 + \left(65x^2 + x - 1\right)C - x,$$

- $c_{m,3,2} = -36C^{18}x^{20} + \left(-1020x^{19} + 1269x^{18}\right)C^{17} + \left(-17360x^{18} + 1251x^{17} + 1885x^{16}\right)C^{16} + \left(-32758x^{17} + 6213x^{16} + 3671x^{15} + 248x^{14}\right)C^{15} + \left(-231764x^{16} - 17013x^{15} + 32948x^{14} + 858x^{13} - 296x^{12}\right)C^{14} + \left(-318348x^{15} + 17674x^{14} + 70665x^{13} - 6751x^{12} - 655x^{11} + 35x^{10}\right)C^{13} + \left(-872275x^{14} + 51537x^{13} + 125067x^{12} - 14549x^{11} - 449x^{10} + 39x^9 - x^8\right)C^{12} + \left(-848404x^{13} + 221033x^{12} + 130768x^{11} - 28194x^{10} - 400x^9 + 29x^8\right)C^{11} + \left(-1218035x^{12} + 259768x^{11} + 118600x^{10} - 27838x^9 - 171x^8 + 18x^7\right)C^{10} + \left(-816278x^{11} + 338667x^{10} + 68331x^9 - 26232x^8 - 325x^7 + 14x^6\right)C^9 + \left(-771601x^{10} + 241122x^9 + 42131x^8 - 16792x^7 - 140x^6\right)C^8 + \left(-352539x^9 + 188745x^8 + 10483x^7 - 9961x^6 - 105x^5\right)C^7 + \left(-240998x^8 + 85455x^7 + 4443x^6 - 3613x^5 - 44x^4\right)C^6 + \left(-70096x^7 + 43003x^6 - 212x^5 - 1396x^4\right)C^5 + \left(-37134x^6 + 11377x^5 + 127x^4 - 189x^3\right)C^4 + \left(-5675x^5 + 3549x^4 - 23x^3 - 34x^2\right)C^3 + \left(-2563x^4 + 356x^3 + 56x^2\right)C^2 + \left(-91x^3 + 62x^2 + 3x\right)C - 52x^2 - x + 1,$

- $c_{m,3,3} = -360C^{17}x^{20} + \left(4153x^{19} - 7039x^{18}\right)C^{16} + \left(166542x^{18} + 3561x^{17} - 20705x^{16}\right)C^{15} + \left(168940x^{17} + 70259x^{16} - 20724x^{15} - 9354x^{14}\right)C^{14} + \left(1522267x^{16} + 239377x^{15} - 231679x^{14} - 25654x^{13} + 4639x^{12}\right)C^{13} + \left(1241093x^{15} + 502078x^{14} - 416659x^{13} - 3244x^{12} + 9270x^{11} - 431x^{10}\right)C^{12} + \left(3997247x^{14} + 477810x^{13} - 824595x^{12} + 26600x^{11} + 13427x^{10} - 446x^9 + 10x^8\right)C^{11} + \left(2725115x^{13} + 224239x^{12} - 816911x^{11} + 97097x^{10} + 12213x^9 - 474x^8\right)C^{10} + \left(4399222x^{12} - 94300x^{11} - 846200x^{10} + 104040x^9 + 10101x^8 - 180x^7\right)C^9 + \left(2177263x^{11} - 351904x^{10} - 482782x^9 + 100472x^8 + 6651x^7 - 138x^6\right)C^8 + \left(2240901x^{10} - 315473x^9 - 342649x^8 + 63068x^7 + 3618x^6\right)C^7 + \left(737034x^9 - 265363x^8 - \right.$

$$109353x^7 + 36098x^6 + 1182x^5\big)C^6 + \big(540761x^8 - 115118x^7 - 54462x^6 +$$
$$11868x^5 + 468x^4\big)C^5 + \big(100294x^7 - 52965x^6 - 8300x^5 + 4270x^4\big)C^4 +$$
$$\big(58133x^6 - 11766x^5 - 3160x^4 + 480x^3\big)C^3 + \big(3590x^5 - 2702x^4 - 80x^3 +$$
$$54x^2\big)C^2 + \big(2171x^4 - 180x^3 - 103x^2\big)C - 60x^3 + 5x^2,$$

