① class→03 Date: 30.09.23

(Ch→13.3) [Anton]

(Partial derivative refers to the derivative with respect to one variable in a multivariable function.)

Definition (13.3.1):

If $z = f(x,y)$ and (x_0, y_0) is a point in domain of f. the partial derivative of f with respect to x at (x_0, y_0) is the der. at x_0 when $y = y_0$ is fixed and x is to vary.

The partial der. is denoted by,

$f_x(x_0, y_0)$:

$$f_x(x_0, y_0) = \frac{d}{dx}\left[f(x, y_0)\right]_{x=x_0}$$

(1st order Partial derivative)

similarly, $f_y(x_0, y_0) = \frac{d}{dy}\left[f(x_0, y)\right]_{y=y_0}$

→ first order Partial derivati
→ 2nd order Partial derivati
3rd or higher

Problem: (Example: 02) Find $f_x(1,3)$ and $f_y(1,3)$

for $f(x,y) = 2x^3y^2 + 2y + 4x$

Soln:

$\frac{d}{dx}[2x^3y^2 + 2y + 4x]$

$f_x(x,y) = 6x^2y^2 + 4$

$\therefore f_x(1,3) = 6 \times 1 \times 3^2 + 4 = 54 + 4 = 58$

$\frac{d}{dy}[2x^3y^2 + 2y + 4x]$

and $f_y(x,y) = 2x^3 2y + 2$

$\therefore f_y(1,3) = 2 \times (1)^3 \times 2 \times (3) + 2$

$= 14$

[Ans.]

* Partial derivative Notation:

If $z = f(x,y)$, then partial derivatives f_x , f_y are denoted by symbols

$$\frac{\partial f}{\partial x} , \frac{\partial z}{\partial x} .$$

(Example : 03)

Problem: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = x^4 \sin($

Solⁿ: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left[x^4 \sin(xy3) \right]$

$= x^4 . \cos(xy^3) \cdot y3 + \sin(xy^3) 4x^{}$

$= x^4 y^3 \cos(xy^3) + 4x^3 \sin(xy^3)$

and $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \left[x^4 \sin(xy^3) \right]$

$= x^4 \cos(xy^3) . 3xy^2 + 0$

$= 3\cos(xy^3) x^5 y^2$

[Ans.]

(Example: 5)

~~Problem:~~ Let: $f(x,y) = x^2y + 5y^3$ *//

a) Find the slope of surface $z = f(x,y)$ in the x-direction at $(1,-2)$

b) Find also in the y-direction at $(1,-2)$.

Soln: a) $f_x(x,y) = 2xy + 0 = 2xy$

The slope in the x-direction $f_x(1,-2) = 2 \times 1 \times (-2)$
$= -4$

z is decreasing at the rate of 4 units

b) $f_y(x,y) = x^2 + 15y^2$

The slope in the y-direction is $f_y(1,-2)$
$= 1 + (15 \times 4)$

$= 61$

z is increasing at the rate of 61 units.

* Partial derivative of functions with more than two variable

Problem: (Ex: 10)

If $f(x,y,z) = x^3y^2z^4 + 2xy + z$

then, Find f_x, f_y and f_z.

Soln: $f_x(x,y,z) = 3x^2y^2z^4 + 2y$

$f_y(x,y,z) = 2x^3yz^4 + 2x$

and $f_z(x,y,z) = 4x^3y^2z^3 + 1$

[Ans.]

(Example : 11)

Problem: If $(\rho, \theta, \phi) = \rho^2 \cos\phi \sin\theta$

Find f_ρ, f_θ and f_ϕ

Soln: $f_\rho = 2\rho \cos\phi \sin\theta$

$f_\theta = \rho^2 \cos\phi \cos\theta$

and $f_\phi = -\rho^2 \sin\theta \sin\phi$

[Ans.]

*Higher order Partial Derivatives:

2nd order partial derivatives of f, which are defined by,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

mixed 2nd order P.D.

