Waves and Oscillation

Course- PHY 2105 / PHY 105

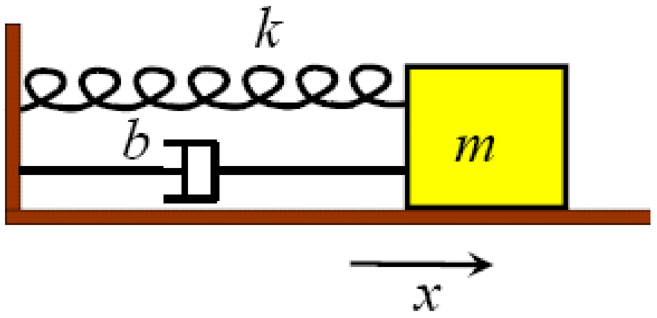
Lecture 6

Md Shafqat Amin Inan

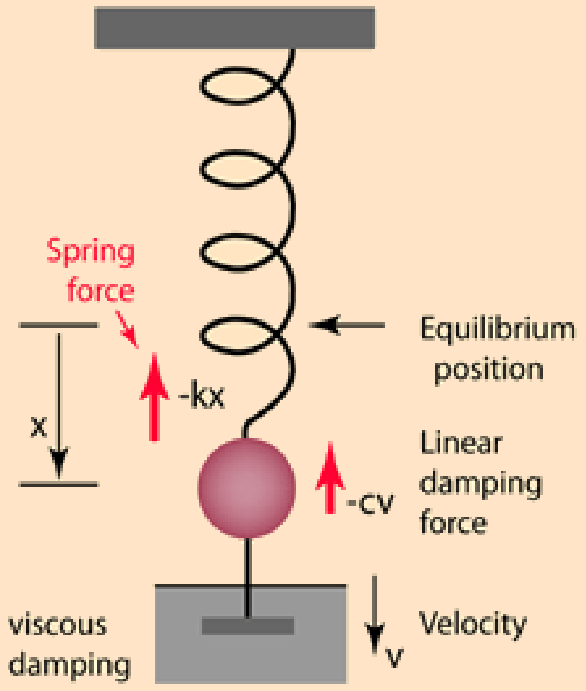
UNITED INTERNATIONAL UNIVERSITY

DHM Eqn

In spring-mass oscillator



the damping force is proportional to the velocity and acts against the direction of motion



For horizontal forces on the mass: $ma = -kx - bv$

or $m\dfrac{d^2x}{dt^2} = -kx - b\dfrac{dx}{dt}$

or $\dfrac{d^2x}{dt^2} + \gamma\dfrac{dx}{dt} + \omega_0^{\,2}x = 0$ where $\begin{cases} \omega_0 = \sqrt{\dfrac{k}{m}} \\ \gamma = \dfrac{b}{m} \end{cases}$

γ : "damping constant" unit: s^{-1}

- "life time" $= \dfrac{1}{\gamma}$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0{}^2 x = 0$$

Let $\quad x = Be^{pt}$

Then $\quad \frac{dx}{dt} = Bpe^{pt} \quad$ and $\quad \frac{d^2x}{dt^2} = Bp^2e^{pt}$

Substituting into DE: $\quad Bp^2e^{pt} + \gamma Bpe^{pt} + \omega_0{}^2 Be^{pt} = 0$

Thus $\quad p^2 + \gamma p + \omega_0{}^2 = 0$

$$\therefore\ p = \tfrac{1}{2}\left\{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0{}^2}\right\}$$

