Foundations of Statistical Modeling

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Exercise sheet 3, submit on Monday March 18th, 2024 on Teams
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1. Basic Operations on Sets [5 points]

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Let $S_1 = \{2,3,4\}$ and $S_2 = \{3,4,5\}$ which are both in the DVS $S = \{1,2,3,4,5,6\}$.
a) Find their union $S_1 \cup S_2$ and their intersection $S_1 \cap S_2$.
b) Find the union and the intersection of S_1^c and S_2^c. Check de Morgan's laws: 1. $(S_1 \cup S_2)^c = S_1^c \cap S_2^c$. 2. $(S_1 \cap S_2)^c = S_1^c \cup S_2^c$.
c) Define $S_a = S_1 \times S_2$ and $S_b = S_2 \times S_1$. Find the union and the intersection of S_a and S_b.

a/ $S_1 \cup S_2 = \{2,3,4,5\}$, $S_1 \cap S_2 = \{3,4\}$

b) $S_1^c = \{1,5,6\}$, $S_2^c = \{1,2,6\}$, therefore $S_1^c \cup S_2^c = \{1,2,5,6\}$, $S_1^c \cap S_2^c = \{1,6\}$

De Morgan's Law.

1. $(S_1 \cup S_2)^c = S_1^c \cap S_2^c$
$\{2,3,4,5\}^c = \{1,6\}$
$\boxed{\{1,6\} = \{1,6\}}$

2. $(S_1 \cap S_2)^2 = S_1^c \cup S_2^c$
$\{3,4\}^c = \{1,2,5,6\}$
$\boxed{\{1,2,5,6\} = \{1,2,5,6\}}$

c/ $S_a = S_1 \times S_2 = \{(2,3),(2,4),(2,5),(3,3),(3,4),(3,5),(4,3),(4,4),(4,5)\}$
$S_b = S_2 \times S_1 = \{(3,2),(3,3),(3,4),(4,2),(4,3),(4,4),(5,2),(5,3),(5,4)\}$
$\boxed{S_a \cup S_b = \{(2,3),(2,4),(2,5),(3,2),(3,3),(3,4),(3,5),(4,2),(4,3),(4,4),(4,5),(5,2),(5,3),(5,4)\}}$
$\boxed{S_a \cap S_b = \{(3,3),(3,4),(4,3),(4,4)\}}$

2. Sigma Fields [5 points]

Let $S = \{3,4,5,6\}$.
a) Give the power set $\mathbf{Pot}(S)$.
b) Consider the sets of sets $F_1 = \{\emptyset, S\}$ and $F_2 = \{\emptyset, \{3,4\}, \{5,6\}, S\}$ on S. Show that they are both sigma fields on S. Find the union and the intersection of the sigma fields F_1 and F_2.
c) Find two other sigma fields F_3 and F_4 on S such that their union $G = F_3 \cup F_4$ is not a sigma field. Check whether their intersection is then a sigma field.

a/. $Pot(S) = \{\phi, \{3\},\{4\},\{5\},\{6\},\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\},$
$\{4,5,6\},\{3,4,5,6\}\}$

b/ For F_1
(i) S is in F_1
(ii) $S^c = \phi$ is in F_1
(iii) $S \cup \phi$ are in F_1

For F_2
(i) S is in F_2
(ii) $S^c = \phi \in F_2$, $\{3,4\}^c = \{5,6\} \in F_2$, $\{5,6\}^c = \{3,4\} \in F_2$
(iii) $\phi \cup S \cup \{3,4\} \cup \{5,6\} \in F_2$

Sigma conditions

$F_1 \cup F_2 = \{\phi, \{3,4\}, \{5,6\}, S\}$
$F_1 \cap F_2 = \{\phi, S\}$

c/. $F_3 = \{\phi, \{3,4\}, \{5,6\}, S\}$
$F_4 = \{\phi, \{3,5\}, \{4,6\}, S\}$
$F_3 \cap F_4 = \{\phi, S\}$ - Yes.

