★ بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ ★

IN THE NAME OF ALLAH

رَبِّ زِدْنِي عِلْمًا

Rabbi Zidni Ilma

O my Lord, Increase me in Knowledge

لَا إِلَٰهَ إِلَّا ٱللَّهُ مُحَمَّدٌ رَسُولُ ٱللَّهِ

LĀĀĀ-ILĀHA IL-LAL-LĀHU MUHAM-MADUR RASŪLUL-LĀ
THERE IS NO OTHER GOD BUT ALLAH MUHAMMAD IS THE MESSENGER OF ALLAH

United International **University**
QUEST FOR EXCELLENCE

COURSE INFORMATION'S

1	Course Title	Linear Algebra & Differential Equations
2	Course Code	MAT 2105
3	Trimester and Year	Spring-2023
4	Pre-requisites	Math 1101, Math 1103
5	Credit Hours	3.00
6	Section	B
7	Class Hours	11.11AM-12.30PM
8	Classroom	701
9	Instructor's Name	Muhaiminul Islam Adnan (MIA) Lecturer, INS, UIU
10	Email	adnan@ins.uiu.ac.bd , 01911611301
11	Office	619
13	TextBook	1. Contemporary Linear Algebra, Howard Anton, Robert C. Busby (HR). 2. Elementary Differential Equations, Boyce &Diprima (BD) [9th Edition].
14	Reference	1. A First Course in Differential Equations with Modeling Applications, Dennis G. Zill (Zill) [10th Edition]. 2. Engineering Mathematics, H. K. Das (HKD) [15th Edition].

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COURSE DESCRIPTION

Midterm Examination Linear Algebra	Final Examination Ordinary Differential Equations and Partial differential equations
➢ Introduction to the system of linear equations, solutions and their applications.(HR 2.1, 2.2, 2.3) ➢ Matrices, Matrix Algebra and Determinants (HR 3.3, 3.2,4.3) ➢ Eigen values and Eigen vectors(HR 4.4, 8.2) ➢ Linear combinations, independence of vectors.(HR 3.4) ➢ Linear transformations(HR 6.1, 6.2, 6.3)	➢ Classification of differential equations.(BD 1.3) ➢ Solutions and applications of first order and second order differential equations by various methods (BD 2.1, 2.2, 2.6,DG 2.5, 3.1,HKD 3.14-3.18,BD 3.4 (part), 3.7, 7.5) ➢ Solutions with boundary and initial conditions of partial differential equations.(BD 10.1) ➢ Heat equation and Wave equation(BD 10.5, 10.6, 10.7)

LINEAR EQUATION

A *linear equation* in unknowns $x_1, x_2, \ldots, x_n$ is an equation that can be put in the *standard form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$, and b are constants. The constant a_k is called the *coefficient* of x_k, and b is called the *constant term* of the equation.

Example of Linear Equation

Observe that a linear equation does not involve any products or roots of variables. All variables occur only to the first power and do not appear, for example, as arguments of trigonometric, logarithmic, or exponential functions. The following are linear equations:

$$x + 3y = 7 \qquad x_1 - 2x_2 - 3x_3 + x_4 = 0$$

$$\tfrac{1}{2}x - y + 3z = -1 \qquad x_1 + x_2 + \cdots + x_n = 1$$

EXAMPLE OF NON-LINEAR EQUATION

The following are not linear equations:

$$x + 3y^2 = 4 \qquad 3x + 2y - xy = 5$$

$$\sin x + y = 0 \qquad \sqrt{x_1} + 2x_2 + x_3 = 1$$

Homogeneous Linear Equation:

A *linear equation* in unknowns $x_1, x_2, \ldots, x_n$ is an equation that can be put in the *standard form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$, and b are constants. The constant a_k is called the *coefficient* of x_k, and b is called the *constant term* of the equation.

In the special case where $b = 0$, the equation has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

which is called a ***homogeneous linear equation***.

Example: i) $x + 2y + z = 0$ ii) $x_1 + 5x_2 - x_3 + 2x_4 = 0$

SYSTEM OF LINEAR EQUATIONS

A finite set of linear equations is called a ***system of linear equations*** or a ***linear system***. The variables in a linear system are called the ***unknowns***. For example,

$$\begin{aligned} 4x_1 - x_2 + 3x_3 &= -1 \\ 3x_1 + x_2 + 9x_3 &= -4 \end{aligned} \tag{5}$$

is a linear system of two equations in the three unknowns x_1, x_2, and x_3. A general linear system of m equations in the n unknowns $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{array}{ccccccc} a_{11}x_1 & + a_{12}x_2 & + & \cdots & + a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + a_{22}x_2 & + & \cdots & + a_{2n}x_n & = & b_2 \\ \vdots & \vdots & & & \vdots & & \vdots \\ a_{m1}x_1 & + a_{m2}x_2 & + & \cdots & + a_{mn}x_n & = & b_m \end{array}$$

$$\begin{cases} -2x_1 + 3x_2 + x_3 - x_4 = -2 \\ x_1 + x_3 - 4x_4 = 1 \\ 3x_1 - x_2 - x_4 = 3 \end{cases}$$

Examples:

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + 4x_3 + 3x_4 &= 5 \\ 2x_1 + 3x_2 + x_3 - 2x_4 &= 1 \\ x_1 + 2x_2 - 5x_3 + 4x_4 &= 3 \end{aligned}$$

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 + 15x_6 &= 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

AUGMENTED MATRIX

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}$$

we can abbreviate the system by writing only the rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

This is called the ***augmented matrix*** for the system. For example, the augmented matrix for the system of equations

$$\begin{array}{r} x_1 + x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - 3x_3 = 1 \\ 3x_1 + 6x_2 - 5x_3 = 0 \end{array} \quad \text{is} \quad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Examples: Linear System

Augmented Matrix

$$\begin{cases} x_1 - x_2 - 2x_3 + x_4 = 0 \\ 2x_1 - x_2 - 3x_3 + 2x_4 = -6 \\ -x_1 + 2x_2 + x_3 + 3x_4 = 2 \\ x_1 + x_2 - x_3 + 2x_4 = 1 \end{cases}$$

$$\left[\begin{array}{rrrr|r} 1 & -1 & -2 & 1 & 0 \\ 2 & -1 & -3 & 2 & -6 \\ -1 & 2 & 1 & 3 & 2 \\ 1 & 1 & -1 & 2 & 1 \end{array}\right]$$

ELEMENTARY ROW OPERATIONS

Elementary Row Operations

Suppose A is a matrix with rows $R_1, R_2, \ldots, R_m$. The following operations on A are called *elementary row operations*.