or $\quad p = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0{}^2}$

$$p = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0{}^2}$$

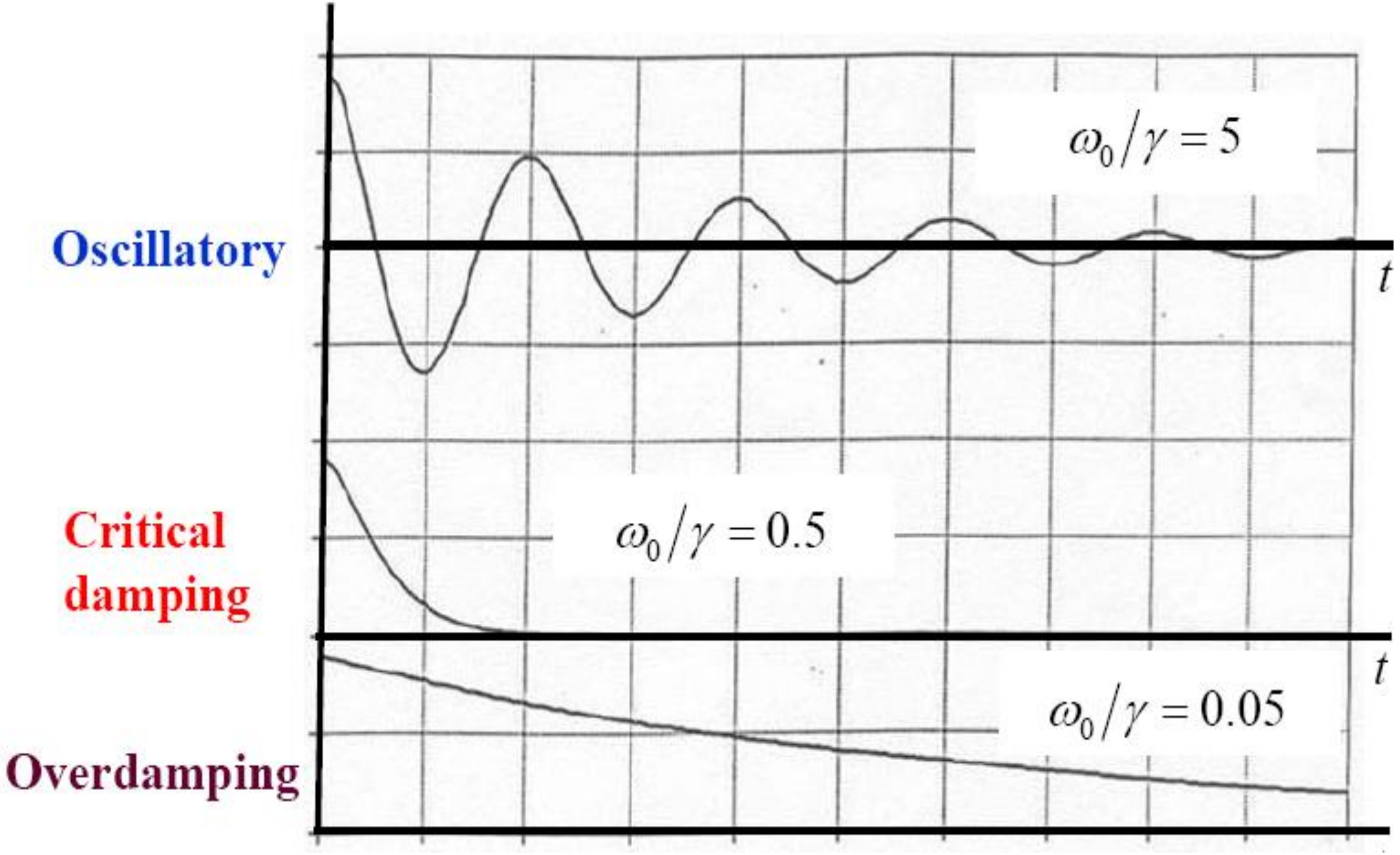
We can distinguish three cases:

