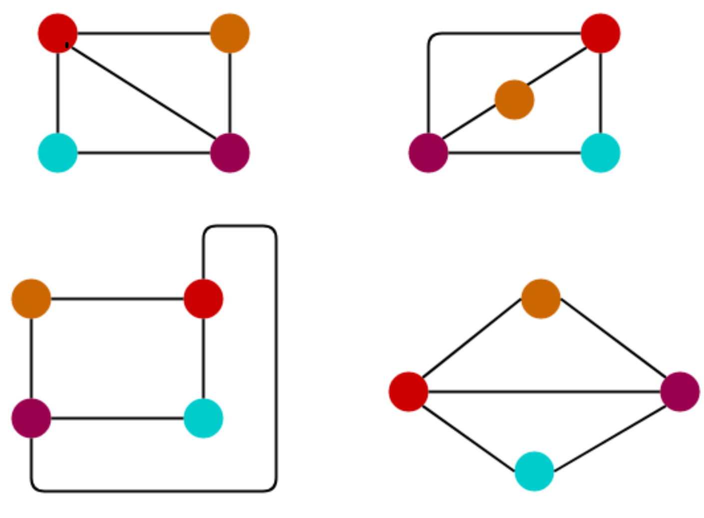
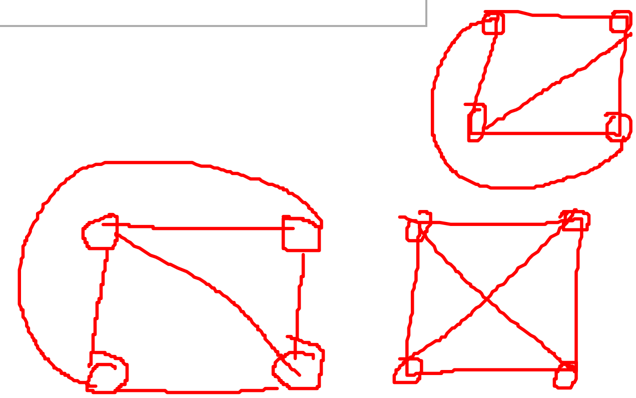
1. **Graph Isomorphism**
2. **Bipartite Graph**
3. **Complement Of Graph**

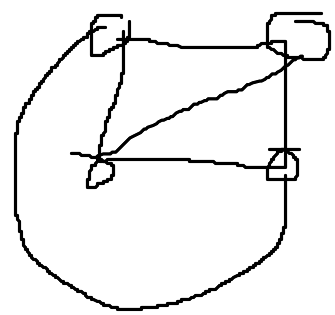
1. Graph Isomorphism

Graph Isomorphism is a phenomenon of existing the same graph in more than one forms. Such graphs are called as **Isomorphic graphs**.

Graph Isomorphism Example-







Graph Isomorphism Example

Here,

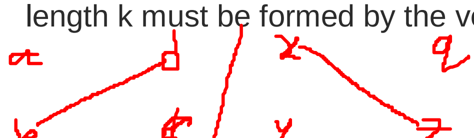
- The same graph exists in multiple forms.
- Therefore, they are **Isomorphic graphs**.

Graph Isomorphism Conditions-

For any two graphs to be isomorphic, following 4 conditions must be satisfied-

Necessary Condition

1. Number of vertices in both the graphs must be same.
2. Number of edges in both the graphs must be same.
3. Degree sequence of both the graphs must be same.
4. If a cycle of length k is formed by the vertices { v_1 , v_2 , ….. , v_k } in one graph, then a cycle of same length k must be formed by the vertices { $f(v_1)$, $f(v_2)$, ….. , $f(v_k)$ } in the other graph as well.



$\{3,3,2,2\}$

	G1	G2	St...
Con 1	V=4	V=4	V
C 2	e=5	e=5	V
3	3,3,2,2	3,3,2,2	V

Degree Sequence

Degree sequence of a graph is defined as a sequence of the degree of all the vertices in ascending order.

Important Points-

- The above 4 conditions are just the necessary conditions for any two graphs to be isomorphic.
- They are not at all sufficient to prove that the two graphs are isomorphic.
- If all the 4 conditions satisfy, even then it can't be said that the graphs are surely isomorphic.
- However, if any condition violates, then it can be said that the graphs are surely not isomorphic.

Sufficient Conditions-

The following conditions are the sufficient conditions to prove any two graphs isomorphic.

