Foundations of Statistical Modeling

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Exercise sheet 4, submit on Wednesday April 10th 2024 on TEAMS

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1. Probability Space, Conditional Probability [2 Points]

Answered in next page.

You have 2 fair coins. Derive the following probabilities, using probability theory.
a) Tossing the first one, you find "tail". What is then the probability to get "tail" when tossing the second one.
b) You toss 2 coins. When you know somehow that one of them shows "tail", what is the probability that the other one also shows "tail"?
Hint: First identify the Universe Ω and DVS S of this data generating scenario. Then, identify the reduced Universe Ω' with the respective condition, as defined in a) and b), Then use the definition of conditional probability.

2. Probability Space, Independence [2 Points]

Write down the probability space for a fair dice, with numbers $S = \{1, ..., 6\}$. Define A to be the subset of S with even numbers, only, B to be the subset of S with numbers smaller or equal than 4 and C the subset $C = \{2, 3\}$ Find out which of the corresponding events are independent and which are dependent.

3. Conditional Probability [2 Points]

Is the following statement correct? $P(X \in A, Y \in B, Z \in C, W \in D) = P(X \in A)P(Y \in B \mid X \in A)P(Z \in C \mid X \in A, Y \in B)P(W \in D \mid X \in A, Y \in B, Z \in C)$. Use the definition of conditional probability to check that.

4. Probability Space, Conditional Probability [2 Points]

A fair coin is tossed giving randomly $k = 1$ if it shows "head" and $k = 2$ if it shows "tail". Next, a fair dice is thrown exactly these k times.
Now, if you know that the dice shows 5 all these k times, but you do not know k. What is then the probability that the coin showed "head"?
Hint: Define the probability spaces, and the events described above, then use Bayes rule to calculate the conditional probability.

5. Probability Space, Conditional Probability [2 Points]

a) Three fair dice are rolled. If you find that two of them show a 3, what is then the probability that the remaining dice shows a 3, as well? b) Three fair dice are rolled. If you know that any two of them show a 3 (but you do not know which ones of the three dice), what is then the probability that the remaining dice shows a 3, as well, if you do not know which dice that is?
c) Now you take a single fair dice and roll it 2 times, recording each time the result, getting thereby a time series. If you find that all 2 times you recorded it you got a 3, what is then the probability that the next time the dice shows a 3, again?

Question 1:

a/ Since the coins are fair, the probability of the second toss to be tails is independent of the first, Therefore the probability remains at $1/2$

$\Omega : x \to S,\ S = \{H, T\}$ $\text{Prob} = \frac{\{T\}}{\{T, H\}} = \frac{1}{2}$

b/ $\Omega : x \longrightarrow S_i$ such that $S_i = \{H, T\}$

DV space for two coins $= \{HH, HT, TH, TT\}$

Knowing that one is tail, reduced universe $\Omega' = \{HT, TH, TT\}$

Therefore $P(t_2 = T \mid t_1 = T) = \dfrac{1/4}{3/4} = 1/3$

Question 2:

Sample Space $S = \{1, 2, 3, 4, 5, 6\}$

$A = \{2, 4, 6\}$ $B = \{1, 2, 3, 4\}$ $C = \{2, 3\}$

i/ Event A and B are independent because the definitions of the subsets do not impact each other. The outcome of A does not impact the outcome of B based on their definitions.

ii/ A and C are dependent because C is the subset of A. If C is rolled out and #2 is the result, this affects the known probability of A which contains even numbers and vice versa.

iii/. B and C are also dependent because C is a subset of B. If the value of C are rolled, we can determine the outcome of B. That is we can compute the probabilities for B.

Question 3:

Product Rule: $P(A \cap B) = P(A)\,P(B \mid A)$

Therefore: $P(\underbrace{\underbrace{\underbrace{X \in A, Y \in B}_{1}, Z \in C}_{2}, W \in D}_{3})$

first term $P(X \in A) \cdot P(Y \in B \mid X \in A)$

second term $P(X \in A) \cdot P(Y \in B \mid X \in A) \cdot P(Z \in C \mid X \in A, Y \in B)$

third term $\underline{P(X \in A) \cdot P(Y \in B \mid X \in A) \cdot P(Z \in C \mid X \in A, Y \in B) \cdot P(W \in D \mid X \in A, Y \in B, Z \in C)}$.

Question 4:

Defining the Events

A_1: The coin = Head $(k=1)$
A_2: The coin = Tail $(k=2)$
B: The dice shows 5 at all k.

Probabilities:

$P(A_1) = \frac{1}{2}$, $S = \{H, T\}$
$P(A_2) = \frac{1}{2}$, $S = \{H, T\}$

$P(B|A_1) = \frac{1}{6}$ for $k=1$
$P(B|A_2) = \frac{1}{6} \cdot \frac{1}{6}$ for $k=2$

Using Bayes Rule

$$P(A_1|B) = \frac{P(B|A_1) \cdot P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)}$$

$$P(A_1|B) = \frac{\frac{1}{6} \cdot \frac{1}{2}}{\left(\frac{1}{6} \cdot \frac{1}{2}\right) + \left(\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{2}\right)} = \frac{\frac{1}{12}}{\frac{1}{12} + \frac{1}{72}}$$

$$= \frac{\frac{1}{12}}{\frac{6}{72} + \frac{1}{72}} = \frac{\frac{1}{12}}{\frac{7}{72}} = \frac{1}{12} \times \frac{72}{7}$$

$$= \frac{\cancel{72}\,(6)}{\cancel{12} \times 7\,(1)} = \boxed{\frac{6}{7}}$$

Question 5:

a/. Since the dice is fair (ie independent) $P(d_3 = 3) = \frac{1}{6}$.

b/. Since the dice are fair, then the $P(d=3) = \frac{1}{6}$ remains the same.

c/. Again the dice remain fair, therefore $P(d_3 = 3) = \frac{1}{6}$.