(1)

[BD] => ch → 2.2

Seperable equation: (Separable Method)

$$M(x)\,dx + N(y)\,dy = 0$$

This eqn is called seperable eqn.

$$\frac{dy}{dx} = f(x)g(y)$$

$$\frac{dy}{g(y)} = f(x)\,dx$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

Problem: (Example-1)

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

$$\Rightarrow x^2\,dx = (1-y^2)\,dy$$

$$\Rightarrow \frac{x^3}{3} = y - \frac{y^3}{3} + C \quad \text{[By integration]}$$

$$\therefore x^3 = 3y - y^3 + C'$$

[Ans.]

Problem: (Example-2)

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)} \; ; \; y(0) = -1$$

$$\Rightarrow (2y-2)\,dy = (3x^2+4x+2)\,dx$$

$$\Rightarrow 2\frac{y^2}{2} - 2y = 3\frac{x^3}{3} + 4\frac{x^2}{2} + 2x + C$$

$$\therefore y^2 - 2y = x^3 + 2x^2 + 2x + C$$

[Ans.]

$y(0) = -1$

$$\Rightarrow 0+0+0+C = (-1)^2 - 2(-1)$$

$$\therefore C = 3$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

[Ans.]

Problem: (Example-3) $\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$

$$\Rightarrow (4 + y^3)\,dy = (4x - x^3)\,dx$$

$$\Rightarrow 4y + \frac{y^4}{4} = 4 \cdot \frac{x^2}{2} - \frac{x^4}{4}$$

$$\Rightarrow y^4 + 16y = 8x^2 - x^4 + c$$

[Ans.]

Exercise: (2.2)

3) $y' + y^2 \sin x = 0$

$$\Rightarrow \frac{dy}{dx} + y^2 \sin x = 0$$

$$\Rightarrow \frac{dy}{dx} = -y^2 \sin x$$

$$\Rightarrow \frac{dy}{y^2} = -\sin x\, dx$$

$$\Rightarrow \frac{y^{-1}}{-1} = -(-\cos x) + c$$

$$\Rightarrow y^{-1} = -\cos x + c$$

$$\therefore y^{1} + \cos x = C$$

[Ans.]

5) $y' = (\cos^2 x)(\cos^2 2y)$

$\Rightarrow \frac{dy}{dx} = (\cos^2 x)(\cos^2 2y)$

$\Rightarrow \cos^2 x \, dx = \frac{dy}{\cos^2(2y)}$

$\Rightarrow (\cos x)^2 \, dx = \int \sec^2 2y \, dy$

$\Rightarrow \frac{1}{2}\int \cos 2x \, dx + \frac{1}{2}\int dx = \frac{1}{2}\tan(2y) + C$

$\cos(2x) = 2\cos^2 x - 1$

$\frac{\cos(2x)+1}{2} = \cos^2 x$

$2\cos^2 x = 1 + \cos 2x$

$\cos^2 x = \frac{1}{2}(1+$

$\Rightarrow \frac{1}{4}\sin 2x + \frac{x}{2} - \frac{1}{2}\tan(2y) + C = 0$

[Ans.]

7) $\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y}$

$\Rightarrow (y + e^y)\,dy = (x - e^{-x})\,dx$

$\Rightarrow y\,dy + e^y\,dy = x\,dx - e^{-x}\,dx$

$\Rightarrow \frac{y^2}{2} + e^y = \frac{x^2}{2} + e^{-x} + C$

[Ans]

3)

9) $y' = (1-2x)\,y^2 \quad ; \quad y(0) = -1/6$

Now, $\dfrac{dy}{dx} = (1-2x)\,y^2$

$\Rightarrow \dfrac{dy}{y^2} = (1-2x)\,dx$

$\Rightarrow -y^{-1} = x - 2\cdot\dfrac{x^2}{2} + C$

$\Rightarrow -\dfrac{1}{y} = x - x^2 + C$

Now, $y(0) = -1/6$

$\Rightarrow -\dfrac{1}{(-1/6)} = 0 + C$

$\Rightarrow 6 = 0 + C$

$\therefore C = 6$

$\therefore -y^2 = x - x^2 + 6$ [Ans.]

13) $\dfrac{dy}{dx} = \dfrac{2x}{y + x^2 y} \quad ; \quad y(0) = -2$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2x}{y(x^2+1)}$

$\Rightarrow y\,dy = \dfrac{2x\,dx}{x^2+1}$

$$\Rightarrow \frac{y^2}{2} = \ln(x^2+1) + c$$

$$\int \frac{f'(x)}{f(x)} dx = \ln(f(x))$$

Now, $\frac{(-2)^2}{2} = \ln(0+1) + c$

$$\Rightarrow 2 = c$$

$$\therefore \frac{y^2}{2} = \ln(x^2+1) + 2$$

[Ans.]

