1 class:(05+06) Date: 08.10.2023

(Ch - 1.3) [BD]

Classification of differential equation;

A differential equation is any equation which contain derivatives, either ordinary or partial derivative

$$\text{D.E} \rightarrow \text{Linear DE.}$$
$$\text{D.E} \rightarrow \text{Non linear DE.}$$

Order and degree of a differential equation:

Order = highest derivative in the equation

Degree = Degree of derivative of highest order in the equation.

Example:

1) $$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0$$

Order = 2

Degree = 1

2) $$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 1$$

Order = 2

Degree = 3

3) $\frac{dy}{dx} + y = 1$

Order = 1

Degree = 1

4) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{2/3} = \frac{d^2y}{dx^2}$

~~Order = 2~~

~~Degree = 1~~

both side cube,

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^2 = \left(\frac{d^2y}{dx^2}\right)^3$$

$$\Rightarrow 1 + 2\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^4 = \left(\frac{d^2y}{dx^2}\right)^3$$

Order = 2

Degree = 3

5) $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 1 + x$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = (1+x)^2$$

Order = 1

Degree = 2

xy, y^2, $(y')^2$

Identify Linear / Non-Linear DE:

1) y and y' is in multiplication

2) y and y' took power: $y^2, (y')^2$ etc — dependent variable and it's derivative has power greater than 1.

3) Dependent and independent variable cannot be the argument of trigonometric / exponential / logarithmic function.

(Book → BD)

Exercise (1.3) [1 - 27] (odd number).

Problem: Determine the order of given differential equation ; also state whether the equation is linear or non-linear :

1) $t^2 \left(\dfrac{d^2y}{dt^2}\right) + t\dfrac{dy}{dt} + 2y = \sin t$

→ Independent variable

Order = 2 ; Degree = 1 .

Linear DE .

3) $\dfrac{d^4y}{dt^4} + \dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 1$

Order = 4 ; Degree = 1

Linear DE .

5) $\dfrac{d^2y}{dt^2} + \underbrace{\sin(t + y)} = \sin t$

Order = 2 ; Degree = 1

Non - Linear DE .

7) $\underbrace{(1 + y^2)\dfrac{d^2y}{dt^2}} + t\dfrac{dy}{dt} + y = e^t$

Order = 2 ; Degree = 1

Non - Linear DE

4) $\dfrac{dy}{dt} + ty^2 = 0$

Non - Linear DE

Order = 1

? 6) $\dfrac{d^3y}{dt^3} + t\dfrac{dy}{dt} + (\cos^2 t)\, y = t^3$

(Linear DE)

Order = 3

Degree = 1

(*) Verify that each given function is a solu[tion] of the differential equation:

7) Given, $y_1(t) = e^t$; $y_2(t) = \cosh(t)$

and $y'' - y = 0$

$y_1 = e^t$

$y_1' = e^t$

$y_1'' = e^t$

$e^t - e^t = 0$.

$y_2' = \sinh(t)$

$y_2'' = \cosh(t)$

$\cosh(t) - \cosh(t) = 0$

[Verified]

9) $y = 3t + t^2$; $ty' - y = t^2$

$y' = 3 + 2t$

Now, $t(3 + 2t) - 3t - t^2$ ~~$= t^2$~~

$= \cancel{3t} + 2t^2 - \cancel{3t} - t^2$

$= 2t^2 - t^2$

$= t^2$

[Verified]

(13) $y'' + y = \sec t \quad ; 0 < t < \pi/2$

$y = (\cos t) \ln \cos t + t \sin t$

$y' = \cos t \ln(\cos t) + t \sin t$

$= \cos t \frac{1}{\cos t} \cdot (-\sin t) + \ln(\cos t)(-\sin t)$
$\quad + t \cos t + \sin t$

$= -\cancel{\sin t} - \sin t \ln(\cos t) + t \cos t + \cancel{\sin t}$

$= t \cos t - \sin t \ln(\cos t)$

$y'' = -t \sin t + \cos t - \sin t \frac{1}{\cos t}(-\sin t)$
$\quad - \ln(\cos t) \cos t$

