Waves and Oscillation

Course- PHY 2105 / PHY 105

Lecture 5

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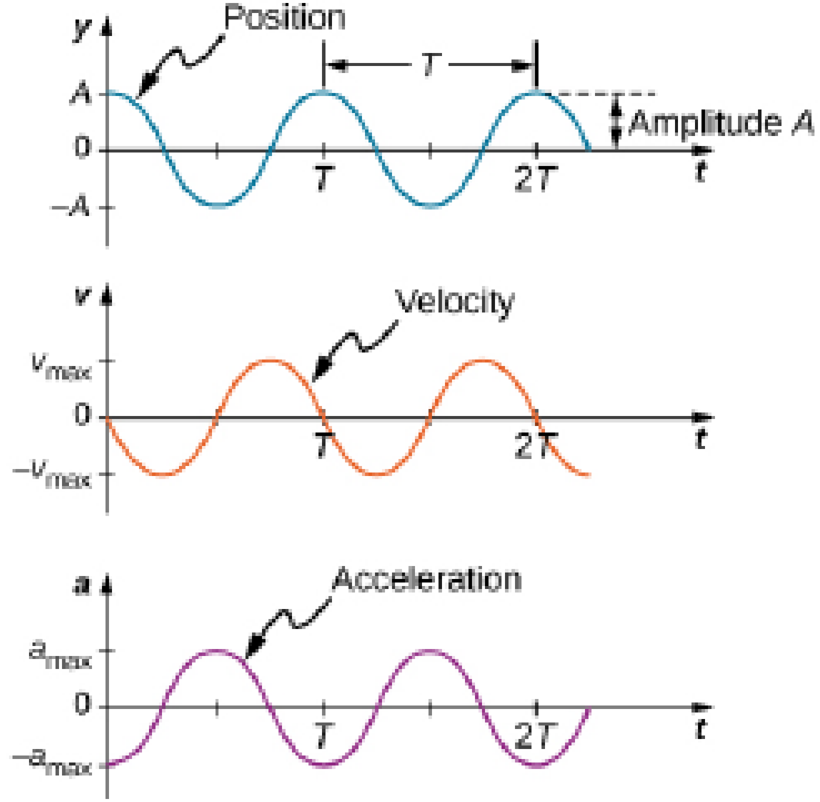
Equations

Equation of SHM: $\frac{d^2x}{dt^2} = -\frac{k}{m}x$

$$x(t) = A\cos(\omega t + \phi)$$
$$v(t) = -v_{max}\sin(\omega t + \phi)$$
$$a(t) = -a_{max}\cos(\omega t + \phi)$$
$$x_{max} = A$$
$$v_{max} = A\omega$$
$$a_{max} = A\omega^2.$$

$$\overbrace{x(t)}^{\text{Displacement at time } t} = \underbrace{\mathbf{A}}_{\text{Amplitude}}\ \mathbf{cos}\overbrace{(\underset{\text{Angular frequency}}{\omega}\,\underset{\text{Time}}{t} + \underset{\text{Phase constant or phase angle}}{\phi})}^{\text{Phase}}$$



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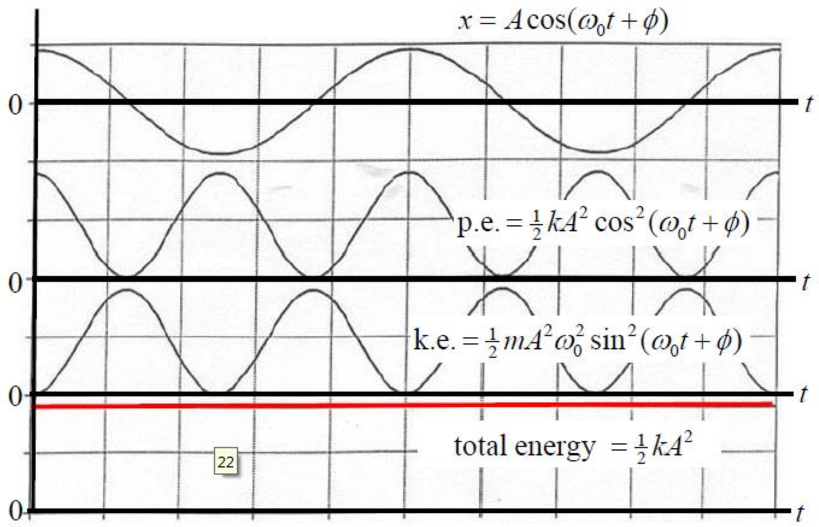
Energy

