Formulae for LC Circuit

$\frac{d^2 i}{dt^2} + \frac{1}{LC} i = 0\ ;\ {\omega_0}^2 = \frac{1}{LC}$	$T = 2\pi\sqrt{LC} \qquad f = \frac{1}{2\pi\sqrt{LC}}$
$Q(t) = Q_0 \cos(\omega_0 t + \varphi)$	

Formulae for RLC Circuit

$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$ $damping\ constant, \gamma = \frac{R}{L}$ Resonant angular frequency, $\omega_0 = \sqrt{\frac{1}{LC}}$ $Resonant\ frequency, f_0 = \frac{\omega_0}{2\pi}$	$Oscillatory$ ${\omega_0}^2 > \frac{\gamma^2}{4} \qquad or\ \frac{1}{LC} > \frac{R^2}{4L^2}$ $Critical\ damping$ ${\omega_0}^2 = \frac{\gamma^2}{4} \qquad or\ \frac{1}{LC} = \frac{R^2}{4L^2}$ $Overdamping$ ${\omega_0}^2 < \frac{\gamma^2}{4} \qquad or\ \frac{1}{LC} < \frac{R^2}{4L^2}$

$$Damping\ angular\ frequency, \omega_d = \sqrt{{\omega_0}^2 - \frac{\gamma^2}{4}} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$Time\ period, T = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}$$

$$Frequency, f = \frac{1}{T} = \frac{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}{2\pi}$$

$$Lifetime, \tau = \frac{1}{\gamma}$$

$$Q(t) = Q_0\, e^{-\frac{\gamma}{2}t} \cos(\omega_d t + \varphi)$$

1. **A capacitor 1.0μF, an inductor 0.2 H and a resistance 800Ω are joined in series. Is the circuit oscillatory? Find the frequency of oscillation.**

L= 0.2 H C= 1 μF = 1×10^{-6} F and R = 800 Ω

$$\omega_0^2 = \frac{1}{LC}$$

$$= \frac{1}{0.2 \times 1 \times 10^{-6}} = 5 \times 10^{6}\ {rad^2}/{s^2}$$

$$\frac{\gamma^2}{4} = \frac{R^2}{4L^2}$$

$$= \frac{800^2}{4 \times 0.2^2}$$

$$= 4 \times 10^{6}\ {rad^2}/{s^2}$$

Since $\omega_0^2 > \frac{\gamma^2}{4}$, circuit is oscillatory.

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = 159.15\ Hz$$

2. **Labid wants to constructed an RLC circuit that produce critical damping. He has a capacitor and inductor with value, C = 0.003 μF, L = 0.1 mH respectively.**

   i) What is the value of resistance he must connect to make his desired circuit?

   ii) If R = 800 Ω, is the circuit oscillatory? If oscillatory, find the frequency of oscillation.

Here, $C = 0.003\ \mu F = 0.003 \times 10^{-6} F$

$L = 0.1\ mH = 0.1 \times 10^{-3} H$

i) For critical damping,

$$\omega_0^2 = \frac{\gamma^2}{4}$$

$$\frac{1}{LC} = \frac{R^2}{4L^2}$$

$$R = \sqrt{\frac{4L}{C}} = \sqrt{\frac{4 \times 0.1 \times 10^{-3}}{0.003 \times 10^{-6}}} = 11.55\ \Omega$$

ii)

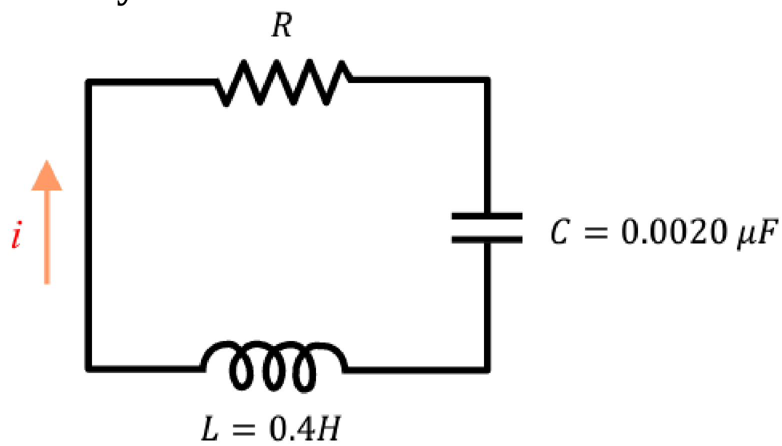
$$\omega_0^2 = \frac{1}{LC} = \frac{1}{0.1 \times 10^{-3} \times 0.003 \times 10^{-6}} = 3.3 \times 10^9\ \frac{rad^2}{s^2}$$