If any one of these conditions satisfy, then it can be said that the graphs are surely isomorphic.

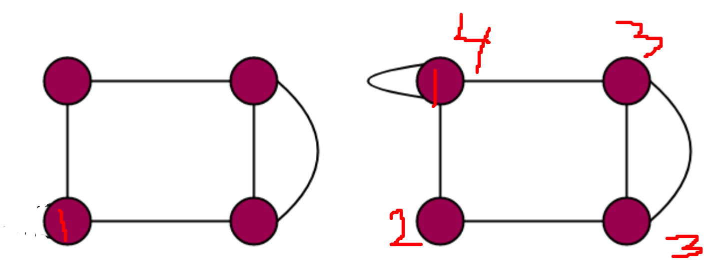
- Two graphs are isomorphic if and only if their complement graphs are isomorphic. 4
- Two graphs are isomorphic if their adjacency matrices are same.
- Two graphs are isomorphic if their corresponding sub-graphs obtained by deleting some vertices of one graph and their corresponding images in the other graph are isomorphic.

PRACTICE PROBLEMS BASED ON GRAPH ISOMORPHISM-

Problem-01:

Are the following two graphs isomorphic?

	G1	G2	Status
1.	v=4	v=4	✓
2.	e=5	e=6	X



G1 G2

$\therefore G_1, G_2$ not isomorphic

$\{2,3,3,4\}$

Solution-

Checking Necessary Conditions-

Condition-01:

- Number of vertices in graph G1 = 4
- Number of vertices in graph G2 = 4

Here,

- Both the graphs G1 and G2 have same number of vertices.
- So, Condition-01 satisfies.

	G1	G2	Satisfy
con 1	V =	V =	✓
c 2	e =	e =	X
Con 3			
Con 4	—		

Condition-02:

- Number of edges in graph G1 = 5
- Number of edges in graph G2 = 6

Here,

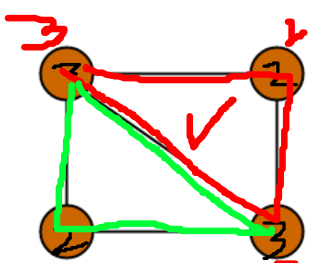
- Both the graphs G1 and G2 have different number of edges.
- So, Condition-02 violates.

Since Condition-02 violates, so given graphs cannot be isomorphic.

∴ G1 and G2 are not isomorphic graphs.

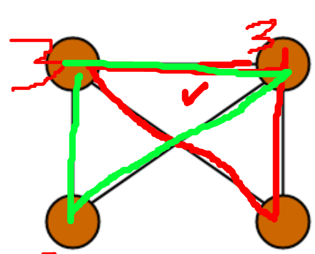
Problem-02:

Which of the following graphs are isomorphic?



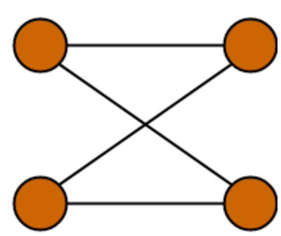
G1

{2,2,3,3}

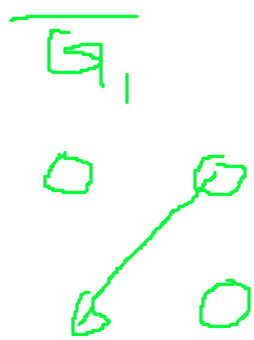


G2

{2,2,3,3}



G3



	G1	G2	G3	Satisfy
1	V = 4	4	4	✓
2	e = 5	5	X 4	G3 not
3	{2,2,3,3}	{2,2,3,3}	-	G1, G2 ✓
4	✓	✓		✓

Solution-

Checking Necessary Conditions-

Condition-01: ✓

- Number of vertices in graph G1 = 4
- Number of vertices in graph G2 = 4
- Number of vertices in graph G3 = 4

Here,

- All the graphs G1, G2 and G3 have same number of vertices.
- So, Condition-01 satisfies. ✓

Condition-02: ✓

- Number of edges in graph G1 = 5
- Number of edges in graph G2 = 5
- Number of edges in graph G3 = 4

Here,

- The graphs G1 and G2 have same number of edges.
- So, Condition-02 satisfies for the graphs G1 and G2.
- However, the graphs (G1, G2) and G3 have different number of edges.
- So, Condition-02 violates for the graphs (G1, G2) and G3.

Since Condition-02 violates for the graphs (G1, G2) and G3, so they cannot be isomorphic.