Energy of the mass-spring simple harmonic oscillator



For the mass-spring system: $x = A\cos(\omega_0 t + \phi)$

Potential energy $= \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)$

k.e. $= \frac{1}{2}mv^2 = \frac{1}{2}m[-A\omega_0\sin(\omega_0 t + \phi)]^2 = \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \phi)$

Total energy $=$ p.e. + k.e

$$= \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi) + \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \phi)$$

$$= \frac{1}{2}kA^2 \quad \left(= \frac{1}{2}m\omega_0{}^2A^2\right) \qquad (\therefore E \propto A^2)$$

We can now write: $\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2$

$$\therefore v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} \qquad \text{or} \qquad v(x) = \pm\omega_0\sqrt{A^2 - x^2}$$

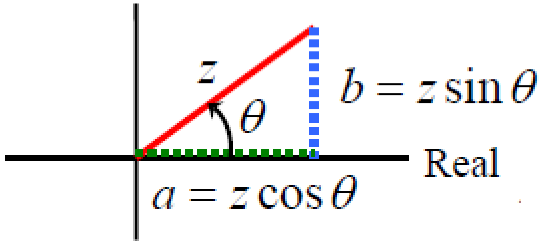
Complex Numbers

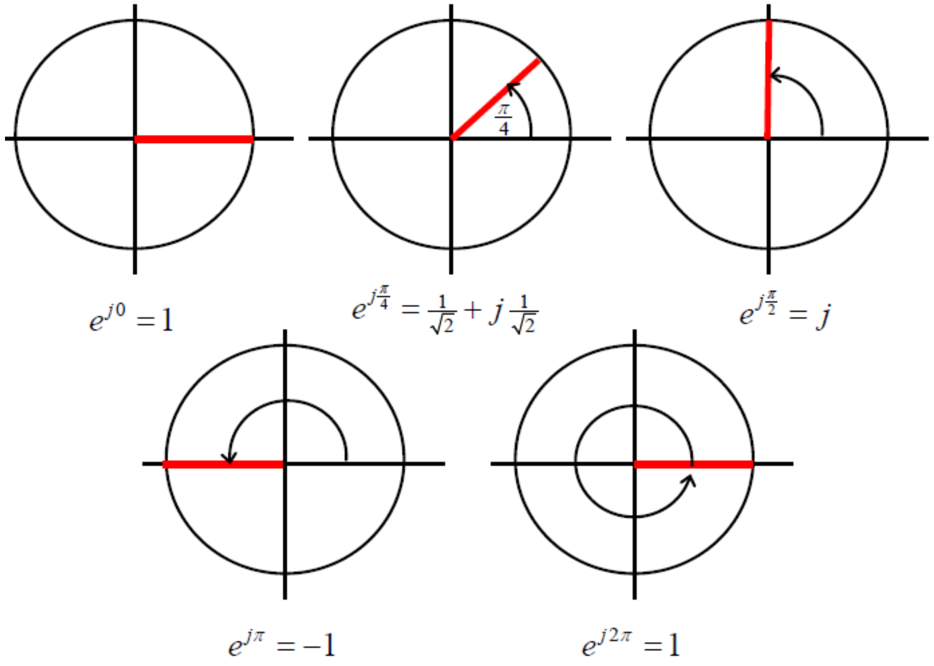
Hence $\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$ **Euler relation**

Then $z = a + jb \;= |z|\, e^{j\theta}$

where $|z| = \sqrt{a^2 + b^2}$

$\tan\theta = b/a$





Now write: $Ae^{j(\omega t+\phi)} = A\cos(\omega t + \phi) + jA\sin(\omega t + \phi)$

... and remember that the physical quantity x (e.g. a displacement) is the real part of z :

i.e. $x = \mathrm{Re}[z]$

Damped Harmonic Motion

When oscillating bodies do not move back and forth between precisely fixed limits because frictional force dissipate the energy and amplitude of oscillation **decreases** with time and finally die out. Such harmonic motion is called Damped Harmonic Motion.