$= -t \sin t + \cos t + \sin t \tan t - \ln(\cos t) \cos t$

Now, $\cancel{-t \sin t} + \cos t + \sin t \tan t - \cancel{\cos t \ln(\cos t)}$
$\quad + \cancel{t \sin t} + \cancel{\cos t \ln(\cos t)}$

$= \cos t + \sin t \tan t$

$= \cos t + \sin t \frac{\sin t}{\cos t}$

$= \frac{\cos^2 t + \sin^2 t}{\cos t}$

$= \frac{1}{\cos t}$

$= \sec t$

[verified]

(4) Determine the values of r for wh[ich] the given DE has solutions of the fo[rm] $y = e^{rt}$

(15) $y' + 2y = 0$ → given DE.

and $y = e^{rt}$

Now, $y' = re^{rt}$

Then, $re^{rt} + 2e^{rt} = 0$

$\Rightarrow e^{rt}(r+2) = 0$

$e^{rt} = 0$; $r+2=0$

$\Rightarrow$ No solution $r = -2$

[Ans]

(17) $y'' + y' - 6y = 0$

$y = e^{rt}$

$y' = re^{rt}$

$y'' = r^2 e^{rt} + e^{rt}$

Then, $r^2 e^{rt} + e^{rt} + re^{rt} - 6e^{rt} = 0$

$\Rightarrow r^2 e^{rt} - 5e^{rt} = 0$

$\Rightarrow e^{rt}(r^2 - 5) = 0$ $\therefore r = \pm\sqrt{5}$

19) $t^2y'' + 4ty' + 2y = 0$

$y = t^r$

$y' = rt^{r-1}$

$y'' = r(r-1)t^{r-2}$

then, $t^2 r(r-1)t^{r-2} + 4t\, r t^{r-1} + 2t^r = 0$

$\Rightarrow r(r-1)t^r + 4t^r r + 2t^r = 0$

$\Rightarrow t^r(r^2 - r + 4r + 2) = 0$

$\Rightarrow t^r(r^2 + 3r + 2) = 0$

$r^2 + 3r + 2 = 0$

$\Rightarrow r^2 + 2r + r + 2 = 0$

$\Rightarrow r(r+2) + 1(r+2) = 0$

$\therefore r = -2, -1$

Determine the order of given PDE Also state whether linear / ~~no~~ non-linear

21) $u_{xx} + u_{yy} + u_{zz} = 0$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Order = 2

Linear PDE.

$$\left(\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0\right)$$

(23) $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$

Order = 4

Linear PDE

(22) $u_{xx} + u_{yy} + \underbrace{u u_x}_{} + u u_y + u = 0$

Dependent variable

Order = 2

Non- Linear PDE.

$\frac{\partial u}{\partial x}$ (u = Dependent)

(25) $u_{xx} + u_{yy} = 0$ [Verify that $u_1(x,y)$ and $u_2(x,y)$ satisfy the DE]

$u_1(x,y) = \cos x \cosh y$

$u_x = -\sin x \cosh y$, $u_{xx} = -\cos x \cosh y$

$u_y = \cos x \sinh y$; $u_{yy} = \cos x \cosh y$

Now, $-\cos x \cosh y + \cos x \cosh y = 0$ [Verified]

$u_2(x,y) = \ln(x^2+y^2)$

$u_x = \frac{1}{x^2+y^2} \cdot 2x$

$u_{xx} = \frac{2(x^2+y^2) - 2x \cdot 2x}{(x^2+y^2)^2}$

$= \frac{2y^2 - 2x^2}{(x^2+y^2)^2}$

$u_y = \frac{2y}{x^2+y^2}$

$u_{yy} = \frac{(x^2+y^2)2 - 2y}{(x^2+y^2)^2}$

$= \frac{-2y^2+2x^2}{(x^2+y^2)^2}$

Now, $u_{xx} + u_{yy}$

$= \frac{2y^2 - 2x^2 - 2y^2 + 2x^2}{(x^2+y^2)^2}$

$= 0$ [Verified]

* Solving Linear equation (Method of Integrating Factors)

Rule

First order linear equation:

$$\frac{dy}{dt} + p(t)\,y = g(t) \quad \text{[standard form]}$$

Integrating Factor: $I = e^{\int p(t)\,dt}$

to make equation integrable easily.

