Class-04 Date: 03.10.2023

$\boxed{\text{ch: 13.5}}$

... the derivative of a multivariable composite function. This is called Partial derivative chain rule.

$\boxed{*}$ The chain Rule: (13.5.1) (chain Rule for derivative)

If $x = x(t)$ and $y = y(t)$ are differentiable at t and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$ is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$z = f(x, y)$ Two variable function

one variable function

Ordinary derivatives are evaluated at t and Partial derivatives are evaluated at x.

similarly,

$x = x(t)$, $y = y(t)$ and $z = z(t)$

Then,
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$\begin{tikzcd}
 & w \arrow[dl, "\frac{\partial w}{\partial x}"'] \arrow[d, "\frac{\partial w}{\partial y}"] \arrow[dr, "\frac{\partial w}{\partial z}"] & \\
x \arrow[d, "\frac{dx}{dt}"'] & y \arrow[d, "\frac{dy}{dt}"] & z \arrow[d, "\frac{dz}{dt}"] \\
t & t & t
\end{tikzcd}$$

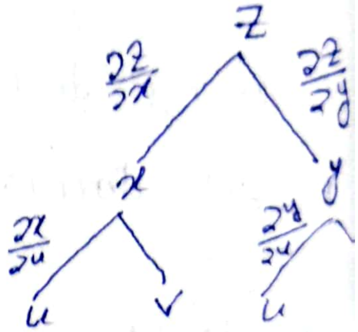
$\frac{dw}{dt}$

$$\begin{tikzcd}
 & z \arrow[dl, "\frac{\partial z}{\partial x}"'] \arrow[dr, "\frac{\partial z}{\partial y}"] & \\
x \arrow[d, "\frac{dx}{dt}"'] & & y \arrow[d, "\frac{dy}{dt}"] \\
t & & t
\end{tikzcd}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Theorem (13.5.2): (Chain Rule for partial de

If $x = x(u,v)$ and $y = y(u,v)$ have
order partial derivatives at the point (u
and if $z = f(x,y)$ is differentiable at
point $(x,y) = (x(u,v), y(u,v))$ then, $z =$
$(x(u,v), y(u,v))$ has first order derivati
at the point (u,v) given by,



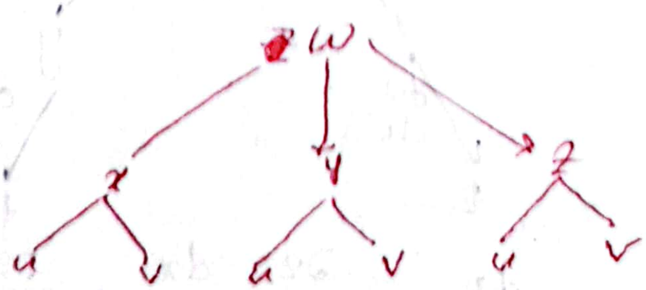
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

similarly,

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$



Explicit $y=f(x)$ $f(x,y)=0$

$f(x,y)=0$ [general form of implicit function]

(*) implicit function contains two or more than two variables.

[*] Theorem (13.5.3):

Implicit differentiation (differentiate each side of an equation with two variables by treating one of the variables as a function of the other)

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

[*] $0 = f(x,y)$

$\frac{d}{dx}$: $0 = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$

$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$

f: x, y

If the equation $f(x,y,z) = c$ defines z implicitly,

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$$

Problem: Suppose that,

$$z = x^2 y \;, \quad x = t^2 \quad \text{and} \quad y = t^3$$

Use chain rule to find $\frac{dz}{dt}$.

Express z as a function of t.

(Soln:)

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= 2xy\cdot 2t + x^2\cdot 3t^2$$

$$= 4xyt + 3x^2t^2$$

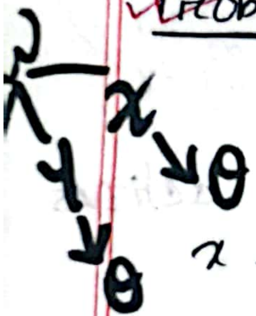
$$\frac{dz}{dt} = 4t^2 \cdot t^3 \cdot t + 3(t^2)^2 \cdot t^2$$

$$= 4t^6 + 3t^6$$

$$= 7t^6$$

[Ans.]