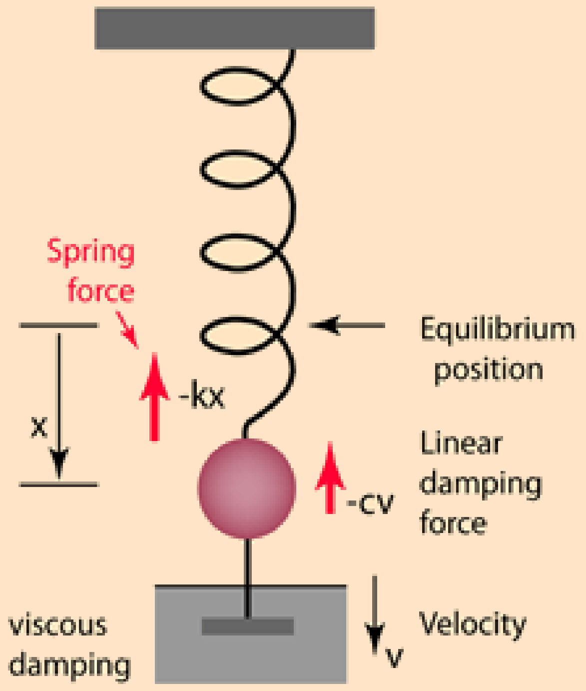
The decrease in amplitude caused by dissipative forces is called **Damping**, and the corresponding motion is called **Damped Oscillation**.

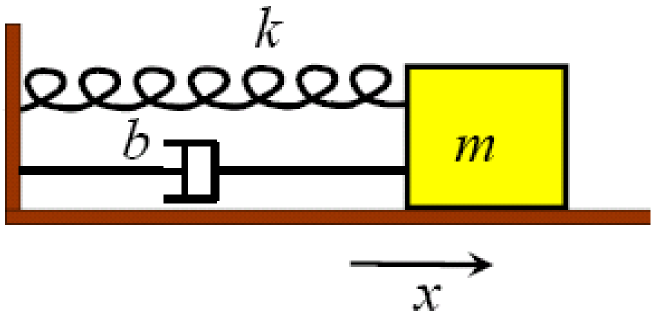
This occurs because the non-conservative damping force **removes** energy from the system, usually in the form of thermal energy

DHM Eqn

In spring-mass oscillator

the damping force is proportional to the velocity and acts against the direction of motion





For horizontal forces on the mass: $ma = -kx - bv$

or $$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

or $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0{}^2 x = 0$$ where $$\begin{cases} \omega_0 = \sqrt{\dfrac{k}{m}} \\ \gamma = \dfrac{b}{m} \end{cases}$$

γ : "damping constant" unit: s^{-1}

- "life time" $= \dfrac{1}{\gamma}$

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0{}^2x=0$$

Let $x = Be^{pt}$

Then $\dfrac{dx}{dt} = Bpe^{pt}$ and $\dfrac{d^2x}{dt^2} = Bp^2e^{pt}$

Substituting into DE: $Bp^2e^{pt} + \gamma Bpe^{pt} + \omega_0{}^2Be^{pt} = 0$

Thus $p^2 + \gamma p + \omega_0{}^2 = 0$

$$\therefore\ p = \tfrac{1}{2}\left\{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0{}^2}\right\}$$

or $p = -\dfrac{\gamma}{2} \pm \sqrt{\dfrac{\gamma^2}{4} - \omega_0{}^2}$

$$p = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0{}^2}$$

We can distinguish three cases:

(i) $\omega_0{}^2 > \dfrac{\gamma^2}{4}$ **Oscillatory behaviour**

(ii) $\omega_0{}^2 = \dfrac{\gamma^2}{4}$ **Critical damping**

(iii) $\omega_0{}^2 < \dfrac{\gamma^2}{4}$ **Overdamping**

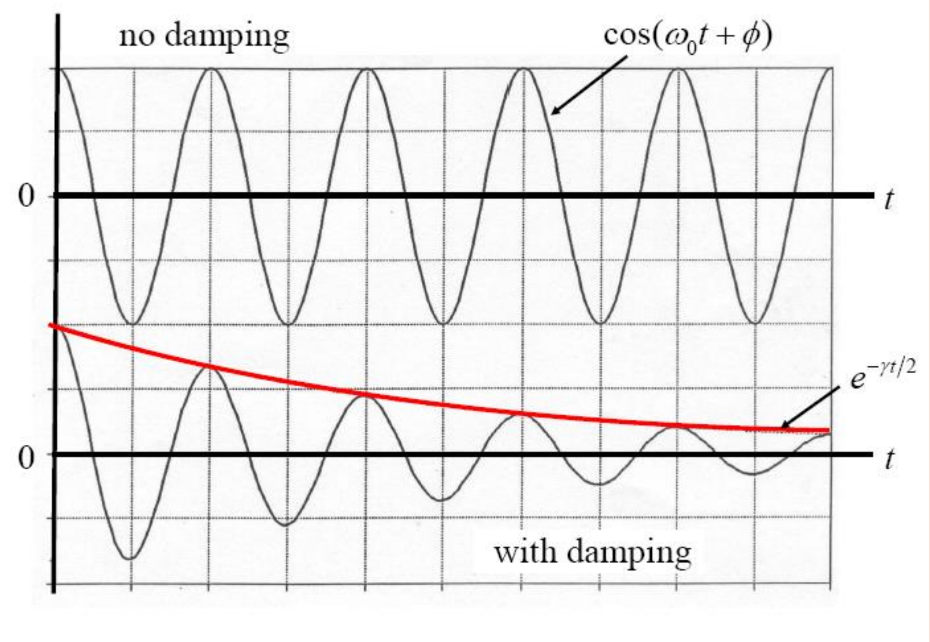
Case (i): $\omega_0^{\,2} > \dfrac{\gamma^2}{4}$

