Waves and Oscillation

Course- PHY 2105 / PHY 105

Lecture 4

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Equations

Equation of SHM: $\dfrac{d^2x}{dt^2} = -\dfrac{k}{m}x$

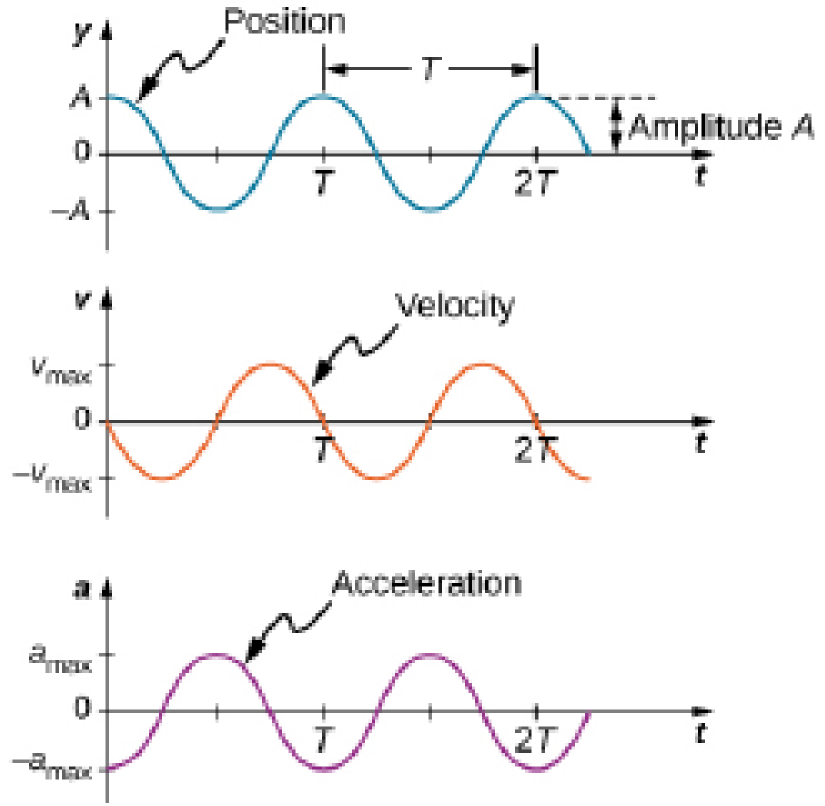
$$x(t) = A\cos(\omega t + \phi)$$
$$v(t) = -v_{\max}\sin(\omega t + \phi)$$
$$a(t) = -a_{\max}\cos(\omega t + \phi)$$
$$x_{\max} = A$$
$$v_{\max} = A\omega$$
$$a_{\max} = A\omega^2.$$

$$\overbrace{x(t)}^{\text{Displacement at time } t} = \underbrace{\mathbf{A}}_{\text{Amplitude}} \cos(\overbrace{\omega t + \phi}^{\text{Phase}})$$