<u>**Problem:**</u> Suppose that,



$$w = \sqrt{x^2+y^2+z^2}$$

$x = \cos\theta$, $y = \sin\theta$ and $z = \tan\theta$.

Find $\frac{dw}{d\theta}$ when $\theta = \pi/4$

<u>soln:</u>

$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\cdot\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\cdot\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\cdot\frac{dz}{d\theta}$$

$x = \cos(\pi/4) = 1/\sqrt{2}$

$y = \sin(\pi/4) = 1/\sqrt{2}$

$z = \tan(\pi/4) = 1$

$$= \frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x \cdot (-\sin\theta) + \frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2y \cdot \cos\theta + \frac{1}{2\sqrt{x^2+y^2+z^2}}\, 2z \cdot \sec^2\theta$$

Now, $\left.\frac{dw}{d\theta}\right|_{\theta=\pi/4} = \sqrt{2}$

[Ans.]

$\boxed{\text{Exercise set : 13-5}}$

(1-10, 17-34, 41-44, 50-54)

Problem: (2) Suppose, $z = \ln(2x^2 + y)$;

$x = \sqrt{t}$; $y = t^{2/3}$

Find $\dfrac{dz}{dt}$

$$\begin{tikzcd} & x \arrow[dr] & \\ z \arrow[ur] \arrow[dr] & & t \\ & y \arrow[ur] & \end{tikzcd}$$

Soln:
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1}{2x^2+y}\cdot 4x \cdot \frac{1}{2\sqrt{t}} + \frac{1}{2x^2+y}\cdot \frac{2}{3}t^{-1/3}$$

$$= \frac{1}{2x^2+y}\left[\frac{2x}{\sqrt{t}} + \frac{2}{3}t^{-1/3}\right]$$

$$= \frac{1}{2(\sqrt{t})^2 + t^{2/3}}\left[2 + \frac{2}{3}t^{-1/3}\right]$$

$$= \frac{1}{2t + t^{2/3}}\left[2 + \frac{2}{3}t^{-1/3}\right]$$

$$= \frac{1}{2t + t^{2/3}} \times \frac{6 + 2t^{-1/3}}{3}$$

$$= \frac{2(3 + t^{-1/3})}{3(2t + t^{2/3})}$$ [Ans.]

Problem: (3) Given, $z = 3\cos x - \sin(xy)$

$x = \frac{1}{t}$; $y = 3t$

Solⁿ:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= \{-3\sin x - \cos(xy) \cdot y\} \cdot -t^{2} \cdot \{-\cos(xy) \cdot x\} \cdot 3$$

$$= \{-3\sin\left(\tfrac{1}{t}\right) - 3t\cos\left(\tfrac{1}{t}\times 3t\right) \cdot\} \cdot -\frac{3}{t}\cos\left(3t\times 1/t\right)$$

$$= -\frac{1}{t^2}\{-3\sin(1/t) \cancel{-3t} \cos(3) - \frac{3}{t}\cos(3)$$

$$= \frac{3}{t^2}\sin(1/t) + \cancel{\frac{3}{t}\cos(3) - \frac{3}{t}\cos}$$

$$= 3t^2\sin(1/t)$$

[Ans.]

Problem: (21) $z = \sqrt{1+x-2xy^4}$

$x = \ln t \quad ; \quad y = t$

Soln: $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$$= \frac{1}{2\sqrt{1+x-2xy^4}} \cdot \frac{1}{t} (1-2y^4) +$$

$$\frac{1}{2\sqrt{1+x-2xy^4}} \cdot (-8xy^3)$$

$$= \frac{1}{2\sqrt{1+x-2xy^4}} \left\{ \frac{1}{t} - \frac{2y^4}{t} - 8xy^3 \right\}$$

$$= \frac{1}{2\sqrt{1+\ln t - 2\ln t \cdot t^4}} \left\{ \frac{1}{t} - 2t^3 - 8\ln t \cdot t^3 \right\}$$

$$= \frac{1 - 2t^4 - 8t^4 \ln t}{2t\sqrt{1+\ln t - 2t^4 \ln t}}$$

[Ans.]

