Graph Basics

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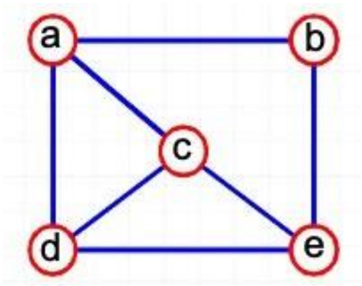
What is a Graph?

- A graph is a pair (*V, E*), where
  - *V* is a set of nodes, called vertices
  - *E* is a collection of pairs of vertices, called edges
- V(G) and E(G) represent the sets of vertices and edges of G, respectively
- Example:



A tree is a special type of graph!

V= {a, b, c, d, e}

E= {(a,b), (a,c), (a,d), (b,e), (c,d), (c,e), (d,e)}

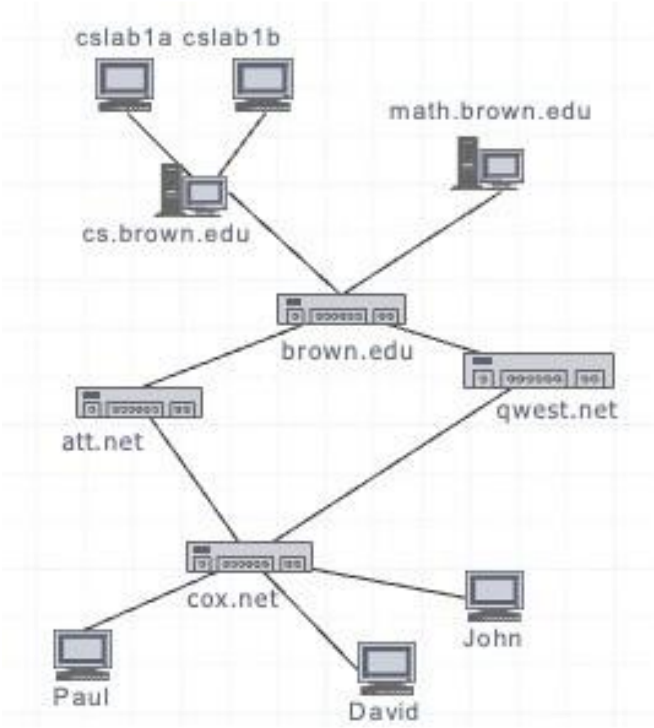
Applications

Electronic circuits
- Printed circuit board
- Integrated circuit

Transportation networks
- Highway network
- Flight network

Computer networks
- Local area network
- Internet
- Web

Databases
- Entity-relationship diagram

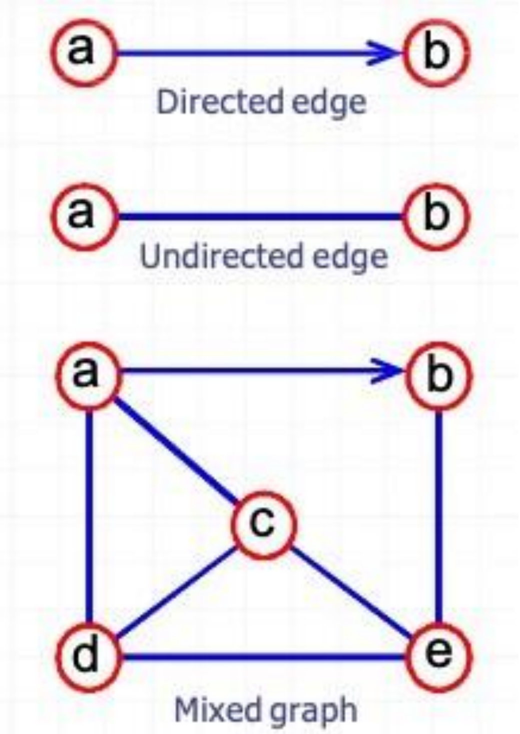


What can we do with graph?