$$\therefore \sqrt{\gamma^2/4-\omega_0^{\,2}} = \sqrt{-(\omega_0^{\,2}-\gamma^2/4)}$$

Put $\omega_1^{\,2} = \omega_0^{\,2} - \gamma^2/4$

$$\therefore p = -\frac{\gamma}{2} \pm \sqrt{-\omega_1^{\,2}} = -\frac{\gamma}{2} \pm j\omega_1$$

… leading to $x(t) = Ae^{-\frac{\gamma t}{2}}\cos(\omega_1 t + \phi)$



This is an **oscillatory solution** $A\cos(\omega_1 t + \phi)$ multiplied by a damping factor $e^{-\gamma t/2}$.

As $\gamma \to 0$ we approach our undamped oscillator.

Case (ii): $\omega_0^2 = \dfrac{\gamma^2}{4}$

The two roots coincide: $p = -\dfrac{\gamma}{2}$

The solution will be $x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$

The condition $\omega_0^2 = \gamma^2/4$ is referred to as the "**critical damping**" condition.

If $\omega_0^2 < \gamma^2/4$ a system released from rest will oscillate.

As γ is increased the oscillations decay more rapidly, until at $\omega_0^2 = \gamma^2/4$ oscillation no longer occurs.

Case (iii): $\omega_0^2 < \dfrac{\gamma^2}{4}$

$$p = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

$$= -\frac{\gamma}{2} \pm \lambda \quad \text{say}$$

The solution will be $x(t) = B_1 e^{\left(-\frac{\gamma}{2}+\lambda\right)t} + B_2 e^{\left(-\frac{\gamma}{2}-\lambda\right)t}$

The condition $\omega_0^2 < \dfrac{\gamma^2}{4}$ is referred to as **overdamp**

… a slower approach to the rest position is observed.

