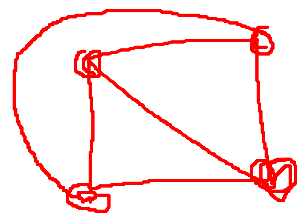
# Planar Graph-

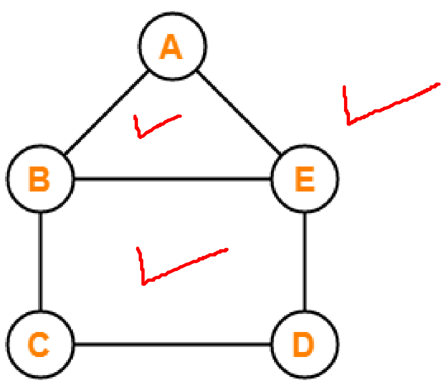
A planar graph may be defined as-

> In graph theory,
>
> Planar graph is a graph that can be drawn in a plane such that none of its edges cross each other.

## Planar Graph Example-



The following graph is an example of a planar graph-




Example of Planar Graph

Here,

In this graph, no two edges cross each other.
Therefore, it is a planar graph.

# Regions of Plane-

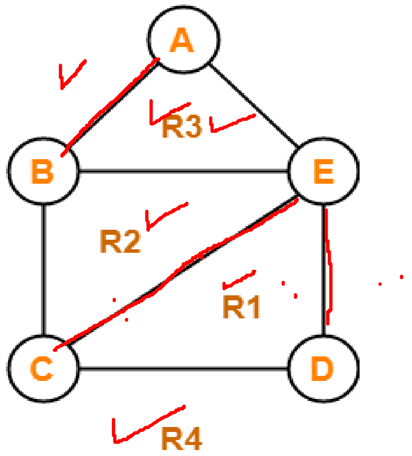
The planar representation of the graph splits the plane into connected areas called as **Regions of the plane**.

Each region has some degree associated with it given as-

- Degree of Interior region = Number of edges enclosing that region
- Degree of Exterior region = Number of edges exposed to that region

**Example-**

Consider the following planar graph-




Regions of Plane

$e - v + 2$
$= 7 - 5 + 2 = 4 = R$

Here, this planar graph splits the plane into 4 regions- R1, R2, R3 and R4 where-

- Degree (R1) = 3
- Degree (R2) = 3
- Degree (R3) = 3
- Degree (R4) = 5

$14 = 2e$

## Planar Graph Chromatic Number-

- Chromatic Number of any planar graph is always less than or equal to 4.
- Thus, any planar graph always requires maximum 4 colors for coloring its vertices.

## Planar Graph Properties-

**Property-01:** In any planar graph, Sum of degrees of all the vertices = 2 x Total number of edges in the graph

Number of vertices x Degree of each vertex = 2 x Total number of edges

$$\boxed{\sum_{i=1}^{n} \deg(v_i) = 2\,|\,E\,|}$$

**Property-02:** In any planar graph, Sum of degrees of all the regions = 2 x Total number of edges in the graph

Number of regions x Degree of each region = 2 x Total number of edges

$$\boxed{\sum_{i=1}^{n} \deg(r_i) = 2\,|\,E\,|}$$

## Special Cases

### Case-01:

In any planar graph, if degree of each region is K, then-

$$K \times |R| = 2 \times |E|$$

### Case-02:

In any planar graph, if degree of each region is at least K (>=K), then-

$$K \times |R| <= 2 \times |E|$$

R

### Case-03:

In any planar graph, if degree of each region is at most K (<=K), then-

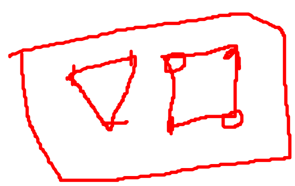
$$K \times |R| >= 2 \times |E|$$

### Property-03:

If G is a connected planar simple graph with 'e' edges, 'v' vertices and 'r' number of regions in the planar representation of G, then- $\boxed{r = e - v + 2}$

This is known as **Euler's Formula**. It remains same in all the planar representations of the graph.

### Property-04:



If G is a planar graph with k components, then-

$$r = e - v + (k + 1)$$

## PRACTICE PROBLEMS BASED ON PLANAR GRAPH IN GRAPH THEORY-

# Problem-01:

v e

Let G be a connected planar simple graph with 25 vertices and 60 edges. Find the number of regions in G.

R = ?

## Solution-

Given-

- Number of vertices (v) = 25
- Number of edges (e) = 60

By Euler's formula, we know r = e – v + 2.

Substituting the values, we get-

Number of regions (r)

= 60 – 25 + 2

= 37

Thus, Total number of regions in G = 37.

## Problem-02:

Let G be a planar graph with 10 vertices, 3 components and 9 edges. Find the number of regions in G.