Angular frequency

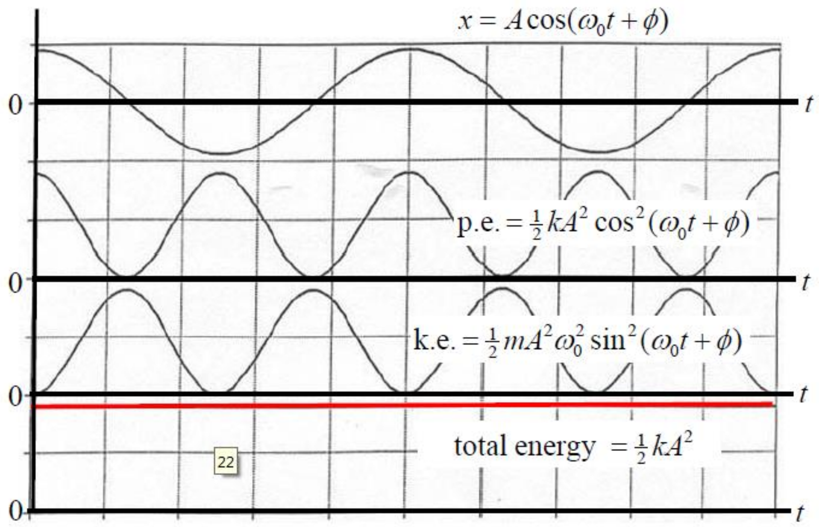
Time

Phase constant or phase angle



Energy

Energy of the mass-spring simple harmonic oscillator



For the mass-spring system: $x = A\cos(\omega_0 t + \phi)$

Potential energy $= \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)$

k.e. $= \frac{1}{2}mv^2 = \frac{1}{2}m[-A\omega_0\sin(\omega_0 t + \phi)]^2 = \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \phi)$

Total energy = p.e. + k.e

$$= \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi) + \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \phi)$$

$$= \frac{1}{2}kA^2 \quad \left(= \frac{1}{2}m\omega_0^2 A^2\right) \qquad (\therefore E \propto A^2)$$

We can now write: $\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2$

$$\boxed{\therefore v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} \qquad \text{or} \qquad v(x) = \pm\omega_0\sqrt{A^2 - x^2}}$$

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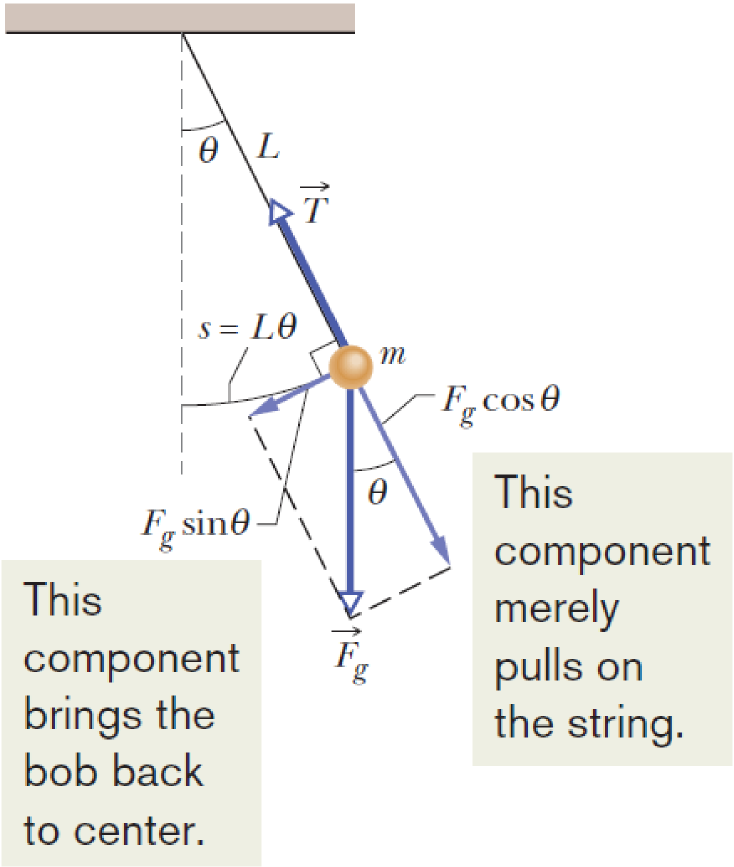
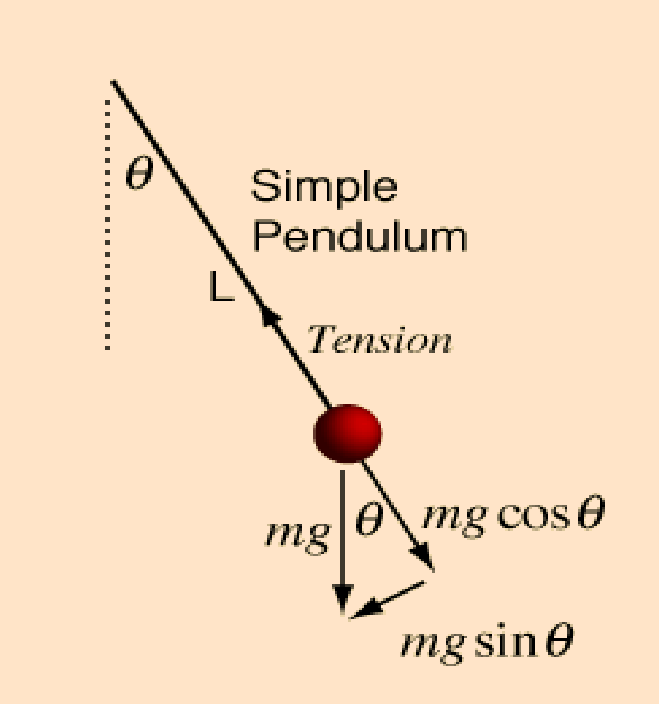
Simple Pendulum