Alternate conditioning:-

$$b = \pm 2\sqrt{mk}$$

There is no oscillation in overdamped conditions

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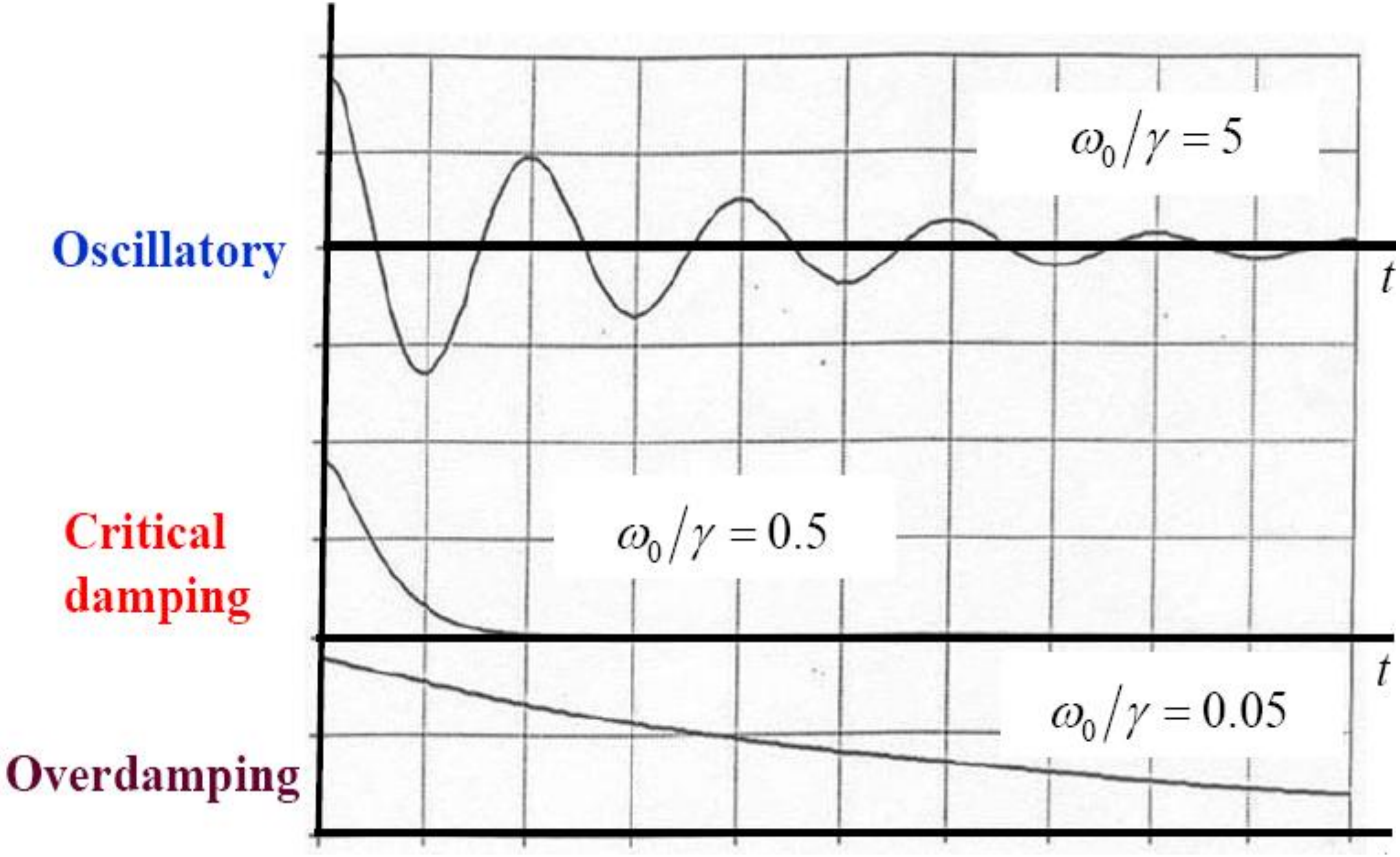