[E_1] (Row Interchange): Interchange rows R_i and R_j. This may be written as

"Interchange R_i and R_j" or "$R_i \longleftrightarrow R_j$"

[E_2] (Row Scaling): Replace row R_i by a nonzero multiple kR_i of itself. This may be written as

"Replace R_i by kR_i $(k \neq 0)$" or "$kR_i \rightarrow R_i$"

[E_3] (Row Addition): Replace row R_j by the sum of a multiple kR_i of a row R_i and itself. This may be written as

"Replace R_j by $kR_i + R_j$" or "$kR_i + R_j \rightarrow R_j$"

The arrow $\rightarrow$ in E_2 and E_3 may be read as "replaces."

ECHELON FORMS / MATRIX

Echelon Matrices

A matrix A is called an *echelon matrix*, or is said to be in *echelon form*, if the following two conditions hold (where a *leading nonzero element* of a row of A is the first nonzero element in the row):

(1) All zero rows, if any, are at the bottom of the matrix.

(2) Each leading nonzero entry in a row is to the right of the leading nonzero entry in the preceding row.

Example:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

REDUCED ROW ECHELON FORMS / MATRIX

Row Canonical Form

A matrix A is said to be in *row canonical form* (or *row-reduced echelon form*) if it is an echelon matrix—that is, if it satisfies the above properties (1) and (2), and if it satisfies the following additional two properties:

(3) Each pivot (leading nonzero entry) is equal to 1.

(4) Each pivot is the only nonzero entry in its column.

The major difference between an echelon matrix and a matrix in row canonical form is that in an echelon matrix there must be zeros below the pivots [Properties (1) and (2)], but in a matrix in row canonical form, each pivot must also equal 1 [Property (3)] and there must also be zeros above the pivots [Property (4)].

The zero matrix 0 of any size and the identity matrix I of any size are important special examples of matrices in row canonical form.

Examples:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

PROCEDURE TO FIND ECHELON FORM

Example1:Find the echelon form of the following Matrix $A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$

Solution: First let us reduce the matrix A to echelon form by the elementary row operations. We multiply 1st row by 2 and subtract from 2nd row, then replace 2nd row

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix} \xrightarrow[replace\ R_2\ by\ R_2 - 2R_1]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$$

Now multiply 1st row by 3 then subtract from 3rd row

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix} \xrightarrow[replace\ R_3\ by\ R_3 - 3R_1]{R_3 - 3R_1 \to R_3} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 2 \end{bmatrix} \xrightarrow[\text{replace } R_3 \text{ by } 3R_3 - 5R_2]{3R_3 - 5R_2 \to R_3} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & -6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & -6 & 1 \end{bmatrix}$$ This matrix is in echelon form

Example 2:

Consider the matrix $A = \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{bmatrix}$. reduce A to an echelon form.

Solution:

$$\begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{bmatrix} \xrightarrow[\text{Replace } R_2 \text{ by } -2R_1 + R_2]{\text{Replace } R_3 \text{ by } -3R_1 + R_3.} \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 6 & 7 \end{bmatrix} \xrightarrow{\text{Replace } R_3 \text{ by } -\frac{3}{2}R_2 + R_3.} \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

PROCEDURE TO FIND REDUCER ROW ECHELON FORM

Example1:Find the reduced row echelon form of the following Matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$$

Solution: First let us reduce the matrix A to echelon form by the elementary row operations. We multiply 1st row by 2 and subtract from 2nd row, then replace 2nd row

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix} \xrightarrow[\textit{replace } R_2 \textit{ by } R_2 - 2R_1]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$$

Now multiply 1st row by 3 then subtract from 3rd row

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix} \xrightarrow[\textit{replace } R_3 \textit{ by } R_3 - 3R_1]{R_3 - 3R_1 \to R_3} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 2 \end{bmatrix} \xrightarrow[\text{replace } R_3 \text{ by } 3R_3 - 5R_2]{3R_3 - 5R_2 \to R_3} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & -6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & -6 & 1 \end{bmatrix}$$ This matrix is in echelon form

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & -6 & 1 \end{bmatrix} \xrightarrow[\text{replace } R_2 \text{ by } R_2 - R_3]{R_2 - R_3 \to R_2} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -6 & 1 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{3} R_2 \;\; \& \;\; R_3 \to \frac{1}{-6} R_3} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} \end{bmatrix} \xrightarrow[\text{replace } R_1 \text{ by } R_1 - 2R_3]{R_1 - 2R_3 \to R_1} \begin{bmatrix} 1 & 2 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} \end{bmatrix}$$

Example 2:

Consider the matrix $A = \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{bmatrix}$. reduce A to its row canonical form.

Solution:

$$\begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{bmatrix} \xrightarrow[\text{Replace } R_2 \text{ by } -2R_1 + R_2]{\text{Replace } R_3 \text{ by } -3R_1 + R_3} \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 6 & 7 \end{bmatrix} \xrightarrow{\text{Replace } R_3 \text{ by } -\frac{3}{2}R_2 + R_3} \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

The matrix is now in echelon form.

$$\begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{\text{Multiply } R_3 \text{ by } -\frac{1}{2}} \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Replace } R_1 \text{ by } -2R_3 + R_1]{\text{Replace } R_2 \text{ by } -6R_3 + R_2} \begin{bmatrix} 1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Multiply } R_2 \text{ by } \frac{1}{2}} \begin{bmatrix} 1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Replace } R_1 \text{ by } 3R_2 + R_1} \begin{bmatrix} 1 & 2 & 0 & 7 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The last matrix is the row canonical form of A.

