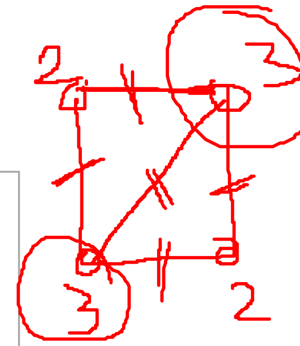
Handshaking Theorem



Handshaking Theorem is also known as **Handshaking Lemma** or **Sum of Degree Theorem**.

In Graph Theory,

Handshaking Theorem states in any given graph,

Sum of degree of all the vertices is twice the number of edges contained in it.

$$\sum_{i=1}^{n} d(v_i) = 2 \times |E|$$

$\Sigma d = 10$

$\Sigma d = 20$

Handshaking Theorem

$2e = 10$

$e = 5$

The following conclusions may be drawn from the Handshaking Theorem. In any graph,

- The sum of degree of all the vertices is always even.
- The sum of degree of all the vertices with odd degree is always even.
- The number of vertices with odd degree are always even.

PRACTICE PROBLEMS BASED ON HANDSHAKING THEOREM IN GRAPH THEORY-

Problem-01:

A simple graph G has 24 edges and degree of each vertex is 4. Find the number of vertices.

$n = ?$

Solution-

Given-

- Number of edges = 24
- Degree of each vertex = 4

Let number of vertices in the graph = n.

Using Handshaking Theorem, we have- Sum of degree of all vertices = 2 x Number of edges

Substituting the values, we get-

$\boxed{n \times 4 = 2 \times 24}$

n = 2 x 6

∴ n = 12

Thus, Number of vertices in the graph = 12.

Problem-02:

A graph contains 21 edges, 3 vertices of degree 4 and all other vertices of degree 2. Find total number of vertices.

Solution-

Given-

- Number of edges = 21
- Number of degree 4 vertices = 3
- All other vertices are of degree 2

(n-3)

Let number of vertices in the graph = n.

Using Handshaking Theorem, we have-

Sum of degree of all vertices = 2 x Number of edges

Substituting the values, we get-

$3 \times 4 + \boxed{(n-3)} \times 2 = 2 \times 21$

$12 + 2n - 6 = 42$

$2n = 42 - 6$

$2n = 36$

$\therefore n = 18$

Thus, Total number of vertices in the graph = 18.

Problem-03:

A simple graph contains 35 edges, four vertices of degree 5, five vertices of degree 4 and four vertices of degree 3. Find the number of vertices with degree 2.

n = ?

x

Solution-

Given-

- Number of edges = 35
- Number of degree 5 vertices = 4
- Number of degree 4 vertices = 5
- Number of degree 3 vertices = 4

Let number of degree 2 vertices in the graph = n.

Using Handshaking Theorem, we have- Sum of degree of all vertices = 2 x Number of edges

Substituting the values, we get-

4 x 5 + 5 x 4 + 4 x 3 + n x 2 = 2 x 35

20 + 20 + 12 + 2n = 70

52 + 2n = 70 ; 2n = 70 – 52 ; 2n = 18

∴ n = 9

Thus, Number of degree 2 vertices in the graph = 9.

Problem-04:

A graph has 24 edges and degree of each vertex is k, then which of the following is possible number of vertices?

(a) 20, (b) 15, (c) 10, (d) 8

Solution-

Given-

- Number of edges = 24
- Degree of each vertex = k

Let number of vertices in the graph = n.

Using Handshaking Theorem, we have- Sum of degree of all vertices = 2 x Number of edges

Substituting the values, we get-

n x k = 2 x 24

k = 48 / n

Now,

- It is obvious that the degree of any vertex must be a whole number.
- So in the above equation, only those values of 'n' are permissible which gives the whole value of 'k'.

