Gale-Shapley Stable Matching Algorithm

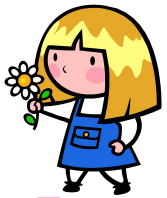
Motivations & Results

- Cheating Strategies in the Stable Marriage problem
  - Gale-Shapley algorithm
    - Deterministic/Randomized strategies
    - Strengthening of Dubins-Freedman theorem
  - Random Stable Matching
    - Group strategies ensuring that every cheating man has a probability which majorizes the original one

Here Comes the Story…

Adam

Geeta, Heiki, Irina, Fran



Fran

Adam, Bob, Carl, David

Bob

Irina, Fran, Heiki, Geeta

Geeta

Carl, David, Bob, Adam

Carl

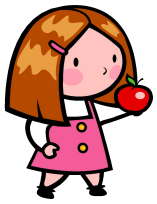
Geeta, Fran, Heiki, Irina

Heiki

Carl, Bob, David, Adam

David

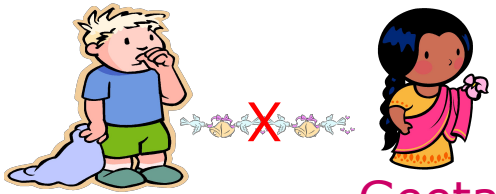
Irina, Heiki, Geeta, Fran

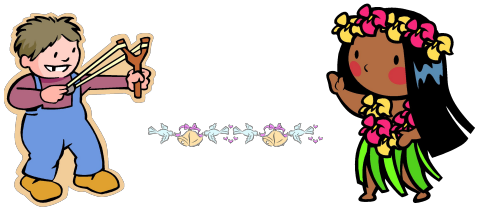


Irina

Adam, Carl, David, Bob

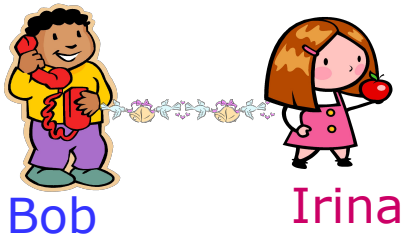
Search for a Matching

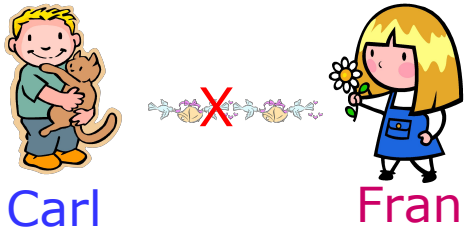




Geeta prefers Carl to Adam!

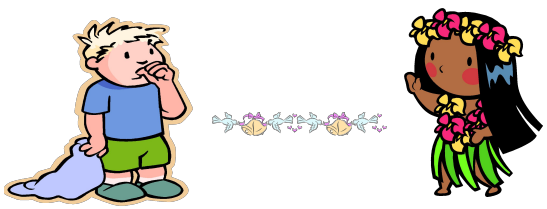
Blocking Pair

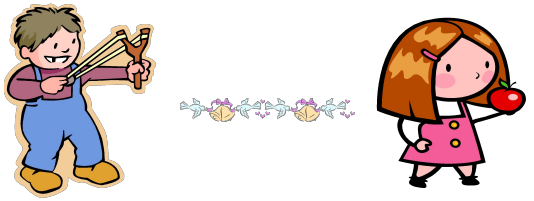




Carl likes Geeta better than Fran!

Stable Matching

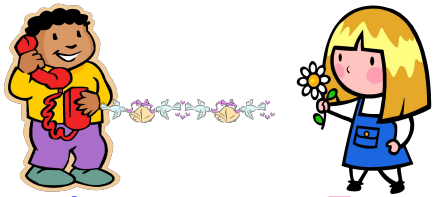


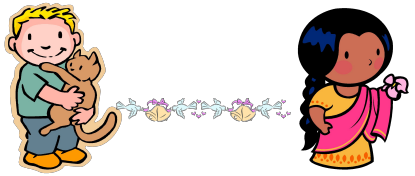


Unfortunately, Irina loves David better!

Bob and Irina are got a blocking pair

Stable Matching: a matching without blocking pairs





Bob likes Irina better than Fran!