(Example : 12)

Problem: Find the 2nd order partial derivatives of $f(x,y) = x^2y^3 + x^4y$

Soln:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left[2xy^3 + 4x^3y\right]$$

$$= 2y^3 + 12x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left[3x^2y^2 + x^4\right]$$

$$= 6x^2y + 0$$

$$= 6x^2y.$$

Then, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$

f_{xy}

$$= \frac{\partial}{\partial x}\left[\cancel{2}\cdot 3x^2y^2 + x^4\right]$$

$$= 6xy^2 + 4x^3$$

and $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[2xy^3 + 4x^3y\right]$

$$= 6xy^2 + 4x^3$$

f_{yx}

(→ Example: 17)

*//

Problem: $f(x,y) = y^2 e^x + y$. Find f_{xy}

Soln: $f_{xyy} = \frac{\partial^2}{\partial y^2}\left[\frac{\partial f}{\partial x}\right]$

$= \frac{\partial^2}{\partial y^2}\left[y^2 e^x\right]$

$= \frac{\partial}{\partial y}\left(2ye^x\right)$

$= 2e^x$

[Ans.]

OR, $\frac{\partial}{\partial y}\left[f_{xy}\right]$

$= \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]$

$= \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(y^2 e^x\right)\right]$

$= \frac{\partial}{\partial y}\left[2ye^x\right]$

$= 2e^x$

Exercise: 13.3 (1-13, 25-52, 85-92, 95-

Example : (1-5)

Problem: (2) Let, $z = e^{2x} \sin y$ Find

(d) $\left.\frac{\partial z}{\partial x}\right|_{(x,0)} = 2e^{2x}\sin y = 0$

(h) $\left.\frac{\partial z}{\partial y}\right|_{(\ln 2,0)} = e^{2x}\cos y = e^{2\ln}$

(f) $\left.\frac{\partial z}{\partial y}\right|_{(x,0)} = e^{2x}\cos y = e^{2x}$

Problem: (4) $f(x,y) = 10x^2y^4 - 6xy^2 + 10x^2$

$$f_x(x,y) = 20xy^4 - 6y^2 + 20x$$

$$f_y(x,y) = 40x^2y^3 - 12xy.$$

Problem: (6) $f(x,y) = \dfrac{1}{xy^2 - x^2y}$

$$f_x = -(xy^2 - x^2y)^{-2}\,[y^2 - 2xy]$$

$$f_y = -(xy^2 - x^2y)^{-2}\,[2xy - x^2]$$

*

Problem: (8) $\dfrac{\partial}{\partial x}\left[x e^{\sqrt{15xy}}\right]$

$$= x\,e^{\sqrt{15xy}} \cdot \frac{1}{2\sqrt{15xy}} \cdot 15y + e^{\sqrt{15xy}}$$

and $\dfrac{\partial}{\partial y}\left[x e^{\sqrt{15xy}}\right]$

$$= x\,e^{\sqrt{15xy}} \cdot \frac{1}{2\sqrt{15xy}} \cdot 15x + 0$$

$$= x\,e^{\sqrt{15xy}}\,\frac{1}{2}\,\frac{1}{\sqrt{15xy}} \cdot 15x$$

[Ans.]

Problem: (12) Let, $f(x,y) = xe^{-y} + 5y$

a) Find the slope $z = f(x,y)$ in x-di
at $(3,0)$

b) Find the slope $z = f(x,y)$ in y-direc
at $(3,0)$.

Soln: (a) $f_x(x,y) = e^{-y}$

$f_x(3,0) = e^0 = 1$

slope $= 1$

(b) $f_y(x,y) = -xe^{-y} + 5$

$f_y(3,0) = -3 + 5 = 2$

slope $= 2$

Problem: (27) */ Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Given,
$z = x^3 \text{Ln}(1 + xy^{-3/5})$

$$\frac{\partial z}{\partial x} = 3x^2 \text{Ln}(1 + xy^{-3/5}) + x^3 \frac{1}{1 + xy^{-3/5}} (y^{-3/5})$$

$$\frac{\partial z}{\partial y} = x^3 \cdot \frac{1}{(1+xy^{-3/5})} \cdot \left[x \times -\tfrac{3}{5} y^{-8/5}\right]$$

$$= -\frac{3}{5} x^4 / (y^{8/5} + xy)$$ [Ans.]