[Used to solve dE] ← be multiplied to make the function integrable

Another form:

$$\frac{dy}{dt} + \overset{\text{const}}{a}\,y = g(t)$$

Integrating Factor: $I = \mu(t) = e^{at}$

Example: (01)

Solve the differential equation

$$\frac{dy}{dt} + \underbrace{\frac{1}{2}y}_{ay} = \underbrace{\frac{1}{2}e^{t/3}}_{g(t)} \longrightarrow (1)$$

Soln: $I.F. = e^{\frac{1}{2}t}$

Then, $e^{1/2\,t}\dfrac{dy}{dt} + e^{\frac{1}{2}t}\,\frac{1}{2}y = \frac{1}{2}e^{\frac{1}{2}t}\,e^{t/3}$

$\Rightarrow \dfrac{d}{dt}\left[y e^{\frac{1}{2}t}\right] = \frac{1}{2}e^{5/6\,t}$

Then, Integrating, $y e^{1/2\,t} = \frac{6}{5}\times\frac{1}{2}\times\cancel{6}\; e^{5/6\,t} + c$

$\Rightarrow y e^{1/2\,t} = \frac{3}{5}e^{5/6\,t}$

$y = \frac{3}{5}e^{t/3} + c\,e^{-t/2}$ [Ans.]

Example: (2) Solve the DE

$$\frac{dy}{dt} - 2y = 4 - t$$

<u>Sol^n:</u> $\quad I.F. = e^{-2t}$.

Now, $\quad e^{-2t}\dfrac{dy}{dt} - 2ye^{-2t} = e^{-2t}(4-t)$

$$\Rightarrow \frac{d}{dt}\left(ye^{-2t}\right) = 4e^{-2t} - te^{-2t}$$

Then, Integrating,

$$ye^{-2t} = \frac{4}{-2}e^{-2t} - \left[t\,\frac{e^{-2t}}{-2} - \int\left\{\frac{d}{dt}(t)\int e^{-2t}\,dt\right\}dt\right]$$

$$\Rightarrow ye^{-2t} = -2e^{-2t} + \frac{t}{2}e^{-2t} + \int \frac{e^{-2t}}{-2}\,dt$$

$$= -2e^{-2t} + \tfrac{1}{2}te^{-2t} - \frac{1}{2}\int e^{-2t}\,dt$$

$$\Rightarrow \cancel{ye^{-2t}} = \cancel{-\frac{3}{2}e^{-2t}} + \cancel{\frac{1}{4}e^{-2t}} + c$$

$$\Rightarrow \cancel{ye^{-2t}} = \cancel{-\frac{5}{4}e^{-2t}} + c$$

$$\Rightarrow ye^{-2t} = -2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t}$$

$$\Rightarrow ye^{-2t} = -\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t} + c$$

$$\therefore y = -\frac{7}{4} + \tfrac{1}{2}t + ce^{2t}$$ [Ans.]

$$t\int e^{-2t}\,dt - \int\left\{\frac{d}{dt}(t)\int e^{-2t}\,dt\right\}dt = t\,\frac{e^{-2t}}{-2} - \int \frac{e^{-2t}}{-2}\,dt$$

$\frac{dy}{dt} + p(t)\, y = g(t)$

Example : 3 Solve the initial value problem

$$ty' + 2y = 4t^2 \quad ; \quad y(1) = 2$$

Soln: ~~I.F. = e^{2t}~~

~~Now, $te^{2t}\frac{dy}{dt} + 2ye^{2t} = 4t^2e^{2t}$~~

~~$\Rightarrow \frac{d}{dt}[$~~

Divide the equation by t, we get,

$$y' + \left(\frac{2}{t}\right)y = 4t \longrightarrow \frac{dy}{dt} + P(t)\,y = g(t)$$