Goal

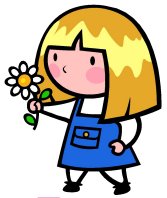
Adam

Bob

Carl

David

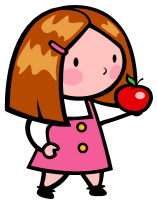
The stable marriage problem (Gale and Shapley, American Mathematical Monthly, 1962)



Fran

Geeta

Heiki



Irina

Deciding a Stable Matching

- Gale-Shapley Stable Matching algorithm
  - Men Propose, women accept/reject
- Random Stable Matching

Gale-Shapley Algorithm

Adam

Geeta, Heiki, Irina, Fran

Fran

Bob

Irina, Fran, Heiki, Geeta

This is a stable matching

Carl > Adam

Geeta

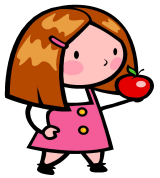
Carl

Geeta, Fran, Heiki, Irina

Heiki

David

Irina, Heiki, Geeta, Fran



David > Bob

Irina

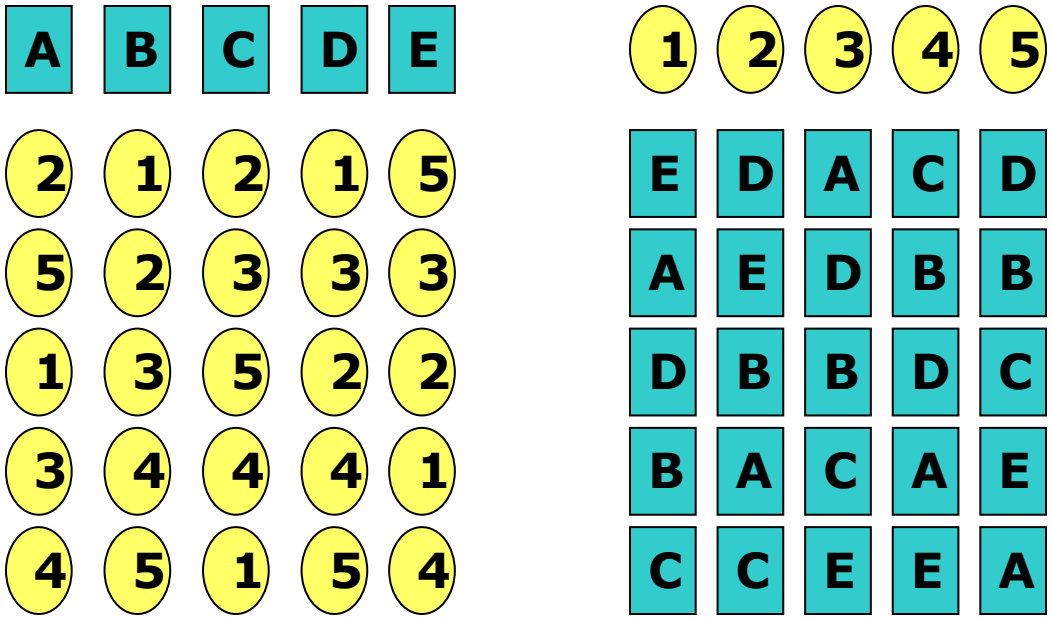
Stable Marriage Problem

Stable Marriage Problem

- Given N men and N women, each person list in order of preference all the people of the opposite sex who would like to marry.
- Problem:
  - Engage all the women to all the men in such a way as to respect all their preferences as much as possible.

Stable?

- A set of marriages is unstable if
  - two people who are not married both prefer each other than their spouses
- E.g. Suppose we have A1 B3 C2 D4 E5. This is unstable since
  - A prefer 2 more than 1
  - 2 prefer A more than C



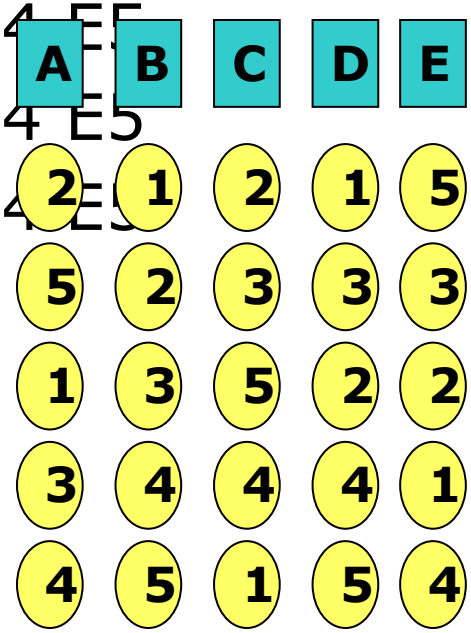
Naïve solution

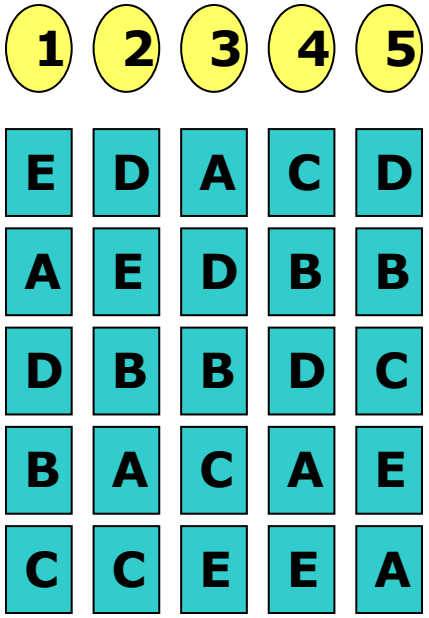
- Starting from a feasible solution.
- Check if it is stable.
  - If yes, done!
  - If not, remove an unstable couple.