Now, let us check all the options one by one-

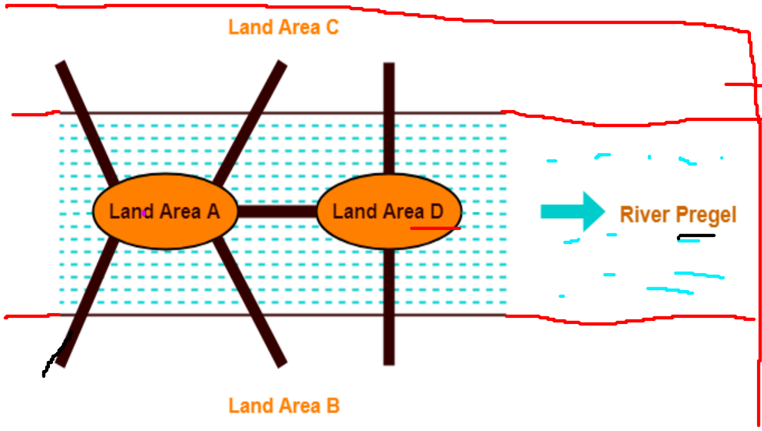
- For n = 20, k = 2.4 which is not allowed.
- For n = 15, k = 3.2 which is not allowed.
- For n = 10, k = 4.8 which is not allowed.
- For n = 8, k = 6 which is allowed.

Thus, Option (D) is correct.

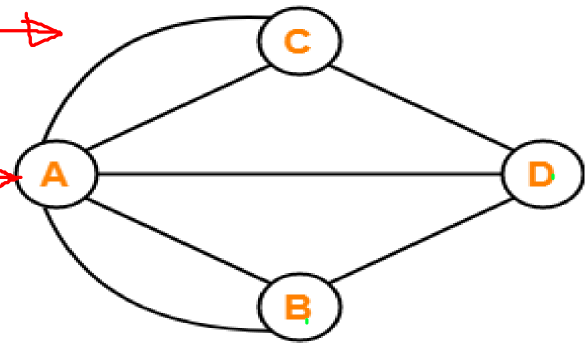
Konigsberg Bridge Problem in Graph Theory

Konigsberg-

- Konigsberg is the former name of a German city that is now in Russia.
- The following picture shows the inner city of Konigsberg with the river Pregel.
- The river Pregel divides the city into four land areas A, B, C and D.
- In order to travel from one part of the city to another, there exists seven bridges.



Konigsberg Bridge Problem



Graph Representation

Konigsberg Bridge Problem-

Konigsberg Bridge Problem may be stated as-

"Starting from any of the four land areas A, B, C, D, is it possible to cross each of the seven bridges exactly once and come back to the starting point without swimming across the river?"

Konigsberg Bridge Problem Solution-

In 1735,

- A Swiss Mathematician Leon hard Euler solved this problem.
- He provided a solution to the problem and finally concluded that such a walk is not possible.

Euler represented the given situation using a graph as shown above.

In this graph,

- Vertices represent the landmasses.
- Edges represent the bridges.

Euler observed that when a vertex is visited during the process of tracing a graph,

- There must be one edge that enters into the vertex.
- There must be another edge that leaves the vertex.
- Therefore, order of the vertex must be an even number.

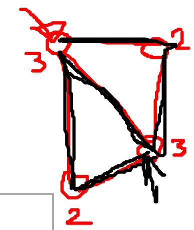
Based on this observation, Euler discovered that it depends on the number of odd vertices present in the network whether any network is traversable or not.

Euler found that only those networks are traversable that have either-

- No odd vertices (then any vertex may be the beginning and the same vertex will also be the ending point)
- Or exactly two odd vertices (then one odd vertex will be the starting point and other odd vertex will be the ending point)

Now,

- Since the Konigsberg network has four odd vertices, therefore the network is not traversable.
- Thus, it was finally concluded that the desired walking tour of Konigsberg is not possible.



NOTE

If the citizens of Konigsberg decide to build an eighth bridge from A to C, then-

- It would be possible to walk without traversing any bridge twice.
- This is because then there will be exactly two odd vertices.

However, adding a ninth bridge will again make the walking tour once again impossible.

Euler Graph

Euler Path

Euler path is also known as **Euler Trail** or **Euler Walk**.

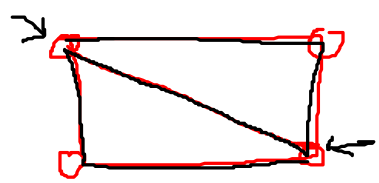
- If there exists a **Trail** in the connected graph that contains all the edges of the graph, then that trail is called as an Euler trail.

OR

- If there exists a walk in the connected graph that visits every edge of the graph exactly once with or without repeating the vertices, then such a walk is called as an Euler walk.