SOLUTION OF LINEAR EQUATION

A *linear equation* in unknowns $x_1, x_2, \ldots, x_n$ is an equation that can be put in the *standard form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{3.1}$$

where $a_1, a_2, \ldots, a_n$, and b are constants. The constant a_k is called the *coefficient* of x_k, and b is call *constant term* of the equation.

A solution of the linear equation (3.1) is a list of values for the unknowns or, equivalently, a vector u in K^n, say

$$x_1 = k_1, \quad x_2 = k_2, \quad \ldots, \quad x_n = k_n \qquad \text{or} \qquad u = (k_1, k_2, \ldots, k_n)$$

such that the following statement (obtained by substituting k_i for x_i in the equation) is true:

$$a_1k_1 + a_2k_2 + \cdots + a_nk_n = b$$

In such a case we say that u *satisfies* the equation.

EXAMPLE 3.1 Consider the following linear equation in three unknowns x, y, z:

$$x + 2v - 3z = 6$$

We note that $x = 5, y = 2, z = 1$, or, equivalently, the vector $u = (5, 2, 1)$ is a solution of the equation. That is,

$$5 + 2(2) - 3(1) = 6 \qquad \text{or} \qquad 5 + 4 - 3 = 6 \qquad \text{or} \qquad 6 = 6$$

On the other hand, $w = (1, 2, 3)$ is not a solution, because on substitution, we do not get a true statement:

$$1 + 2(2) - 3(3) = 6 \qquad \text{or} \qquad 1 + 4 - 9 = 6 \qquad \text{or} \qquad -4 = 6$$

SOLUTION OF SYSTEM OF LINEAR EQUATIONS

EXAMPLE 3.2 Consider the following system of linear equations:

$$
\begin{aligned}
x_1 + \quad\quad + 4x_3 + 3x_4 &= 5 \\
2x_1 + 3x_2 + \ \ x_3 - 2x_4 &= 1 \\
x_1 + 2x_2 - 5x_3 + 4x_4 &= 3
\end{aligned}
$$

It is a 3×4 system because it has three equations in four unknowns. Determine whether (a) $u = (-8, 6, 1, 1)$ and (b) $v = (-10, 5, 1, 2)$ are solutions of the system.

(a) Substitute the values of u in each equation, obtaining

$$
\begin{array}{rcrcr}
-8 + 6 + 4(1) + 3(1) = 5 & \text{or} & -8 + 6 + 4 + 3 = 5 & \text{or} & 5 = 5 \\
2(-8) + 3(6) + 1 - 2(1) = 1 & \text{or} & -16 + 18 + 1 - 2 = 1 & \text{or} & 1 = 1 \\
-8 + 2(6) - 5(1) + 4(1) = 3 & \text{or} & -8 + 12 - 5 + 4 = 3 & \text{or} & 3 = 3
\end{array}
$$

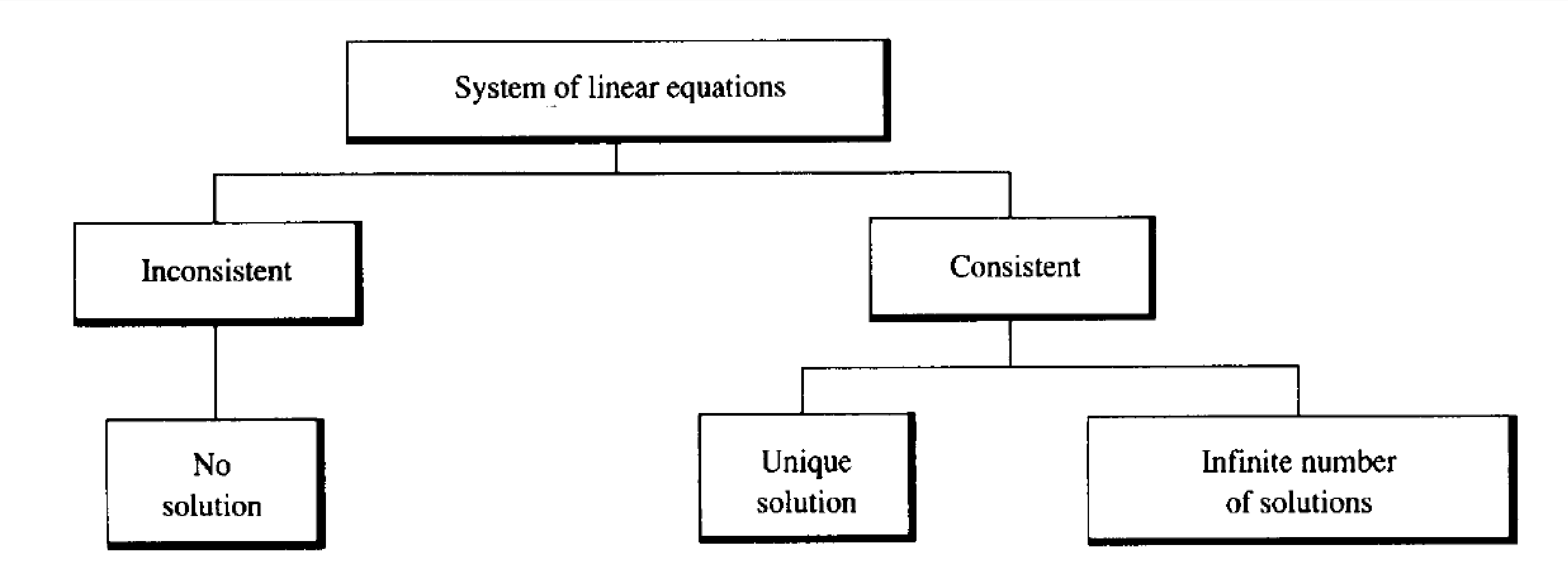
Yes, u is a solution of the system because it is a solution of each equation.

