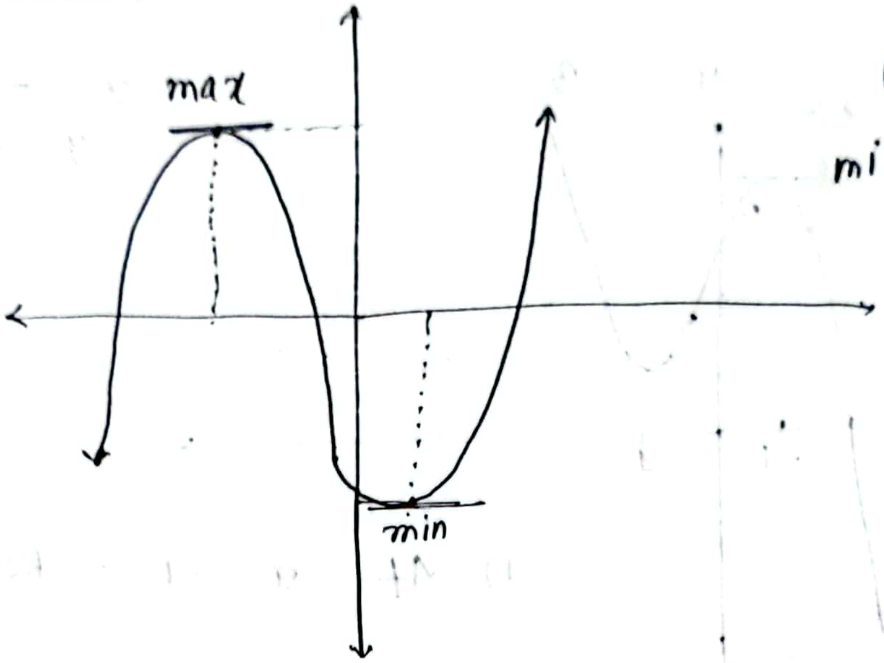
Class → 02

Date: 26.09.23

(Ch-4.2)

Relative Maxima and Minima:

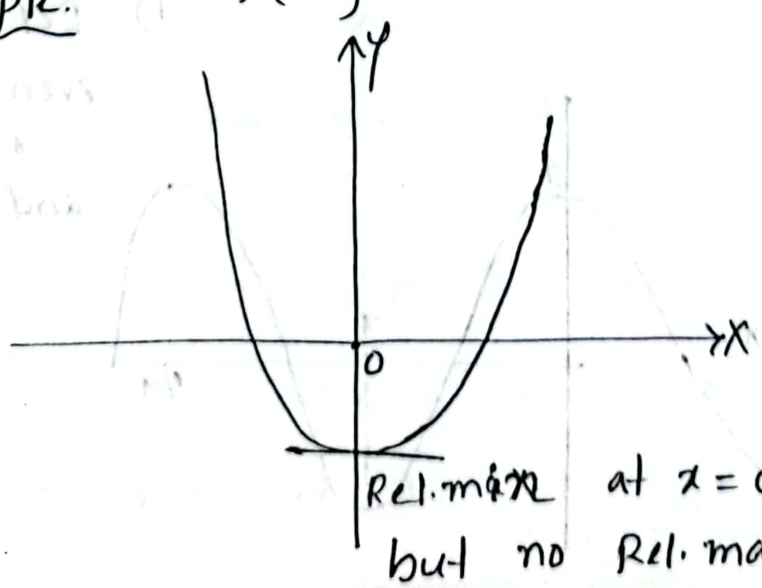


(Rel max and min occur at that point where derivative is zero / slope is zero.)

Def^n: A function f is said to have a Rel. max if an open interval containing x_0 on which $f(x_0) \geq f(x)$.

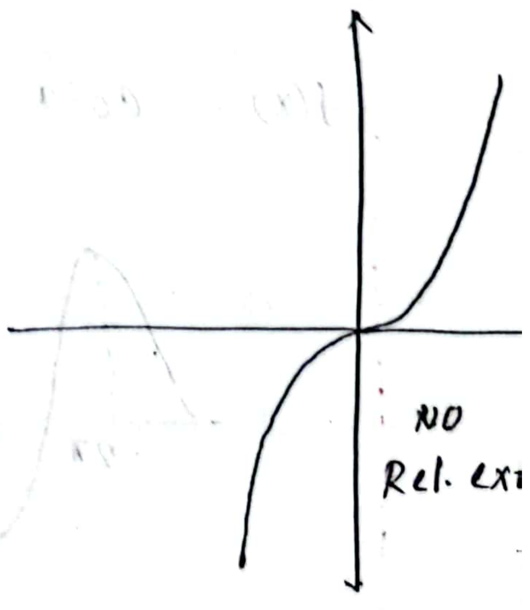
f is said to have a Rel. min if $f(x_0) \leq f(x$