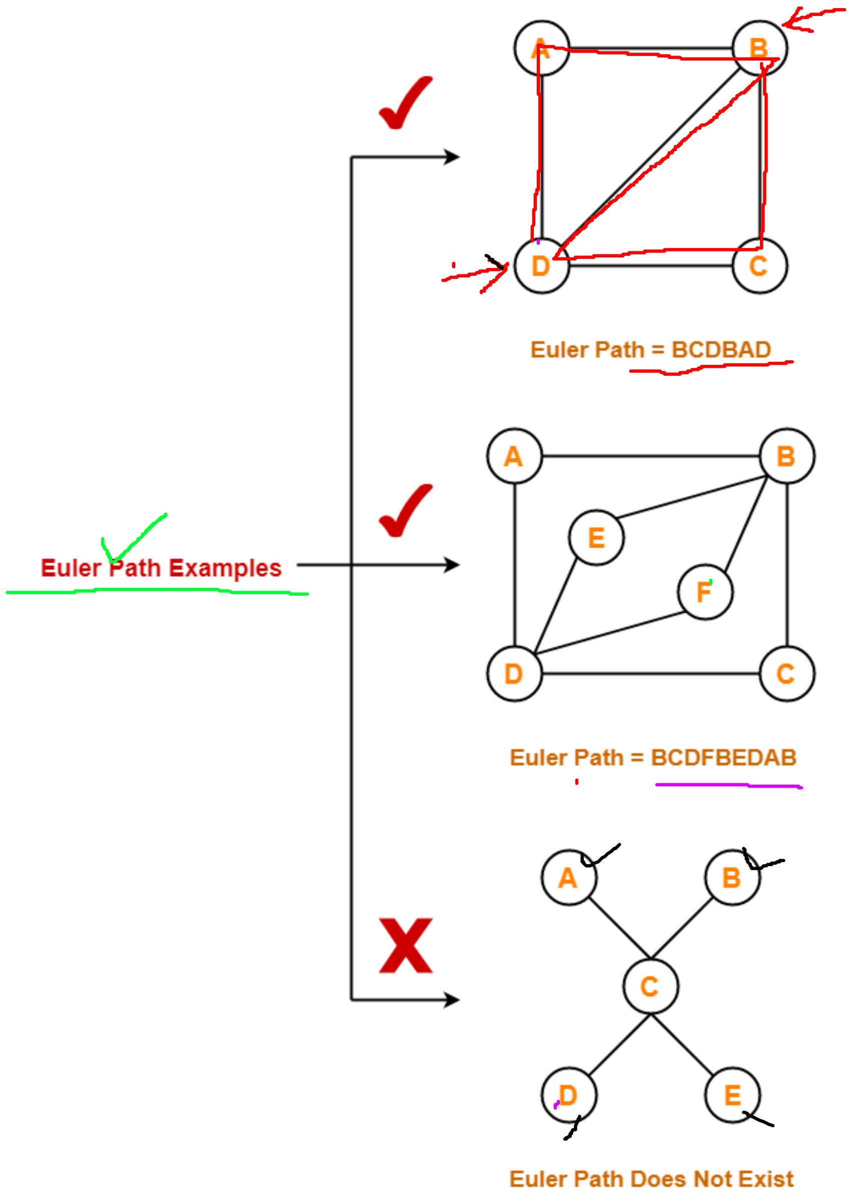
NOTE

A graph will contain an Euler path if and only if it contains at most two vertices of odd degree.



Euler Path Examples-

Examples of Euler path are as follows-



Euler Circuit-

Euler circuit is also known as **Euler Cycle** or **Euler Tour**.

- If there exists a **Circuit** in the connected graph that contains all the edges of the graph, then that circuit is called as an Euler circuit.

OR

- If there exists a walk in the connected graph that starts and ends at the same vertex and visits every edge of the graph exactly once with or without repeating the vertices, then such a walk is called as an Euler circuit.

OR

- An Euler trail that starts and ends at the same vertex is called as an Euler circuit.

