62

ASSIGNMENT PROBLEMS

62.1 INTRODUCTION

Imagine, if in a printing press there is one machine and one operator is there to operate. How would you employ the worker?

Your immediate answer will be, the available operator will operate the machine.

Again suppose there are two machines in the press and two operators are engaged at different rates to operate them. Which operator should operate which machine for maximising profit?

Similarly, if there are *n* machines available and *n* persons are engaged at different rates to operate them. Which operator should be assigned to which machine to ensure maximum efficiency?

While answering the above questions we have to think about the interest of the press, so we have to find such an assignment by which the press gets maximum profit on minimum investment. Such problems are known as "**assignment problems**".

In this lesson we will study such problems.

62.2 OBJECTIVES

After completion of this lesson you will be able to:

- formulate the assignment problem
- know Hungarian method to find proper assignment
- employ Hungarian method to find proper assignment

62.3 FORMULATION OF THE PROBLEM

Let there are n jobs and n persons are available with different skills. If the cost of doing j^{th} work by i^{th} person is c_{ij}. Then the **cost matrix** is given in the table 1 below:

Table 1

Jobs / Persons	1	2	3	…….. j	…….. n
1	c_{11}	c_{12}	c_{13}	…….. c_{1j}	…….. c_{1n}
2	c_{21}	c_{22}	c_{23}	…….. c_{2j}	…….. c_{2n}
. . . . i	. . . . c_{i1}	. . . . c_{i2}	. . . . c_{i3}	………… ………… ………… ………… …….. c_{ij}	…………. …………. …………. …………. …….. c_{in}
. . . . n	. . . . c_{n1}	. . . . c_{n2}	. . . . c_{n3}	…………. …………. …………. …….. c_{nj}	………… ………… …………. …….. c_{nn}

Now the problem is which work is to be assigned to whom so that the cost of completion of work will be minimum.

Mathematically, we can express the problem as follows:

$$\text{To minimize } z \text{ (cost)} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}; \quad [\, i=1,2,\ldots n; \; j=1,2,\ldots n] \quad \ldots(1)$$

$$\text{where } x_{ij} = \begin{cases} 1; & \text{if } i^{th} \text{ person is assigned } j^{th} \text{ work} \\ 0; & \text{if } i\text{th person is not assigned the } j^{th} \text{ work} \end{cases}$$

with the restrictions

(i) $\sum_{i=1}^{n} x_{ij} = 1$; $j=1,2,\ldots n.$, i.e., i^{th} person will do only one work.

(ii) $\sum_{j=1}^{n} x_{ij} = 1$; $i=1,2,\ldots n.$,i.e., j^{th} work will be done only by one person.

62.4 ASSIGNMENT ALGORITHM

(The Hungarian Method)

In order to find the proper assignment it is essential for us to know the Hungarian method. This method is dependent upon two vital theorems, stated as below.

Theorem 1: If a constant is added (or subtracted) to every element of any row (or column) of the cost matrix [c_{ij}] in an assingment problem then an assingment which minimises the total cost for the new matrix will also minimize the total cost matrix.

Theorem 2: If all $c_{ij} \geq 0$ and there exists a solution

$$x_{ij} = X_{ij} \text{ such that } \sum_i \qquad c_{ij} \ x_{ij} = 0.$$

then this solution is an optimal solution, i.e., minimizes *z*.

The computational proecdure is given as under:

Step I (A) Row reduction:

Subtract the minimum entry of each row from all the entires of the respective row in the cost matrix.

$\sum_j$ **(B) Column reduction:**

After completion of row reduction, subtract the minimum entry of each column from all the entires of the respective column.

Step II Zero assignment:

(A) Starting with first row of the matrix received in first step, examine the rows one by one until a row containing exactly one zero is found. Then an experimental assignment indicated by '$\boxed{\checkmark}$' is marked to that zero. Now cross all the zeros in the column in which the assignment is made. This procedure should be adopted for each row assignment.

(B) When the set of rows has been completely examined, an identical procedure is applied successively to columns. Starting with column 1, examine all columns until a column containing exactly one zero is found. Then make an experimental assignment in that position and cross other zeros in the row in which the assignment was made.

Continue these successive operations on rows and columns until all zero's have either been assigned or corssed-out.