Problem: (8) Given, $w = \ln(3x^2 - 2y + 4z^3)$

$x = t^{1/2}$; $y = t^{2/3}$; $z = t^{-2}$

Soln:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

$$= \frac{1}{3x^2 - 2y + 4z^3} \cdot 6x \cdot \frac{1}{2} t^{-1/2} + \frac{1}{3x^2 - 2y + 4z^3} \cdot (-2) \cdot \frac{2}{3} t^{-1/3} + \frac{1}{3x^2 - 2y + 4z^3} \cdot 12z^2 \cdot (-2) t^{-3}$$

$$= \frac{1}{3x^2 - 2y + 4z^3} \left[3x t^{-1/2} - \frac{4}{3} t^{-1/3} - 24 z^2 t^{-3} \right]$$

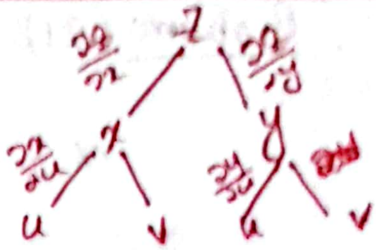
$$= \frac{1}{3t - 2t^{2/3} + 4t^{-6}} \left[3 - \frac{4}{3} t^{-1/3} - 24 t^{-4} t^{-3} \right]$$

$$= \frac{9 - 4t^{-1/3} - 72t^{-7}}{3\left(3t - 2t^{2/3} + 4t^{-6}\right)}$$ [Ans.]

Problem: (18) Given,

$$z = x^2 - y\tan x$$

$$x = \frac{u}{v} \quad ; \quad y = u^2 v^2$$



Soln:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= (2x - y\sec^2 x) \cdot \frac{1}{v} + (-\tan x) \cdot 2uv^2$$

$$= \left\{\frac{2u}{v} - y\sec^2\left(\frac{u}{v}\right)\right\}\frac{1}{v} - \tan\left(\frac{u}{v}\right) \cdot 2uv^2$$

$$= \frac{2u}{v^2} - u^2 v \sec^2 \frac{u}{v} - 2\tan\left(\frac{u}{v}\right) uv^2$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= -(2x - y\sec^2 x) \cdot uv^{-2} + (-\tan x) \cdot 2u^2 v$$

$$= -\left(2\frac{u}{v} - u^2v^2\sec^2\frac{u}{v}\right)uv^{-2} - \tan\left(\frac{u}{v}\right) \cdot 2u^2 v$$

$$= -\frac{2u^2}{v^3} + u^3\sec^2\left(\frac{u}{v}\right) - 2\tan\left(\frac{u}{v}\right) \cdot u^2 v$$

[Ans.]

Problem: (21) Given, $z = e^{x^2 y}$.

$x = \sqrt{uv}$; $y = \frac{1}{v}$

Sol^n:
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= e^{x^2 y}.2xy\cdot\frac{1}{2\sqrt{uv}}\cdot v + e^{x^2 y}.$$

$$= 2xy e^{x^2 y}\frac{1}{2\sqrt{uv}}\cdot v$$

$$= \cancel{\sqrt{uv}}\cdot\frac{1}{v}e^{uv.1/v}\cdot\frac{1}{\cancel{\sqrt{uv}}}$$

$$= \frac{v}{v}e^{u} = e^{u}. \text{[Ans.]}$$

Problem: (23) $T = x^2 y - xy^3 + 2$

$x = r\cos\theta$; $y = r\sin\theta$

Sol^n:
$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= (2xy - y^3)\cos\theta + (x^2 - 3xy^2)\sin\theta$$

$$= (2r^2\sin\theta\cos\theta - r^3\sin^3\theta)\cos\theta + (r^2\cos^2\theta$$
$$3r^3\cos\theta\sin^2\theta)\sin\theta$$