Problem: (29) $z = \dfrac{xy}{x^2+y^2}$

$$\frac{\partial z}{\partial x} = \frac{(x^2+y^2)\cdot y - \overset{xy\cdot 2x}{\cancel{xy\cdot 2(x^2+y^2)\cdot 2x}}}{(x^2+y^2)^2}$$

$$= \frac{y(x^2+y^2) - \overset{2x^2y}{4\cancel{x^2y(x^2+y^2)}}}{(x^2+y^2)^2} = \frac{y}{x^2+y^2} - \frac{2yx^2}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x^2+y^2)\cdot x - \overset{xy\cdot 2y}{2\cancel{xy^2(x^2+y^2)}}}{(x^2+y^2)^2} = \frac{x}{x^2+y^2} - \frac{2xy^2}{(x^2+y^2)^2}$$ [Ans]

Problem: (30) $z = \dfrac{x^2y^3}{\sqrt{x+y}}$

$$\frac{\partial z}{\partial x} = \frac{\sqrt{x+y}\cdot 2y^3x - x^2y^3 \cdot \frac{1}{2\sqrt{x+y}}}{x+y}$$

$$\frac{\partial z}{\partial y} = \frac{\sqrt{x+y}\cdot 3x^2y^2 - x^2y^3\,\frac{1}{2\sqrt{x+y}}}{x+y}$$

[Ans.]

Problem: (33) Find $f_x(x,y)$ and $f_y(x,y)$

Given, $f_x(x,y) = y^{-3/2} \tan^{-1}\left(\frac{x}{y}\right)$

$$= y^{-3/2} \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{y-0}{y^2}$$

$$= y^{-5/2} \frac{y^2}{x^2+y^2}$$

$$= \frac{y^{-1/2}}{x^2+y^2}$$

$$f_y(x,y) = -\frac{3}{2} y^{-5/2} \tan^{-1}\left(\frac{x}{y}\right) + y^{-3/2} \frac{y^2}{x^2+y^2} \cdot x(-1)\cdot \bar{y}$$

$$= -\frac{3}{2} y^{-5/2} \tan^{-1}\left(\frac{x}{y}\right) - x y^{-3/2} \frac{x}{x^2+y^2}$$

[Ans.]

Problem: (36) Given,

$$f(x,y) = \cosh(\sqrt{x}) \sinh^2(xy^2)$$

Now, $f_x(x,y) = \sinh(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \sinh^2(xy^2) + \cosh(\sqrt{x})\, 2\sinh(xy^2) \cdot \cosh(xy^2)\, y^2$

and $f_y(x,y) = \cosh(\sqrt{x}) \cdot 2\sinh(xy^2)\cosh(xy^2)\, 2yx$.

[Ans]

Problem: (38) $f(x,y) = x^2 y e^{xy}$

Evaluate $\left.\frac{\partial f}{\partial x}\right|_{(1,1)}$; $\left.\frac{\partial f}{\partial y}\right|_{(1,1)}$

Soln: $\left.\frac{\partial f}{\partial x}\right|_{(1,1)} = 2x\, y e^{xy} + x^2 [y e^{xy}]$

Fixed Y

$= 2e + e$

$= 3e$

and $\left.\frac{\partial f}{\partial y}\right|_{(1,1)} = x^2 [y e^{xy} \cdot x + e^{xy}]$

Fixed x

$= e + e$

$= 2e$ [Ans.]

Problem: (41) Given, $f(x,y,z) = x^2y^4z^3 + xy,$

Now,

(a) $f_x(x,y,z) = 2xy^4z^3 + y.$

(c) $f_z(x,y,z) = 3x^2y^4z^2 + 2z$

(e) $f_y(1,2,z) = 4x^2y^3z^3 + x.$

$$= 4(1)^2 \cdot 2^3 z^3 + 1$$

$$= 32z^3 + 1$$

Problem: (43) Given, $f(x,y,z) = z\ln(x^2y\,co$

Now, $f_x = z\,\frac{1}{x^2y\cos z} \cdot 2xy\cos z + 0$

$$\therefore f_x = \frac{2z}{x}$$

$$f_y = z\,\frac{1}{x^2y\,\cancel{\cos z}} \cdot \cancel{x^2\cos z}.$$

$$\therefore f_y = \frac{z}{y}$$

and $f_z = z\,\frac{x^2y}{x^2y\cos z}(-\sin z) + \ln(x^2y\cos z)$

$$= \overset{-z\tan z}{-\cancel{\frac{z}{x^2y}\tan z}} + \ln(x^2y\cos z)$$

Problem: (48) Given, $w = \dfrac{x^2 - y^2}{y^2 + z^2}$

Now,

$$\frac{\partial w}{\partial x} = \frac{(y^2+z^2)\,2x - (x^2-y^2).0}{(y^2+z^2)^2}$$