? v K e

## Solution-

Given-

- Number of vertices (v) = 10
- Number of edges (e) = 9
- Number of components (k) = 3

By Euler's formula, we know $\boxed{r = e – v + (k+1).}$

Substituting the values, we get-

Number of regions (r)

= 9 – 10 + (3+1)

= -1 + 4

= 3

Thus, Total number of regions in G = 3.

# Problem-03:

Let G be a connected planar simple graph with 20 vertices and degree of each vertex is 3. Find the number of regions in G.

v

$\Sigma d = 60$

$e = 30$

# Solution-

Given-

- Number of vertices (v) = 20
- Degree of each vertex (d) = 3

## Calculating Total Number Of Edges (e)-

By sum of degrees of vertices theorem, we have-

Sum of degrees of all the vertices = 2 x Total number of edges

Number of vertices x Degree of each vertex = 2 x Total number of edges

20 x 3 = 2 x e

∴ $\boxed{e = 30}$

Thus, Total number of edges in G = 30.

## Calculating Total Number Of Regions (r)-

By Euler's formula, we know $\boxed{r = e - v + 2.}$

Substituting the values, we get-

Number of regions (r)

= 30 – 20 + 2

= 12

Thus, Total number of regions in G = 12.

# Problem-04:

Let G be a connected planar simple graph with 35 regions, degree of each region is 6. Find the number of vertices in G.

## Solution-

Given-

- Number of regions (n) = 35
- Degree of each region (d) = 6

### Calculating Total Number Of Edges (e)-

By sum of degrees of regions theorem, we have-

Sum of degrees of all the regions = 2 x Total number of edges

Number of regions x Degree of each region = 2 x Total number of edges

35 x 6 = 2 x e

$\boxed{\therefore e = 105}$

Thus, Total number of edges in G = 105.

### Calculating Total Number Of Vertices (v)-

By Euler's formula, we know r = e – v + 2.

Substituting the values, we get-

35 = 105 – v + 2

$\therefore \boxed{v = 72}$

Thus, Total number of vertices in G = 72.

## Problem-05:

Let G be a connected planar graph with 12 vertices, 30 edges and degree of each region is k. Find the value of k.

## Solution-

Given-

- Number of vertices (v) = 12
- Number of edges (e) = 30
- Degree of each region (d) = k

## Calculating Total Number Of Regions (r)-

By Euler's formula, we know r = e – v + 2.

Substituting the values, we get-

Number of regions (r)

= 30 – 12 + 2 = 20

Thus, Total number of regions in G = 20.

## Calculating Value Of k-

By sum of degrees of regions theorem, we have-

Sum of degrees of all the regions = 2 x Total number of edges

Number of regions x Degree of each region = 2 x Total number of edges

20 x k = 2 x 30

∴ k = 3 Thus, Degree of each region in G = 3.

# Problem-06:

What is the maximum number of regions possible in a simple planar graph with 10 edges?

# Solution-

In a simple planar graph, degree of each region is >= 3.

So, we have 3 x |R| <= 2 x |E|.

Substituting the value |E| = 10, we get-

3 x |R| <= 2 x 10

|R| <= 6.67

|R| <= 6

Thus, Maximum number of regions in G = 6.

## Problem-07:

What is the minimum number of edges necessary in a simple planar graph with 15 regions?

## Solution-

In a simple planar graph, degree of each region is >= 3.

So, we have 3 x |R| <= 2 x |E|.

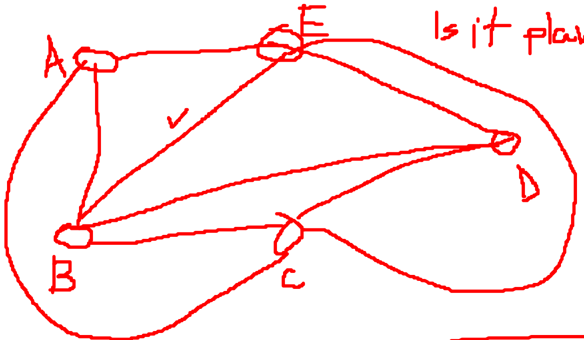
Substituting the value |R| = 15, we get-

3 x 15 <= 2 x |E|

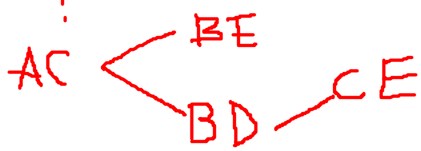
|E| >= 22.5

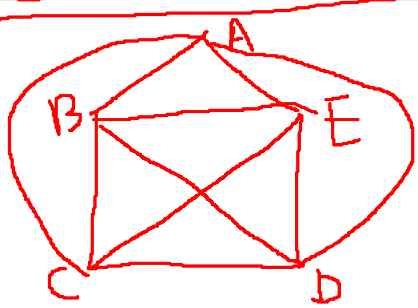
|E| >= 23

Thus, Minimum number of edges required in G = 23.




Is it planar?




$\boxed{K_4}$