(i) $\quad \omega_0{}^2 > \frac{\gamma^2}{4} \quad$ **Oscillatory behaviour**

(ii) $\quad \omega_0{}^2 = \frac{\gamma^2}{4} \quad$ **Critical damping**

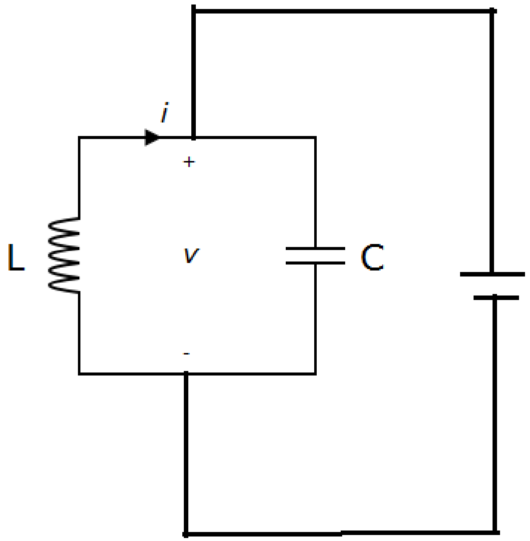
(iii) $\quad \omega_0{}^2 < \frac{\gamma^2}{4} \quad$ **Overdamping**

UNITED INTERNATIONAL UNIVERSITY



Damping conditions

LC Circuit



An **LC circuit**, also called a **resonant circuit**, **tank circuit**, or **tuned circuit**, consists of an inductor, represented by the letter L, and a capacitor, represented by the letter C. When connected together, they can act as an electrical resonator.

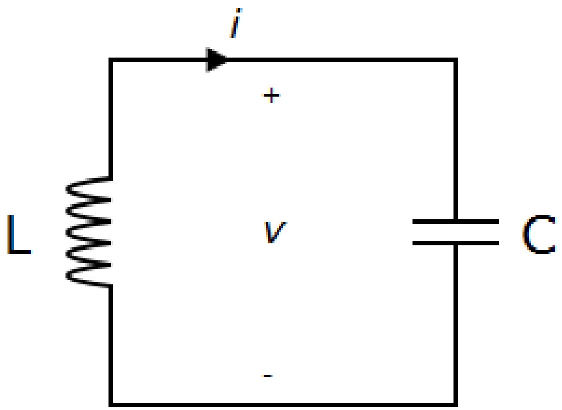
Voltage across capacitor

$$V_C = \frac{Q}{C}$$

Voltage across inductor

$$V_L = L\frac{di}{dt}$$

Q is the charge on the capacitor and
C is the capacitance of capacitor.



Kirchhoff's voltage law

$$\frac{Q}{C} + L\frac{di}{dt} = 0$$

$$\boxed{\frac{d^2 i}{dt^2} + \frac{1}{LC} i = 0}$$

$$\boxed{\omega_0{}^2 = \frac{1}{LC}}$$

$$\boxed{T = 2\pi\sqrt{LC}}$$

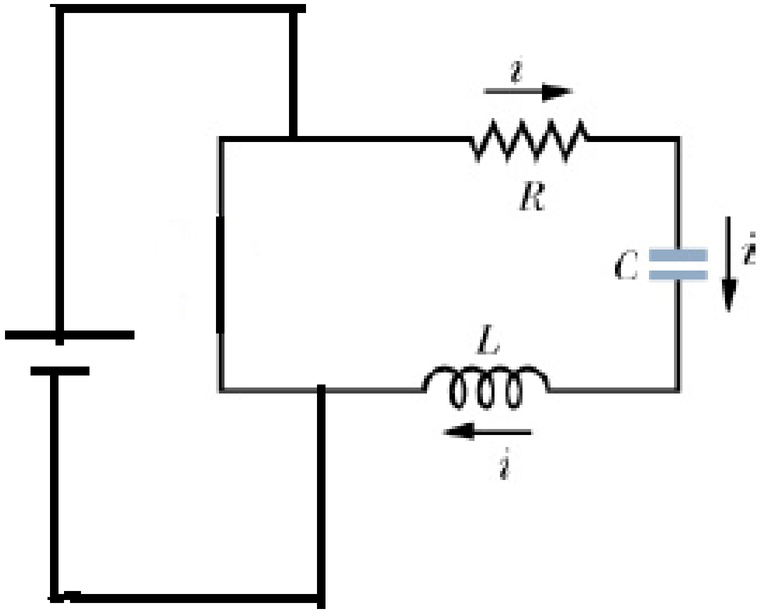
$$\boxed{f = \frac{1}{2\pi\sqrt{LC}}}$$

Similar to differential equation of SHM

$$\frac{d^2 x}{dt^2} + \omega_0 x = 0,$$

$$\boxed{\begin{aligned} Q(t) &= Q_0 \cos(\omega_0 t + \phi) \\ i(t) &= -i_0 \sin(\omega_0 t + \phi) \end{aligned}}$$