- $c_{m,3,4} = 4329C^{16}x^{20} + \big(9425x^{19} - 1787x^{18}\big)C^{15} + \big(-1108097x^{18} - 72194x^{17} + 122066x^{16}\big)C^{14} + \big(-324262x^{17} - 1034353x^{16} + 56981x^{15} + 95015x^{14}\big)C^{13} + \big(-7006402x^{16} - 1286161x^{15} + 918969x^{14} + 206902x^{13} - 32003x^{12}\big)C^{12} + \big(-2678623x^{15} - 4043860x^{14} + 1350659x^{13} + 260708x^{12} - 56540x^{11} + 2299x^{10}\big)C^{11} + \big(-13033010x^{14} - 3066453x^{13} + 2900597x^{12} + 179020x^{11} - 93326x^{10} + 2159x^9 - 41x^8\big)C^{10} + \big(-5060396x^{13} - 3982797x^{12} + 2574022x^{11} - 11686x^{10} - 79304x^9 + 2507x^8\big)C^9 + \big(-10952844x^{12} - 1590800x^{11} + 2842715x^{10} - 134367x^9 - 67051x^8 + 738x^7\big)C^8 + \big(-3313725x^{11} - 1134717x^{10} + 1424820x^9 - 151008x^8 - 36205x^7 + 560x^6\big)C^7 + \big(-4322123x^{10} - 119420x^9 + 1052939x^8 - 113004x^7 - 19724x^6\big)C^6 + \big(-828684x^9 - 11055x^8 + 291611x^7 - 62344x^6 - 4950x^5\big)C^5 + \big(-769281x^8 + 52790x^7 + 145379x^6 - 19693x^5 - 1908x^4\big)C^4 + \big(-63616x^7 + 18169x^6 + 18349x^5 - 6572x^4\big)C^3 + \big(-53052x^6 + 6062x^5 + 5948x^4 - 645x^3\big)C^2 + \big(342x^5 + 434x^4 + 19x^3 - 27x^2\big)C - 787x^4 + 115x^3 + 45x^2,$

- $c_{m,3,5} = -14688C^{15}x^{20} + \big(-250167x^{19} + 316794x^{18}\big)C^{14} + \big(5426861x^{18} + 274808x^{17} - 391142x^{16}\big)C^{13} + \big(-1206135x^{17} + 6559718x^{16} - 19910x^{15} - 540250x^{14}\big)C^{12} + \big(23787797x^{16} + 3737941x^{15} - 2033843x^{14} - 923380x^{13} + 129299x^{12}\big)C^{11} + \big(1164309x^{15} + 17175976x^{14} - 2532657x^{13} - 1558886x^{12} + 198049x^{11} - 6925x^{10}\big)C^{10} + \big(31924838x^{14} + 8411087x^{13} - 6004081x^{12} - 1208852x^{11} + 340204x^{10} - 5728x^9 + 88x^8\big)C^9 + \big(3776491x^{13} + 15252490x^{12} - 4622339x^{11} - 942493x^{10} + 263841x^9 - 6916x^8\big)C^8 + \big(19918789x^{12} + 4885145x^{11} - 5549081x^{10} - 203442x^9 + 219197x^8 - 1584x^7\big)C^7 + \big(2154670x^{11} + 5007725x^{10} - 2254004x^9 - 87325x^8 + 100373x^7 - 1194x^6\big)C^6 + \big(5812968x^{10} + 908374x^9 - 1777996x^8 + 76950x^7 + 53551x^6\big)C^5 + \big(323596x^9 + 609297x^8 - 374322x^7 + 40596x^6 + 10674x^5\big)C^4 + \big(711938x^8 + 34962x^7 - 192175x^6 + 16502x^5 + 4050x^4\big)C^3 + \big(-3646x^7 + 25425x^6 - 16499x^5 + 5110x^4\big)C^2 + \big(26801x^6 - 1805x^5 - 4638x^4 + 486x^3\big)C - 871x^5 + 376x^4 + 3x^3,$