(b) Substitute the values of v into each successive equation, obtaining

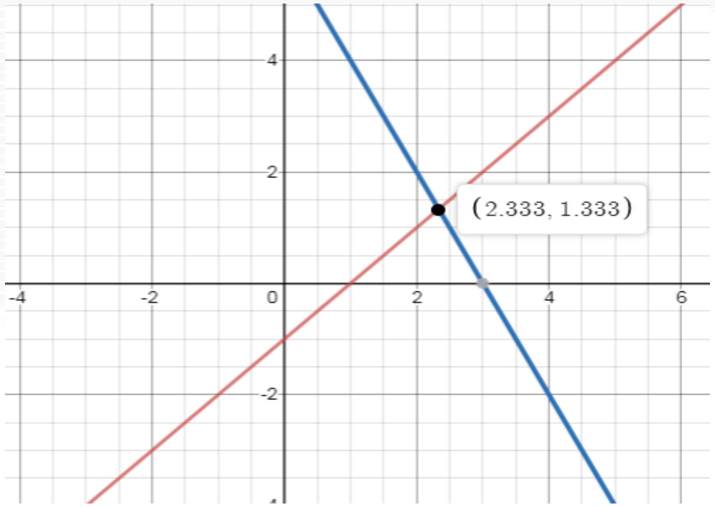
$$
\begin{array}{rcrcr}
-10 + 5 + 4(1) + 3(2) = 5 & \text{or} & -10 + 5 + 4 + 6 = 5 & \text{or} & 5 = 5 \\
2(-10) + 3(5) + 1 - 2(2) = 1 & \text{or} & -20 + 15 + 1 - 4 = 1 & \text{or} & -8 = 1
\end{array}
$$

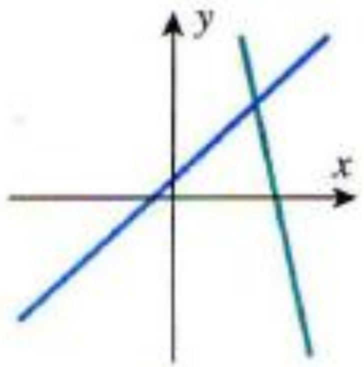
No, v is not a solution of the system, because it is not a solution of the second equation. (We do not need to substitute v into the third equation.)

DIFFERENT TYPES OF SOLUTION



CONSISTENT SYSTEM → UNIQUE SOLUTION





One solution

EXAMPLE A Linear System with One Solution

Solve the linear system

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the simplified system

$$\begin{aligned} x - y &= 1 \\ 3y &= 4 \end{aligned}$$

From the second equation we obtain $y = \frac{4}{3}$, and on substituting this value in the first equation we obtain $x = 1 + y = \frac{7}{3}$. Thus, the system has the unique solution $x = \frac{7}{3}$, $y = \frac{4}{3}$. Geometrically, this means that the lines represented by the equations in the system intersect at the single point $\left(\frac{7}{3}, \frac{4}{3}\right)$. We leave it for you to check this by graphing the lines. ■

CONSISTENT SYSTEM → UNIQUE SOLUTION

System	Augmented Matrix
$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$	$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$
Add -2 times the first equation to the second to obtain	Add -2 times the first row to the second to obtain
$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned}$	$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$
Add -3 times the first equation to the third to obtain	Add -3 times the first row to the third to obtain
$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$	$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$
Multiply the second equation by $\frac{1}{2}$ to obtain	Multiply the second row by $\frac{1}{2}$ to obtain
$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$	$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$
Add -3 times the second equation to the third to obtain	Add -3 times the second row to the third to obtain
$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned}$	$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$
Multiply the third equation by -2 to obtain	Multiply the third row by -2 to obtain
$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$	$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Example: Solve

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

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Multiply the third equation by -2 to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Multiply the third row by -2 to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add -1 times the second equation to the first to obtain

$$\begin{aligned} x \qquad + \tfrac{11}{2}z &= \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Add -1 times the second row to the first to obtain

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $-\frac{11}{2}$ times the third equation to the first and $\frac{7}{2}$ times the third equation to the second to obtain

$$\begin{aligned} x \qquad &= 1 \\ y \quad &= 2 \\ z &= 3 \end{aligned}$$

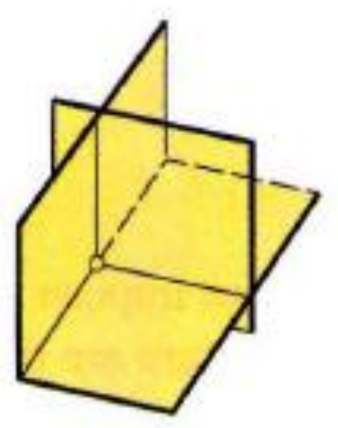
Add $-\frac{11}{2}$ times the third row to the first and $\frac{7}{2}$ times the third row to the second to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution

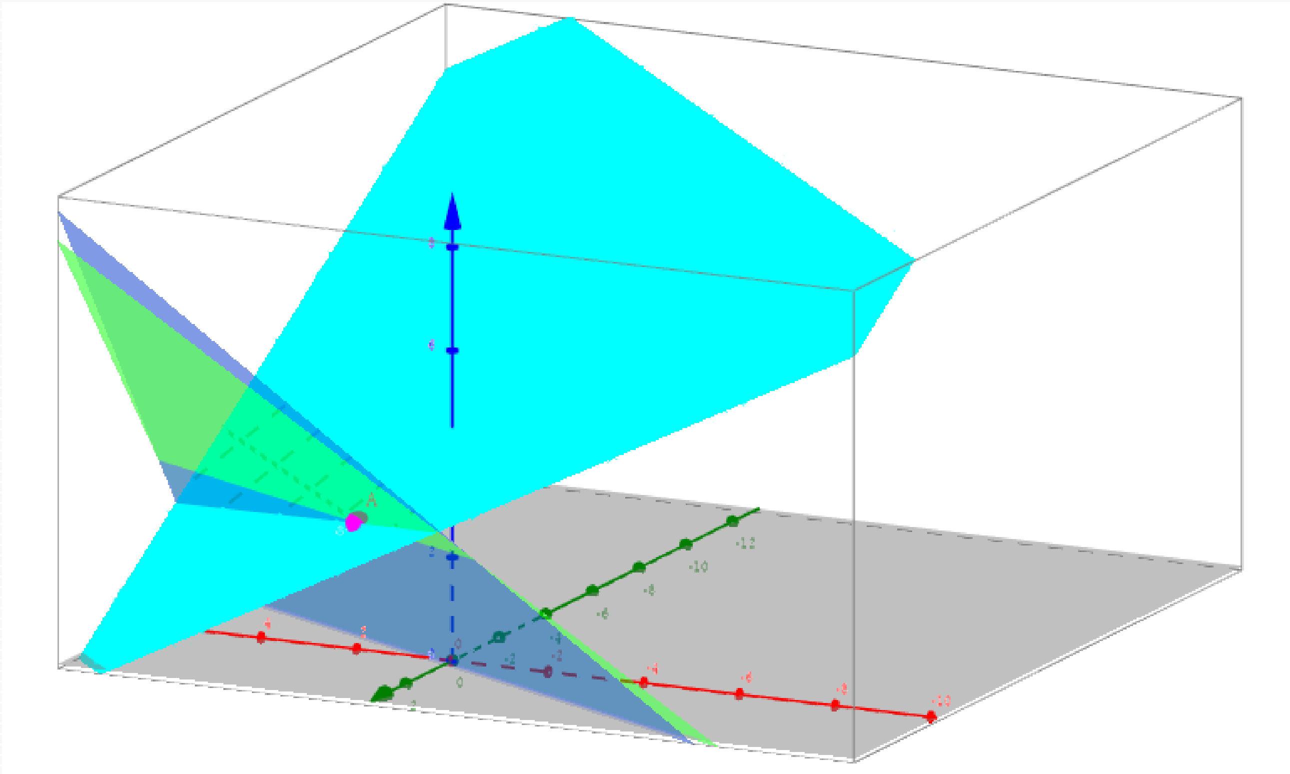
$$x = 1, \quad y = 2, \quad z = 3$$

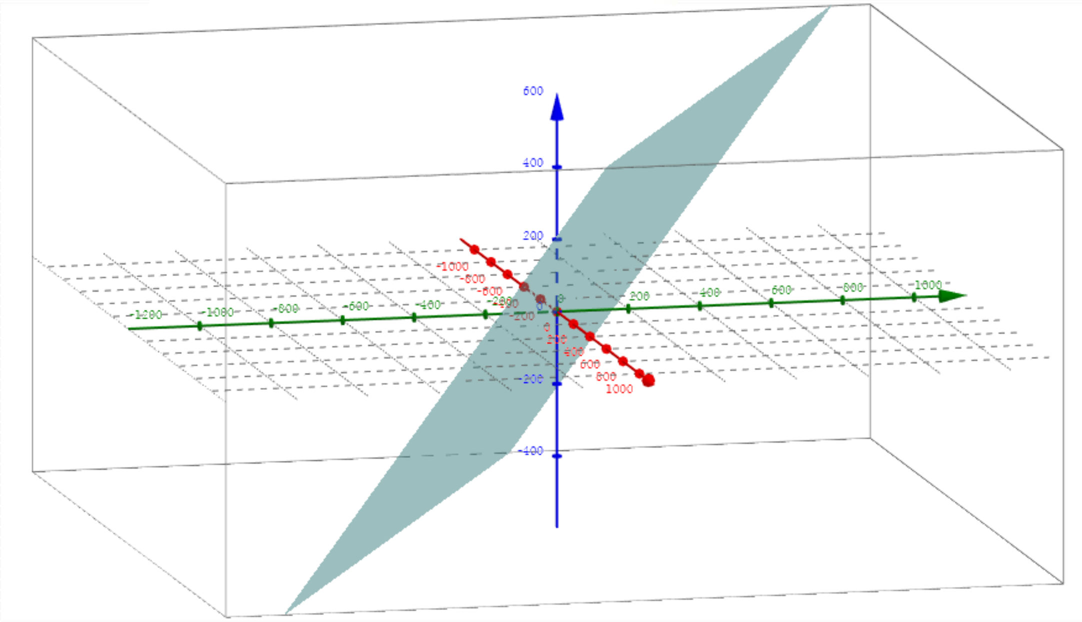
is now evident. Geometrically, this means that the planes represented by the equations in the system intersect at the single point (1, 2, 3) in R^3. ■



One solution
(intersection is a point)

$$
\begin{aligned}
x + y + 2z &= 9 \\
2x + 4y - 3z &= 1 \\
3x + 6y - 5z &= 0
\end{aligned}
$$





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EXAMPLES

Example: Solve the following system of linear equations by Gaussian elimination method

$$\begin{aligned} x + 2y - z &= 3 \\ x + 3y + z &= 5 \\ 3x + 8y + 4z &= 17 \end{aligned}$$

Solution: Augmented matrix of the given system is $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 1 & 5 \\ 3 & 8 & 4 & 17 \end{bmatrix}$

Now reduce the augmented matrix A to echelon form as follows

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 1 & 5 \\ 3 & 8 & 4 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

Now write down the corresponding triangular system

$$\begin{aligned} x + 2y - z &= 3 \\ y + 2z &= 2 \\ 3z &= 4 \end{aligned}$$

and solve by back-substitution to obtain the unique solution

$$x = \tfrac{17}{3}, \quad y = -\tfrac{2}{3}, \quad z = \tfrac{4}{3} \qquad \text{or} \qquad u = \left(\tfrac{17}{3}, -\tfrac{2}{3}, \tfrac{4}{3}\right)$$

Alternative Solution process

Example: Solve the following system of linear equations by Gauss-Jordan elimination method

$$\begin{aligned} x + 2y - z &= 3 \\ x + 3y + z &= 5 \\ 3x + 8y + 4z &= 17 \end{aligned}$$

Solution: Augmented matrix of the given system is $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 1 & 5 \\ 3 & 8 & 4 & 17 \end{bmatrix}$

