# Connectivity

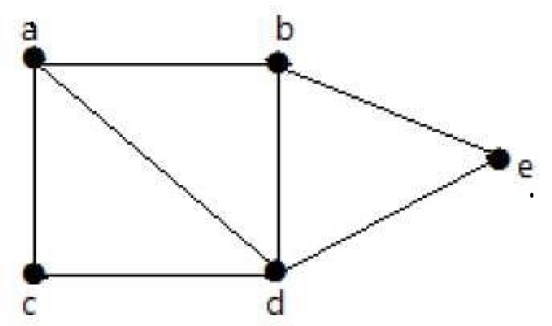
Whether it is possible to traverse a graph from one vertex to another is determined by how a graph is connected. Connectivity is a basic concept in Graph Theory. Connectivity defines whether a graph is connected or disconnected. It has subtopics based on edge and vertex, known as edge connectivity and vertex connectivity. Let us discuss them in detail.

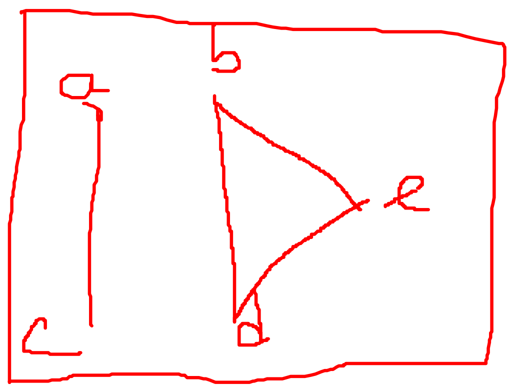
## Connectivity

A graph is said to be **connected if there is a path between every pair of vertex**. From every vertex to any other vertex, there should be some path to traverse. That is called the connectivity of a graph. A graph with multiple disconnected vertices and edges is said to be disconnected.

**Example 1**

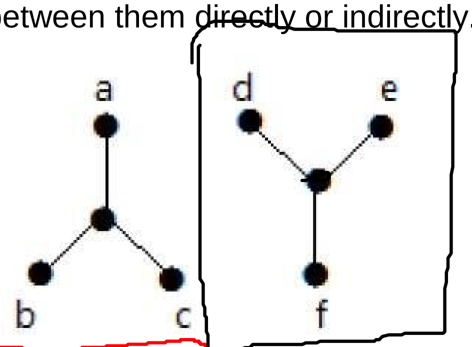
In the following graph, it is possible to travel from one vertex to any other vertex. For example, one can traverse from vertex ‘a’ to vertex ‘e’ using the path ‘a-b-e’.




**Example 2**

In the following example, traversing from vertex ‘a’ to vertex ‘f’ is not possible because there is no path between them directly or indirectly. Hence it is a disconnected graph.




## Cut Vertex

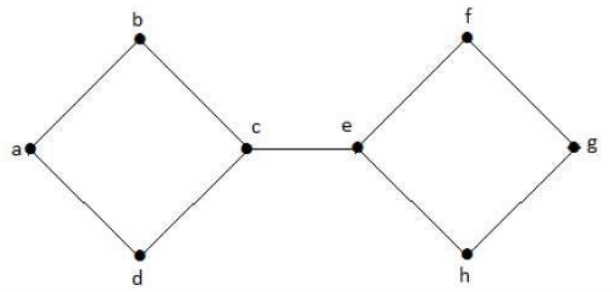
Let ‘G’ be a connected graph. A vertex V ∈ G is called a cut vertex of ‘G’, if ‘G-V’ (Delete ‘V’ from ‘G’) results in a disconnected graph. Removing a cut vertex from a graph breaks it in to two or more graphs.

**Note** − Removing a cut vertex may render a graph disconnected.

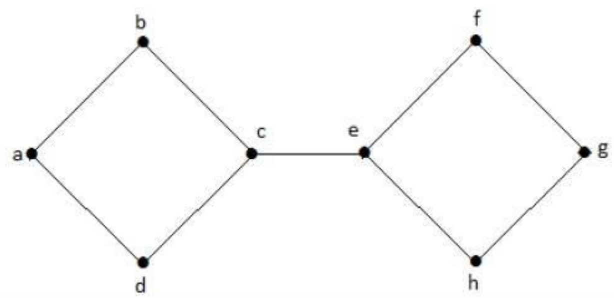
A connected graph ‘G’ may have at most (n–2) cut vertices.