$$\frac{\partial T}{\partial r} = 2r^2 \sin\theta \cos^2\theta - r^3 \sin^3\theta \cos\theta + r^2 \cos^2\theta \sin\theta - 3r^3 \cos\theta \sin^3\theta$$

$$= 3r^2 \cos^2\theta \sin\theta - 4r^3 \sin^3\theta \cos\theta$$

$$= r^2 \sin\theta \cos\theta \left(3\cos\theta - 4r \sin^2\theta\right)$$

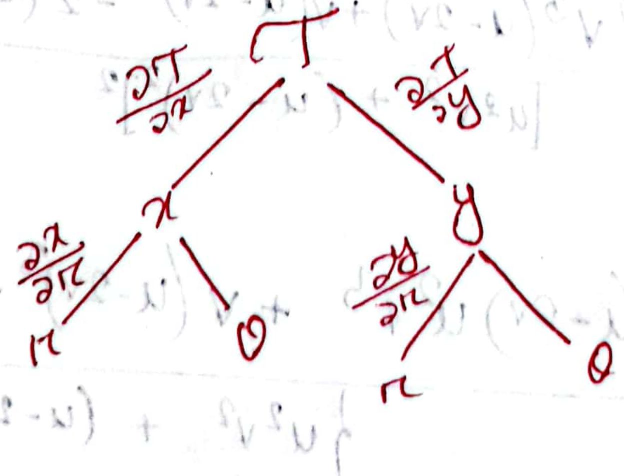
$$\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= -(2xy - y^3) \cdot r\sin\theta + (x^2 - 3xy^2) \cdot r\cos\theta$$

$$= -(2r^2 \sin\theta \cos\theta - r^3 \sin^3\theta) r\sin\theta + (r^2\cos^2\theta - 3r^3 \cos\theta \sin^2\theta)\, r\cos\theta$$

$$= -2r^3 \sin^2\theta \cos\theta + r^4 \sin^4\theta + r^3 \cos^3\theta - 3r^4 \cos^2\theta \sin^2\theta$$

[Ans.]



Problem: (26) Let, $w = \dfrac{rs}{r^2+s^2}$

$r = uv \quad ; \quad s = u - 2v$

soln:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial u} + \frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial u}$$

$$= \frac{(r^2+s^2)\,s - rs.2r}{(r^2+s^2)^2}\cdot v + \frac{(r^2+s^2)\cdot r - rs.2s}{(r^2+s^2)^2}\cdot 1$$

$$= \frac{(r^2 s + s^3 - 2sr^2)v + r^3 + s^2 r - 2s^2 r}{(r^2+s^2)^2}$$

$$= \frac{u^2v^3(u-2v) + (u-2v)^3 v - 2(u-2v)\cdot u^2v^3 + u^3v^3 + (u-2v)^2 uv - 2(u-2v)^2 uv}{(u^2v^2 + u^2 - 2uv + 4v^2)^2}$$

$$= \frac{u^2v^3(u-2v) + v(u-2v)^3 - 2(u-2v)u^2v^3 + u^3v^3 - (u-2v)^2 uv}{[u^2v^2 + (u-2v)^2]^2}$$

$$= \frac{-(u-2v)u^2v^3 + v(u-2v)^3 - (u-2v)^2 uv + u^3v^3}{\{u^2v^2 + (u-2v)^2\}^2}$$

Problem: Given that, $z = e^{xy}$

$x = 2u + v$ and $y = u/v$

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Soln:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= e^{xy} \cdot y \cdot 2 + e^{xy} \cdot x \cdot 1/v$$

$$= 2ye^{xy} + \frac{x}{v} e^{xy}$$

$$= \left[2y + \frac{x}{v}\right] e^{xy}$$

$$= \left(2y + \frac{2u+v}{v}\right) e^{(2u+v)\cdot\left(\frac{u}{v}\right)}$$

$$= \left(2u/v + \frac{2u}{v} + 1\right) e^{(2u+v)\cdot\left(\frac{u}{v}\right)}$$

$$\frac{\partial z}{\partial u} = \left(\frac{4u}{v} + 1\right) e^{(2u+v)\left(u/v\right)}$$