17) $y' = \frac{3x^2 - e^x}{2y-5}$; $y(0) = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2 - e^x}{2y-5} ;$$

$$\Rightarrow (2y-5)\,dy = (3x^2 - e^x)\,dx$$

$$\Rightarrow 2.\frac{y^2}{2} - 5y = 3.\frac{x^3}{3} - e^x + c$$

$$\Rightarrow y^2 - 5y = x^3 - e^x + c$$

Now, $1 - 5 = 0 - e^0 + c$

$$\Rightarrow -4 = -1 + c$$

$$\therefore c = -3$$

$\therefore y^2 - 5y = x^3 - e^x - 3$ [Ans.]

19) $\sin 2x \, dx + \cos 3y \, dy = 0 \; ; \; y(\pi/2) =$

$$\Rightarrow -\frac{\cos 2x}{2} + \frac{\sin 3y}{3} = 0$$

$$\therefore \frac{1}{3}\sin(3y) - \frac{1}{2}\cos(2x) \cancel{=0} + C = 0$$

$y(\pi/2) = \pi/3$

$$\Rightarrow \frac{1}{3}\sin(3 \times \pi/3) - \frac{1}{2}\cos(2 \times \pi/2) \cancel{=0} + C = 0$$

$$\Rightarrow \frac{1}{3}\sin(\pi) - \frac{1}{2}\cos(\pi) \cancel{=0} + C = 0$$

$$\Rightarrow -\frac{1}{2}\cos(180) \cancel{=0} + C = 0$$

$$\therefore C = -1/2$$

$$\therefore \frac{1}{3}\sin(3y) - 1/2 \cos(2x) - 1/2 = 0$$

[Ans]

[BD] Ch → 2.6 (when equation is neither Linear nor separable)

Exact equation and Integrating Factor:

$M(x,y)\,dx + N(x,y)\,dy = 0$

* First we've to identify whether it is exact / not.

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then it will be exact equation.

Steps:
1) Find the form
2) See whether it is exact / not.
3) Find the function

(Example → 02)

Problem: Solve the DE.

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$$

$$\Rightarrow (y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)\frac{dy}{dx} = 0$$

Now, $y\cos x + 2xe^y = -(\sin x + x^2e^y - 1)\frac{dy}{dx}$

$$\Rightarrow (y\cos x + 2xe^y)\,dx + (\sin x + x^2e^y - 1)\,dy = 0$$

Then, $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[y\cos x + 2xe^y]$

$= \cos x + 2xe^y$

and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[\sin x + x^2e^y - 1]$

$= \cos x + 2xe^y$

So, exact.

$\int M\,dx$ $\int N\,dy$

Now,

$$\int M\,dx = \int \left(y\cos x + 2xe^y\right) dx$$

$$= \int y\cos x\,dx + 2\int x e^y\,dx$$

$$= y\sin x + 2\frac{x^2}{2}e^y + c$$

$$= y\sin x + x^2e^y + c \checkmark$$

and $\int N\,dy = \int \left(\sin x + x^2e^y - 1\right) dy$

$$= y\sin x + x^2e^y - y + c.$$

Hence, The solution is:

$$y\sin x + x^2e^y - y + c = 0$$

[Ans.]

Book method: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ [Exact]

to Find out

There is a function $\psi(x,y)$ such that

$$\psi_x(x,y) = y\cos x + 2xe^y \longrightarrow (1)$$

holding y constant

$$\psi_y(x,y) = \sin x + x^2e^y - 1 \checkmark$$

Integrating (1) w.r.t x, $y\sin x + x^2e^y + h(y) = \psi(x,y)$

or (2) ← Derivative, ψ. with r. to y, $\psi_y(x,y) = \sin x + x^2e^y + h'(y)$

Comparing, $h'(y) = -1$ $\therefore h(y) = -y$ *

* Therefore, $\psi(x,y) = y\sin x + x^2e^y - y$

$\therefore y\sin x + x^2e^y - y = C$ [Ans.]

(Example → 01)

Problem: Solve the D.E.

$$2x + y^2 + 2xyy' = 0$$

Soln: $2x + y^2 + 2xy\, y' = 0$

Now, $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[2x + y^2]$

$= 2y$

and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[2xy]$

$= 2y$

Now, $\int M\, d\cancel{y}x = \int (2x + y^2)\, dx$

$= 2 \cdot \frac{x^2}{2} + y^2 x$

$= x^2 + y^2 x + C$

and $\int N\, dy = \int 2xy\, dy$

$= 2x \cdot \frac{y^2}{2} + C$

$= y^2 x + C$

Therefore, $x^2 + y^2 x + C = 0$

Degree $= 3$

Exercise: (2.6)

Problems: (1) $(2x+3)+(2y-2)y'=0$

Now, $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[2x+3]$

$= 0$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[2y-2]$

$= 0$

Then, $\int M\,dx = \int (2x+3)\,dx$

$= 2\cdot\frac{x^2}{2} + 3x + c$

$= x^2+3x+c$

and $\int N\,dy = \int (2y-2)\,dy$

$= 2\cdot\frac{y^2}{2} - 2y + c$

$= y^2-2y+c$

$\therefore$ The soln is $x^2+3x+y^2-2y=c$

[Ans.]