A simple pendulum consists of a particle of mass *m* (called the *bob* of the pendulum) suspended from one end of an unstretchable, massless string of length *L* that is fixed at the other end.

The only forces acting on the bob are the force of gravity (i.e., the weight of the bob) and tension from the string.



From the above figure restoring force

$$F = -mg\sin\theta$$

If the angle θ is very small sinθ is very nearly equal to θ, The displacement along the arc is-

$$x = L\theta$$

$$F = -mg\theta$$

Acceleration $\dfrac{d^2x}{dt^2} = L\dfrac{d^2\theta}{dt^2}$ $\quad Force = mL\dfrac{d^2\theta}{dt^2}$

$$mL\frac{d^2\theta}{dt^2} = -mg\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Therefore, $\boxed{\omega^2 = \dfrac{g}{L} \quad \text{And} \quad T = 2\pi\sqrt{\dfrac{L}{g}}}$

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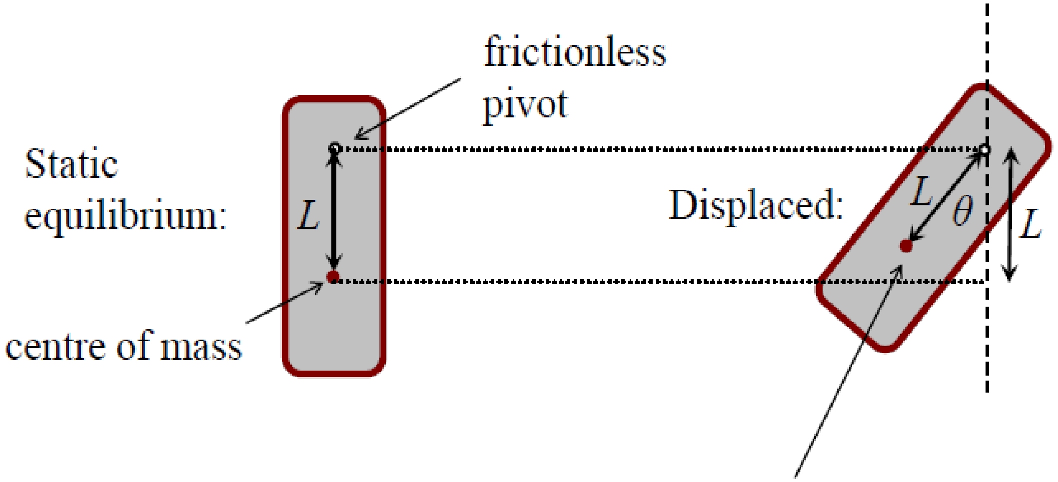
Physical Pendulum

A **physical pendulum** is any object whose oscillations are similar to those of the simple pendulum, but cannot be modeled as a point mass on a string, and the mass distribution must be included into the equation of motion.



In displaced position, centre of mass is $L - L\cos\theta$ above the equilibrium position.