Example: $f(x) = x^2$



Rel. min at $x = 0$ but no Rel. max

$f(x) = x^3$

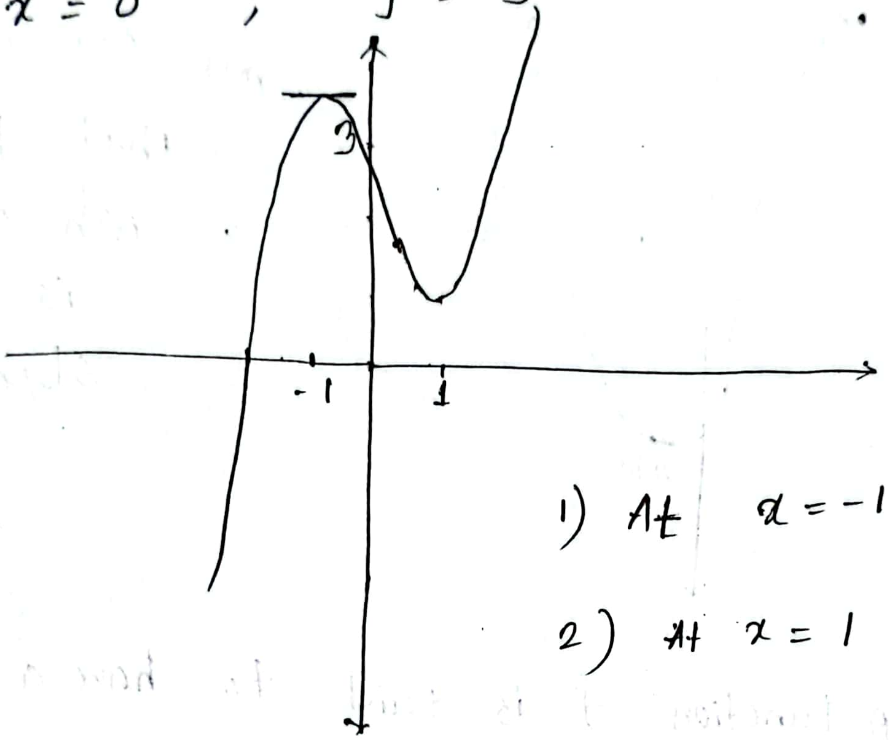


NO Rel. ext

$\boxed{*}$ $f(x) = x^3 - 3x + 3$ $x^3 - 3x + 3 =$

$x = 0$; $y = 3$ $x = -2.1$

$y = 0$



1) At $x = -1$; Rel. max

2) At $x = 1$; Rel min.