Problem: (3) $(3x^2 - 2xy + 2)dx + (6y^2 - x^2 + 3)dy = 0$

Soln: $\quad \dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[3x^2 - 2xy + 2\right]$

$= -2x$

$\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[6y^2 - x^2 + 3\right]$

$= -2x$

Then, $\displaystyle\int M\,dx = \int (3x^2 - 2xy + 2)\,dx$

$= 3\dfrac{x^3}{3} - 2y \cdot \dfrac{x^2}{2} + 2x + C$

$= x^3 - x^2y + 2x + C$

and $\displaystyle\int N\,dy = \int (6y^2 - x^2 + 3)\,dy$

$= 6 \cdot \dfrac{y^3}{3} - x^2y + 3y + C$

$= 2y^3 - x^2y + 3y + C$

$\therefore \quad x^3 - x^2y + 2x + 2y^3 + 3y = C$

[Ans.]

(7) $(e^x \sin y - 2y\sin x)\,dx + (e^x \cos y + 2\cos x)\,dy = 0$

Soln:
$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[e^x \sin y - 2y\sin x\right]$$

$$= e^x \cos y - 2\sin x.$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[e^x \cos y + 2\cos x\right]$$

$$= e^x \cos y - 2\sin x$$

Now, $\int M\,dx = \int (e^x \sin y - 2y\sin x)\,dx$

$$= e^x \sin y + 2y\cos x + c$$

and $\int N\,dy = \int (e^x \cos y + 2\cos x)\,dy$

$$= e^x \sin y + 2y\cos x + c$$

$\therefore\ e^x \sin y + 2y\cos x = c$

[Ans.]

9) $\left(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x\right)dx + (xe^{xy}\cos 2x - 3)\,dy = 0$

Soln:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x\right]$$

$$= \cos 2x\{ye^{xy}.x + e^{xy}\} - 2\sin 2x.e^{xy}.x$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[xe^{xy}\cos 2x - 3\right]$$

$$= \cos 2x.\{e^{xy} + xe^{xy}.y\} + xe^{xy}(-2\sin 2x)$$

Now,

$$\int M\,dx = \int(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)\,dx$$

$$= y\int e^{xy}\cos 2x\,dx - 2\int e^{xy}\sin 2x\,dx + 2\int x\,dx$$

$$= y\left[\cos 2x.\frac{e^{xy}}{y} - \int\left\{\frac{d}{dx}(\cos 2x)\int e^{xy}dx\right\}dx\right]$$

$$- 2\left[\sin 2x.\frac{e^{xy}}{y} - \int\left\{\frac{d}{dx}(\sin 2x)\int e^{xy}dx\right\}dx\right] + x^2 + c$$

$$= \cancel{e^{xy}\cos 2x + \frac{2y\sin 2x}{2} - e^{xy} - \frac{2}{y}\sin 2x . e^{xy} + 4\cos 2x + \frac{2e^{xy}}{y}}$$

$$= e^{xy}\cos 2x + y\int 2\sin 2x \cdot e^{xy}\, dx - \frac{2}{y}\sin 2x\, e^{xy} + \int 4\cos 2x \cdot \frac{2e^{xy}}{y}\, dx + C$$

$$= e^{xy}\cos 2x + 2y\int \sin 2x\, e^{xy}\, dx - \frac{2}{y}\sin 2x . e^{xy} + \frac{4}{y}\int \cos 2x\, e^{xy} dx$$

$$x^2 + C$$

Now,

Let, $\int e^{xy}\cos 2x\, dx = I$

Then, $I = e^{xy}\cos 2x + 2y \left[\sin 2x \cdot \frac{e^{xy}}{y} - \int 2\cos 2x \cdot \frac{e^{xy}}{y}\, dx\right]$

$$= e^{xy}\cos 2x + 2\cancel{y}\sin 2x . e^{xy} - \cancel{4y\cos 2x\, 2e^{xy}}\ 4\int e^{xy}\cos 2x\, dx$$

$$\Rightarrow 5I = e^{xy}\cos 2x + 2e^{xy}\sin 2x + C$$

similarly,

Find $\int N \, dy = T]$

[Ans.]

(13) $(2x-y)\,dx + (2y-x)\,dy = 0$; y (1)

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2x-y) = -1$

$\frac{\partial N}{\partial x} = -1$

Then, $\int M\,dx = \int (2x-y)\,dx$

$= 2\cdot\frac{x^2}{2} - yx + c$

$= x^2 - yx + c$

and $\int N\,dy = \int (2y-x)\,dy$

$= 2\cdot\frac{y^2}{2} - xy + c$

$= y^2 - xy + c$

The solⁿ is : $x^2 + y^2 - xy + c = c$

[Ans.]