$$\frac{\gamma^2}{4} = \frac{R^2}{4L^2} = \frac{800^2}{4 \times (0.1 \times 10^{-3})^2} = 1.6 \times 10^{13}\ \frac{rad^2}{s^2}$$

Since $\omega_0^2 < \frac{\gamma^2}{4}$

$$Therefore, the\ circuit\ is\ not\ oscullatory\ (overdampming).$$

3. Draw an LRC series circuit using L = 0.4 h, C = 0.0020 µF components. What is the maximum resistance for which circuit will be oscillatory?



$C = 0.0020\ \mu F$
$\quad = 0.002 \times 10^{-6}\ F$
$L = 0.4\ H$
$R = ?$

Let, this circuit will be oscillatory for maximum resistance R.

Now,

$$\frac{R^2}{4L^2} = \frac{1}{LC}$$

$$\text{or, } R^2 = \frac{4L^2}{LC}$$

$$\text{or, } R = \sqrt{\frac{4L}{C}}$$

$$\text{or, } R = \sqrt{\frac{4 \times 0.4}{0.002 \times 10^{-6}}}$$

$$\therefore R = 28284.27\ \Omega$$

DHM spring-mass system

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \qquad damping\ constant, \gamma = \frac{b}{m} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$$Displacement, \qquad x(t) = A_0\, e^{-\frac{\gamma}{2}t}\cos(\omega_d t + \varphi)$$

$$Amplitude, A(t) = A_0 e^{-\frac{\gamma}{2}t}$$

$$Damping\ angular\ frequency,$$

$$\omega_d = \sqrt{{\omega_0}^2 - \frac{\gamma^2}{4}} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

$$Time\ period, T = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

$$Frequency, f = \frac{1}{T} = \frac{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}{2\pi}$$

$$Lifetime, \tau = \frac{1}{\gamma}$$

$$Oscillatory$$

$${\omega_0}^2 > \frac{\gamma^2}{4} \quad or \quad \sqrt{4mk} > b$$

$$Critical\ damping$$

$${\omega_0}^2 = \frac{\gamma^2}{4} \quad or \quad \sqrt{4mk} = b$$

$$Overdamping$$

$${\omega_0}^2 < \frac{\gamma^2}{4} \quad or \quad \sqrt{4mk} < b$$

4. **For a damped oscillator, $m = 250\ g, k = 85\ N/m, and\ b = 70\ g/s$.**

(a) What is the period of the motion?

(b) How long does it take for the amplitude of the damped oscillations to drop to half its initial value?

(c) How long does it take for the mechanical energy to drop to one-half its initial value?

(d) What is its lifetime? How many oscillations does it complete in life time?

(e) The maximum displacement of undamped oscillator is 35 cm. If the damping is stopped after five cycles, what is the damping energy?

Here, $m = 250\ g = 0.25\ kg,\ k = 85\ N/m\ \ and\ \ b = 70\ g/s = 0.07\ kg/s$

(a)

$Damping\ angular\ frequenc$

$$\omega_d = \sqrt{\omega_0{}^2 - \frac{\gamma^2}{4}}$$
$$= \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$
$$= \sqrt{\frac{85}{0.25} - \frac{0.07^2}{40.25^2}}$$
$$= 18.44\ rad/s$$

$$Time\ period, T = \frac{2\pi}{\omega_d}$$
$$= \frac{2\pi}{18.44}$$
$$= 0.34\ s$$

(b)

$$A(t) = A_0 e^{-\frac{\gamma}{2}t}$$

$$\frac{1}{2}A_0 = A_0 e^{-\frac{\gamma}{2}t}$$ Amplitude drops to half

$$\frac{1}{2} = e^{-\frac{\gamma}{2}t}$$
$$ln\frac{1}{2} = lne^{-\frac{\gamma}{2}t}$$
$$ln\frac{1}{2} = -\frac{\gamma}{2}t$$
$$t = \frac{-2ln\frac{1}{2}}{\gamma} = \frac{-2ln\frac{1}{2}}{\frac{b}{m}} = \frac{-2ln\frac{1}{2}}{\frac{0.07}{0.25}} = 5\ s$$