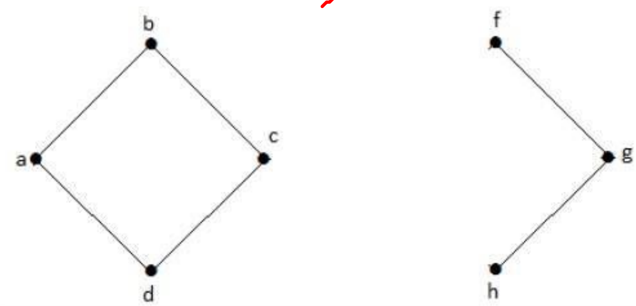
**Example**

In the following graph, vertices ‘e’ and ‘c’ are the cut vertices.




By removing ‘e’ or ‘c’, the graph will become a disconnected graph.




Without ‘g’, there is no path between vertex ‘c’ and vertex ‘h’ and many other. Hence it is a disconnected graph with cut vertex as ‘e’. Similarly, ‘c’ is also a cut vertex for the above graph.

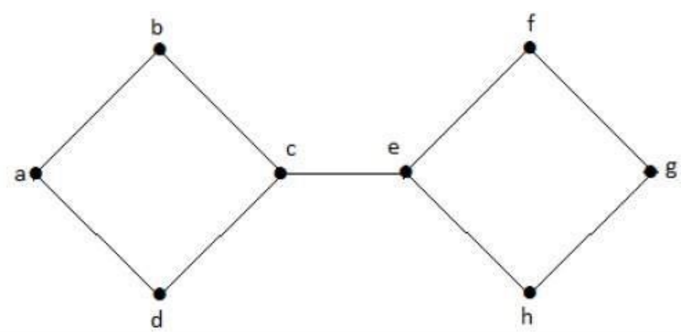
## Cut Edge (Bridge)

Let ‘G’ be a connected graph. An edge ‘e’ ∈ G is called a cut edge if ‘G-e’ results in a disconnected graph.

If removing an edge in a graph results in to two or more graphs, then that edge is called a Cut Edge.

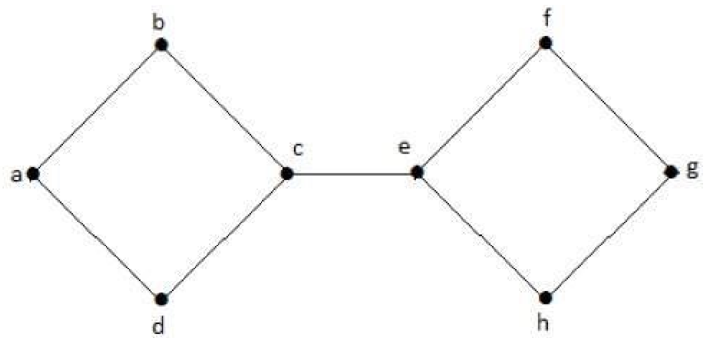
**Example**

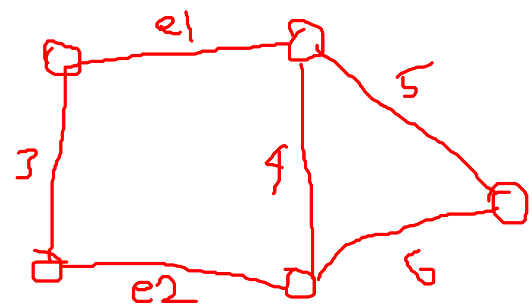
In the following graph, the cut edge is [(c, e)].

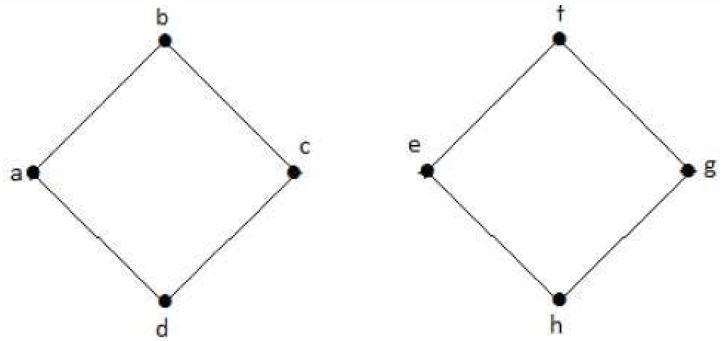



By removing the edge (c, e) from the graph, it becomes a disconnected graph.