OR

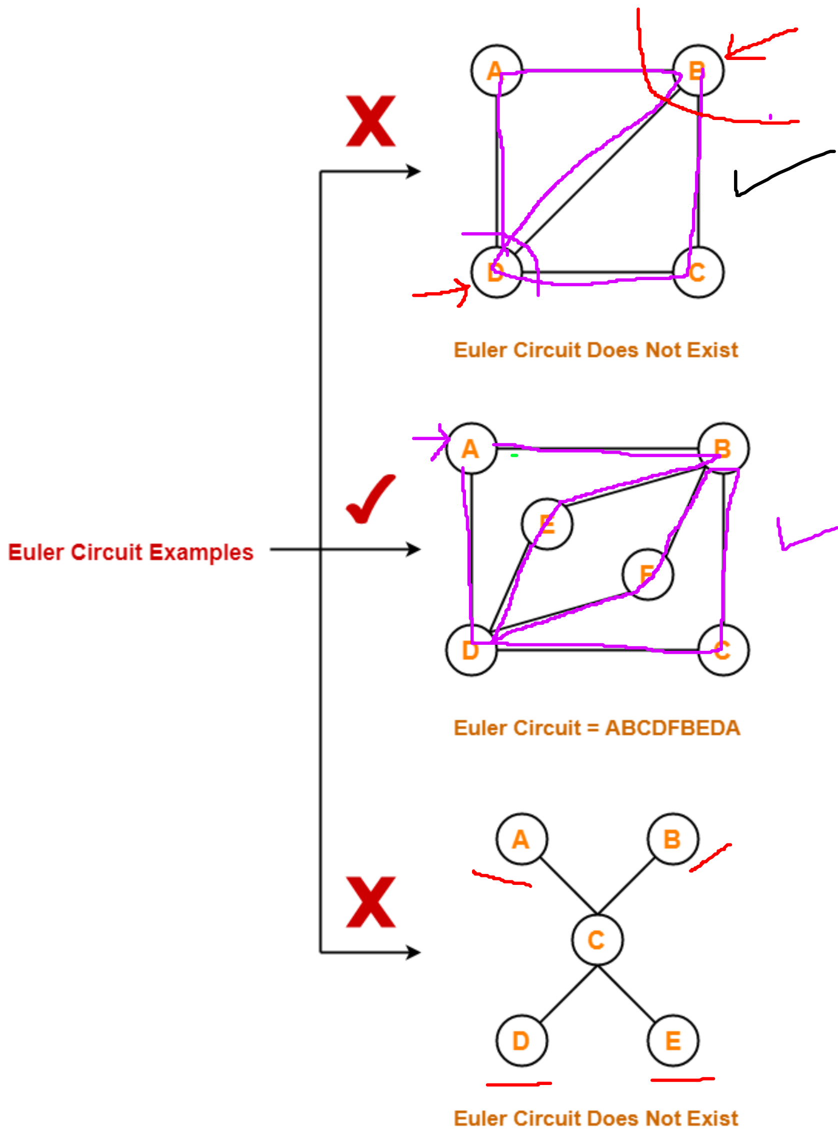
- A closed Euler trail is called as an Euler circuit.

NOTE

A graph will contain an Euler circuit if and only if all its vertices are of even degree.

Euler Circuit Examples-

Examples of Euler circuit are as follows-



$(n \geq 3)$ for cycle

Euler Graph

Semi-Euler Graph-

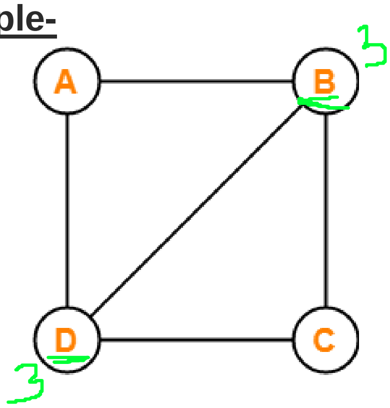
If a connected graph contains an Euler trail but does not contain an Euler circuit, then such a graph is called as a semi-Euler graph.

Thus, for a graph to be a semi-Euler graph, following two conditions must be satisfied-

- Graph must be connected.
- Graph must contain an Euler trail.

Trail ⇒ open walk

Example-



Semi-Euler Graph

Here,

- This graph contains an Euler trail BCDBAD.
- But it does not contain an Euler circuit.
- Therefore, it is a semi-Euler graph.

Important Notes-

Note-01: Euler Graph

To check whether any graph is an Euler graph or not, any one of the following two ways may be used-

- If the graph is connected and contains an Euler circuit, then it is an Euler graph.
- If all the vertices of the graph are of even degree, then it is an Euler graph.