∴ **G3 is neither isomorphic to G1 nor G2.**

Since Condition-02 satisfies for the graphs G1 and G2, so they may be isomorphic.

∴ **G1 may be isomorphic to G2.**

Now, let us continue to check for the graphs G1 and G2.

Condition-03: ✓

- Degree Sequence of graph G1 = { 2 , 2 , 3 , 3 }
- Degree Sequence of graph G2 = { 2 , 2 , 3 , 3 }

Here,

- Both the graphs G1 and G2 have same degree sequence.
- So, Condition-03 satisfies.

Condition-04:

- Both the graphs contain two cycles each of length 3 formed by the vertices having degrees { 2 , 3 , 3 }
- It means both the graphs G1 and G2 have same cycles in them.
- So, Condition-04 satisfies.

Thus,

- All the 4 necessary conditions are satisfied.
- So, graphs G1 and G2 may be isomorphic.

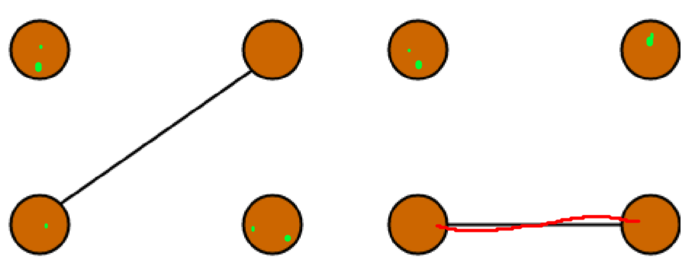
    Now, let us check the sufficient condition.

Checking Sufficient Condition-

We know that two graphs are surely isomorphic if and only if their complement graphs are isomorphic.

So, let us draw the complement graphs of G1 and G2.

The complement graphs of G1 and G2 are-

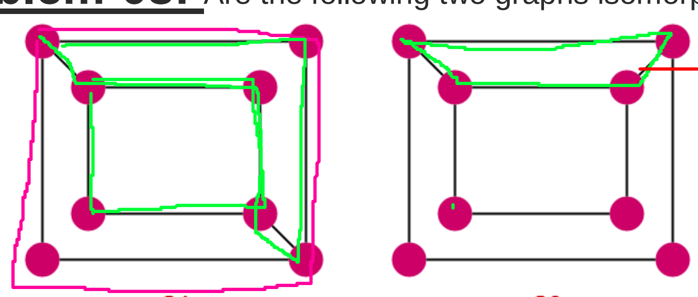


$\overline{G1}$ $\overline{G2}$

Clearly, Complement graphs of G1 and G2 are isomorphic.

∴ Graphs G1 and G2 are isomorphic graphs.

Problem-03: Are the following two graphs isomorphic?



(3,3,3,3)

G1 G2

1,3 3,3

2,2,3,3

3,3,3,3

Solution-

Checking Necessary Conditions-

Condition-01:

- Number of vertices in graph G1 = 8
- Number of vertices in graph G2 = 8

Here,

- Both the graphs G1 and G2 have same number of vertices.
- So, Condition-01 satisfies.

Condition-02:

- Number of edges in graph G1 = 10
- Number of edges in graph G2 = 10

Here,

- Both the graphs G1 and G2 have same number of edges.
- So, Condition-02 satisfies.

Condition-03:

- Degree Sequence of graph G1 = { 2 , 2 , 2 , 2 , 3 , 3 , 3 , 3 }
- Degree Sequence of graph G2 = { 2 , 2 , 2 , 2 , 3 , 3 , 3 , 3 }

Here,

- Both the graphs G1 and G2 have same degree sequence.
- So, Condition-03 satisfies.

Condition-04:

- In graph G1, degree-3 vertices form a cycle of length 4.
- In graph G2, degree-3 vertices do not form a 4-cycle as the vertices are not adjacent.

Here,

- Both the graphs G1 and G2 do not contain same cycles in them.
- So, Condition-04 violates.

Since Condition-04 violates, so given graphs can not be isomorphic.