Damping conditions

Damping conditions

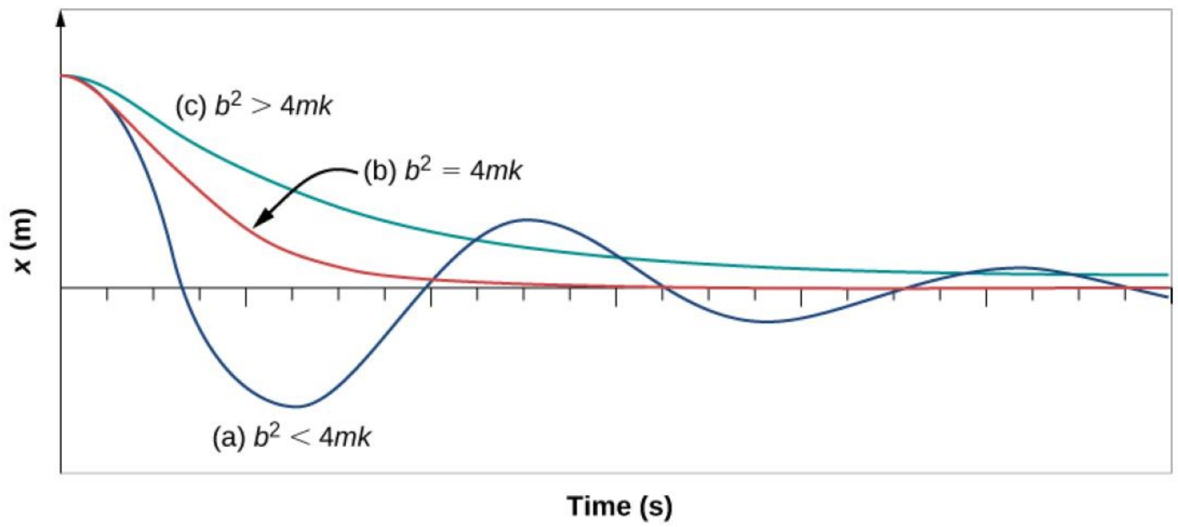
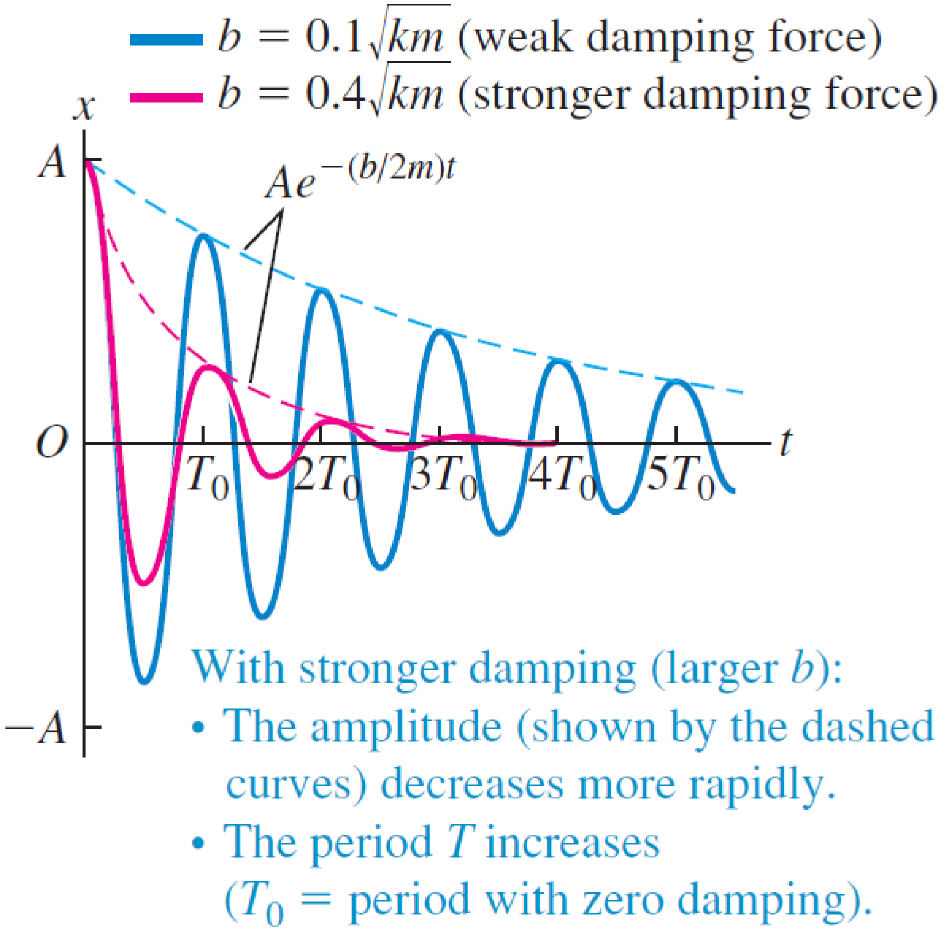


Figure 15.27 The position versus time for three systems consisting of a mass and a spring in a viscous fluid. (a) If the damping is small $\left(b < \sqrt{4mk}\right)$, the mass oscillates, slowly losing amplitude as the energy is dissipated by the non-conservative force(s). The limiting case is (b) where the damping is $\left(b = \sqrt{4mk}\right)$. (c) If the damping is very large $\left(b > \sqrt{4mk}\right)$, the mass does not oscillate when displaced, but attempts to return to the equilibrium position.



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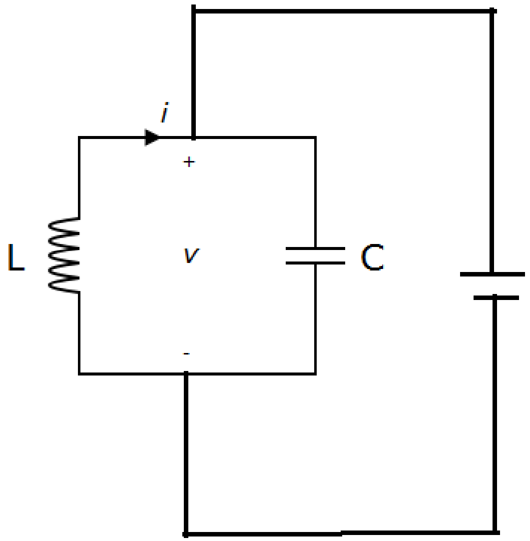
LC Circuit



An **LC circuit**, also called a **resonant circuit**, **tank circuit**, or **tuned circuit**, consists of an inductor, represented by the letter L, and a capacitor, represented by the letter C. When connected together, they can act as an electrical resonator.