Note-02: Euler circuit

To check whether any graph contains an Euler circuit or not,

- Just make sure that all its vertices are of even degree.
- If all its vertices are of even degree, then graph contains an Euler circuit otherwise not.

Note-03: semi-Euler graph

To check whether any graph is a semi-Euler graph or not,

- Just make sure that it is connected and contains an Euler trail.
- If the graph is connected and contains an Euler trail, then graph is a semi-Euler graph otherwise not.

Note-04: Euler trail

To check whether any graph contains an Euler trail or not,

- Just make sure that the number of vertices in the graph with odd degree are not more than 2.
- If the number of vertices with odd degree are at most 2, then graph contains an Euler trail otherwise not.

Note-05:

- A graph will definitely contain an Euler trail if it contains an Euler circuit.
- A graph may or may not contain an Euler circuit if it contains an Euler trail.

Note-06:

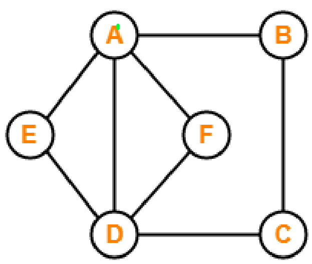
- An Euler graph is definitely be a semi-Euler graph.
- But a semi-Euler graph may or may not be an Euler graph.

PRACTICE PROBLEMS BASED ON EULER GRAPHS IN GRAPH THEORY-

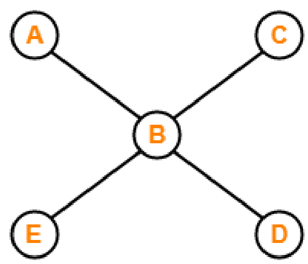
Problems-

Which of the following is / are Euler Graphs?

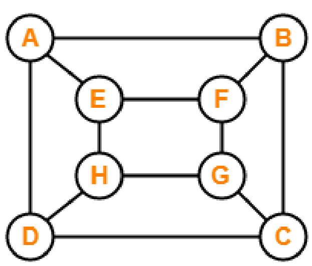
A)



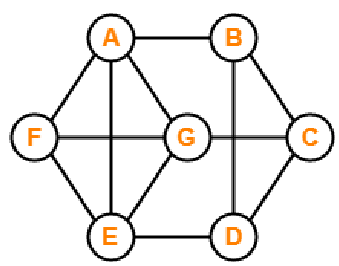
B)



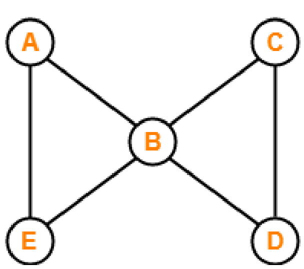
C)



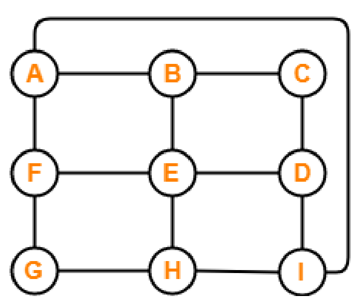
D)



E)



F)



Solutions-

If all the vertices of a graph are of even degree, then graph is an Euler Graph otherwise not.

Using the above rule, we have-

A) It is an Euler graph.

B) It is not an Euler graph.

C) It is not an Euler graph.

D) It is not an Euler graph.

E) It is an Euler graph.

F) It is not an Euler graph.

Planar Graph

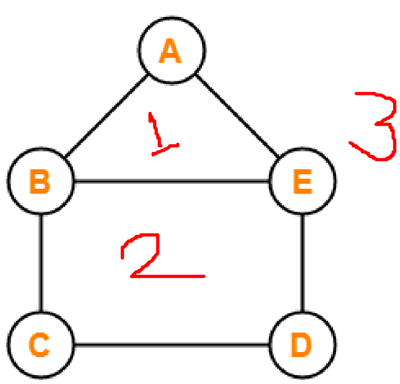
A planar graph may be defined as-

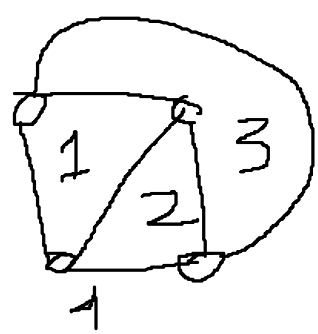
In graph theory,

Planar graph is a graph that can be drawn in a plane such that none of its edges cross each other.