{ce, eh} {bc, ce, eh}




{e1, e2}.

{1, 4, 6}.

In the above graph, removing the edge (c, e) breaks the graph into two which is nothing but a disconnected graph. Hence, the edge (c, e) is a cut edge of the graph.

**Note −** Let ‘G’ be a connected graph with ‘n’ vertices, then

Thm

- a cut edge e ∊ G if and only if the edge ‘e’ is not a part of any cycle in G.
- the maximum number of cut edges possible is ‘n-1’.
- whenever cut edges exist, cut vertices also exist because at least one vertex of a cut edge is a cut vertex.
- if a cut vertex exists, then a cut edge may or may not exist.

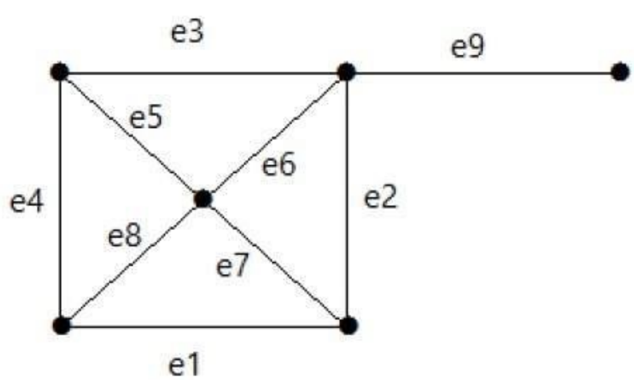
## Cut Set of a Graph

Let ‘G’= (V, E) be a connected graph. A subset E’ of E is called a cut set of G if deletion of all the edges of E’ from G makes G disconnect.

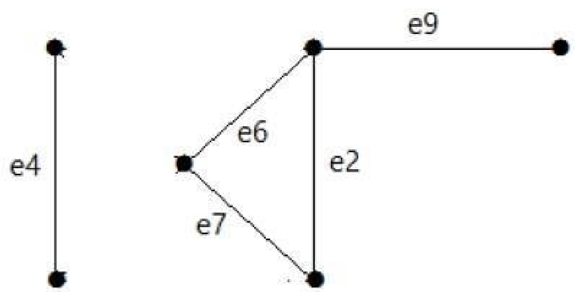
If deleting a certain number of edges from a graph makes it disconnected, then those deleted edges are called the cut set of the graph.

**Example**

Take a look at the following graph. Its cut set is E1 = {e1, e3, e5, e8}.



After removing the cut set E1 from the graph, it would appear as follows −




Similarly, there are other cut sets that can disconnect the graph −

- E3 = {e9} – Smallest cut set of the graph.
- E4 = {e3, e4, e5}

## Edge Connectivity

Let ‘G’ be a connected graph. The minimum number of edges whose removal makes ‘G’ disconnected is called edge connectivity of G.

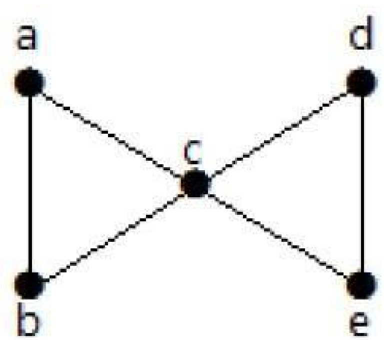
**Notation** − λ(G)

In other words, the **number of edges in a smallest cut set of G** is called the edge connectivity of G.

