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United International University

Name (Optional)

ID No.

Section

Invigilator's Signature with date

Course Code

Trimester / Semester : Spring / Summer / Fall, 20........

Name of Exam : Class Test / Mid-term / Final

Date: ....

☒ Homogenous differential equation examples:

A first order differential equation is said to be homogenous if both co-efficients M and N are homogenous functions of same degree. — Def^n:

OR,
Homogenous D.E.
Form:

$$\frac{dy}{dx} = f(x, y)$$

$$\boxed{M(x,y)}\,dx + \boxed{N(x,y)}\,dy = 0$$

Example:

1) $f(x, y) = x^3 + y^3$ — (homogenous function)

2) $f(x, y) = x^3 + y^3 + 1$. ✗ (not homogenous function)

3) $(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$ ✓ (homogenous D.E)

4) $\frac{dy}{dx} = x^2 + y^2$ (Not homogenous D.E.)

$(x^2 + y^2)\,dx = dy$

Here, degree of

[Zill] ch → 2.5

$\frac{dy}{dx} + x^2 = 0$ $f' + f^2 = 0$ $\frac{d(f)}{dx} + f^2 = 0$

Homogenous Equation: (Function)

$$f(tx, ty) = t^{\alpha} f(x, y)$$

A homog. d.E is an equation containing a differentiation and a function with variables

suppose, $M(x,y)\,dx + N(x,y)\,dy = 0$

$$M(tx, ty) = t^{\alpha} M(x,y)$$

$$N(tx, ty) = t^{\alpha} N(x,y)$$

substitution : $y = ux$

same degree in x and y.

$\frac{dy}{dx} = u + x\frac{du}{dx}$

$\Rightarrow dy = u\,dx + x\,du$

(product Rule)

<u>Problem:</u> 1) $(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$

<u>Soln:</u> Here, $M(x,y) = x^2 + y^2$

$$(tx)^2 + (ty)^2 = t^2 f(x,y)$$

$$(tx)^2 - (tx)(ty) = t^2 f(x,y)$$

So, homogenous.

Now, Let, $y = ux$ $\therefore dy = u\,dx + x\,du$

Then, $(x^2 + u^2x^2)\,dx + (x^2 - ux^2)(u\,dx + x\,du) = 0$

$$\Rightarrow x^2\,dx + \cancel{u^2x^2\,dx} + x^2u\,dx + x^3\,du - \cancel{u^2x^2\,dx} - ux^3\,du = 0$$

$$\Rightarrow x^2(1+u)\,dx + x^3(1-u)\,du = 0$$ } use Separable

$$\Rightarrow -\frac{x^2\,dx}{x^3} = \int \frac{(1-u)\,du}{1+u} \Rightarrow \int \frac{dx}{x} = \int \frac{u-1}{u+1}\,du$$

$$\int \frac{-1+u}{1+u}\,du$$

$$= \int \frac{u-1}{1+u}\,du$$

$$= \int \frac{u+1-2}{u+1}\,du$$

$$= \int 1 - \frac{2}{1+u}\,du$$

$$= u - 2\ln|1+u|$$

Therefore, $u - 2\ln|1+u| = \ln x -$

$$\Rightarrow y/x - 2\ln\left|1+\frac{y}{x}\right|$$

$$= \ln x + c$$

[Ans]

Exercise : (2.5)

Problem: (1) $(x-y)\,dx + x\,dy = 0$

$M(x,y) = x-y$; $N(x,y) = x$

Then, $M(tx, ty) = tx - ty = t\,f(x,y)$

$N(tx, ty) = tx = t\,f(x,y)$

Now, Let, $y = ux$ $\therefore dy = u\,dx + x\,du$.

Then, $(x-ux)\,dx + x\,[u\,dx + x\,du] = 0$.

$\Rightarrow x\,dx - \cancel{ux\,dx} + \cancel{xu\,dx} + x^2\,du = 0$

$\Rightarrow \cancel{x(1+u)\,dx}$ $x\,dx + x^2\,du = 0$

$\Rightarrow \dfrac{x\,dx}{x^2} = -du$

$\Rightarrow \dfrac{1}{x}\,dx = -du$

$\Rightarrow \ln x = -u + C$ $\therefore \ln x + \dfrac{y}{x} = C$

[Ans.]