Voltage across capacitor

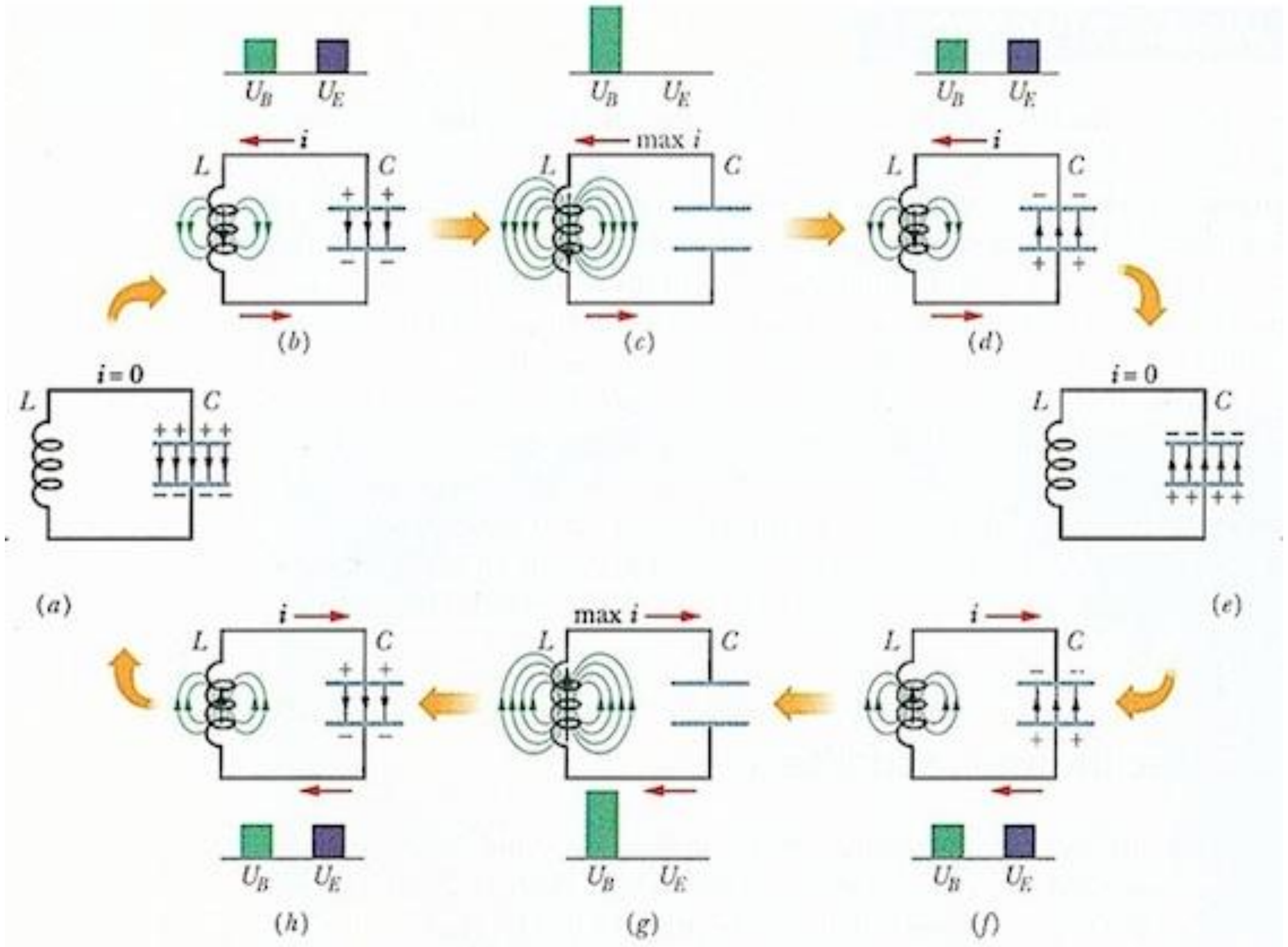
$$V_C = \frac{Q}{C}$$

Voltage across inductor

$$V_L = L\frac{di}{dt}$$

Q is the charge on the capacitor and
C is the capacitance of capacitor.