Now reduce the augmented matrix A to reduced row echelon form as follows

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{4}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & \frac{13}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{bmatrix}$$

$$x = \tfrac{17}{3},\ y = -\tfrac{2}{3},\ z = \tfrac{4}{3} \qquad \text{or} \qquad u = \left(\tfrac{17}{3}, -\tfrac{2}{3}, \tfrac{4}{3}\right)$$

EXAMPLE 3 Solve the linear system by transforming the augmented matrix to reduced row echelon form.

$$\begin{cases} x_1 - x_2 - 2x_3 + x_4 = 0 \\ 2x_1 - x_2 - 3x_3 + 2x_4 = -6 \\ -x_1 + 2x_2 + x_3 + 3x_4 = 2 \\ x_1 + x_2 - x_3 + 2x_4 = 1 \end{cases}$$

Solution The augmented matrix of the linear system is

$$\left[\begin{array}{rrrr|r} 1 & -1 & -2 & 1 & 0 \\ 2 & -1 & -3 & 2 & -6 \\ -1 & 2 & 1 & 3 & 2 \\ 1 & 1 & -1 & 2 & 1 \end{array}\right]$$

To transform the matrix into reduced row echelon form, we first use the leading 1 in row 1 as a pivot to eliminate the terms in column 1 of rows 2, 3, and 4. To do this, we use the three row operations

$$\begin{aligned} -2R_1 + R_2 &\to R_2 \\ R_1 + R_3 &\to R_3 \\ -R_1 + R_4 &\to R_4 \end{aligned}$$

in succession, transforming the matrix

$$\left[\begin{array}{rrrr|r} 1 & -1 & -2 & 1 & 0 \\ 2 & -1 & -3 & 2 & -6 \\ -1 & 2 & 1 & 3 & 2 \\ 1 & 1 & -1 & 2 & 1 \end{array}\right] \quad \text{to} \quad \left[\begin{array}{rrrr|r} 1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & -6 \\ 0 & 1 & -1 & 4 & 2 \\ 0 & 2 & 1 & 1 & 1 \end{array}\right]$$

For the second step we use the leftmost 1 in row 2 as the pivot and eliminate the term in column 2 above the pivot, and the two terms below the pivot. The required row operations are

$$\begin{aligned} R_2 + R_1 &\to R_1 \\ -R_2 + R_3 &\to R_3 \\ -2R_2 + R_4 &\to R_4 \end{aligned}$$

reducing the matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & -6 \\ 0 & 1 & -1 & 4 & 2 \\ 0 & 2 & 1 & 1 & 1 \end{array}\right] \quad \text{to} \quad \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -6 \\ 0 & 1 & 1 & 0 & -6 \\ 0 & 0 & -2 & 4 & 8 \\ 0 & 0 & -1 & 1 & 13 \end{array}\right]$$

Notice that each entry in row 3 is evenly divisible by 2. Therefore, a leading 1 in row 3 is obtained using the operation $-\frac{1}{2}R_3 \to R_3$, which results in the matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -6 \\ 0 & 1 & 1 & 0 & -6 \\ 0 & 0 & 1 & -2 & -4 \\ 0 & 0 & -1 & 1 & 13 \end{array}\right]$$

Now, by using the leading 1 in row 3 as a pivot, the operations

$$\begin{aligned} R_3 + R_1 &\to R_1 \\ -R_3 + R_2 &\to R_2 \\ R_3 + R_4 &\to R_4 \end{aligned}$$

row-reduce the matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -6 \\ 0 & 1 & 1 & 0 & -6 \\ 0 & 0 & 1 & -2 & -4 \\ 0 & 0 & -1 & 1 & 13 \end{array}\right] \quad \text{to} \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & -1 & 9 \end{array}\right]$$

Using the operation $-R_4 \to R_4$, we change the signs of the entries in row 4 to obtain the matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 1 & -9 \end{array}\right]$$

Finally, using the leading 1 in row 4 as the pivot, we eliminate the terms above it in column 4. Specifically, the operations

$$\begin{aligned} R_4 + R_1 &\to R_1 \\ -2R_4 + R_2 &\to R_2 \\ 2R_4 + R_3 &\to R_3 \end{aligned}$$

applied to the last matrix give

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -19 \\ 0 & 1 & 0 & 0 & 16 \\ 0 & 0 & 1 & 0 & -22 \\ 0 & 0 & 0 & 1 & -9 \end{array}\right]$$

which is in reduced row echelon form.

The solution can now be read directly from the reduced matrix, giving us

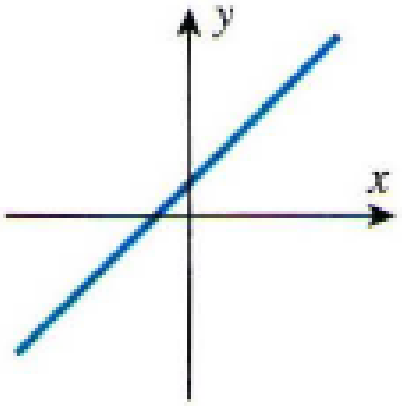
$$x_1 = -19 \qquad x_2 = 16 \qquad x_3 = -22 \qquad \text{and} \qquad x_4 = -9.$$

CONSISTENT SYSTEM → INFINITE SOLUTION

EXAMPLE 4 A Linear System with Infinitely Many Solutions

Solve the linear system

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$



Infinitely many solutions (coincident lines)