RLC Circuit



Voltage across resistor R $$V_R = iR$$

Voltage across capacitor C $$V_C = \frac{Q}{C}$$

Voltage across inductor L $$V_L = L\frac{di}{dt}$$

According to **Kirchhoff's Voltage Law**:

$$iR + \frac{Q}{C} + L\frac{di}{dt} = 0$$

Rewrite the equation

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$$

Comparing with the equation

$$\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0{}^2 x = 0$$

Where

$$\boxed{\gamma = \frac{R}{L}} \qquad \boxed{\omega_0 = \sqrt{\frac{1}{LC}}}$$

Three distinguish cases are

i) $\frac{1}{LC} > \frac{R^2}{4L^2}$ Oscillatory behavior

ii) $\frac{1}{LC} = \frac{R^2}{4L^2}$ Critical damping

iii) $\frac{1}{LC} < \frac{R^2}{4L^2}$ Over damping

Resemblance between systems

Mechanical	Electrical
displacement x	charge Q
velocity v	current I
mass m	inductance L
spring constant k	$\frac{1}{\text{capacitance}}$ $\frac{1}{C}$
$\omega_0 = \sqrt{\frac{k}{m}}$	$\omega_0 = \sqrt{\frac{1}{LC}}$
potential energy: $\frac{1}{2}kx^2$	Electric energy stored in capacitor: $\frac{1}{2}\frac{Q^2}{C}$
kinetic energy: $\frac{1}{2}mv^2$	Magnetic energy stored in inductor: $\frac{1}{2}LI^2$

UNITED INTERNATIONAL UNIVERSITY

Case 1

$$\frac{1}{LC} > \frac{R^2}{4L^2}$$

Solution of the differential equation

$$Q(t) = Ae^{-\frac{R}{2L}t}\cos(\omega_1 t + \phi)$$

Where $\omega_1 = \sqrt{(\frac{1}{LC} - \frac{R^2}{4L^2})}$

Frequency of oscillation

$$f = \frac{1}{2\pi}\sqrt{(\frac{1}{LC} - \frac{R^2}{4L^2})}$$

Example 6.1

A capacitor 1.0μF, an inductor 0.2h and a resistance 800Ω are joined in series. Is the circuit oscillatory?
Find the frequency of oscillation.

Example 6.2

Find whether the discharge of capacitor through the following inductive circuit is oscillatory.

$C = 0.1\mu F$, $L = 10mh$, R $200\ \Omega$

If Oscillatory, find the frequency of oscillation.

Example 6.3

For a damped oscillator, m =250 g, k =85 N/m, and b =70 g/s.

(a) What is the period of the motion?

(b) How long does it take for the amplitude of the damped oscillations to drop to half its initial value?

(c) How long does it take for the mechanical energy to drop to one-half its initial value?

Because $b \ll \sqrt{km} = 4.6$ kg/s, the period is approximately that of the undamped oscillator.

(a)

Calculation: From Eq. 15-13, we then have

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.25\text{ kg}}{85\text{ N/m}}} = 0.34\text{ s.}$$

(b)

$$t = \frac{-2m\ln\frac{1}{2}}{b} = \frac{-(2)(0.25\text{ kg})(\ln\frac{1}{2})}{0.070\text{ kg/s}}$$

$$= 5.0\text{ s.} \qquad \text{(Answer)}$$

Because $T = 0.34$ s, this is about 15 periods of oscillation.