∴ G1 and G2 are not isomorphic graphs

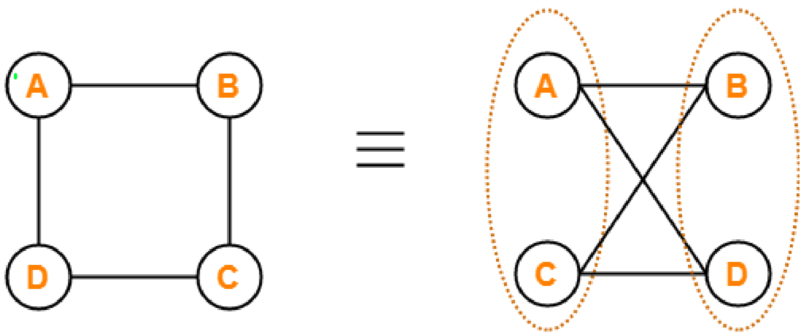
2. Bipartite Graph

A bipartite graph is a special kind of graph with the following properties-

- It consists of two sets of vertices X and Y.
- The vertices of set X join only with the vertices of set Y.
- The vertices within the same set do not join.

Bipartite Graph Example-

The following graph is an example of a bipartite graph-



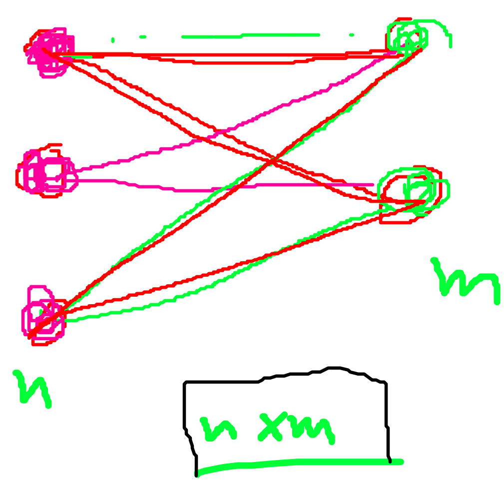
Example of Bipartite Graph

Here,

- The vertices of the graph can be decomposed into two sets.
- The two sets are X = {A, C} and Y = {B, D}.
- The vertices of set X join only with the vertices of set Y and vice-versa.
- The vertices within the same set do not join.
- Therefore, it is a bipartite graph.

Complete Bipartite Graph-

A complete bipartite graph may be defined as follows-



A bipartite graph where every vertex of set X is joined to every vertex of set Y is called as complete bipartite graph.

OR

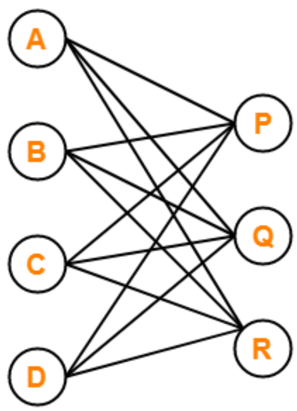
Complete bipartite graph is a bipartite graph which is complete.

OR

Complete bipartite graph is a graph which is bipartite as well as complete.

Complete Bipartite Graph Example-

The following graph is an example of a complete bipartite graph-



Example of Complete Bipartite Graph

Here,

- This graph is a bipartite graph as well as a complete graph.
- Therefore, it is a complete bipartite graph.
- This graph is called as $\mathbf{K_{4,3}}$.

Bipartite Graph Chromatic Number-

To properly color any bipartite graph,

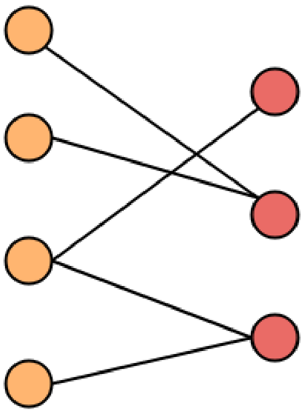
- Minimum 2 colors are required.
- This ensures that the end vertices of every edge are colored with different colors.
- Thus, bipartite graphs are 2-colorable.

Note

If graph is bipartite with no edges, then it is 1-colorable.

Example-

The chromatic number of the following bipartite graph is 2-



Chromatic Number = 2

Bipartite Graph Properties-

Few important properties of bipartite graph are-

- Bipartite graphs are 2-colorable.
- Bipartite graphs contain no odd cycles.
- Every sub graph of a bipartite graph is itself bipartite.
- There does not exist a perfect matching for a bipartite graph with bipartition X and Y if $|X| \neq |Y|$.
- In any bipartite graph with bipartition X and Y,

Sum of degree of vertices of set X = Sum of degree of vertices of set Y

Bipartite Graph Perfect Matching-

Number of complete matchings for $K_{n,n} = n!$

Given a bipartite graph G with bipartition X and Y,

- There does not exist a perfect matching for G if $|X| \neq |Y|$.
- A perfect matching exists on a bipartite graph G with bipartition X and Y if and only if for all the subsets of X, the number of elements in the subset is less than or equal to the number of elements in the neighborhood of the subset.