Next,
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= ye^{xy} + xe^{xy} \cdot u\left(\frac{-1}{v^2}\right)$$

$$= y e^{xy} - ux e^{xy} \frac{1}{v^2}$$

$$= e^{xy}\left[y - \frac{xu}{v^2}\right]$$

$$= e^{(2u+v)\left(\frac{u}{v}\right)}\left[\frac{u}{v} - \frac{(2u+v)u}{v^2}\right]$$

$$= e^{(2u+v)\left(\frac{u}{v}\right)}\left[\frac{u}{v} - \frac{2u^2}{v^2} - \frac{uv}{v^2}\right]$$

$$= e^{(2u+v)\left(\frac{u}{v}\right)}\left[\frac{u}{v} - \frac{2u^2}{v^2} - \frac{u}{v}\right]$$

$$= -\frac{2u^2}{v^2}\, e^{(2u+v)\frac{u}{v}}$$

[Ans.]

Problem: $w = x^2 + y^2 - z^2$

$x = \rho \sin\phi \cos\theta$; $y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$. Find $\frac{\partial w}{\partial \rho}$ and $\frac{\partial w}{\partial \theta}$.

Soln:

$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \rho}$$

$$= 2x \cdot \sin\phi\cos\theta + 2y \cdot \sin\phi \sin\theta - 2z \cdot \cos\phi$$

$[x = \rho \sin\phi \cos\theta]$ $[y = \rho \sin\theta \sin\phi]$ $z = \rho\cos\phi$

$$= 2\rho \sin^2\phi \cos^2\theta + 2\rho \sin^2\phi \sin^2\theta - 2\rho\cos^2\phi$$

$$= 2\rho \sin^2\phi (\cos^2\theta + \sin^2\theta) - 2\rho\cos^2\phi$$

$$= 2\rho (\sin^2\phi - \cos^2\phi) = -2\rho(\cos^2\phi - \sin^2\phi)$$

$$= -2\rho\cos 2\phi \qquad [\because -\sin^2\theta + \cos^2\theta = \cos 2\theta]$$

And

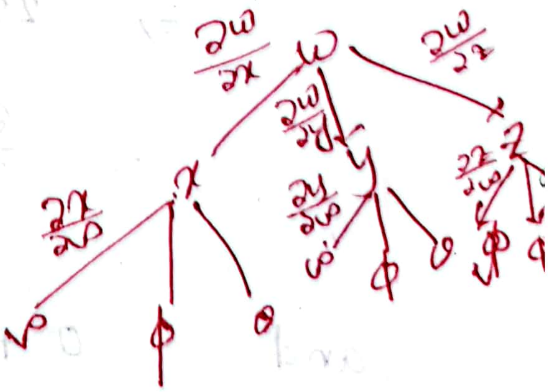
$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \theta}$$

$$= -2x \cdot \rho \sin\phi \sin\theta + 2y \cdot \rho \sin\phi \cos\theta - 2z \cdot 0$$

$$= -2\rho^2 \sin^2\phi \sin\theta\cos\theta + 2\rho^2 \sin^2\phi \sin\theta\cos\theta$$

$$\frac{\partial w}{\partial \theta} = 0$$

[Ans.]



$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-3x^2+y^2}{0+2yx}$$

$$\frac{-(3x^2+y^2)}{2yx}$$

Problem: Given, $x^3 + y^2x - 3 = 0$

Find $\frac{dy}{dx}$.