(c)

Calculations: The mechanical energy has the value $\frac{1}{2}kx_m^2$ at $t = 0$. Thus, we must find the value of t for which

$$\tfrac{1}{2}kx_m^2\, e^{-bt/m} = \tfrac{1}{2}(\tfrac{1}{2}kx_m^2).$$

If we divide both sides of this equation by $\frac{1}{2}kx_m^2$ and solve for t as we did above, we find

$$t = \frac{-m\ln\frac{1}{2}}{b} = \frac{-(0.25\text{ kg})(\ln\frac{1}{2})}{0.070\text{ kg/s}} = 2.5\text{ s.} \qquad \text{(Answer)}$$

UNITED INTERNATIONAL UNIVERSITY

Forced Oscillation

Forced vibrations occur when there is a periodic driving force. This force may or may not have the same period as the natural frequency of the system.

If the frequency is the same as the natural frequency, the amplitude can become quite large. This is called resonance.

Equation of Forced Harmonic Motion

Consider what happens when we apply a time-dependent force $F(t)$ to a system that normally would carry out SHM with an angular frequency ω_0.

Assume the external force $F(\mathrm{t}) = F_0 \sin(\omega t)$. The equation of motion can now be written as

$$\frac{d^2x}{dt^2} = -\omega_0{}^2 x + F_0 \sin(\omega t)$$

The steady state motion of this system will be harmonic motion with an angular frequency equal to the angular frequency of the driving force.

FHM is also sometimes called a Driven Harmonic Motion

UNITED INTERNATIONAL UNIVERSITY

Consider the general solution

$$x(t) = A\cos(\omega t + \phi)$$

The parameters in this solution must be chosen such that the equation of motion is satisfied. This requires that

This equation can be rewritten as

$$-\omega^2 A\cos(\omega t + \phi) + \omega_0{}^2 A\cos(\omega t + \phi) - F_0 \sin(\omega t) = 0$$

$$\left(\omega_0{}^2 - \omega^2\right)A\cos(\omega t)\cos(\phi) -$$
$$\left(\omega_0{}^2 - \omega^2\right)A\sin(\omega t)\sin(\phi) - F_0 \sin(\omega t) = 0$$

Our general solution must thus satisfy the following condition:

$$\left(\omega_0{}^2 - \omega^2\right)A\cos(\omega t)\cos(\phi) - \left\{\left(\omega_0{}^2 - \omega^2\right)A\sin(\phi) - F_0\right\}\sin(\omega t) = 0$$

Since this equation must be satisfied at all time, we must require that the coefficients of cos(ωt) and sin(ωt) are 0. This requires that:

$$\left(\omega_0{}^2 - \omega^2\right)A\cos(\phi) = 0$$

$$\left(\omega_0{}^2 - \omega^2\right)A\sin(\phi) - F_0 = 0$$

The interesting solutions are solutions where $A \neq 0$ and $\omega \neq \omega_0$. In this case, our general solution can only satisfy the equation of motion if

$\cos(\phi) = 0$ and $\left(\omega_0{}^2 - \omega^2\right)A\sin(\phi) - F_0 = \left(\omega_0{}^2 - \omega^2\right)A - F_0 = 0$