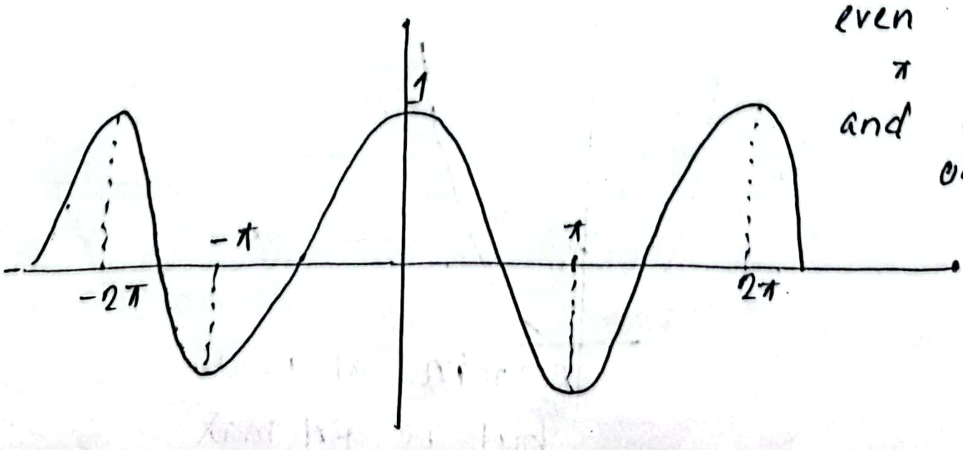
$f'(x) = 3x^2 - 3$

$3(x^2 - 1) = 0$

$x^2 = 1$

$x = 1$; $y = 1$

$f(x) = \cos x$

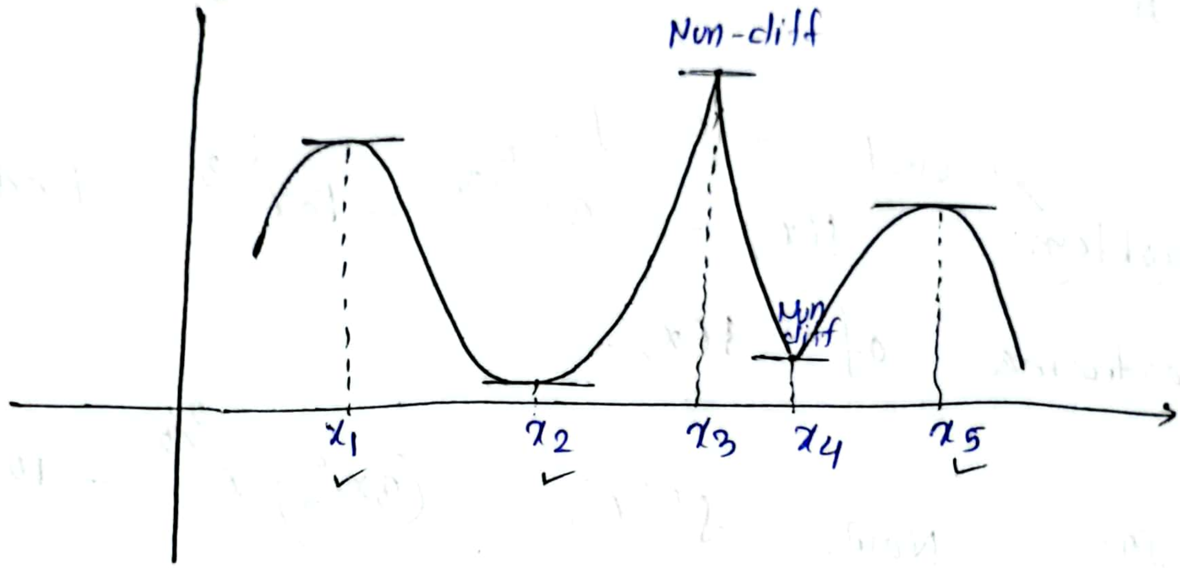


1) Rel. max whe even multiples π and Rel. min odd multi of

* Critical point is one where the derivative is either 0 or undefined.

* Stationary point is where the derivative is 0 and only zero.

* Stationary point:



Critical points → x_1, x_2, x_3, x_4, x_5.

stationary points → x_1, x_2 and x_5.

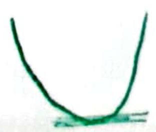
Relative Extrema Math Technique:

$\boxed{\text{First Derivative test:}}$

suppose f is conts at a critical point x_0.

$\boxed{a}$ If $f'(x) > 0$ extending left from x_0 and $f'(x) < 0$ ext. right from x_0 f has a Rel. max at x_0.

$\boxed{b}$ If $f'(x) < 0$ extending left from x_0 and $f'(x) > 0$ ext. right from x_0 f has a Rel. min at x_0



$\boxed{c)}$ If $f'(x)$ same sign $\rightarrow$ no Rel. max / min.

→ Book Example → 04

Problem: $f(x) = 3x^{5/3} - 15x^{2/3}$: Find Rel. extrema of $f(x)$.

Soln: Now, $f'(x) = (3 \times \frac{5}{3}) x^{2/3} - 10x^{-1/3}$

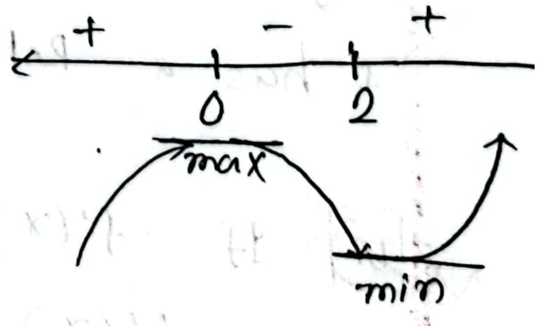
Then, $\cancel{3x}\ 5x^{2/3} - 10x^{-1/3} = 0$

$\Rightarrow 5x^{-1/3}(x-2) = 0$

$\Rightarrow \dfrac{5(x-2)}{x^{1/3}} = 0$

$\therefore x = 0 \; ; \; x = 2$

Interval	$f'(x)$
$x > 2$	+ve
$0 < x < 2$	-ve
$x < 0$	+ve



At $x = 0$; Rel. max

At $x = 2$; Rel. min

At $x = 0$; $y = 0$ [Rel. max]

At $x = 2$; $y = -14.28$ [Rel. min]

[Ans.]