Soln: $x^3 + y^2x - 3 = 0$

$$\Rightarrow 3x^2 + y^2 + x\,2y\frac{dy}{dx} - 0 = 0$$

$$\Rightarrow 3x^2 + y^2 = -x\,2y\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{3x^2+y^2}{-2xy}$$

[Ans.]

$$\frac{d}{dx}\left[x^3 + y^2x - 3\right] = \frac{d}{dx}(0)$$

$$\Rightarrow 3x^2 + \frac{d}{dx}(y^2x) - \frac{d}{dx}(3) = 0$$

$$y^2\frac{d}{dx}(x) + \frac{d(y^2x)}{dy}\cdot\frac{dy}{dx}$$

$$= y^2\cdot 1 + 2y\cdot x\cdot\frac{dy}{dx}$$

Important

Problem: $x^2 + y^2 + z^2 = 1 \rightarrow F(x,y,z) = C$

$$\Rightarrow 2x + \cancel{2y\frac{dy}{dx}}\,0 + 2z\left(\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow 2x = -2z\frac{\partial z}{\partial x}$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{2x}{2z} = -x/z$$

and $0 + 2y\,\cancel{\frac{dy}{dy}} + 2z\left(\frac{\partial z}{\partial y}\right) = 0$

$$\Rightarrow \frac{\partial z}{\partial y} = -\frac{2y}{2z} = -\frac{y}{z}$$

[Ans.]

Problem: Let, $w = r^2 - r\tan\theta$

$r = \sqrt{s}$; $\theta = \pi s$

Soln:

$$\frac{dw}{ds} = \frac{\partial w}{\partial r} \cdot \frac{dr}{ds} + \frac{\partial w}{\partial \theta} \cdot \frac{d\theta}{ds}$$

$$= (2r - \tan\theta) \cdot \frac{1}{2\sqrt{s}} + (-r\sec^2\theta) \cdot \pi$$

$$= \left\{2\sqrt{s} - \tan(\pi s)\right\} \frac{1}{2\sqrt{s}} + -\sqrt{s}\sec^2(\pi s) \cdot \pi$$

$\frac{\partial w}{\partial \theta}$ $\theta \xrightarrow{\frac{d\theta}{ds}} s$

$$= 1 - \tan(\pi/4) \cdot \frac{1}{2\sqrt{1/4}} - \pi\sqrt{1/4}\,\sec^2(\pi/4)$$

$$= 1 - \frac{1}{2 \cdot 1/2} - \left(\pi \cdot \frac{1}{2}\right)\sec^2 \pi/4$$

$$= 1 - 1 - \frac{\pi}{2} \cdot 2$$

$$= -\pi$$

[Ans.]

Problem: (41) Given,

$$x^2y^3 + \cos y = 0$$

Now, $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

$= -\frac{2xy^3}{3x^2y^2 - \sin y}$

$= \frac{2xy^3}{-3x^2y^2 + \sin y}$

$= 2xy^3 + x^2 \cdot 3y^2 \cdot \frac{dy}{dx} - \sin y \frac{dy}{dx} = 0$

$\Rightarrow 2xy^3 + \frac{dy}{dx}(x^2 3y^2 - \sin y)$

$\Rightarrow \frac{dy}{dx} = \frac{-2xy^3}{3x^2y^2 - \sin y}$

Using Implicit differentiation,

$$2xy^3 + 3x^2y^2\frac{dy}{dx} - \sin y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(3x^2y^2 - \sin y) = -2xy^3$$

$$\therefore dy/dx = \frac{2xy^3}{\sin y - 3x^2y^2}$$

[checked]

$\frac{d}{dx}(x^2y^3)$

$= y^3 \cdot 2x + x^2 \cdot \frac{d}{dx}(y^3)$

$= y^3 2x + x^2 \cdot \frac{d(y^3)}{dy} \frac{dy}{dx}$

Problem:

Soln:

Problem: (43) $e^{xy} + ye^{y} = 1$

$$\Rightarrow e^{xy} + ye^{y} - 1 = 0$$

Soln: $$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{e^{xy}\cdot y + 0}{e^{xy}\cdot x + ye^{y} + e^{y}}$$

$$= \frac{-ye^{xy}}{xe^{xy} + e^{y}(y+1)}$$

Using Implicit differentiation,

$$\cancel{ye^{xy}\cdot\frac{dy}{dx} + ye^{y}\cdot\frac{dy}{dx} + e^{y} - 0 = 0}$$

$$\Rightarrow \cancel{\frac{dy}{dx}\left[e^{xy} + ye^{y}\right]}$$

$$\Rightarrow \cancel{\frac{dy}{dx} =}$$

$$e^{xy}\cdot\left(y + x\,\frac{dy}{dx}\right) + e^{y}\frac{dy}{dx} + ye^{y}\frac{dy}{dx} = 0$$

$$\Rightarrow ye^{xy} + \frac{dy}{dx}\left[e^{y} + ye^{y}\right] + xe^{xy}\frac{dy}{dx} =$$

$$\therefore \frac{dy}{dx} = \frac{-ye^{xy}}{xe^{xy} + e^{y}(y+1)}$$

[Ans.]