- Find a path from one place to another
- Find the shortest path from one place to another
- Determine connectivity
- Find the "weakest link" (min cut)
    - check amount of redundancy in case of failures
- Find the amount of flow that will go through them

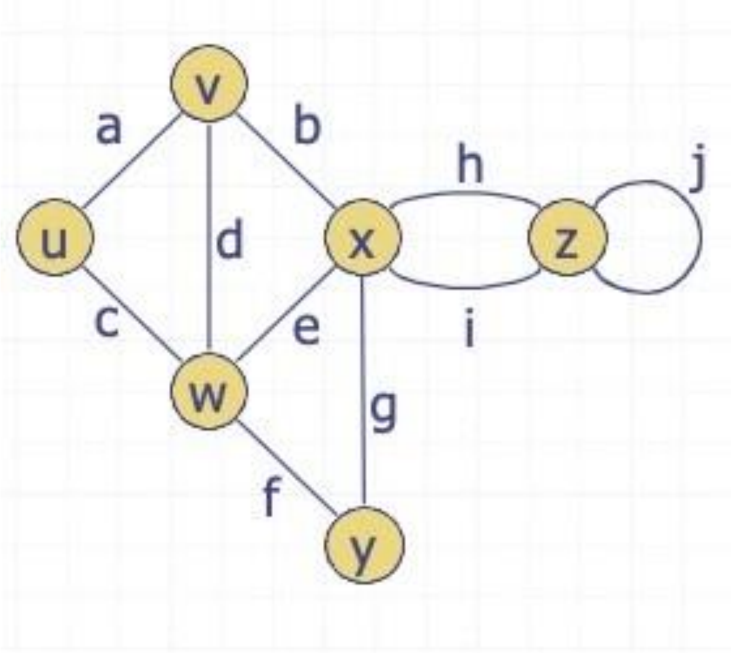
Edge and Graph Types

- Directed edge
  - ordered pair of vertices (***u***,***v***)
  - first vertex ***u*** is the origin
  - second vertex ***v*** is the destination
- Undirected edge
  - unordered pair of vertices (***u***,***v***)
- Directed graph (Digraph)
  - all the edges are directed
  - e.g., route network
- Undirected graph
  - all the edges are undirected
  - e.g., flight network
- Mixed graph
  - some edges are undirected and some edges are directed
  - e.g., a graph modeling a city map



Terminology

- End vertices (or endpoints) of an edge
  - u and v are the endpoints of *a*
- Edges incident to a vertex
  - a, d, and b are incident to v
- Adjacent vertices
  - u and v are adjacent
- Degree of a vertex
  - x has degree 5
- Parallel edges
  - h and i are parallel edges
- Self-loop
  - j is a self-loop

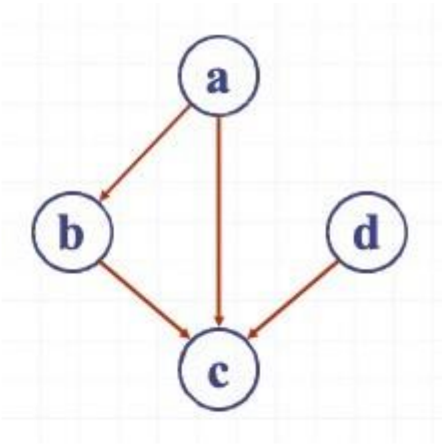


Terminology

- Out-degree : Outgoing edges of a vertex
  - (a, b) and (a, c) are outgoing edges of vertex a
- In-degree : Incoming edges of a vertex
  - (b, c), (d, c) and (a, c) are incoming edges of vertex c