The amplitude of the motion is thus equal to

$$A = \frac{F_0}{\left(\omega_0{}^2 - \omega^2\right)}$$

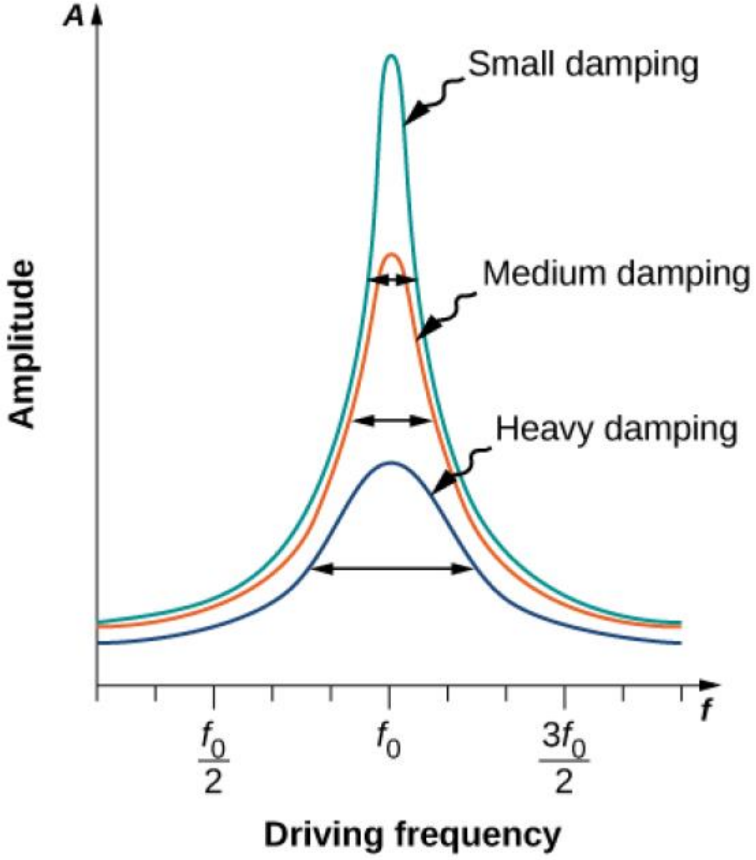
UNITED INTERNATIONAL UNIVERSITY

The amplitude of the motion is thus equal to

$$A = \frac{F_0}{\left(\omega_0{}^2 - \omega^2\right)}$$

If the driving force has a frequency close to the natural frequency of the system, the resulting amplitudes can be very large even for small driving amplitudes. The system is said to be in resonance.

In realistic systems, there will also be a damping force. Whether or not resonance behavior will be observed will depend on the strength of the damping term.



Resonance occurs when the driving frequency equals the natural frequency, and the greatest response is for the least amount of damping. The narrowest response is also for the least damping