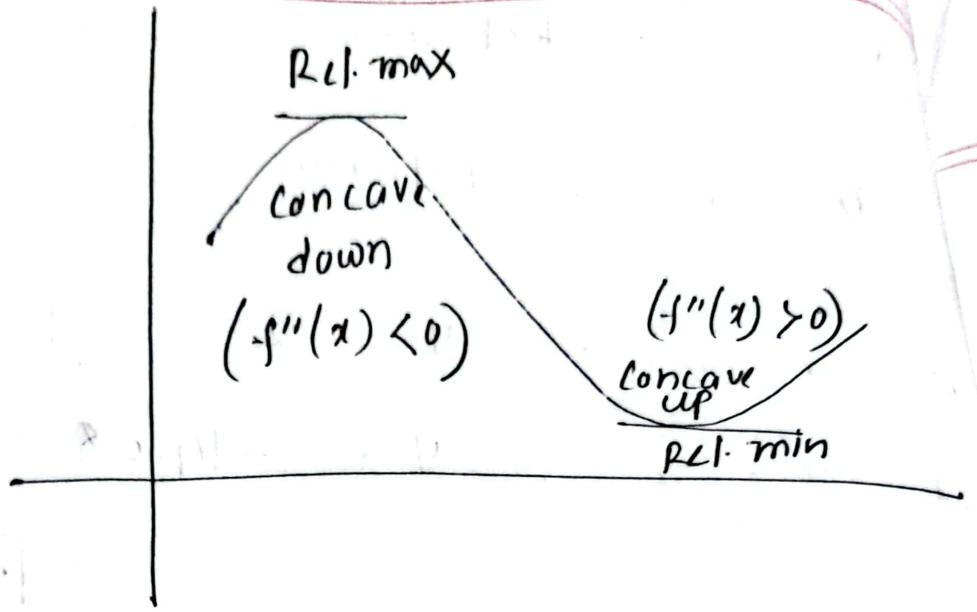
Second Derivative Test:

Theorem: Suppose that f is twice diff. at the point x_0.

a) If $f'(x_0) = 0$ and $f''(x_0) > 0$ then f has a relative min at x_0.

b) If $f'(x_0) = 0$ and $f''(x_0) < 0$ then f has a Rel. max at x_0

c) If $f''(x_0) = 0$ then test is inconclusive. f may have a Rel. max, Rel. min or neither at x_0.



→ Book Example → 5

Problem: Find the Relative Extrema of $f(x) = 3x^5 - 5x^3$. Use 2nd D.T.

Soln: we have,

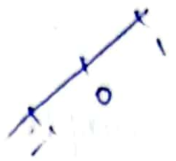
$$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2-1)$$
$$= 15x^2(x+1)(x-1)$$

Stationary point, $f'(x) = 0$

$$\Rightarrow 15x^2(x+1)(x-1) = 0$$
$$f''(x) = 30x(2x^2-1)$$
$$x = 0, 1, -1$$

S. Point	$f''(x)$	2nd D.T.
$x=-1$	-ve	Rel. max
$x=0$	0	inconclusive
$x=1$	+ve	Rel. min

~~$x > 1$~~
~~$-1 < x < 0$~~
~~$0 < x < 1$~~
$x < -1$



Since At $x = 0$, the test is inconclusive we will try the 1st derivative at that point.