Planar Graph Example-

The following graph is an example of a planar graph-





Example of Planar Graph

Here,

- In this graph, no two edges cross each other.
- Therefore, it is a planar graph.

Regions of Plane-

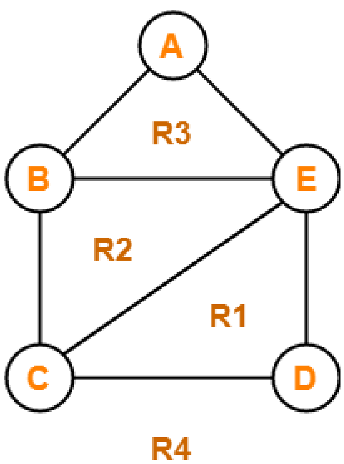
The planar representation of the graph splits the plane into connected areas called as **Regions of the plane**.

Each region has some degree associated with it given as-

- Degree of Interior region = Number of edges enclosing that region
- Degree of Exterior region = Number of edges exposed to that region

Example-

Consider the following planar graph-



Regions of Plane

Here, this planar graph splits the plane into 4 regions- R1, R2, R3 and R4 where-

- Degree (R1) = 3
- Degree (R2) = 3
- Degree (R3) = 3
- Degree (R4) = 5

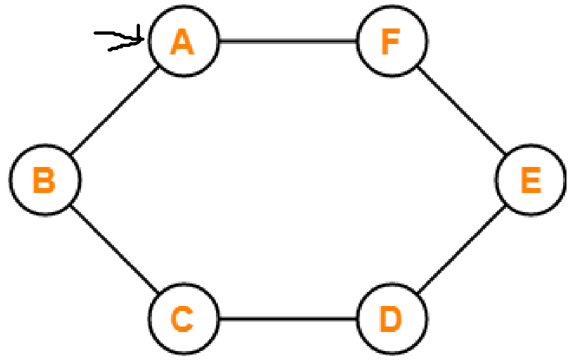
Hamiltonian Graph

A Hamiltonian graph may be defined as-

> If there exists a closed walk in the connected graph that visits every vertex of the graph exactly once (except starting vertex) without repeating the edges, then such a graph is called as a Hamiltonian graph.
>
> **OR**
>
> Any connected graph that contains a Hamiltonian circuit is called as a Hamiltonian Graph.

Hamiltonian Graph Example-

The following graph is an example of a Hamiltonian graph-



Example of Hamiltonian Graph

Here,

- This graph contains a closed walk ABCDEFA.
- It visits every vertex of the graph exactly once except starting vertex.
- The edges are not repeated during the walk.
- Therefore, it is a Hamiltonian graph.

Alternatively, there exists a Hamiltonian circuit ABCDEFA in the above graph, therefore it is a Hamiltonian graph.

Hamiltonian Path-

- If there exists a walk in the connected graph that visits every vertex of the graph exactly once without repeating the edges, then such a walk is called as a Hamiltonian path.

OR

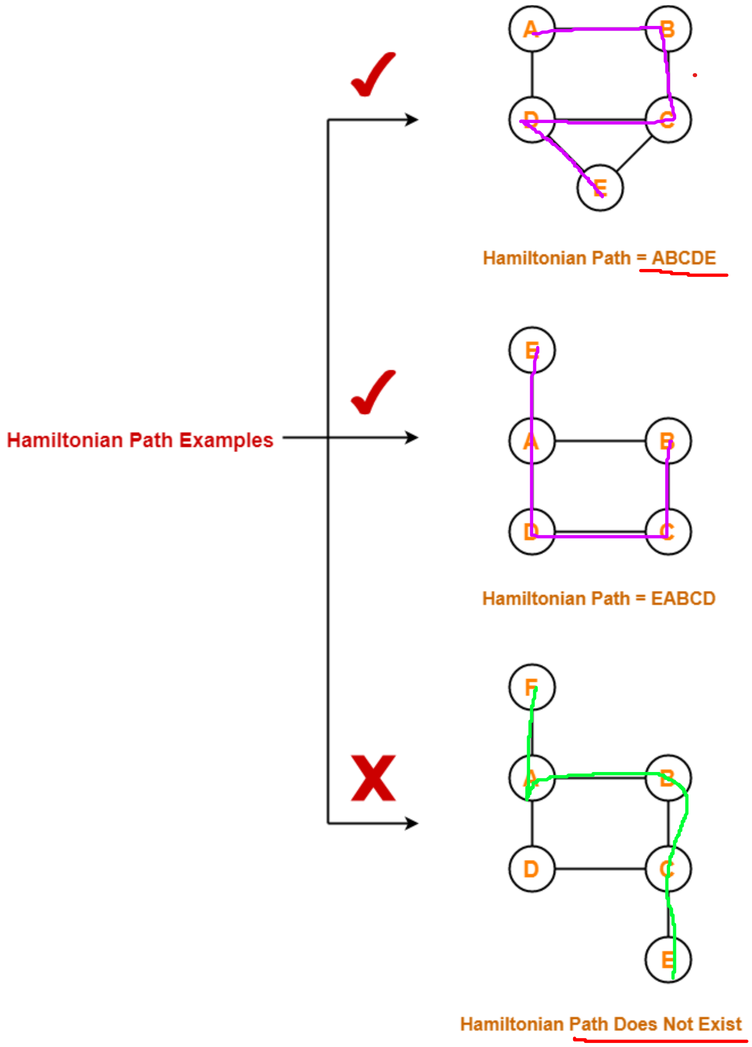
- If there exists a **Path** in the connected graph that contains all the vertices of the graph, then such a path is called as a Hamiltonian path.