Maximum Number Of Edges-

- Any bipartite graph consisting of 'n' vertices can have at most $(1/4) \times n^2$ edges.
- Maximum possible number of edges in a bipartite graph on 'n' vertices = $(1/4) \times n^2$.

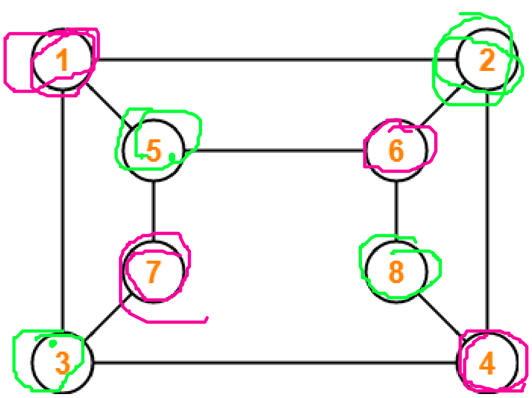
Explanation

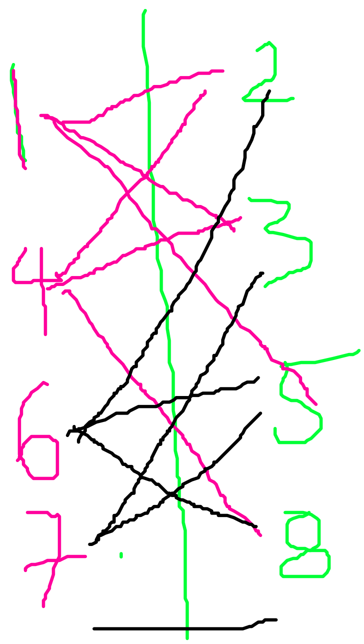
- Suppose the bipartition of the graph is (V_1, V_2) where $|V_1| = k$ and $|V_2| = n-k$.
- The number of edges between V_1 and V_2 can be at most $k(n-k)$ which is maximized at $k = n/2$.
- Thus, maximum $1/4\ n^2$ edges can be present.
- Also, for any graph G with n vertices and more than $1/4\ n^2$ edges, G will contain a triangle.
- This is not possible in a bipartite graph since bipartite graphs contain no odd cycles.

PRACTICE PROBLEMS BASED ON BIPARTITE GRAPH IN GRAPH THEORY-

Problem-01:

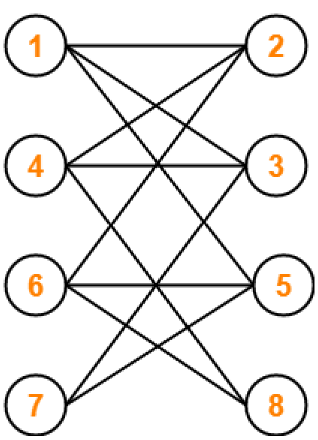
Is the following graph a bipartite graph?





Solution-

The given graph may be redrawn as-



Here,

- This graph consists of two sets of vertices.
- The two sets are X = {1, 4, 6, 7} and Y = {2, 3, 5, 8}.
- The vertices of set X are joined only with the vertices of set Y and vice-versa.
- Also, any two vertices within the same set are not joined.
- This satisfies the definition of a bipartite graph.

Therefore, Given graph is a bipartite graph.

Problem-02:

The maximum number of edges in a bipartite graph on 12 vertices is _________?

Solution-

We know, Maximum possible number of edges in a bipartite graph on 'n' vertices = $(1/4) \times n^2$.

Substituting n = 12, we get- Maximum number of edges in a bipartite graph on 12 vertices

$= (1/4) \times (12)^2$ $= (1/4) \times 12 \times 12$ $= 36$

Therefore, Maximum number of edges in a bipartite graph on 12 vertices = 36.