Interval	$f'(x)$
$-1 < x < 0$	$-ve$
$0 < x < 1$	$-ve$

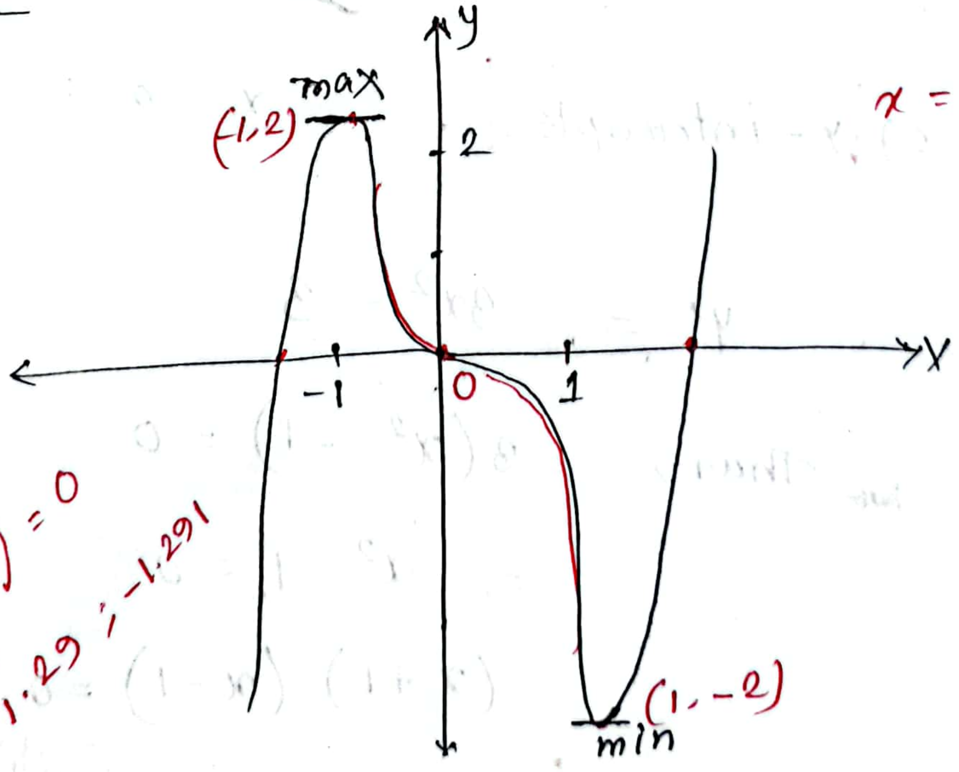
No sign change. So, No Rel. extrema at $x = 0$.

$x = 0$; $f(x) = 0$

$x = 1$; $f(x) = 3 - 5 = -2$

$x = -1$; $f(x) = -3 + 5 = 2$

Graph



$y = 0$

$x^3(x^2 - 5) = 0$

$\therefore x = 0$;

$x = 1.29$; -1.291

(Example : 08)

Problem: sketch the graph of eqn.

$$y = x^3 - 3x + 2$$

and identify the location of intercepts, Rel. Extrema and Inflection points.