Solution We can eliminate x from the second equation by adding -4 times the first equation to the second. This yields the simplified system

$$\begin{aligned} 4x - 2y &= 1 \\ 0 &= 0 \end{aligned}$$

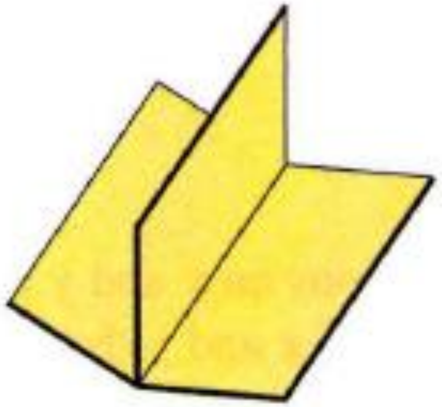
The second equation does not impose any restrictions on x and y and hence can be eliminated. Thus, the solutions of the system are those values of x and y that satisfy the single equation

$$4x - 2y = 1 \qquad x = \tfrac{1}{4} + \tfrac{1}{2}t, \quad y = t \tag{7}$$

EXAMPLE 5 A Linear System with Infinitely Many Solutions

Solve the linear system

$$\begin{aligned} x - y + 2z &= 5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 6z &= 15 \end{aligned}$$



Infinitely many solutions (intersection is a line)

Solution This system can be solved by inspection, since the second and third equations are multiples of the first. Geometrically, this means that the three planes coincide and that those values of x, y, and z that satisfy the equation

$$x - y + 2z = 5 \tag{8}$$

automatically satisfy all three equations. Using the method of Example 7 in Section 1.3, we can express the solution set parametrically as

$$x = 5 + t_1 - 2t_2, \quad y = t_1, \quad z = t_2$$

Example: Solve the following system of linear equations by Gaussian elimination method

$$\begin{aligned} x + y + 3z &= 1 \\ 2x + 3y - z &= 3 \\ 5x + 7y + z &= 7 \end{aligned}$$

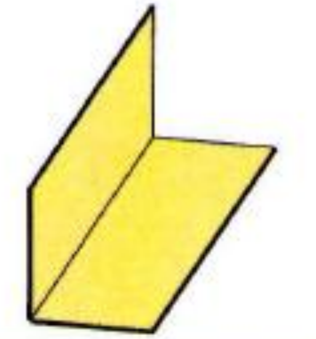
Solution: Augmented matrix of the given system is $A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 3 & -1 & 3 \\ 5 & 7 & 1 & 7 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 3 & -1 & 3 \\ 5 & 7 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & -7 & 1 \\ 0 & 2 & -14 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 10 & 0 \\ 0 & 1 & -7 & 1 \end{bmatrix}$$

$$\begin{aligned} x + 10z &= 0 \\ y - 7z &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x &= -10z \\ y &= 1 + 7z \end{aligned}$$

Here z is the only free variable. The parametric solution, using $z = a$, is as follows:

$$x = -10a, \;\; y = 1 + 7a, \;\; z = a \qquad \text{or} \qquad u = (-10a, \;\; 1 + 7a, \;\; a)$$



Infinitely many solutions (two coincident planes; intersection is a line)

Infinitely many solutions (planes are all coincident; intersection is a plane)

EXAMPLES

Solve the following system using its augmented matrix M:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 - 2x_4 + 4x_5 &= 1 \\ 2x_1 + 5x_2 - 8x_3 - x_4 + 6x_5 &= 4 \\ x_1 + 4x_2 - 7x_3 + 5x_4 + 2x_5 &= 8 \end{aligned}$$

Reduce the augmented matrix M to echelon form and then to row canonical form:

$$M = \begin{bmatrix} 1 & 2 & -3 & -2 & 4 & 1 \\ 2 & 5 & -8 & -1 & 6 & 4 \\ 1 & 4 & -7 & 5 & 2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & -2 & 4 & 1 \\ 0 & 1 & -2 & 3 & -2 & 2 \\ 0 & 2 & -4 & 7 & -2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & -2 & 4 & 1 \\ 0 & 1 & -2 & 3 & -2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 0 & 8 & 7 \\ 0 & 1 & -2 & 0 & -8 & -7 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 24 & 21 \\ 0 & 1 & -2 & 0 & -8 & -7 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

Write down the system corresponding to the row canonical form of M and then transfer the free variables to the other side to obtain the free-variable form of the solution:

$$\begin{aligned} x_1 \quad + x_3 + \quad 24x_5 &= 21 \\ x_2 - 2x_3 - \quad 8x_5 &= -7 \\ x_4 + 2x_5 &= 3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 &= 21 - x_3 - 24x_5 \\ x_2 &= -7 + 2x_3 + 8x_5 \\ x_4 &= 3 - 2x_5 \end{aligned}$$

Here x_1, x_2, x_4 are the pivot variables and x_3 and x_5 are the free variables. Recall that the parametric form of the solution can be obtained from the free-variable form of the solution by simply setting the free variables equal to parameters, say $x_3 = a$, $x_5 = b$. This process yields

$$x_1 = 21 - a - 24b, \quad x_2 = -7 + 2a + 8b, \quad x_3 = a, \quad x_4 = 3 - 2b, \quad x_5 = b$$

or

$$u = (21 - a - 24b, \ -7 + 2a + 8b, \ a, \ 3 - 2b, \ b)$$

which is another form of the solution.

EXAMPLE 4 Solve the linear system.

$$\begin{cases} 3x_1 - x_2 + x_3 + 2x_4 = -2 \\ x_1 + 2x_2 - x_3 + x_4 = 1 \\ -x_1 - 3x_2 + 2x_3 - 4x_4 = -6 \end{cases}$$

Solution The linear system in matrix form is

$$\left[\begin{array}{rrrr|r} 3 & -1 & 1 & 2 & -2 \\ 1 & 2 & -1 & 1 & 1 \\ -1 & -3 & 2 & -4 & -6 \end{array}\right]$$

which can be reduced to

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & 3 & 5 \\ 0 & 0 & 1 & -\frac{20}{3} & -10 \end{array}\right]$$