$$= 2x / y^2+z^2$$

$$\frac{\partial w}{\partial y} = \frac{(y^2+z^2)(-2y) - (x^2-y^2).2y}{(y^2+z^2)^2} = \frac{2y(\cancel{-y^2}-z^2-x^2\cancel{+y^2})}{(y^2+z^2)^2} = \frac{-2y(x^2+z^2)}{(y^2+z^2)^2}$$

Problem: (52) * Given, $f(x,y,z) = \sqrt{x^2+4y^2-z^2}$

(a) $\left.\dfrac{\partial w}{\partial x}\right|_{(2,1,-1)}$

$$= \frac{1}{2\sqrt{x^2+4y^2-z^2}} . 2x$$

$$= \frac{2}{\sqrt{(2)^2+4-1}}$$

$$= \frac{2}{\sqrt{7}}$$ [Ans.]

~~$y^2 e^{xz} . z + 0$~~

~~$= (1)^2 e^{2.(-1)} . (-1)$~~

~~$= -e^{-2}$~~

(b) $\left.\dfrac{\partial w}{\partial y}\right|_{(2,1,-1)}$

$$= \frac{8y}{2\sqrt{x^2+4y^2-z^2}}$$

$$= \frac{4y}{\sqrt{x^2+4y^2-z^2}}$$

$$= \frac{4}{\sqrt{4+4-1}} = \frac{4}{\sqrt{7}}$$ [Ans.]

Problem (88) Confirm that the mixed 2nd Order partial derivatives are same.

Given, $f(x,y) = e^{x-y^2}$

Now, $f_x = e^{x-y^2}$

$f_y = e^{x-y^2}(-2y)$

Then, $f_{xy} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[e^{x-y^2}\right]$

$= e^{x-y^2}.(-2y)$

$= -2y\, e^{x-y^2}$

and $f_{yx} = \frac{\partial}{\partial x}\left[f_y\right] = \frac{\partial}{\partial x}\left[-2y\, e^{x-y^2}\right]$

$= -2\left[\cancel{y\, e^{x-y^2}.(-2y) + e^{x-y^2}}\right]$

$= -2\left[y\, e^{x-y^2} + 0\right]$

$= -2y\, e^{x-y^2}$

$\therefore f_{xy} = f_{yx}$ [Ans.]

Problem: (a1) $f(x,y) = \dfrac{x-y}{x+y}$

Now, $f_x = \dfrac{(x+y).1-(x-y)}{(x+y)^2}$

$= \dfrac{\cancel{x}+y-\cancel{x}+y}{(x+y)^2}$

$= \dfrac{2y}{(x+y)^2}$

and $f_y = \dfrac{(x+y)\cdot(-1)-(x-y)\cdot 1}{(x+y)^2}$

$= \dfrac{-x-\cancel{y}-x+\cancel{y}}{(x+y)^2}$

$= \dfrac{-2x}{(x+y)^2}$

$f_{xy} = \dfrac{\partial}{\partial y}\left[\dfrac{2y}{(x+y)^2}\right]$

$= 2\left[\dfrac{(x+y)^2\cdot 1 - y\cdot 2(x+y)}{(x+y)^4}\right] = \dfrac{2(x-y)}{(x+y)^3}$

$\dfrac{+y)\{x+y-2y\}}{(x+y)^4}$

$=2\,\dfrac{(x+y)^2-2xy-2y^2}{(x+y)^4}$

$\dfrac{x-y}{x+y)^3}$

$f_{yx} = \dfrac{\partial}{\partial x}\left[\dfrac{-2x}{(x+y)^2}\right] = -2\left[\dfrac{(x+y)^2\cdot 1-(x\cdot 2(x+y)}{(x+y)^4}\right]$

[Confirmed] $= \dfrac{2(x-y)}{(x+y)^3} = \dfrac{-2(-x+y)}{(x+y)^3} = -2\left[\dfrac{(x+y)\{x+y-2x\}}{(x+y)^4}\right.$