Soln: 1) $\boxed{x \text{ intercepts :}}$ $x^3 - 3x + 2 = 0$

$x^3 - 3x + 2 = 0$

$\Rightarrow (x-1)^2 (x+2) = 0$

$\Rightarrow x^3 - 2x - x + 2 = 0$

$\Rightarrow x(x^2 - 2) - 1(x-2) = 0$

$\Rightarrow$

$x = -2, \quad x = 1$

2) $\boxed{y\text{-intercepts}}$: $x = 0; \quad y = 2$

Now, $y' = 3x^2 - 3$

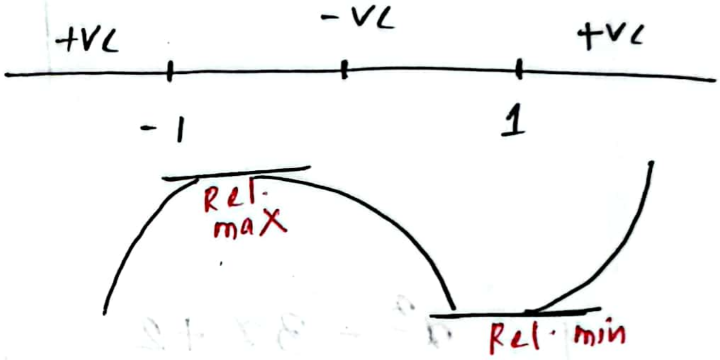
Then, $3(x^2 - 1) = 0$

$\Rightarrow x^2 - 1 = 0$

$(x+1)(x-1) = 0$

$\boxed{x = 1, -1} \Rightarrow$ Critical Points

Interval	$f'(x)$
$x > 1$	$+ve$
$-1 < x < 1$	$-ve$
$x < -1$	$+ve$



At, $x = -1$; Rel. max

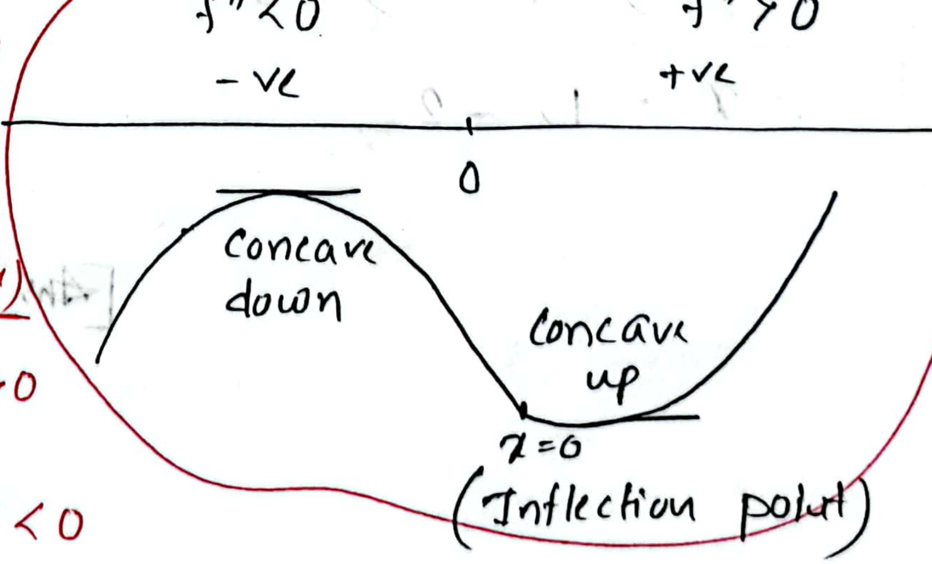
At, $x = 1$; Rel. min.

Then, $y'' = 6x$

$6x = 0$ → Inflection Point

$\therefore \boxed{x = 0}$

Wrong ignore it!



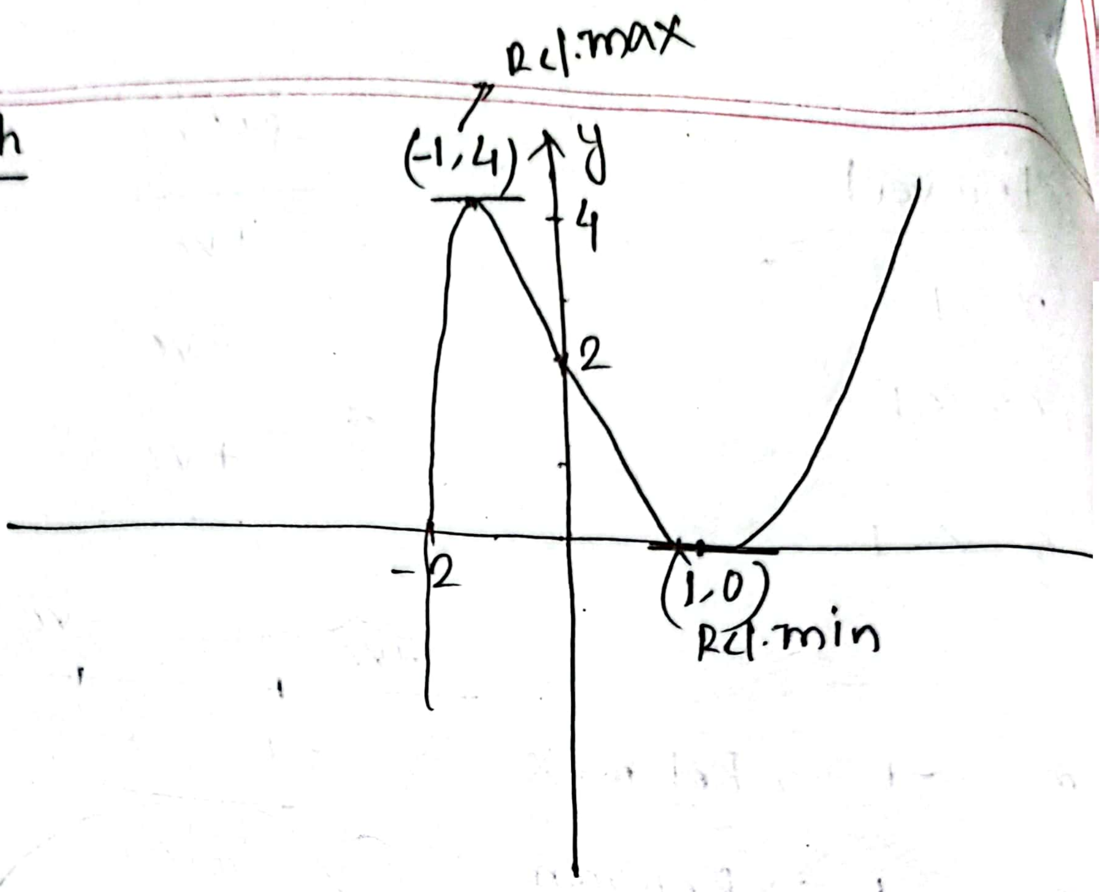
Using 2nd derivative test, to determine Relative Extrema,