$$\text{I.F.} = e^{\int \frac{2}{t} dt}$$

$$= e^{2\ln t}$$

$$= t^2$$

Now, $t^2 y' + \left(\frac{2}{t}\right) \times t^2 y = 4t^3$

$$\Rightarrow t^2\frac{dy}{dt} + 2ty = 4t^3$$

$$\Rightarrow \frac{d}{dt}[yt^2] = 4t^3$$

Now, $yt^2 = \int 4t^3\, dt$

$$\Rightarrow yt^2 = 4\frac{t^4}{4} + c$$

$\Rightarrow \quad yt^2 = t^4 + c$

$\therefore \quad y = t^2 + ct^{-2}$

Then, $\quad y(1) = 2$

Now, $\quad \Rightarrow (1)^2 + c(1)^{-2} = 2$

$\Rightarrow \quad 1 + c = 2$

$\therefore \quad c = 1$

So, $\quad y = t^2 + \frac{1}{t^2} \quad ; t > 0$

[Ans.]

Exercise (2.1):

$\boxed{1}$ $y' + 3y = t + e^{-2t}$

$I.F. = e^{3t}$

Now, $e^{3t}\frac{dy}{dt} + 3e^{3t}y = te^{3t} + e^{t}$

$\Rightarrow \frac{d}{dt}\left[ye^{3t}\right] = e^{t} + te^{3t}$

Integrating,

$$ye^{3t} = e^{t} + \left[t\,\frac{e^{3t}}{3} - \int\left\{\frac{d}{dt}(t)\int\right.\right.$$

$$= e^{t} + \left[\frac{1}{3}te^{3t} - \int\frac{1}{3}e^{3t}\,dt\right]$$

$$\Rightarrow ye^{3t} = e^{t} + \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + c$$

$$\Rightarrow ye^{3t} = e^{t} + \frac{1}{3}te^{3t} - 1/9\,e^{3t} + c$$

$$\Rightarrow y = e^{-3t}\left[e^{t} + \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t}\right] + ce^{-3t}$$

$$\therefore y = e^{t}e^{-3t} + \frac{t}{3} - \frac{1}{9} + ce^{-3t}$$

[Ans.]

$p(t)=1$

$\int dt$

3) $y' + y = te^{-t} + 1$

I.F. $= e^{t}$

Now, $e^{t} y' + e^{t} y = t + e^{t}$

$\Rightarrow \frac{d}{dt}[ye^{t}] = t + e^{t}$

Integrating,

$ye^{t} = \frac{t^2}{2} + e^{t} + c$

$\therefore y = \frac{1}{2}e^{-t} t^2 + 1 + ce^{-t}$

[Ans.]

$\frac{t^2}{2}$

7) $y' + 2ty = 2te^{-t^2}$

I.F. $= e^{\int 2t \cdot dt}$

$= e^{2\frac{t^2}{2}}$

$= e^{t^2}$

Now, $y'e^{t^2} + 2te^{t^2} y = 2t$

$\Rightarrow \frac{d}{dt}[ye^{t^2}] = 2t$

Integrating, $ye^{t^2} = 2\frac{t^2}{2} + c$

$\Rightarrow y = e^{-t^2} t^2 + ce^{-t^2}$

[Ans.]

$\boxed{11}$ $y' + y = 5\sin 2t$

$I.F. = e^{\int dt}$

$= e^t$

Now, $e^t y' + ye^t = 5e^t \sin 2t$

$\Rightarrow \frac{d}{dt}[ye^t] = 5e^t \sin 2t$

IATE then integrating,

$\int 5e^t \sin 2t = 5\left[\sin 2t\, e^t - \int\left\{\frac{d}{dt}(\sin 2t)\int e^t\, dt\right\} dt\right]$

$= 5\left[e^t \sin 2t - \int 2\cos 2t\, e^t\, dt\right]$

$= 5e^t \sin 2t - 10\left[\cos 2t\, e^t - \int -2\sin 2t\, e^t\right]$

$= 5e^t \sin 2t - 10e^t \cos 2t - 20\int e^t \sin 2t$

$\Rightarrow 5I = 5e^t \sin 2t - 10e^t \cos 2t - 20I$

$\Rightarrow 25I = 5e^t \sin 2t - 10e^t \cos 2t$

$\therefore I = \frac{1}{25}\left[5e^t \sin 2t - 10e^t \cos 2t\right]$

$\therefore y = e^{-t}\left[\frac{1}{25}\left\{5e^t \sin 2t - 10e^t \cos 2t\right\}\right]$

[Ans.]