- $c_{m,3,6} = -38970C^{14}x^{20} + (1814532x^{19} - 2545929x^{18})C^{13} + (-19946591x^{18} + 146355x^{17} + 399512x^{16})C^{12} + (10906450x^{17} - 26309309x^{16} - 375986x^{15} + 2007528x^{14})C^{11} + (-61401236x^{16} - 4950542x^{15} + 1212383x^{14} + 2668312x^{13} - 338780x^{12})C^{10} + (12171625x^{15} - 48870102x^{14} + 2344867x^{13} + 5061570x^{12} - 439621x^{11} + 12862x^{10})C^9 + (-60386796x^{14} - 11735717x^{13} + 6607734x^{12} + 3460740x^{11} - 760583x^{10} + 8984x^9 - 104x^8)C^8 + (6112944x^{13} - 34293368x^{12} + 4421030x^{11} + 3417847x^{10} - 524804x^9 + 11092x^8)C^7 + (-27306808x^{12} - 6455728x^{11} + 6444518x^{10} + 1003357x^9 - 422540x^8 + 1872x^7)C^6 + (1691949x^{11} - 9344868x^{10} + 1831612x^9 + 716237x^8 - 160655x^7 + 1406x^6)C^5 + (-5566665x^{10} - 1074834x^9 + 1750598x^8 + 49530x^7 - 83353x^6)C^4 + (327686x^9 - 949135x^8 + 223109x^7 + 27577x^6 - 12630x^5)C^3 + (-419999x^8 - 39567x^7 + 135108x^6 - 5122x^5 - 4752x^4)C^2 + (24607x^7 - 27921x^6 + 5907x^5 - 1593x^4)C - 5731x^6 + 511x^5 + 1314x^4 - 162x^3$,

- $c_{m,3,7} = 567945C^{13}x^{20} + (-8037636x^{19} + 11645484x^{18})C^{12} + (55620860x^{18} - 5158377x^{17} + 2155434x^{16})C^{11} + (-40588654x^{17} + 74027781x^{16} + 1158520x^{15} - 5185962x^{14})C^{10} + (122450834x^{16} - 4863542x^{15} + 7432655x^{14} - 5311948x^{13} + 600931x^{12})C^9 + (-44848901x^{15} + 99987363x^{14} + 310080x^{13} - 10686030x^{12} + 640856x^{11} - 15064x^{10})C^8 + (88963381x^{14} + 3507145x^{13} - 223929x^{12} - 5820076x^{11} + 1097774x^{10} - 8320x^9 + 64x^8)C^7 + (-22331760x^{13} + 52075164x^{12} - 977704x^{11} - 6422584x^{10} + 650248x^9 - 10432x^8)C^6 + (28480459x^{12} + 2841387x^{11} - 3878184x^{10} - 1579420x^9 + 503459x^8 - 1152x^7)C^5 + (-5086975x^{11} + 10421638x^{10} - 372760x^9 - 1235198x^8 + 150392x^7 - 864x^6)C^4 + (3773474x^{10} + 339836x^9 - 952372x^8 - 117020x^7 + 75672x^6)C^3 + (-473833x^9 + 697582x^8 - 20357x^7 - 63864x^6 + 7776x^5)C^2 + (144526x^8 + 2038x^7 - 46089x^6 - 1080x^5 + 2916x^4)C - 8188x^7 + 7823x^6 - 315x^5$,