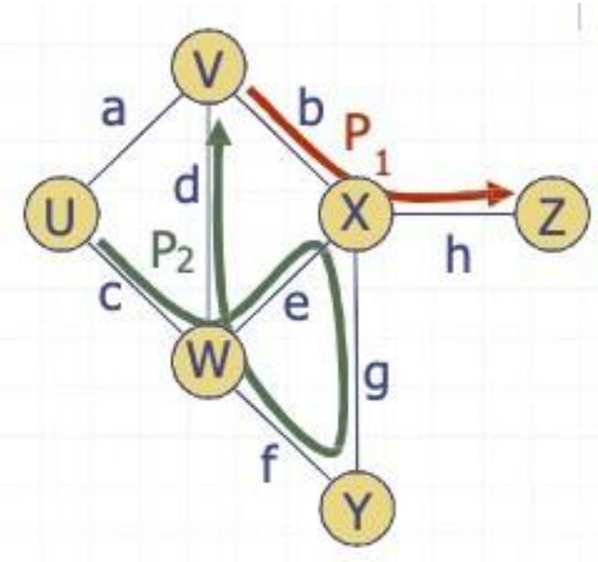
- In-degree of a vertex
  - c has in-degree 3
  - b has in-degree 1

- Out-degree of a vertex
  - a has out-degree 2
  - b has out-degree 1



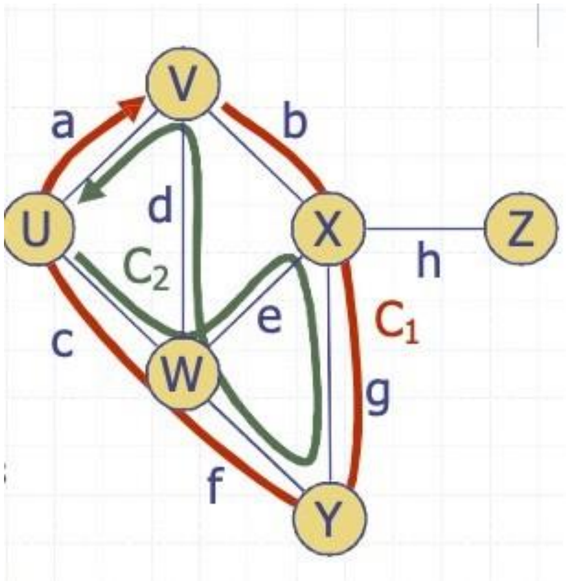
Terminology

- Path
  - sequence of alternating vertices and edges
  - begins with a vertex
  - ends with a vertex
  - each edge is preceded and followed by its endpoints
- Simple path
  - path such that all its vertices and edges are distinc
- Examples
  - P_1=(V, b , X , h , Z)
    is a simple path
  - P_2=(U, c, W, e, X, g, Y, f , W, d , V)
    is a path that is not simple



Terminology

- Cycle
  - A cycle is a path whose start and end vertices are the same
  - each edge is preceded and followed by its endpoints
- Simple cycle
  - A cycle is simple if each edge is distinct and each vertex is distinct,
  - except for the first and the last one
- Examples
  - C_1=(V, b, X, g, Y, f, W, c, U, a, V) is a simple cycle
  - C_2=(U,c,W,e,X,g,Y,f,W,d,V,a,U) is a cycle that is not simple

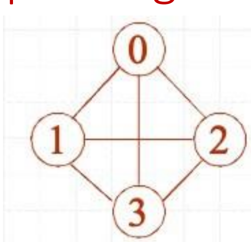


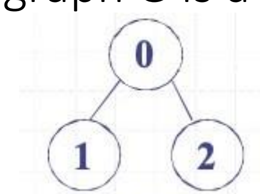
Terminology

- Dense graph: $|E|$ ⍰ $|V|^2$; Sparse graph: $|E|$ ⍰ $|V|$
- A weighted graph associates weights with either the edges or the vertices
- A complete graph is a graph that has the maximum number of edges
  - for undirected graph with n vertices, the maximum number of edges is *n(n-1)/2*
  - for directed graph with n vertices, the maximum number of edges is *n(n-1)*