Critical Point	$f''(x)$
$x = 1$	$6 > 0$
$x = -1$	$-6 < 0$

At, $x = 1$; Relative minimum

At, $x = -1$; Rel. maximum

Graph



$$y = x^3 - 3x + 2$$

$$x = 0 \quad ; y = 2$$

$$x^3 - 3x + 2 = 0$$

$$\Rightarrow (x-1)^2 (x+2) = 0$$

$$\therefore x = 1, -2$$

[Ans.]

$x^4 - 12x^3 = 0$

$\Rightarrow x^3(x - 12) = 0$

$x = 0 \; ; \; 12$

$x = 0 \; ; \; y = 0$

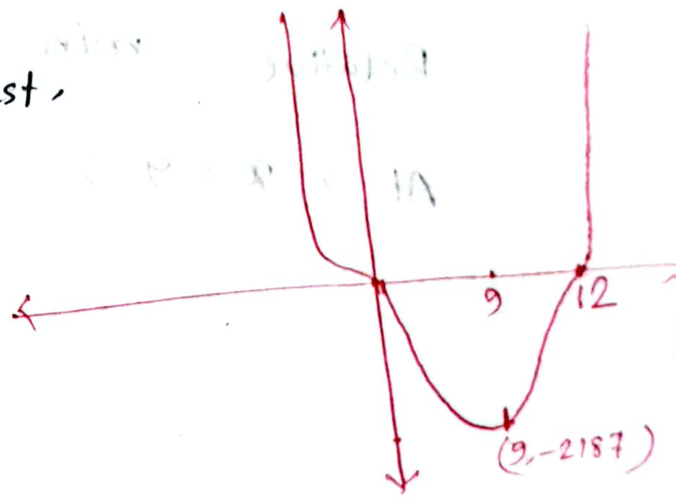
$y = 0 \; ; \; x = 12$

$\boxed{\text{Exercise : 4.2}}$

Problem (39): Find the Relative extrema using both first and second derivative tests.

$f(x) = x^4 - 12x^3$

$f'(x) = 4x^3 - 36x^2$; $f''(x) = 12x^2 - 72x$



Using 1st derivative test,

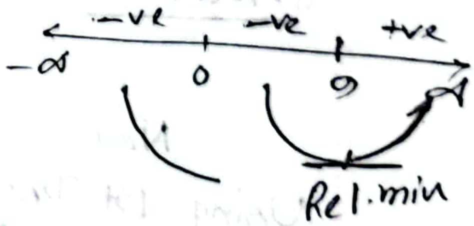
$f'(x) = 0$

$\Rightarrow 4x^3 - 36x^2 = 0$

$\Rightarrow 4x^2(x - 9) = 0$

$\therefore x = 0 \; ; \; 9$

Interval	$f'(x)$
$x > 9$	+ve
$0 < x < 9$	-ve
$x < 0$	-ve



At $x = 9$; $f' > 0$; Rel. min

Using 2nd der. test,

Stationary Point	$f''(x)$
$x = 0$	0
$x = 9$	$324 > 0$

Relative min at $x = 9$

At, $x = 9$; $f(x) = -2187$

[Ans.]

Problem: (36) $f(x) = (x-3)e^x$

Now, $f'(x) = (x-3)e^x + e^x = e^x(x-2)$

Using 1st Der. test,
Then, $f'(x) = 0$

$\Rightarrow e^x [x - 3 + 1] = 0$

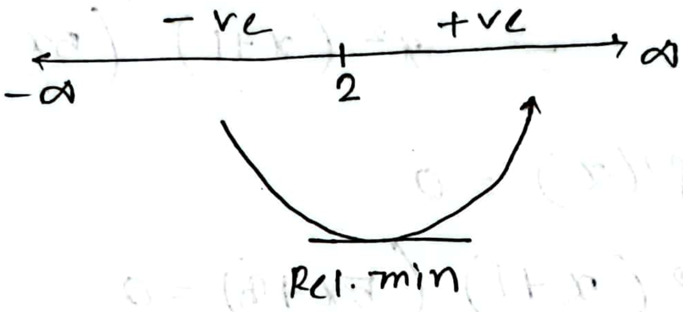
$\Rightarrow e^x (x-2) = 0$

$e^x = 0$; $x = 2$

$\Rightarrow$ (No solution)

~~At, $x = 2$;~~

Interval	$f'(x)$
$x > 2$	+ve
$x < 2$	-ve



At $x = 2$; Rel. min

Using 2nd der. test,

$$f''(x) = (x-2)e^x + e^x$$

~~Interval~~

S. Point	$f''(x)$
$x = 2$	+ve

Rel. min.