NOTE

In Hamiltonian path, all the edges may or may not be covered but edges must not repeat.

Hamiltonian Path Examples-

Examples of Hamiltonian path are as follows-



Hamiltonian Circuit-

Hamiltonian circuit is also known as **Hamiltonian Cycle**.

- If there exists a walk in the connected graph that visits every vertex of the graph exactly once (except starting vertex) without repeating the edges and returns to the starting vertex, then such a walk is called as a Hamiltonian circuit.

OR

- If there exists a **Cycle** in the connected graph that contains all the vertices of the graph, then that cycle is called as a Hamiltonian circuit.

OR

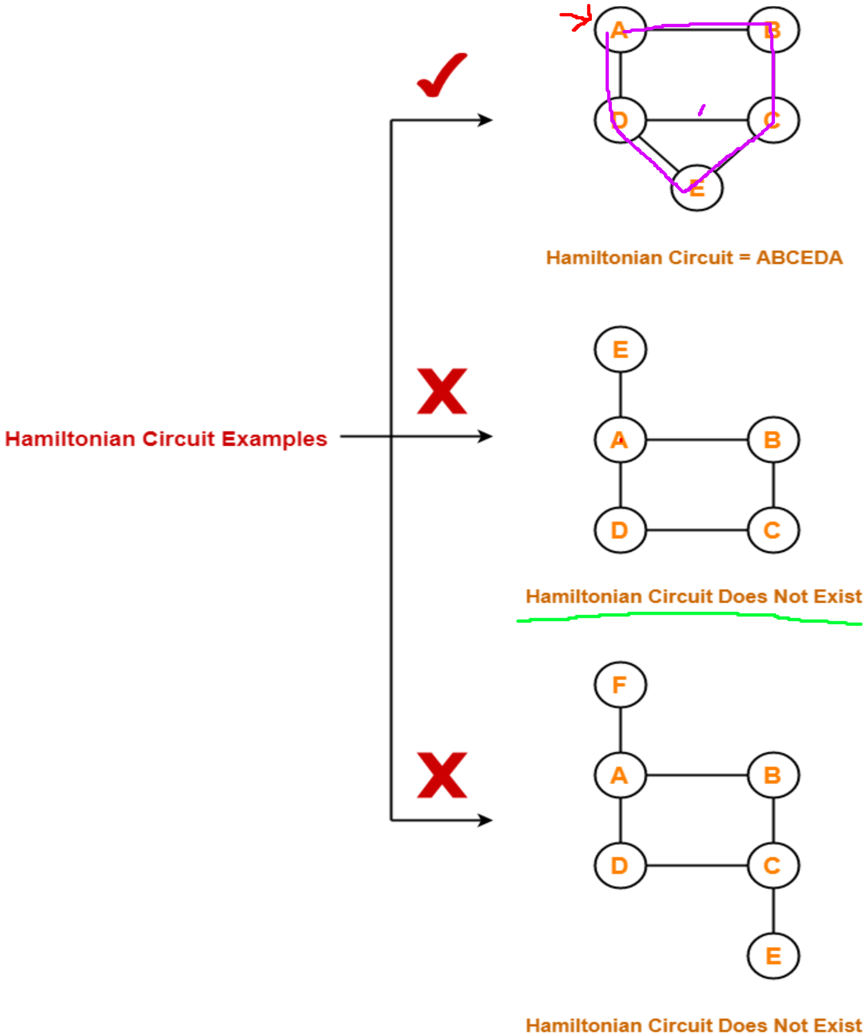
- A Hamiltonian path which starts and ends at the same vertex is called as a Hamiltonian circuit.