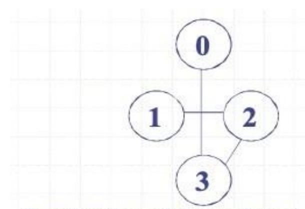
Terminology

- A subgraph of G is a graph G’ such that
  - V(G’) is a subset of V(G) [V(G’) $\subseteq$ V(G)] and
  - E(G’) is a subset of E(G) [E(G’) $\subseteq$ E(G)]
- A spanning subgraph G’ of G is a subgraph of G that contains all the vertices of G, that is
  - V(G’) is equal to V(G) [V(G’) = V(G)] and
  - E(G’) is a subset of E(G) [E(G’) $\subseteq$ E(G)]
- A forest is a graph without cycles.
- A tree is a connected forest, that is, a connected graph without cycles.
- A spanning tree of a graph G is a spanning subgraph that is a (free) tree.





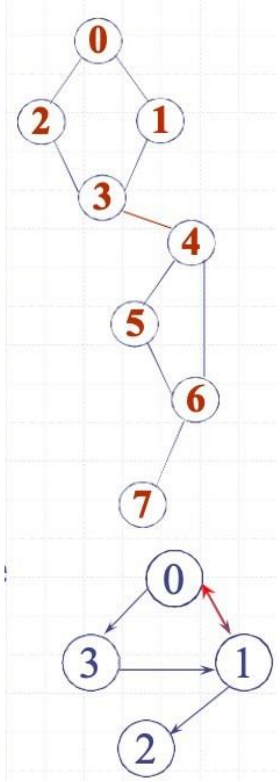
A subgraph



A spanning subgraph (tree)

Terminology

- In a graph G, two vertices, v_0 and v_1, are connected if there is a path in G from v_0 to v_1
- A graph is connected if, for every pair of distinct vertices v_i and v_j, there is a path from v_i to v_j
- A connected component of an undirected graph is a maximal connected subgraph.
- A tree is a graph that is connected and acyclic.
- A directed graph is strongly connected if there is a directed path from v_i to v_j and also from v_j to v_i.
- A strongly connected component is a maximal subgraph that is strongly connected.



Properties

Property 1

For an undirected graph

$S_v \deg(v) = 2m$

Proof: each edge is counted twice

Property 2

For a directed graph

$S_v \text{indeg}(v) = S_v \text{outdeg}(v) = m$

Proof: each for out-degreeedge is counted once for in-degree and once

Property 3

If G is a simple undirected graph, then m ≤ n(n - 1)/2, and if G is a simple directed graph, then m ≤ $\boldsymbol{n}$(n - 1).

Proof: each vertex has degree at most (n - 1). Then use Property 1 and Property 2.

Graph Representations

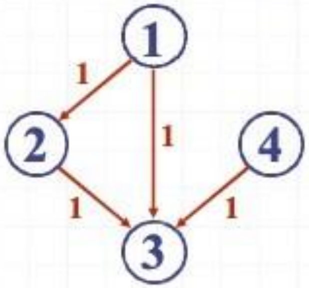
- For graphs to be computationally useful, they have to be conveniently represented in programs
- There are two computer representations of graphs:
  - Adjacency matrix representation
  - Adjacency lists representation

Adjacency Matrix Representation

- Assume V = {1, 2, …, n}
- An adjacency matrix represents the graph as a n xn matrix A:

▪A[i, j] = 1 if edge (i, j) $\in$ E (or weight of edge)