Charging & discharging an LC Circuit



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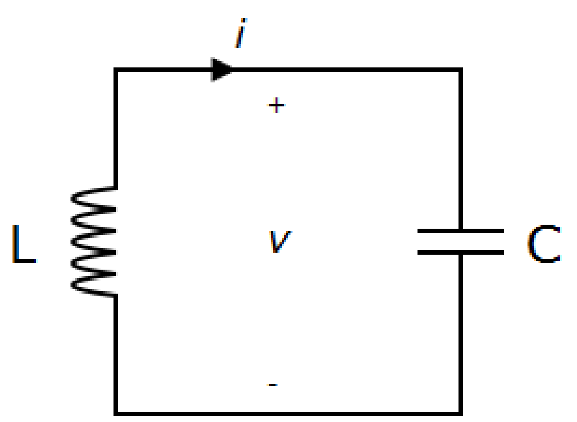


Kirchhoff's voltage law

$$\frac{Q}{C}+L\frac{di}{dt}=0$$

$$\boxed{\frac{d^2 i}{dt^2}+\frac{1}{LC}i=0}$$

$$\boxed{\omega_0{}^2=\frac{1}{LC}}$$

$$\boxed{T=2\pi\sqrt{LC}}$$

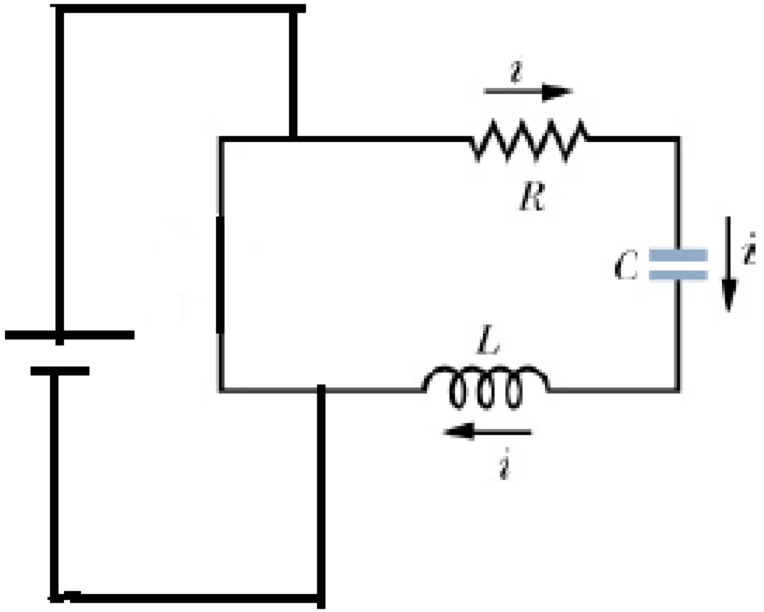
$$\boxed{f=\frac{1}{2\pi\sqrt{LC}}}$$

Similar to differential equation of SHM

$$\frac{d^2 x}{dt^2}+\omega_0 x=0,$$

$$\boxed{\begin{aligned} Q(t)&=Q_0\cos(\omega_0 t+\phi)\\ i(t)&=-i_0\sin(\omega_0 t+\phi)\end{aligned}}$$

RLC Circuit



Voltage across resistor R $$V_R = iR$$

Voltage across capacitor C $$V_C = \frac{Q}{C}$$

Voltage across inductor L $$V_L = L\frac{di}{dt}$$

According to **Kirchhoff's Voltage Law**:

$$iR + \frac{Q}{C} + L\frac{di}{dt} = 0$$

Rewrite the equation

$$\frac{d^2 i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{LC}i=0$$

Comparing with the equation

$$\frac{d^2 x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0{}^2 x=0$$

Where

$$\boxed{\gamma=\frac{R}{L}} \qquad \boxed{\omega_0=\sqrt{\frac{1}{LC}}}$$

Three distinguish cases are

i) $\frac{1}{LC}>\frac{R^2}{4L^2}$ Oscillatory behavior

ii) $\frac{1}{LC}=\frac{R^2}{4L^2}$ Critical damping

iii) $\frac{1}{LC}<\frac{R^2}{4L^2}$ Over damping

Resemblance between systems

Mechanical	Electrical
displacement x	charge Q
velocity v	current I
mass m	inductance L
spring constant k	$\frac{1}{\text{capacitance}}$ $\frac{1}{C}$
$\omega_0 = \sqrt{\frac{k}{m}}$	$\omega_0 = \sqrt{\frac{1}{LC}}$
potential energy: $\frac{1}{2}kx^2$	Electric energy stored in capacitor: $\frac{1}{2}\frac{Q^2}{C}$
kinetic energy: $\frac{1}{2}mv^2$	Magnetic energy stored in inductor: $\frac{1}{2}LI^2$

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