- $c_{m,3,8} = -2607912C^{12}x^{20} + (24619148x^{19} - 36113039x^{18})C^{11} + (-118595294x^{18} + 23069788x^{17} - 11866564x^{16})C^{10} + (96640944x^{17} - 152697354x^{16} - 956283x^{15} + 9571867x^{14})C^9 + (-189613625x^{16} + 36509589x^{15} - 28263100x^{14} + 7467064x^{13} - 730393x^{12})C^8 + (87603164x^{15} - 150853960x^{14} - 3003833x^{13} + 15546254x^{12} - 612924x^{11} + 10856x^{10})C^7 + (-101310738x^{14} + 17428434x^{13} - 11509562x^{12} +$

$6101396x^{11}-1027444x^{10}+4208x^9-16x^8)C^6+(33069580x^{13}-55669300x^{12}-$
$2392089x^{11}+7445639x^{10}-493312x^9+5344x^8)C^5+(-22220231x^{12}+$
$3155332x^{11}+70120x^{10}+1239294x^9-365362x^8+288x^7)C^4+(5009732x^{11}-$
$7373418x^{10}-533101x^9+1086882x^8-76604x^7+216x^6)C^3+(-1724254x^{10}+$
$278147x^9+218498x^8+63616x^7-37368x^6)C^2+(221250x^9-256376x^8-$
$34972x^7+38790x^6-1944x^5)C-21690x^8+4575x^7+5483x^6+702x^5-729x^4,$

- $c_{m,3,9}=7241724C^{11}x^{20}+(-54683163x^{19}+80680950x^{18})C^{10}+(194164482x^{18}-$
$59344476x^{17}+31339494x^{16})C^9+(-161682877x^{17}+235340850x^{16}-2636428x^{15}-$
$12728804x^{14})C^8+(226876656x^{16}-82416766x^{15}+54026044x^{14}-7427800x^{13}+$
$600604x^{12})C^7+(-112328818x^{15}+168321391x^{14}+1858452x^{13}-15841780x^{12}+$
$370592x^{11}-4400x^{10})C^6+(87190157x^{14}-35334852x^{13}+19546951x^{12}-$
$3897088x^{11}+601784x^{10}-896x^9)C^5+(-29384469x^{13}+41959098x^{12}+$
$2396599x^{11}-5490328x^{10}+210192x^9-1152x^8)C^4+(12350890x^{12}-5378782x^{11}+$
$1652515x^{10}-434632x^9+148380x^8)C^3+(-2574582x^{11}+3248664x^{10}+$
$379902x^9-513372x^8+16416x^7)C^2+(466057x^{10}-242582x^9+15998x^8-$
$4176x^7+7776x^6)C-35452x^9+37272x^8+10146x^7-7938x^6,$

- $c_{m,3,10}=-13774113C^{10}x^{20}+(90102670x^{19}-133496968x^{18})C^9+(-$
$243834047x^{18}+102731652x^{17}-53923185x^{16})C^8+(195826358x^{17}-272076084x^{16}+$
$9360429x^{15}+12111470x^{14})C^7+(-206712467x^{16}+111051420x^{15}-66466093x^{14}+$
$5132744x^{13}-319888x^{12})C^6+(98748510x^{15}-136989944x^{14}+2275656x^{13}+$
$11185484x^{12}-128480x^{11}+768x^{10})C^5+(-54566509x^{14}+34500053x^{13}-$
$17609962x^{12}+1350592x^{11}-199984x^{10})C^4+(16370562x^{13}-21600848x^{12}-$
$535854x^{11}+2499896x^{10}-38592x^9)C^3+(-4466714x^{12}+3255674x^{11}-$
$1161812x^{10}+4968x^9-25920x^8)C^2+(665727x^{11}-808126x^{10}-52898x^9+$
$113940x^8)C-53179x^{10}+50388x^9-11385x^8-3888x^7,$

- $c_{m,3,11}=18783225C^9x^{20}+(-111138777x^{19}+165432159x^{18})C^8+(233019392x^{18}-$
$125377962x^{17}+64797483x^{16})C^7+(-172234868x^{17}+233889154x^{16}-$
$14211388x^{15}-8047472x^{14})C^6+(139718603x^{16}-98112718x^{15}+55102586x^{14}-$
$2350720x^{13}+99712x^{12})C^5+(-59049161x^{15}+78756310x^{14}-4504440x^{13}-$
$5246496x^{12}+19456x^{11})C^4+(23379597x^{14}-19564408x^{13}+9591558x^{12}-$