Now there are two possibilities:

(a) Either all the zeros are assigned or crossed out, i.e., we get the maximal assignment.

or

(b) At least two zeros are remained by assignment or by crossing out in each row or column. In this situation we try to exclude some of the zeros by trial and error method.

This completes the second step. After this step we can get two situations.

(i) Total assigned zero's = *n*

The assignment is optimal.

(ii) Total assigned zero's < *n*

Use step III and onwards.

Step III: Draw of minimum lines to cover zero's

In order to cover all the zero's at least once you may adopt the following procedure.

(i) Marks (√) to all rows in which the assignment has not been done.

(ii) See the position of zero in marked (√) row and then mark (√) to the corresponding column.

(iii) See the marked (√) column and find the position of assigned zero's and then mark (√) to the corresponding rows which are not marked till now.

(iv) Repeat the procedure (ii) and (iii) till the completion of marking.

(v) Draw the lines through **unmarked rows** and **marked columns.**

Note: If the above method does not work then make an arbitrary assignment and then follow step IV.

Step IV: Select the smallest element from the uncovered elements.

(i) Subtract this smallest element from all those elements which are not covered.

(ii) Add this smallest element to all those elements which are at the intersection of two lines.

Step V: Thus we have increased the number of zero's. Now, modify the matrix with the help of step II and find the required assignment.

This procedure will be more clear by the following examples.

Example A:

Four persons *A*,*B*,*C* and *D* are to be assigned four jobs *I*, *II*, *III* and *IV*. The cost matrix is given as under, find the proper assignment.

Man / Jobs	*A*	*B*	*C*	*D*
I	8	10	17	9
II	3	8	5	6
III	10	12	11	9
IV	6	13	9	7

10
5
9
6
―
30

Solution :

In order to find the proper assignment we apply the Hungarian algorithm as follows:

I (A) **Row reduction**

Man / Jobs	*A*	*B*	*C*	*D*
I	0	0 ~~2~~	7 ~~9~~	1
II	0	3 ~~5~~	0 ~~2~~	3
III	1	1 ~~3~~	0 ~~2~~	0
IV	0	5 ~~7~~	1 ~~3~~	1

I (B) **Column reduction**

Man / Jobs	*A*	*B*	*C*	*D*
I	0	0	7	1
II	0	3	0	3
III	1	1	0	0
IV	0	5	1	1

I = B → 10
II = C → 5
III = D → 9
IV = A → 6
⇒ = $\boxed{30}$

II(A) and (B) **Zero assignment**

Man / Jobs	A	B	C	D
I	$\cancel{0}$	$\boxed{0}$	7	1
II	$\cancel{0}$	3	$\boxed{0}$	3
III	1	1	$\cancel{0}$	$\boxed{0}$
IV	$\boxed{0}$	5	1	1

In this way all the zero's are either crossed out or assigned. Also total assigned zero's = 4 (i.e., number of rows or columns). Thus, the assignment is optimal.

From the table we get $I \to B$; $II \to C$: $III \to D$ and $IV \to A$.

Example B:

There are five machines and five jobs are to be assigned and the associated cost matrix is as follows. Find the proper assignment.

Machines

		I	II	III	IV	V
	A	6	12	3	11	15
	B	4	2	7	1	10
Jobs	C	8	11	10	7	11
	D	16	19	12	23	21
	E	9	5	7	6	10

Solution:

In order to find the proper assignment, we apply the Hungarian method as follows:

IA (Row reduction)

Machines

		I	II	III	IV	V
	A	3	9	0	8	12
	B	3	1	6	0	9
Jobs	C	1	4	3	0	4
	D	4	7	0	11	9
	E	4	0	2	1	5

IB (Column reduction)

Machines

		I	II	III	IV	V
	A	2	9	$\boxed{0}$	8	8
	B	2	1	6	$\boxed{0}$	5
Jobs	C	$\boxed{0}$	4	3	$\cancel{0}$	$\cancel{0}$
	D	3	7	$\cancel{0}$	11	5
	E	3	$\boxed{0}$	2	1	1

II (Zero assignment)

Machines

		I	II	III	IV	V
	A	2	9	$\boxed{0}$	8	8
	B	2	1	6	$\boxed{0}$	5
Jobs	C	$\boxed{0}$	4	3	$\cancel{0}$	$\cancel{0}$
	D	3	7	$\cancel{0}$	$\cancel{11}$	5
	E	3	$\boxed{0}$	2	1	1