Notice that the system has infinitely many solutions, since from the last row we see that the variable x_4 is a free variable. We can reduce the matrix further, but the solution can easily be found from the echelon form by back substitution, giving us

$$x_3 = -10 + \frac{20}{3}x_4$$

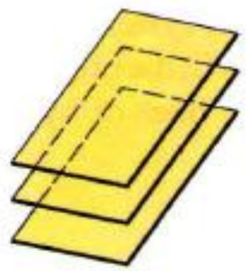
$$x_2 = 5 + x_3 - 3x_4 = 5 + \left(-10 + \frac{20}{3}x_4\right) - 3x_4 = -5 + \frac{11}{3}x_4$$

$$x_1 = 1 - 2x_2 + x_3 - x_4 = 1 - \frac{5}{3}x_4$$

Letting x_4 be the arbitrary parameter t, we see the general solution is

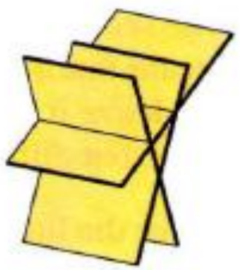
$$S = \left\{ \left(1 - \frac{5t}{3}, -5 + \frac{11t}{3}, -10 + \frac{20t}{3}, t\right) \middle| \, t \in \mathbb{R} \right\}$$

INCONSISTENT SYSTEM → NO SOLUTION

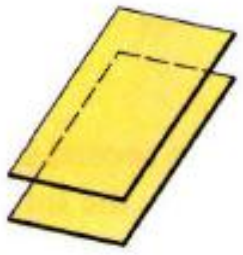


No solutions (three parallel planes; no common intersection)

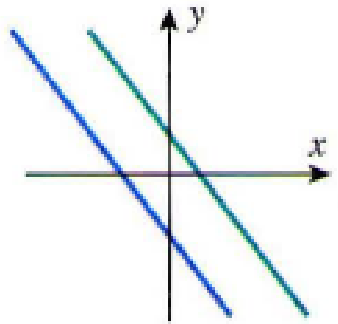
No solutions (two parallel planes; no common intersection)



No solutions (no common intersection)



No solutions (two coincident planes parallel to the third; no common intersection)



No solution

EXAMPLE 3 A Linear System with No Solutions

Solve the linear system

$$\begin{aligned} x + y &= 4 \\ 3x + 3y &= 6 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -3 times the first equation to the second equation. This yields the simplified system

$$\begin{aligned} x + y &= 4 \\ 0 &= -6 \end{aligned}$$

The second equation is contradictory, so the given system has no solution. Geometrically, this means that the lines corresponding to the equations in the original system are parallel and distinct. We leave it for you to check this by graphing the lines or by showing that they have the same slope but different y-intercepts. ■

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EXAMPLES

Example: Solve

$$\begin{aligned} x - 2y + 4z &= 2 \\ 2x - 3y + 5z &= 3 \\ 3x - 4y + 6z &= 7 \end{aligned}$$

Solution:

First reduce the augmented matrix M to echelon form as follows:

$$M = \begin{bmatrix} 1 & -2 & 4 & 2 \\ 2 & -3 & 5 & 3 \\ 3 & -4 & 6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 2 & -6 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The third row corresponds to the degenerate equation $0x + 0y + 0z = 3$, which has no solution. Thus, "DO NOT CONTINUE." The original system also has no solution. (Note that the echelon form indicates whether or not the system has a solution.)

EXAMPLE 5 Solve the linear system.

$$\begin{cases} x + y + z = 4 \\ 3x - y - z = 2 \\ x + 3y + 3z = 8 \end{cases}$$

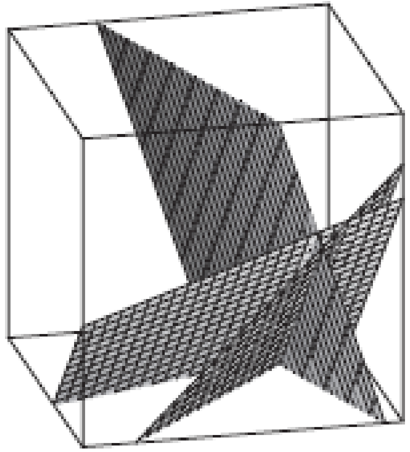
Solution To solve this system, we reduce the augmented matrix to triangular form. The following steps describe the process.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 3 & -1 & -1 & 2 \\ 1 & 3 & 3 & 8 \end{array}\right] \quad \begin{array}{l} -3R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \end{array} \quad \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -4 & -4 & -10 \\ 0 & 2 & 2 & 4 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \quad \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 2 & 2 & 4 \\ 0 & -4 & -4 & -10 \end{array}\right]$$

$$\tfrac{1}{2}R_2 \rightarrow R_2 \quad \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -4 & -4 & -10 \end{array}\right]$$

$$4R_2 + R_3 \rightarrow R_3 \quad \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{array}\right]$$



The third row of the last matrix corresponds to the equation $0 = -2$. As this system has no solution, the system is inconsistent. This can also be seen from the fact that the three planes do not have a common intersection, as shown in Fig.2.

HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

Problem: Use Gauss–Jordan elimination to solve the homogeneous linear system

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0\\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0\\
5x_3 + 10x_4 \qquad\quad + 15x_6 &= 0\\
2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$

Solution:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 2 & 6 & -5 & -2 & 4 & -3 & 0\\ 0 & 0 & 5 & 10 & 0 & 15 & 0\\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned}
x_1 + 3x_2 \qquad + 4x_4 + 2x_5 &= 0\\
x_3 + 2x_4 \qquad &= 0\\
x_6 &= 0
\end{aligned}$$

Solving for the leading variables we obtain

$$\begin{aligned}
x_1 &= -3x_2 - 4x_4 - 2x_5\\
x_3 &= -2x_4\\
x_6 &= 0
\end{aligned}$$

If we now assign the free variables x_2, x_4, and x_5 arbitrary values r, s, and t, respectively, we can express the solution set parametrically as

$$x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s,\ x_5 = t,\ x_6 = 0$$

We leave it for you to show that the solution set can be expressed in vector form as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)$$

or alternatively, as

$$\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6 \end{bmatrix} = r\begin{bmatrix} -3\\ 1\\ 0\\ 0\\ 0\\ 0 \end{bmatrix} + s\begin{bmatrix} -4\\ 0\\ -2\\ 1\\ 0\\ 0 \end{bmatrix} + t\begin{bmatrix} -2\\ 0\\ 0\\ 0\\ 1\\ 0 \end{bmatrix}$$