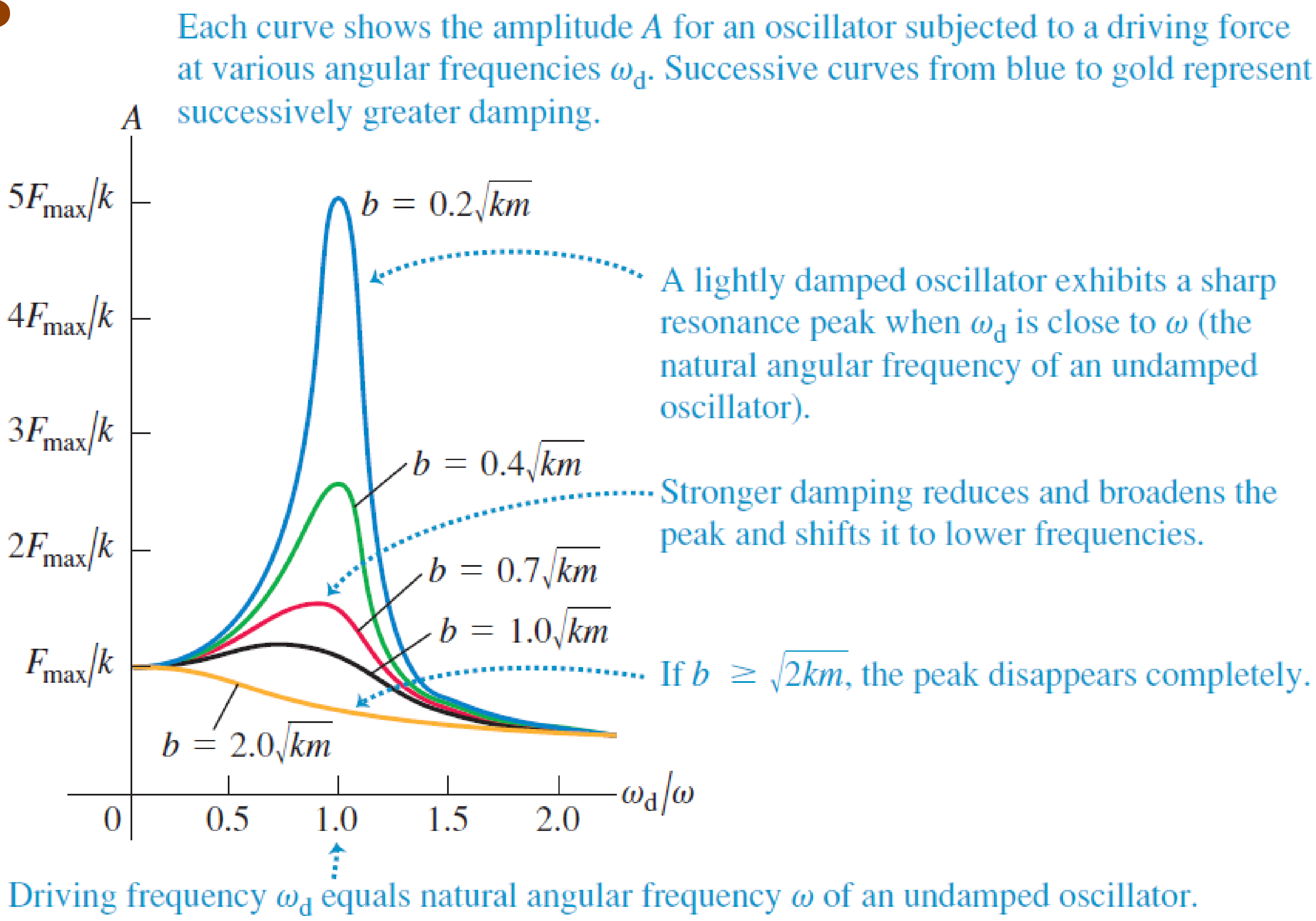
Resonance

If an external driving force with angular frequency ω_d acts on an oscillating system with natural angular frequency ω, the system oscillates with angular frequency ω_d.

The velocity amplitude v_m of the system is greatest when $\omega_d = \omega$, a condition called **resonance**. The amplitude x_m of the system is (approximately) greatest under the same condition.

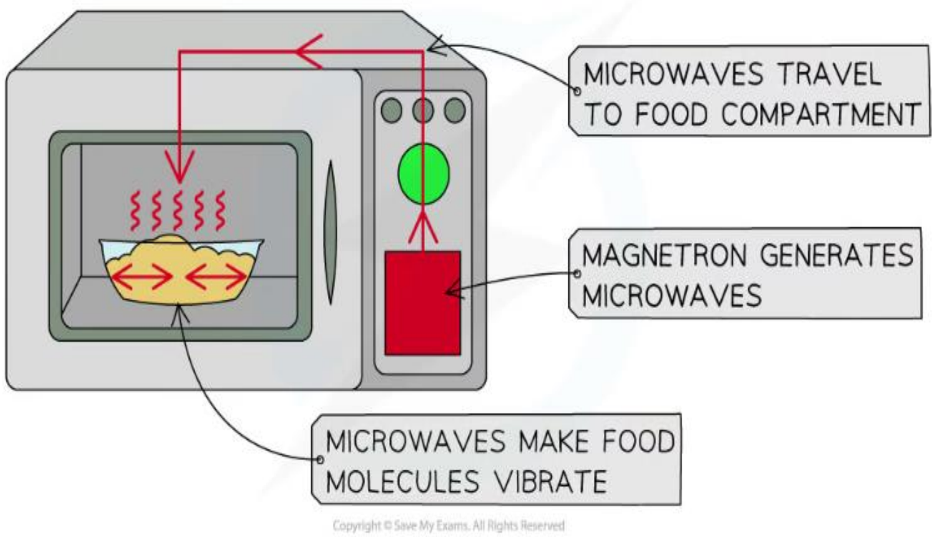
Collapse of a freeway in California, due to the 1989 earthquake