(15) $ty' + 2y = t^2 - t + 1$; $y(1) = 1/2$; $t > 0$

$\Rightarrow y' + \frac{2}{t} y = t - 1 + \frac{1}{t}$

I.F.

$= e^{\int \frac{2}{t} dt}$

$= e^{2 \ln t}$

$= t^2$

Now, $t^2 y' + 2ty = t^3 - t^2 + t$

$\Rightarrow \frac{d}{dt} [yt^2] = t^3 - t^2 + t$

Integrating, $yt^2 = \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + c$

$\Rightarrow y = t^{-2} \left[\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} \right] + ct^{-2}$

$\therefore y = \frac{1}{4} t^2 - \frac{1}{3} t + \frac{1}{2} + ct^{-2}$

[Ans.]

$\frac{1}{3} + \frac{1}{2} + C$

C

$\boxed{19}$ $t^3 y' + 4t^2 y = e^{-t}$; $y(-1) = 0$

$\Rightarrow y' + \frac{4}{t} y = e^{-t} t^{-3}$

$\therefore$ I.F. $= e^{\int \frac{4}{t} dt}$

$= e^{4 \ln t}$

$= t^4$

Then, $y' t^4 + 4t^3 y = e^{-t} t$

$\Rightarrow \frac{d}{dt} [y t^4] = t e^{-t}$

Now, Integrating,

$y t^4 = -t e^{-t} - \int \left\{ \frac{d}{dt}(t) \int e^{-t} dt \right\} dt$

$= -t e^{-t} - \int \frac{e^{-t}}{-1} dt$

$= -t e^{-t} + \int e^{-t} dt$

$\Rightarrow y t^4 = -t e^{-t} - e^{-t} + c$

$\Rightarrow y = t^{-4} [-t e^{-t} - e^{-t}] + c t^{-4}$

$\therefore y = -e^{-t} t^{-3} - e^{-t} t^{-4} + c t^{-4}$

$y(-1) = 0$

$\Rightarrow -e(-1)^{-3} - e(-1)^{-4} + c(-1)^{-4} = 0$

$\Rightarrow e - e + c = 0$

$\therefore c = 0$

So, $y = -e^{-t} t^{-3} - e^{-t} t^{-4}$

[Ans.]

Fall - 2023 (Mid Question)

$x \frac{dy}{dx} + y = 4x + 1 \; ; y(1) = 8$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = 4 + \frac{1}{x}$

Int. Factor $= e^{\int \frac{1}{x} dx}$

$= e^{\ln x} = x$

Now, $x \frac{dy}{dx} + y = 4x + 1$

$\Rightarrow \frac{d}{dx}(yx) = 4x + 1$

Integrating,

$yx = 4\frac{x^2}{2} + x + C$

$\Rightarrow yx = 2x^2 + x + C$

Now, $x = 1$, $y = 8$

$8 \times 1 = 2(1)^2 + 1 + C$

$\Rightarrow 8 = 2 + 1 + C$

$\therefore C = 5$

So, $yx = 2x^2 + x + 5$

$\therefore y = \frac{2x^2 + x + 5}{x} = 2x + 1 + 5/x$

Example:

1) $y^2 \frac{dx}{dy} + 2xy = 3x\sin y$ (dependent variable)

Order = 1 degree = 1

(~~Non~~ - Linear)

2) $\ln x \left(\frac{d^3u}{dx^3}\right)^4 + (x^2+3)\left(\frac{d^2u}{dx^2}\right)^2 = \tan x$

Order = ~~4~~ 3

degree = 4

(Non - Linear)

3) $x\frac{dy}{dx} = 6y + 12x^4 y^{2/3}$

Order = 1

degree = 1

(Linear)

4) $y^2 \frac{dx}{dy} + 2x\cos y = xe^y$ D.V.

Order = 1 degree = 1

~~Non~~ Linear.

5) $(1+x)\frac{dy}{dx} = 6y''' + 12x^4$

Order = 3

degree = 1

(Linear)