Recall $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$ For small angles, $\cos\theta \approx 1 - \frac{\theta^2}{2}$

Gravitational potential energy = $mgL(1-\cos\theta) = mgL\frac{\theta^2}{2}$

Gravitational potential energy = $\frac{1}{2}mgL\theta^2$

Kinetic energy = $\frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2$

Total energy = $\frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}mgL\theta^2$ = constant

$$\therefore I\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgL\theta\frac{d\theta}{dt} = 0$$ … true for all $\frac{d\theta}{dt}$

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{mgL}{I}\theta = -\omega_0^2\theta$$ where $\omega_0 = \sqrt{\frac{mgL}{I}}$

Equation of SHM

The moment of inertia of the pendulum about an passing through the point of suspension is

$$= mK^2 + mL^2$$

K= radius of gyration
L= distance between suspension and oscillation points
= distance of suspension point and Center of gravity

Therefore, $\omega_0 = \sqrt{\dfrac{gL}{K^2 + L^2}}$

Time Period

$$T = 2\pi\sqrt{\frac{K^2 + L^2}{Lg}}$$

What if this had been a Simple Pendulum instead?

Example 2.6

Determine the length of a simple pendulum that will swing back and forth in simple harmonic motion with a period of 1.00 s.

$$L = \frac{T^2 g}{4\pi^2} = \frac{(1.00\ \mathrm{s})^2 (9.80\ \mathrm{m/s^2})}{4\pi^2} = \boxed{0.248\ \mathrm{m}}$$

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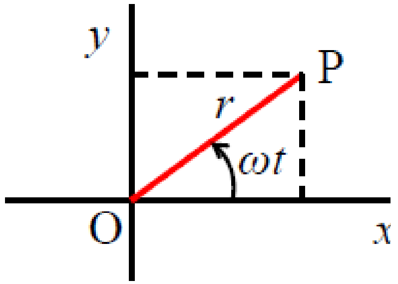
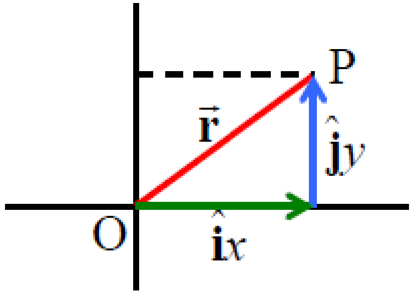
Complex number

Consider a vector $\overrightarrow{OP}$ of length r which rotates with angular velocity ω

The point P has coordinates

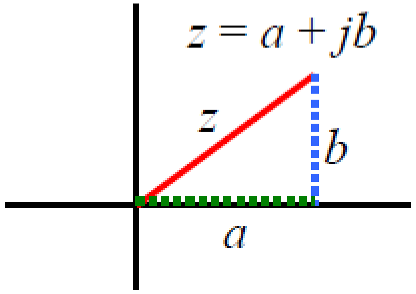
$$x = r\cos\omega t \qquad y = r\sin\omega t$$



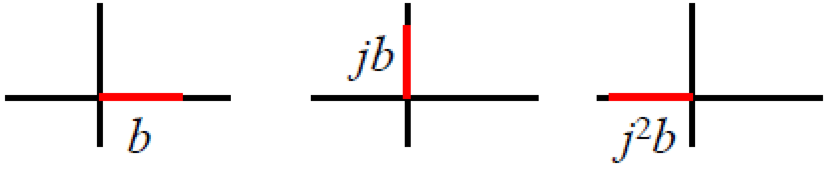


Modify our notation to $z = x + jy$

... where x means a displacement in the x-direction and jy means a displacement in the y-direction

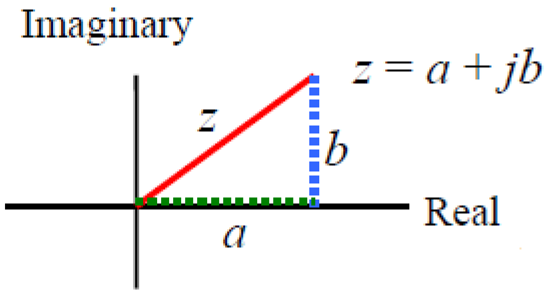


We can also think of j as a rotation through $\pi/2$ anticlockwise



Hence $j^2 = -1$

... really talking about vectors in the complex number plane:



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From Taylor's theorem: $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

therefore $e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots$

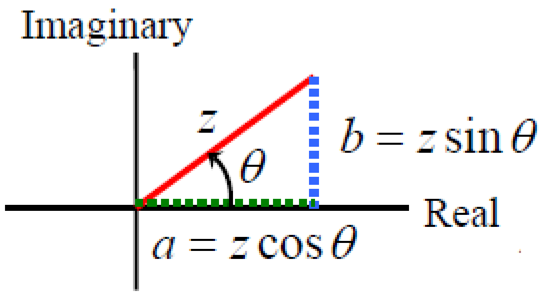
and $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots$ and $j\sin\theta = j\theta - \frac{j\theta^3}{3!} + \ldots$

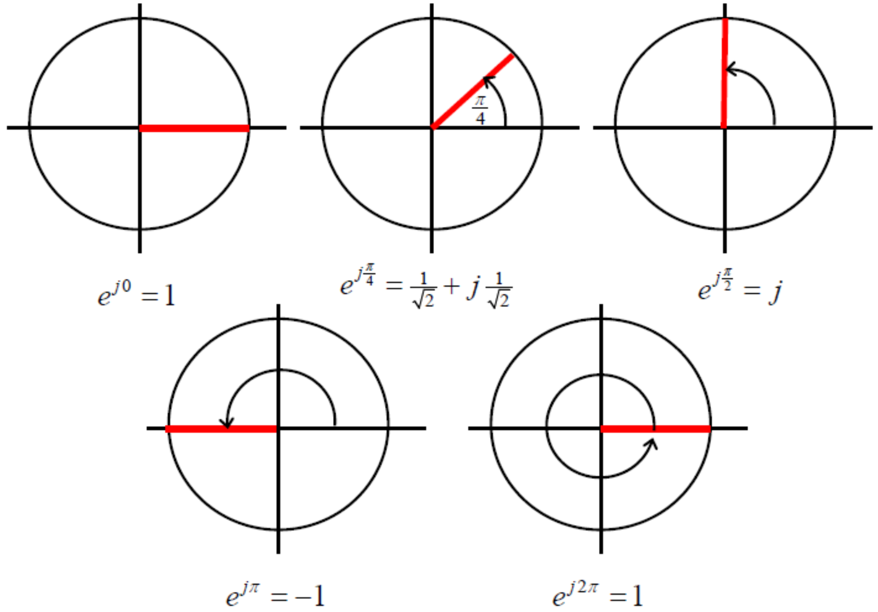
Hence $\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$ **Euler relation**

Then $z = a + jb = |z| e^{j\theta}$

where $|z| = \sqrt{a^2 + b^2}$

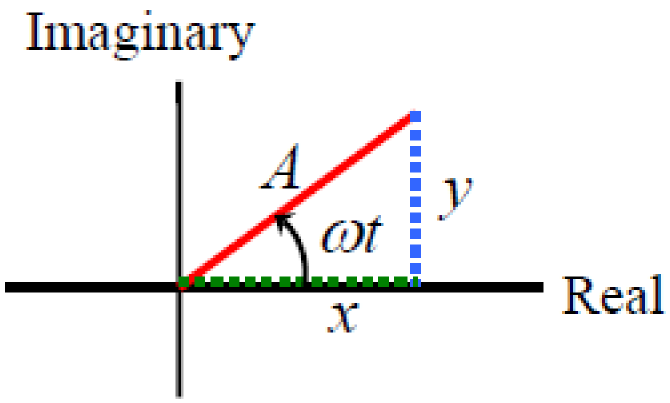
$\tan\theta = b/a$





For our rotating vectors:

$$z = x + jy$$
$$= A\cos\omega t + jA\sin\omega t$$
$$= A(\cos\omega t + j\sin\omega t)$$
$$= Ae^{j\omega t}$$



Now write: $Ae^{j(\omega t+\phi)} = A\cos(\omega t+\phi) + jA\sin(\omega t+\phi)$

... and remember that the physical quantity x (e.g. a displacement) is the real part of z :

i.e. $x = \mathrm{Re}[z]$

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Complex & SHM

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

Using $z = x + jy$ becomes $\frac{d^2z}{dt^2} + \omega_0^2 z = 0$

Try $z = Ae^{j(\omega t + \phi)}$

$$\therefore A(j\omega)^2 e^{j(\omega t + \phi)} + \omega^2 A e^{j(\omega t + \phi)} = 0$$

Therefore $z = Ae^{j(\omega t + \phi)}$ is the most general solution

A and ϕ are determined from the initial conditions.