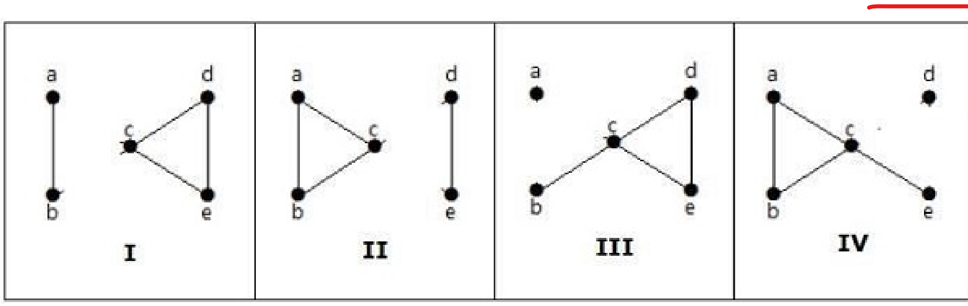
If ‘G’ has a cut edge, then λ(G) is 1. (edge connectivity of G.)

**Example**

Take a look at the following graph. By removing two minimum edges, the connected graph becomes disconnected. Hence, its edge connectivity (λ(G)) is 2.




Here are the four ways to disconnect the graph by removing two edges −



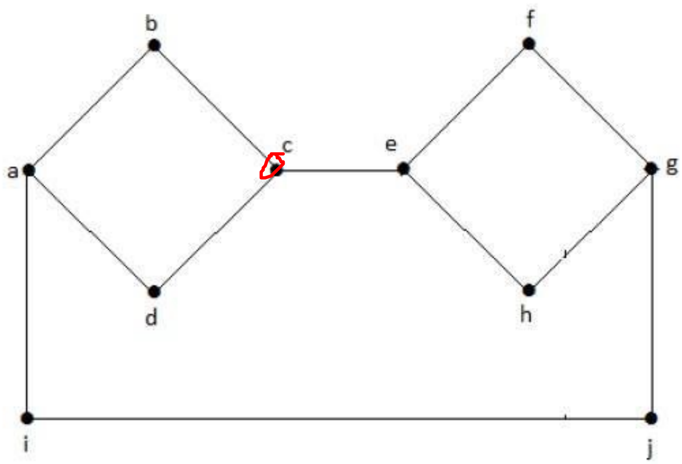

## Vertex Connectivity

Let ‘G’ be a connected graph. The minimum number of vertices whose removal makes ‘G’ either disconnected or reduces ‘G’ in to a trivial graph is called its vertex connectivity.

**Notation** − K(G)

**Example**

In the above graph, removing the vertices ‘e’ and ‘i’ makes the graph disconnected.




If G has a cut vertex, then K(G) = 1.

**Notation** − For any connected graph G,

imp

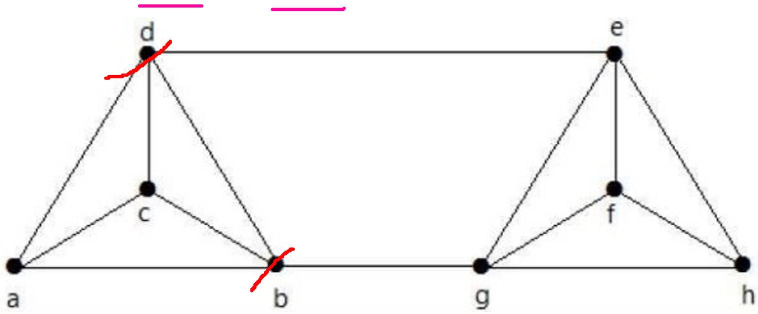
proof H.W.

K(G) ≤ λ(G) ≤ δ(G)

Vertex connectivity (K(G)), edge connectivity (λ(G)), minimum number of degrees of G(δ(G)).

**Example**

Calculate λ(G) and K(G) for the following graph −




$\delta = 3$

$\lambda = 2$

$k = 2$

$\boxed{2 \leq 2 \leq 3}$

**Solution**

From the graph,

δ(G) = 3

K(G) ≤ λ(G) ≤ δ(G) = 3 (1)

K(G) ≥ 2 (2)

Deleting the edges {d, e} and {b, h}, we can disconnect G.

Therefore,  λ(G) = 2

2 ≤ λ(G) ≤ δ(G) = 2 (3)

From (2) and (3), vertex connectivity K(G) = 2

# Preliminaries

**DEFINITIONS**

D1: A graph is connected if there exists a walk between every pair of its vertices.

A graph that is not connected is called disconnected.

D2: The subgraphs of G which are maximal with respect to the property of being connected are called the components of G.