Problem: (b) $(y^2 + xy)\,dx - x^2\,dy = 0$

$$M(tx, ty) = (ty)^2 + tx\,ty$$
$$= t^2y^2 + t^2xy$$
$$= t^2 f(x,y)$$

$$N(tx, ty) = t^2x^2$$
$$= t^2 f(x,y)$$

Then, Let, $y = ux$

$$\Rightarrow dy = u\,dx + x\,du$$

Now, $(u^2x^2 + x\cdot ux)\,dx - x^2(u\,dx + x\,du) = 0$

$$\Rightarrow u^2x^2\,dx + \cancel{x^2u\,dx} - \cancel{x^2u\,dx} - x^3\,du = 0$$

$$\Rightarrow u^2x^2\,dx = x^3\,du$$

$$\Rightarrow \frac{x^2\,dx}{x^3} = \frac{du}{u^2}$$

$$\Rightarrow \ln x = -u^{-1} + c$$

$$\therefore \ln x + \frac{y}{x} = c \quad \text{[Ans.]}$$

7) <u>Problem:</u> (7) $\frac{dy}{dx} = \frac{y-x}{y+x}$

$M(tx, ty)$

$(ty - tx)$

$t(y-x)$

$\Rightarrow (x+y)\,dy = (y-x)\,dx$

Now, $y = ux$

$\therefore dy = u\,dx + x\,du$

Then; $(x + ux)\,[u\,dx + x\,du] = (ux - x)\,dx$

$\Rightarrow x u\,\cancel{dx} + x^2\,du + u^2 x\,dx + u x^2\,du - u\cancel{x}\,dx + x\,dx = 0$

$\Rightarrow (u^2+1)\,x\,dx = -x^2(1+u)\,du$

$\Rightarrow \frac{x\,dx}{-x^2} = \frac{(1+u)\,du}{u^2+1}$

$\int \frac{f'(x)\,dx}{f(x)} = \ln|f(x)|$

$\Rightarrow -\ln x = \int \frac{u}{u^2+1}\,du + \int \frac{du}{u^2+1}$

$\int \frac{dx}{a^2+x^2} = \tan^{-1}\left(\frac{x}{a}\right)$

$\Rightarrow -\ln x = \frac{1}{2}\ln(u^2+1) + \tan^{-1}(u) + C$

$\therefore -\ln x = \frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right) + \tan^{-1}\left(\frac{y}{x}\right) + C$

[Ans.]

$$-y + (x + \sqrt{xy})\frac{dy}{dx} = 0$$

Problem: (a) $-y\,dx + (x + \sqrt{xy})\,dy = 0$

$M(tx, ty) = -ty = t(-y) = tf(x,y)$

$N(tx, ty) = tx + \sqrt{t^2 xy} = t[x + \sqrt{xy}]$
$= tf(x,y)$

Let, $y = ux \quad \therefore dy = u\,dx + x\,du$

then, $-ux\,dx + (x + \sqrt{x.ux})[u\,dx + x\,du]$

$\Rightarrow \cancel{-ux\,dx} + \cancel{xu\,dx} + x^2\,du + xu\sqrt{u}\,dx + x^2\sqrt{u}\,du = 0$

$\Rightarrow (\sqrt{u} + 1)x^2\,du + u^{\frac{3}{2}}x\,dx = 0$

$\Rightarrow \frac{1 + \sqrt{u}}{u^{3/2}}\,du = -\frac{1}{x}\,dx$

$\Rightarrow \frac{du}{u^{3/2}} + \frac{1}{u}\,du = -\ln x.$

$\Rightarrow -2u^{-1/2} + \ln u = -\ln x + c$

$\Rightarrow -2\left(\frac{y}{x}\right)^{-1/2} + \ln(y/x) = -\ln x + c$

[Ans.]