= 0 if edge (i, j) $\notin$ E



A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

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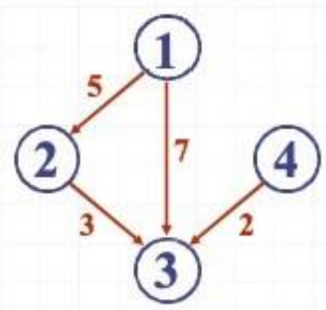
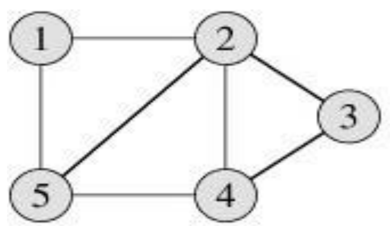
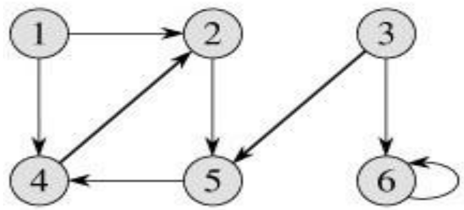


A	1	2	3	4
1	0	5	7	0
2	0	0	3	0
3	0	0	0	0
4	0	0	2	0

Adjacency Matrix Representation



	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0



	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

The adjacency matrix for an undirected graph is symmetric;
The adjacency matrix for a digraph need not be symmetric

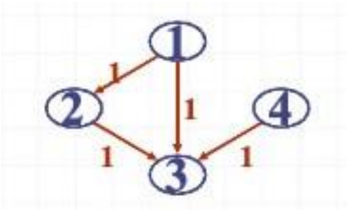
Adjacency Matrix Representation

Pros:

- Simple to implement
- Easy and fast to tell if a pair (i, j) is an edge: simply check if A[i, j] is 1 or 0
- Can be very efficient for small graphs

Cons:

- No matter how few edges the graph has, the matrix takes $O(n^2)$ in memory

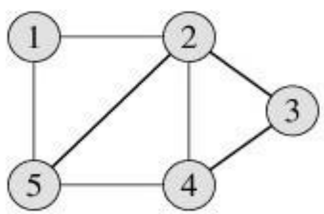


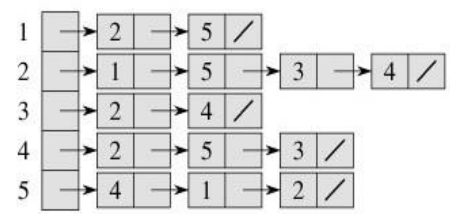
A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

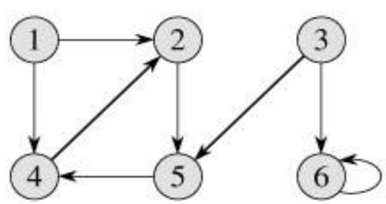
Adjacency List Representation

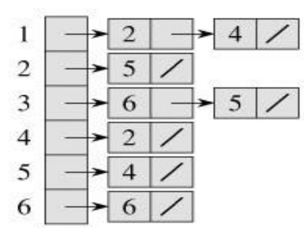
A graph is represented by a one-dimensional array L of linked lists, where

- L[i] is the linked list containing all the nodes adjacent to node i.









Adjacency List Representation

Pros:

- Saves on space (memory): the representation takes $O(|V|+|E|)$ memory.
- Good for large, sparse graphs (e.g., planar maps)

Cons:

- It can take up to $O(n)$ time to determine if a pair of nodes (i, j) is an edge: one would have to search the linked list L[i], which takes time proportional to the length of L[i].

Asymptotic Perfromance

Assumptions: (1) n vertices, m edges, (2) simple graph

Operations	Adjacency List	Adjacency Matrix
Space	$O(n + m)$	$O(n^2)$
incidentEdges(v)	$O(\deg(v))$	$O(n)$
areAdjacent(v, w)	$O(\min(\deg(v), \deg(w)))$	$O(1)$
insertVertex(v)	$O(1)$	$O(n^2)$
insertEdge(e)	$O(1)$	$O(1)$
removeVertex(v)	$O(\deg(v))$	$O(n^2)$
removeEdge(e)	$O(1)$	$O(1)$


THANK YOU