D3: Let G= (V, E ) be a graph and U⊂V. The vertex-deletion subgraph G−U is the graph obtained from G by deleting from G the vertices in U. That is, G−U is the subgraph induced on the vertex subset V−U. If U={u}, we simply write G−u.

D4: Let G= (V, E ) be a graph and F⊂E. The edge-deletion subgraph G−F is the subgraph obtained from G by deleting from G the edges in F. Thus, G−F= (V, E −F).

As in the case of vertex deletion, if F={e}, it is customary to write G−e rather than G− {e}.

D5: A disconnecting (vertex-)set (or vertex-cut) of a connected graph G is a vertex subset U such that G−U has at least two different components.

D6: A vertex v is a cut-vertex of a connected graph Gif {v} is a disconnecting set of G.

D7: A disconnecting edge-set (or edge-cut) of a connected graph G is an edge subset F such that G−F has at least two different components.

D8: An edge e is a bridge (or cut-edge) of a connected graph G if {e} is a disconnecting edge-set of G.

**FACTS**

F1: Every nontrivial connected graph contains at least two vertices that are not cut-vertices.

F2: An edge is a bridge if and only if it lies on no cycle.

**Vertex- and Edge-Connectivity**

The simplest way of quantifying connectedness of a graph is by means of its parameters vertex-connectivity and edge-connectivity.

**DEFINITIONS**

D9: The (vertex-)connectivity κ(G) of a graph G is the minimum number of vertices whose removal from G leaves a disconnected or a trivial graph.

D10: The edge-connectivity λ(G) of a nontrivial graph G is the minimum number of edges whose removal from G results in a disconnected graph.

notation: When the context is clear, we suppress the dependence on G and simply use κ and λ.

notation: In some other sections of the Handbook, $\kappa_v(G)$ and $\kappa_e(G)$ are used instead of κ(G) and λ(G).

**EXAMPLE**

E1: Figure 4.1.1 shows an example of a graph with κ= 2 and λ= 3.

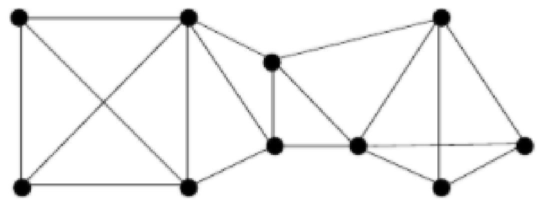


Figure 4.1.1: κ= 2 and λ= 3.

**FACTS**

F3: We have κ= 0 if and only if G is disconnected or $G=K_1$. If G has order n, then κ=n−1 if and only if G is the complete graph $K_n$. In this case, the removal of n−1 vertices results in the trivial graph $K_1$. Moreover, if G $!=K_n$ is a connected graph, then 1 ≤κ≤n−2 and there exists a disconnecting set U of κ vertices.

F4: If G $!=K_1$ we have λ= 0 if G is disconnected. By convention, we set $\lambda(K_1) = 0$.

F5: If G $!=K_1$ is connected, then the removal of λ edges results in a disconnected graph with precisely two components.

F6: The parameters κ and λ can be computed in polynomial time.

**Relationships Among the Parameters**

notation: The minimum degree of a graph G is denoted δ(G). When the context is clear, we simply write δ. (In some other sections of the Handbook, the notation $\delta_{min}$ (G) is used.)

**FACTS**

F7: For any graph, κ ≤ λ ≤ δ.

F8: For all integers a, b, c such that 0 < a ≤ b ≤ c, there exists a graph G with κ = a, λ = b, and δ = c.

**DEFINITIONS**

D11: A graph G is *maximally connected* when κ=λ=δ, and G is *maximally edge-connected* when λ=δ.

D12: A graph G with connectivity κ ≥ k ≥ 1 is called k-connected. Equivalently, G is k-connected if the removal of fewer than k vertices leaves neither a disconnected graph nor a trivial one. Analogously, if λ≥k≥1, G is said to be k-edge-connected.

D13: A connected graph G without cut-vertices (κ > 1 or G=K2) is called a block.

Some Simple Observations

The following facts are simply restatements of the definitions.

**FACTS**

F9: A nontrivial graph is 1-connected if and only if it is connected.

F10: A graph G is k-edge-connected if the deletion of fewer than kedges does not disconnect it.

F11: Every block with at least three vertices is 2-connected.