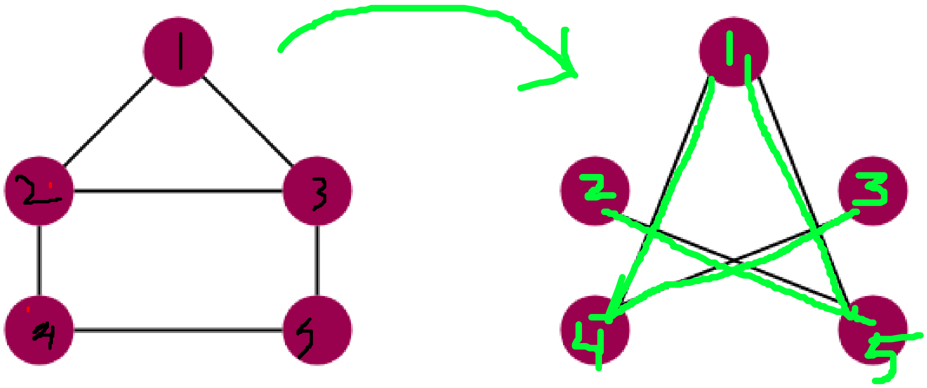
3. Complement Of Graph

Complement of a simple graph G is a simple graph G' having-

- All the vertices of G.
- An edge between two vertices v and w iff there exists no edge between v and w in the original graph G.

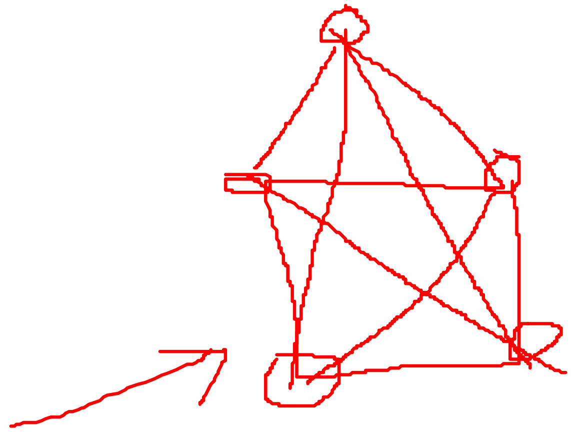
Complement Of Graph Example-

The following example shows a graph G and its complement graph G'-



Graph G   Complement Graph $\overline{G}$

e   e'



$G + \overline{G}$

Relationship Between G & G'-

The following relationship exists between a graph G and its complement graph G'-

1. Number of vertices in G = Number of vertices in G'.

$$|V(G)| = |V(G')|$$

2. The sum of total number of edges in G and G' is equal to the total number of edges in a complete graph.

$$|E(G)| + |E(G')| = C(n,2) = \boxed{n(n-1) / 2}$$

here n = total number of vertices in the graph

Important Terms-

It is important to note the following terms-

- **Order of graph** = Total number of vertices in the graph
- **Size of graph** = Total number of edges in the graph

PRACTICE PROBLEMS BASED ON COMPLEMENT OF GRAPH IN GRAPH THEORY-

Problem-01:

A simple graph G has 10 vertices and 21 edges. Find total number of edges in its complement graph G'.

n e

Solution-

Given-

- Number of edges in graph G, $|E(G)| = 21$
- Number of vertices in graph G, $n = 10$

We know $|E(G)| + |E(G')| = n(n-1) / 2$.

Substituting the values, we get-

$21 + |E(G')| = 10 \times (10-1) / 2$

$|E(G')| = 45 - 21$

$\therefore |E(G')| = 24$

Thus, Number of edges in complement graph G' = 24.

Problem-02:

n

A simple graph G has 30 edges and its complement graph G' has 36 edges. Find number of vertices in G.

Solution-

Given-

- Number of edges in graph G, $|E(G)| = 30$
- Number of edges in graph G', $|E(G')| = 36$

We know $|E(G)| + |E(G')| = n(n-1) / 2$.

Substituting the values, we get-

$30 + 36 = n(n-1) / 2$

$n(n-1) = 132$

$n^2 - n - 132 = 0$

Solving this quadratic equation, we get n = 12.

Thus, Number of vertices in graph G = 12.

Problem-03:

Let G be a simple graph of order n. If the size of G is 56 and the size of G' is 80. What is n?

Solution-

Size of a graph = Number of edges in the graph.

Given-

- Number of edges in graph G, |E(G)| = 56
- Number of edges in graph G', |E(G')| = 80

We know |E(G)| + |E(G')| = n(n-1) / 2.

Substituting the values, we get-

56 + 80 = n(n-1) / 2

n(n-1) = 272

$n^2 – n – 272 = 0$

Solving this quadratic equation, we get n = 17.

Thus, Number of vertices in graph G = 17.

In other words, Order of graph G = 17.