OR

- A closed Hamiltonian path is called as a Hamiltonian circuit.

Hamiltonian Circuit Examples-

Examples of Hamiltonian circuit are as follows-



Important Notes-

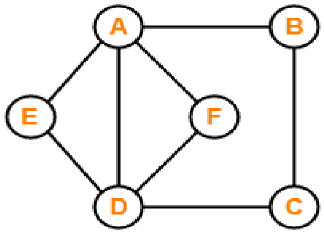
- Any Hamiltonian circuit can be converted to a Hamiltonian path by removing one of its edges.
- Every graph that contains a Hamiltonian circuit also contains a Hamiltonian path but vice versa is not true.
- There may exist more than one Hamiltonian paths and Hamiltonian circuits in a graph.

PRACTICE PROBLEMS BASED ON HAMILTONIAN GRAPHS IN GRAPH THEORY-

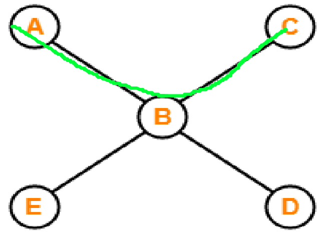
Problems-

Which of the following is / are Hamiltonian graphs?

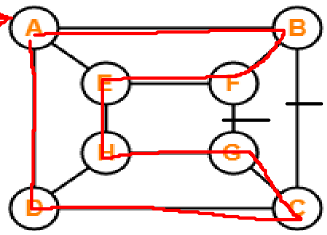
A)



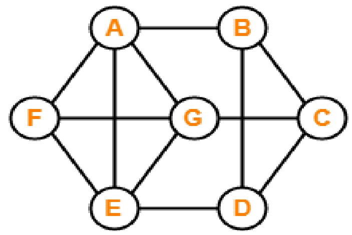
B)



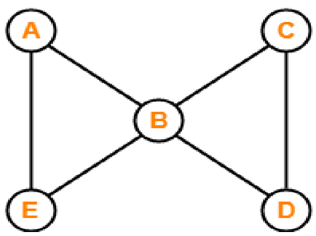
C)



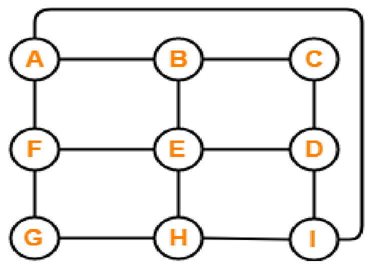
D)



E)



F)



Solutions-

A) The graph neither contains a Hamiltonian path nor it contains a Hamiltonian circuit.
Since graph does not contain a Hamiltonian circuit, therefore **It is not a Hamiltonian Graph**.

B)The graph neither contains a Hamiltonian path nor it contains a Hamiltonian circuit.
Since graph does not contain a Hamiltonian circuit, therefore **It is not a Hamiltonian Graph**.

C)The graph contains both a Hamiltonian path (ABCDHGFE) and a Hamiltonian circuit (ABCDHGFEA).
Since graph contains a Hamiltonian circuit, therefore **It is a Hamiltonian Graph**.

D)The graph contains both a Hamiltonian path (ABCDEFG) and a Hamiltonian circuit (ABCDEFGA).
Since graph contains a Hamiltonian circuit, therefore **It is a Hamiltonian Graph**.

E)The graph neither contains a Hamiltonian path nor it contains a Hamiltonian circuit.
Since graph does not contain a Hamiltonian circuit, therefore **It is not a Hamiltonian Graph**.

F)The graph contains both a Hamiltonian path (ABCDEFGHI) and a Hamiltonian circuit (ABCDEFGHIA)
Since graph contains a Hamiltonian circuit, therefore **It is a Hamiltonian Graph**.