Take real part of z:

$$x = \mathrm{Re}[z] = A\cos(\omega_0 t + \phi)$$

Damped Harmonic Motion

When oscillating bodies do not move back and forth between precisely fixed limits because frictional force dissipate the energy and amplitude of oscillation **decreases** with time and finally die out. Such harmonic motion is called Damped Harmonic Motion.

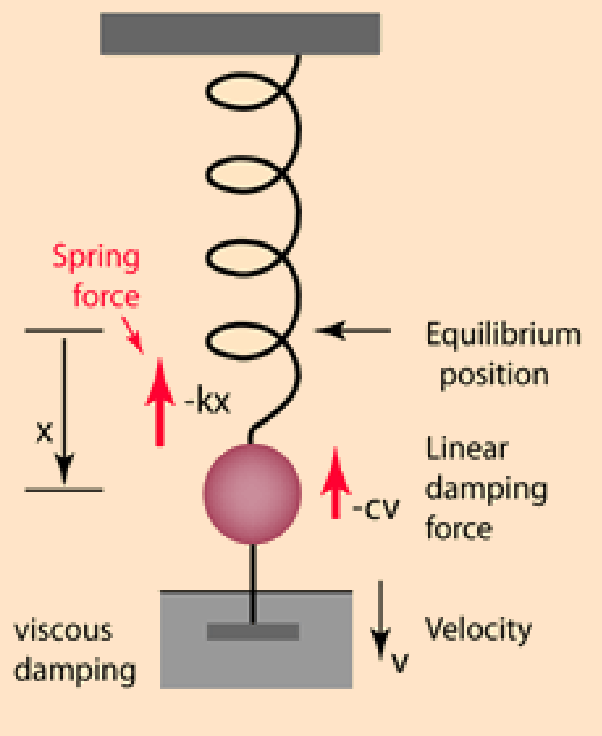
The decrease in amplitude caused by dissipative forces is called **Damping**, and the corresponding motion is called **Damped Oscillation**.

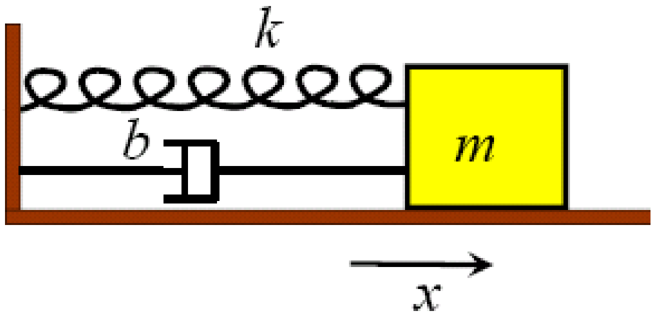
This occurs because the non-conservative damping force **removes** energy from the system, usually in the form of thermal energy

DHM Eqn

In spring-mass oscillator

the damping force is proportional to the velocity and acts against the direction of motion





For horizontal forces on the mass: $ma = -kx - bv$

or $m\dfrac{d^2x}{dt^2} = -kx - b\dfrac{dx}{dt}$

or $\dfrac{d^2x}{dt^2} + \gamma\dfrac{dx}{dt} + \omega_0{}^2 x = 0$ where $\begin{cases} \omega_0 = \sqrt{\dfrac{k}{m}} \\ \gamma = \dfrac{b}{m} \end{cases}$

γ : "damping constant" unit: s^{-1}

- "life time" $= \dfrac{1}{\gamma}$


That's all Folks!