So, At $x = 2$;

Minimum value $= f(2)$
$= -7.38$ [Ans.]

Problem: (39) $f(x) = x^3 (x+1)^2$

$$f'(x) = 3x^2 (x+1)^2 + 2x^3 (x+1)$$

$$= x^2 (x+1) \left(3(x+1) + 2x\right)$$

~~Now~~

$$= x^2 (x+1) \{2x + 3x + 3\}$$

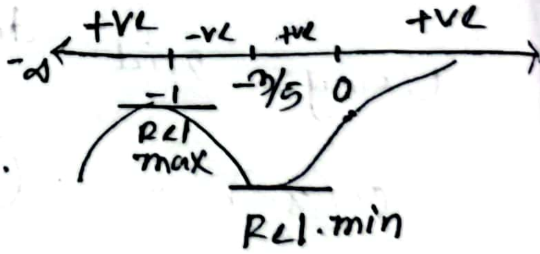
$$= x^2 (x+1) (5x+3)$$

Now, $f'(x) = 0$

$$\Rightarrow x^2 (x+1)(5x+3) = 0$$

$$x = 0, -1, -3/5$$

Using 1st der. test,



Interval	$f'(x)$
$x > 0$	+ve
$x < -1$	+ve
$-1 < x < -3/5$	-ve
$-3/5 < x < 0$	+ve

At $x = -1$; Rel. max

and $x = -3/5$; Rel. min

Using 2nd der. test

$$f''(x) = 5(x^3 + x^2) + (5x+3)(3x^2 + 2x)$$

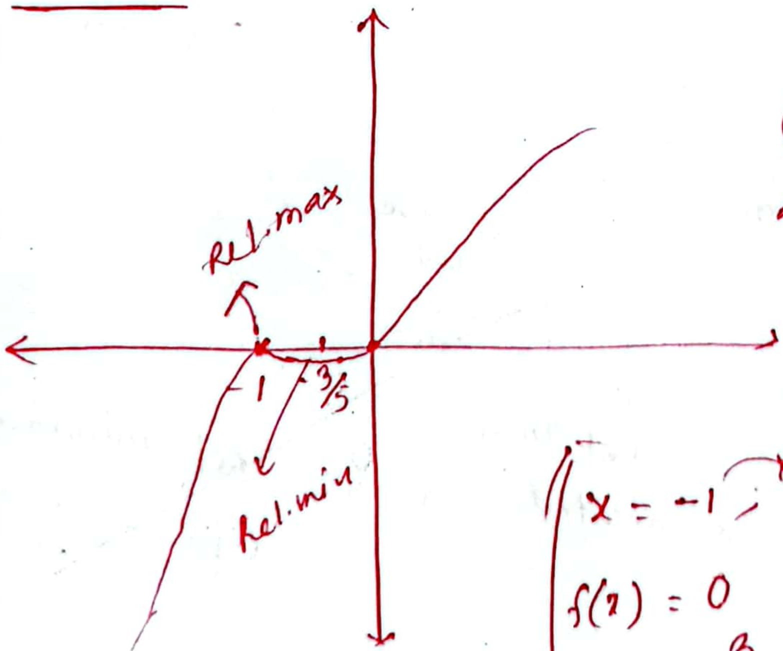
S. Point	$f''(x)$
$x = 0$	0
$x = -1$	$-4 < 0$
$x = -3/5$	$4/25 > 0$

$f''(x) > 0$; At $x = -3/5$; Rel. min

$f''(x) < 0$; At $x = -1$; Rel. max

[Ans.]

Graph:



Exercise : 4.1 (9, 10, 15-20)

Exercise : 4.2 (33-54)

$x = -1$; Rel. max

$f(x) = 0$

$x = -\frac{3}{5} = -0.6$

$f(x) = -0.035$

Rel. min

$x = 0$; $y = 0$

$x^3 + (x+1)^2 = 0$

$x = 0$; $x + 1 = 0$

$\therefore x = -1$