$162048x^{11} + 28672x^{10}\big)C^3 + \big(-5445939x^{13} + 7026258x^{12} - 338140x^{11} -$
$635840x^{10}\big)C^2 + \big(888800x^{12} - 883228x^{11} + 346470x^{10} + 25344x^9\big)C - 63508x^{11} +$
$84400x^{10} - 11576x^9 - 6912x^8,$

- $c_{m,3,12} = -18649620C^8x^{20} + \big(102436371x^{19} - 153416454x^{18}\big)C^7 + \big(-$
  $166538264x^{18} + 108780004x^{17} - 55174962x^{16}\big)C^6 + \big(108196820x^{17} - 146123496x^{16} +$
  $12746172x^{15} + 3547448x^{14}\big)C^5 + \big(-67132286x^{16} + 57016140x^{15} - 30414448x^{14} +$
  $643392x^{13} - 13824x^{12}\big)C^4 + \big(22931116x^{15} - 30156556x^{14} + 3194424x^{13} +$
  $1478784x^{12}\big)C^3 + \big(-6204637x^{14} + 6278846x^{13} - 3047934x^{12} - 17408x^{11}\big)C^2 +$
  $\big(930836x^{13} - 1216446x^{12} + 184304x^{11} + 67968x^{10}\big)C - 65208x^{12} + 79736x^{11} -$
  $39360x^{10},$

- $c_{m,3,13} = 13430007C^7x^{20} + \big(-69617211x^{19} + 105117438x^{18}\big)C^6 + \big(86240233x^{18} -$
  $65944576x^{17} + 32788276x^{16}\big)C^5 + \big(-46653910x^{17} + 63547856x^{16} - 6979712x^{15} -$
  $932352x^{14}\big)C^4 + \big(21344276x^{16} - 20746984x^{15} + 10548000x^{14} - 79872x^{13}\big)C^3 +$
  $\big(-5214558x^{15} + 6892930x^{14} - 1090816x^{13} - 190464x^{12}\big)C^2 + \big(839688x^{14} -$
  $963504x^{13} + 470720x^{12}\big)C - 53304x^{13} + 69312x^{12} - 18432x^{11},$

- $c_{m,3,14} = -6854013C^6x^{20} + \big(33895978x^{19} - 51720628x^{18}\big)C^5 + \big(-30573234x^{18} +$
  $26629848x^{17} - 12973888x^{16}\big)C^4 + \big(12764814x^{17} - 17773308x^{16} + 2169856x^{15} +$
  $110592x^{14}\big)C^3 + \big(-3924213x^{16} + 4173552x^{15} - 2009856x^{14}\big)C^2 + \big(545080x^{15} -$
  $741184x^{14} + 148480x^{13}\big)C - 34304x^{14} + 38912x^{13} - 16384x^{12},$

- $c_{m,3,15} = 2355786C^5x^{20} + \big(-11195308x^{19} + 17306944x^{18}\big)C^4 + \big(6639946x^{18} -$
  $6449152x^{17} + 3078144x^{16}\big)C^3 + \big(-1875960x^{17} + 2721024x^{16} - 294912x^{15}\big)C^2 +$
  $\big(300160x^{16} - 332800x^{15} + 147456x^{14}\big)C - 10240x^{15} + 16384x^{14},$

- $c_{m,3,16} = -489969C^4x^{20} + \big(2248272x^{19} - 3530880x^{18}\big)C^3 + \big(-666432x^{18} +$
  $709632x^{17} - 331776x^{16}\big)C^2 + \big(92160x^{17} - 147456x^{16}\big)C,$

- $c_{m,3,17} = 46656C^3x^{20} + \big(-207360x^{19} + 331776x^{18}\big)C^2.$

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