- Is this work?

Naïve solution (2)

- Does not work!
- E.g.
  - A1 B3 C2 D4 E5
  - A2 B3 C1 D4 E5
  - A3 B2 C1 D4 E5
  - A3 B1 C2 D4 E5

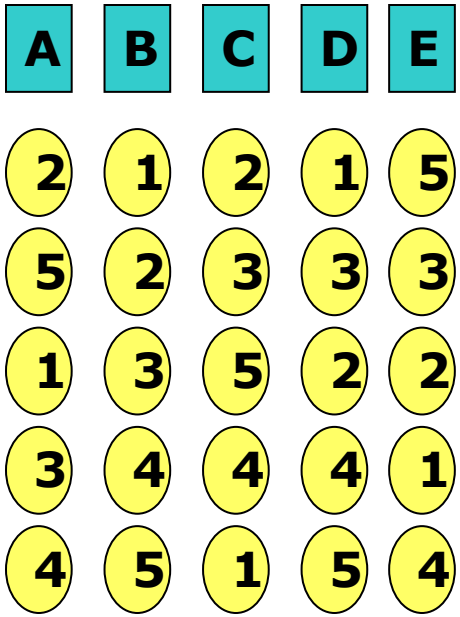


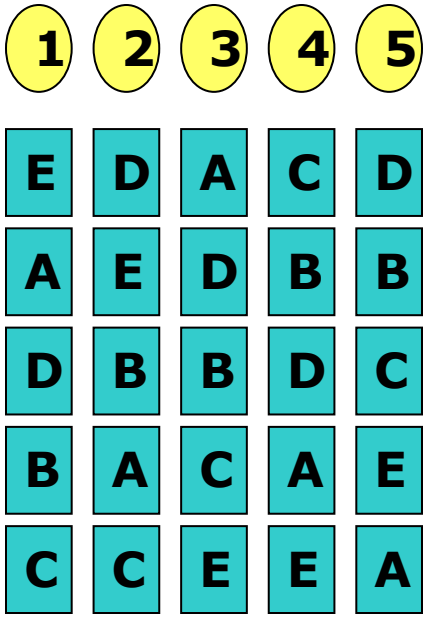


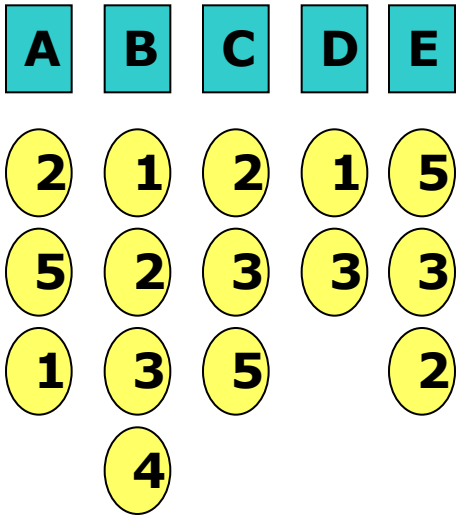
Solution

1. Let X be the first man.
2. X proposes to the best woman in the remaining on his list. (Initially, the first woman on his list!)
3. If α is not engaged
   - Pair up (X, α). Then, set X=next man and goto 1.
4. If α prefers X more than her fiancee Y,
   - Pair up (X, α). Then, set X=Y and goto 1.
5. Goto 1

Example


A B C D E
2 1 2 1 5
5 2 3 3 3
1 3 5 2 2
3 4 4 4 1
4 5 1 5 4


1 2 3 4 5
E D A C D
A E D B B
D B B D C
B A C A E
C C E E A


A B C D E
2 1 2 1 5
5 2 3 3 3
1 3 5 2
4

Time analysis

- If there are N men and N women,
  - $O(N^2)$ time

Algorithm

- prefer[m][s]=w means the woman w is on the s-th position in the preference list of the man m
- Let next[m] be the current best woman in his remaining list. (Initially, next[m]=0)
- fiancee[w]=m means the man m engaged to woman w. (Initially, fiancee[w]=0)
- Let rank[w][m] is the ranking of the man m in the preference list of the woman w.
- For(m=1;m<=N;m++) {
  For(s=m;s!=0;
  }