(c) We know that, mechanical energy, $E(t) = \frac{1}{2}kA(t)^2$

At t = 0, the mechanical energy, $E(0) = \frac{1}{2}kA_0^2$

As per question,

$$E(t) = \frac{1}{2}E(0)$$

$$\frac{1}{2}kA(t)^2 = \frac{1}{2} \times \frac{1}{2}kA_0^2$$

$$A(t)^2 = \frac{1}{2}A_0^2$$

$$[A_0 e^{-\frac{\gamma}{2}t}]^2 = \frac{1}{2}A_0^2$$

$$A_0^2 e^{-\gamma t} = \frac{1}{2}A_0^2$$

$$e^{-\gamma t} = \frac{1}{2}$$

$$ln\frac{1}{2} = lne^{-\gamma t}$$

$$-\gamma t = ln\frac{1}{2}$$

$$t = \frac{-\ln\left(\frac{1}{2}\right)}{\gamma}$$

$$= \frac{-ln\frac{1}{2}}{\frac{b}{m}}$$

$$= \frac{-ln\frac{1}{2}}{\frac{0.07}{0.25}} = 2.5\ s$$

(d) The lifetime,

$$\tau = \frac{1}{\gamma} = \frac{1}{\frac{b}{m}} = \frac{m}{b} = \frac{0.25}{0.07} = 3.57\ s$$

From (a), we get,

Time period, T = 0.34 s (time taken to complete one oscillation)

0.34 s = 1 oscillation

3.57 s = 3.57/0.34 = 10.5 $\approx 10\ oscillations$

(e) Initial mechanical energy,

$E(0) = \frac{1}{2}kA_0^2 = \frac{1}{2} \times 85 \times 0.35^2 = 5.206$ J

From (a), we get,

Time period, T = 0.34 s (time taken to complete one oscillation)

For 5 cycles, t = 5× 0.34 = 1.7 s

Energy after 5 cycles,

$$E(t) = \frac{1}{2}kA(t)^2$$

$$= \frac{1}{2}k[A_0 e^{-\frac{\gamma}{2}t}]^2$$

$$= \frac{1}{2}kA_0^2 e^{-\gamma t}$$

$$= \frac{1}{2}kA_0^2 e^{-\frac{b}{m}t}$$

$$= \frac{1}{2} \times 85 \times 0.35^2 e^{-\frac{0.07}{0.25}\times 1.7}$$

$$= 3.23\ J$$

Therefore,

damping energy (lost energy) = Initial energy – current energy = E(0) – E(t)

$$=5.206 - 3.23 = 1.976\ J$$

5. A mass spring system is undergoing DHM with mass m and the equation of displacement

$$\mathbf{y = 5e^{-2t}\cos 2t}$$

Show that damping energy decreases faster compared to the amplitude using the damping constant. [Use equations to justify your answer].

$$y(t) = 5e^{-2t}\cos(2t)$$

The amplitude is:

$$A(t) = 5e^{-2t}$$

Amplitude decays as e^{-2t}.

The total energy is:

$$E(t) = \frac{1}{2}kA^2 = \frac{1}{2}k(5e^{-2t})^2 = \frac{1}{2}k \cdot 25e^{-4t}$$

Energy decays as e^{-4t}.

Energy $E(t) \propto e^{-4t}$ decays faster than amplitude $A(t) \propto e^{-2t}$.

6. If $\frac{\omega}{\gamma} > 2, determine\ type\ of\ damping.$

Given that,

$$\frac{\omega}{\gamma} < 3$$

$$\frac{\omega^2}{\gamma^2} < 9$$

$$\omega^2 < 9\gamma^2$$

$$\omega^2 < 4 \times \frac{9\gamma^2}{4}$$

$$\omega^2 < 36\frac{\gamma^2}{4}$$

$$Since, \qquad \omega^2 < \frac{\gamma^2}{4} \quad , overdamping.$$