Problem: (50) Suppose, $z = f(u)$; $u = g(x, y)$

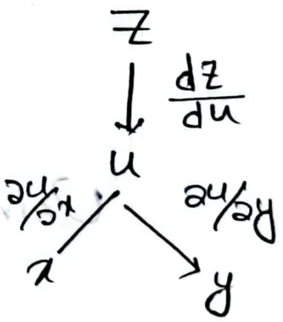
Draw a tree diagram, use it to construct chain rules that express $\partial z/\partial x$, $\partial z/\partial y$ in terms of dz/du, $\partial u/\partial x$, $\partial u/\partial y$

Let, $z = f(x^2 - y^2)$.

show that $y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = 0$

Soln:

$\cancel{\frac{\partial z}{\partial x}} =$



$z = f(u)$

and $u = (x^2 - y^2)$

Now, $\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$

$= 2x \frac{dz}{du}$

$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$

$= -2y \frac{dz}{du}$

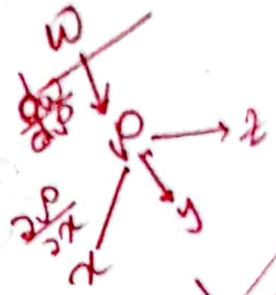
Now,

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$$

$$= y\,2x\frac{dz}{du} - x\,2y\frac{dz}{du}$$

$$= 0$$

[proved]



$$\frac{1}{2\sqrt{x^2+y^2+z^2}} = \frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}$$

Problem: (54) $w = f(\rho)$; $\rho = (x^2+y^2+z^2)^{1/2}$

Soln:

$$\frac{\partial w}{\partial x} = \frac{dw}{d\rho}\cdot\frac{\partial \rho}{\partial x}$$

$$= \frac{dw}{d\rho}\cdot\frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}\cdot 2x$$

$$= x\frac{dw}{d\rho}\left(x^2+y^2+z^2\right)^{-1/2}$$

$$= x\frac{dw}{d\rho}\cdot\frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x}{\rho}\cdot\frac{dw}{d\rho}$$

$$\frac{\partial w}{\partial y} = \frac{dw}{d\rho}\cdot\frac{\partial \rho}{\partial y}$$

$$= y\left(x^2+y^2+z^2\right)^{-1/2}\frac{dw}{d\rho} = \frac{y}{\rho}\frac{dw}{d\rho}$$

$$\frac{\partial w}{\partial z} = \frac{dw}{d\rho}\cdot\frac{\partial \rho}{\partial z}$$

$$= z\left(x^2+y^2+z^2\right)^{-1/2}\cdot\frac{dw}{d\rho} = \frac{z}{\rho}\frac{dw}{d\rho}$$

Now,

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2$$

$$= \frac{x^2}{\rho^2}\left(\frac{dw}{d\rho}\right)^2 + \frac{y^2}{\rho^2}\left(\frac{dw}{d\rho}\right)^2 + \frac{z^2}{\rho^2}\left(\frac{dw}{d\rho}\right)^2$$

$$= \left(\frac{dw}{d\rho}\right)^2 \left\{\frac{x^2+y^2+z^2}{\rho^2}\right\}$$

$$= \left(\frac{dw}{d\rho}\right)^2 \frac{\rho^2}{\rho^2} \quad \left[\rho = \left(x^2+y^2+z^2\right)^{1/2}\right]$$

$$= \left(\frac{dw}{d\rho}\right)^2$$

[proved]