Conditions for DHM & FHM

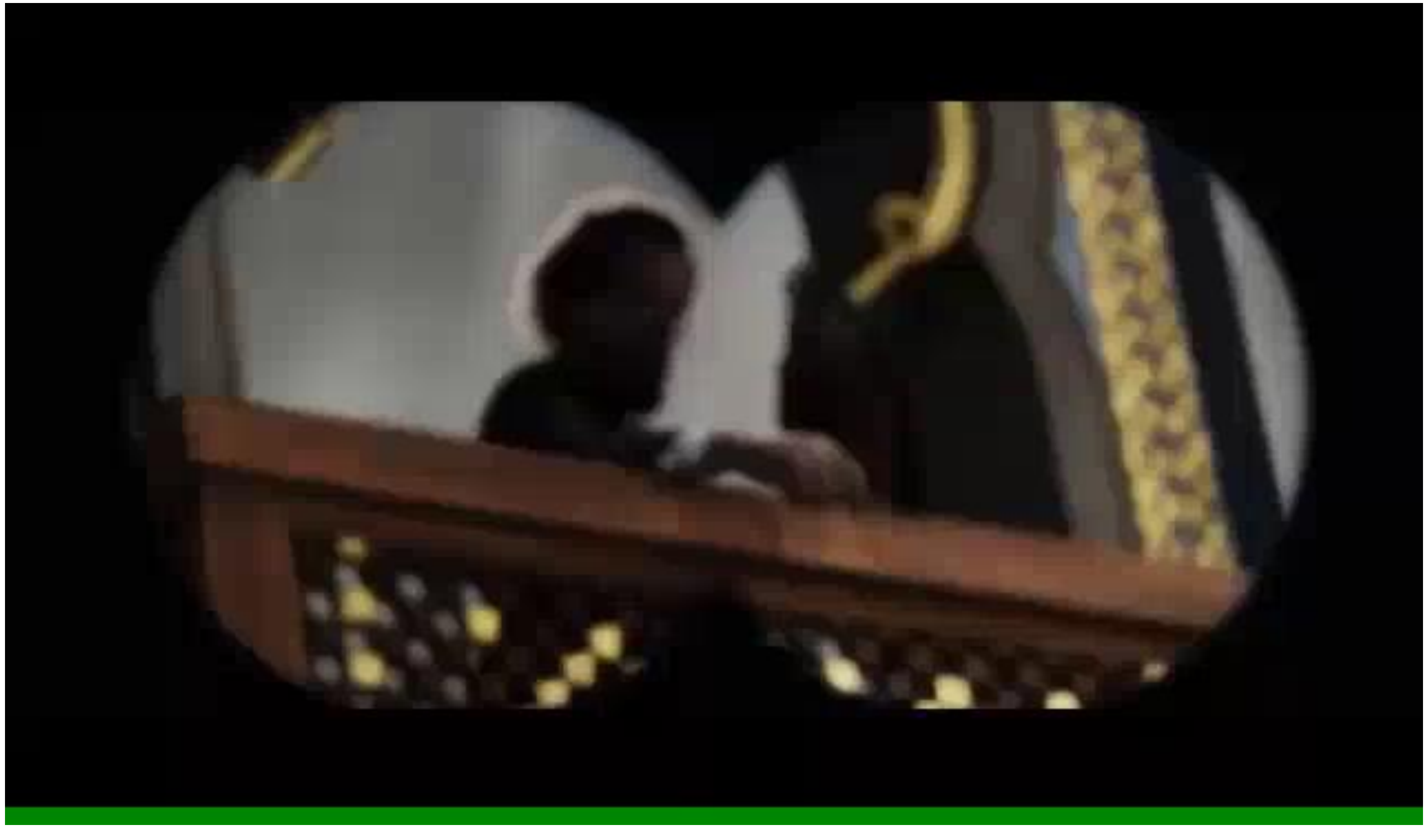


UNITED INTERNATIONAL UNIVERSITY

Resonance in real life



Resonance in reel life



UNITED INTERNATIONAL UNIVERSITY

Lethal(!) resonance

The Tacoma Narrows Bridge near Washington was the world's third-longest suspension bridge by main span, behind the Golden Gate Bridge and the George Washington Bridge. It opened to traffic on July 1, 1940. The bridge's main span dramatically collapsed in 40-mile-per-hour (64 km/h) winds on the morning of November 7, 1940, as the deck oscillated in an alternating twisting motion that gradually increased in amplitude until the deck tore apart. In many physics textbooks, the event is presented as an example of elementary forced mechanical resonance and also aeroelastic flutter.


That's all Folks!