$F_3 \cup F_4 = \{\phi, S, \{3,4\}, \{3,5\}, \{4,6\}, \{5,6\}\}$
↳ No

3. Generation of Sigma Fields and Borel Sigma Fields [5 P]

Let $S = [0,1]$ be the data value space.
a) Generate a sigma field on S from the set $G_1 = \{\emptyset, S, (a,b) \text{ with } a < b \in S\}$.
b) Generate a sigma field on S from the set $G_2 = \{\emptyset, S, (a,b), (c,d), \text{ with } a < b < c < d \in S\}$.
c) Compare $\sigma(G_1)$ and $\sigma(G_2)$ with the Borel σ-field on S.

a/ $\sigma(G_1) = \{\phi, \{1/4, 3/4\}, S\}$ where $a = 1/4$, $b = 3/4$, $a < b \in S$.

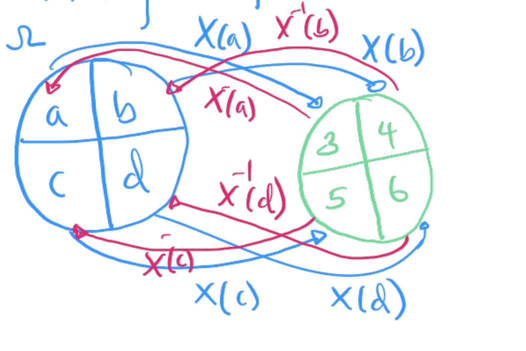
b/ $\sigma(G_2) = \{\phi, (0.2, 0.4), (0.6, 0.8), S\}$ where $a = 0.2$, $b = 0.4$, $c = 0.6$, $d = 0.8$.

c/ Both $\sigma(G_1)$ & $\sigma(G_2)$ are subsets of the Borel $B([0,1])$ which will contain all discrete and continuous values within the open domain $[0,1]$.

4. Sigma Fields and Measurable Spaces [2.5 P]

Let $\Omega = \{a,b,c,d\}$ be the universe and $S = \{3,4,5,6\}$ the data value space. Construct a non-trivial sigma field $\mathscr{F}$ on S ($\{0,S\}$ is trivial!) and a sigma field A on Ω and an RV-function $X : \Omega \to S$ such that X is $A - \mathscr{F}$- measurable.

Modelling the function $X: \Omega - S$



i/ Non-trivial sigma field

$\sigma\text{-}F = \{\phi, S, \{3\}, \{4,5,6\}\}$

ii/ $\sigma\text{-}A = \{\phi, S, \{a\}, \{a,b\}, \{c,d\}, \{b,c,d\}\}$

iii/ The RV function $X: \Omega \to S$ such that X is $A - F$ can be

$X: a \to 3, b \to 4, c \to 5, d \to 6$

Therefore, $X(a) = 3$, $X(b) = 4$, $X(c) = 5$, $X(d) = 6$.

Alternatively: $X: \Omega \to S$ where S is $\{X^{-1}(a), X^{-1}(b), X^{-1}(c), X^{-1}(d)\}$.

5. Sigma Fields and Measurable Functions [2,5 P]

Let $(S_1, \mathscr{F}_1)$, $(S_2, \mathscr{F}_2)$, $(S_3, \mathscr{F}_3)$ be measurable spaces. If $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$ are respectively $\mathscr{F}_1 - \mathscr{F}_2$ and $\mathscr{F}_2 - \mathscr{F}_3$-measurable functions, prove that $f_2 \circ f_1 : S_1 \to S_3$, where $f_2 \circ f_1(x) := f_2(f_1(x))$ is $\mathscr{F}_1 - \mathscr{F}_3$ measurable.

If $f_1 : S_1 \to S_2$ and $f_2 : S \to S_3$, then to show that $f_2 \circ f_1 : S_1 \to S_3$ where $f_2 \circ f_1(x) : f_2(f_1(x))$ is $F_1 - F_2$.

Suppose we have set $A \in S_1$, $B \in S_2$ and $C \in S_3$,

We know that, given $f_2 : S_2 \to S_3$, that we can define $B = f_2^{-1}(C)$.

We also know that given $f_1 : S_1 \to S_2$, that we can define $A = f_1^{-1}(B)$.

Since $B = f_2^{-1}(C)$ and $A = f_1^{-1}(B)$, then $A = f_1^{-1}(f_2^{-1}(C))$

We can then rewrite $A = f_1^{-1}(f_2^{-1}(C))$ as $(f_2 \circ f_1)^{-1}(C)$

Note that we have mapped set $C \in S_3(F_3)$ to set $A \in S_1(F_1)$ using the $(f_2 \circ f_1)^{-1}(C)$. This show that the function $f_2 \circ f_1(A) : S_1 \to S_3$.

Therefore $F_1 - F_3$ is measurable.