From the last table we see that all the zeros are either assigned or crossed out, but the total number of assignment, i.e., 4<5 (number of jobs to be assigned to machines). Therefore, we have to follow step III and onwards as follows:

Step III

Machines

		I	II	III	IV	V	
	A	2	9	$\boxed{0}$	8	8	√
	B	2	1	6	$\boxed{0}$	5	
Jobs	C	$\boxed{0}$	4	3	$\cancel{0}$	$\cancel{0}$	
	D	3	7	$\cancel{0}$	11	5	√
	E	3	$\boxed{0}$	2	1	1	

Step IV: Here, the smallest element among the uncovered elements is 2.

(i) Subtract 2 from all those elements which are not covered.

(ii) Add 2 to those entries which are at the junction of two lines.

Complete the table as under:

Machines

		I	II	III	IV	V
	A	$\boxed{0}$	7	$\cancel{0}$	6	6
	B	2	1	8	0	5
Jobs	C	$\cancel{0}$	4	5	$\cancel{0}$	$\boxed{0}$
	D	1	5	$\boxed{0}$	9	3
	E	3	$\boxed{0}$	4	1	1

Step V. using step II again

A -

Machines

		I	II	III	IV	V
	A	$\boxed{0}$	7	$\cancel{0}$	6	6
	B	2	1	8	$\boxed{0}$	5
Jobs	C	$\cancel{0}$	4	5	$\cancel{0}$	$\boxed{0}$
	D	1	5	$\boxed{0}$	9	3
	E	3	$\boxed{0}$	4	1	1

Thus, we have got five assignments as required by the problem. The assignment is as follows:

Ans. $A \to I,\ B \to IV,\ C \to V,\ D \to III$ and $E \to II$.

This assignment holds for table given in step IV but from theorem 1 it also holds for the original cost matrix. Thus from the cost matrix the minimum cost = 6+1+11+12+5=Rs.35.

Ans

Note:

If we are given a maximization problem then convert it into minimization problem, simply, multiplying by -1 to each entry in the effectiveness matrix and then solve it in the usual manner.

Example C:

Solve the minimal assignment problem whose effectiveness matrix is given by

	A	*B*	*C*	*D*
I	2	3	4	5
II	4	5	6	7
III	7	8	9	8
IV	3	5	8	4

Solution:

In order to find the proper assignment, we apply the Hungarian method as follows:

Step I (A)

	A	*B*	*C*	*D*
I	0	1	2	3
II	0	1	2	3
III	0	1	2	1
IV	0	2	5	1

I(B)

	A	*B*	*C*	*D*
I	0	0	0	2
II	0	0	0	2
III	0	0	0	0
IV	0	1	3	0

Step II:

Since single zero neither exist in columns nor in rows, it is usually easy to make zero assignments.

While examining rows successively, it is observed that row 4 has two zeros in both cells (4,1) and (4,4).

Now, arbitrarily make an experiemental assignment to one of these two cells, say (4,1) and cross other zeros in row 4 and column 1.

The tables given below show the necessary steps for reaching the optimal assignment $I \to B,\ II \to C,\ III \to D,\ IV \to A$.

	A	B	C	D
I	$\cancel{0}$	0	0	2
II	$\cancel{0}$	0	0	2
III	$\cancel{0}$	0	0	0
IV	$\boxed{0}$	1	3	$\cancel{0}$

	A	B	C	D
I	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
II	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
III	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$
IV	$\boxed{0}$	1	3	$\cancel{0}$

	A	B	C	D
I	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
II	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
III	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$
IV	$\boxed{0}$	1	3	$\cancel{0}$

I–B, II–C, III–D, IV–A

Other optimal assignments are also possible in this example.

$$\left.\begin{array}{l} I \to A,\ II \to B,\ III \to C,\ IV \to D \\ I \to C,\ II \to B,\ III \to A,\ IV \to D \\ I \to C,\ II \to B,\ III \to D,\ IV \to A \\ I \to B,\ II \to C,\ III \to A,\ IV \to D \end{array}\right\} \text{(each has cost 20)}$$

INTEXT QUESTIONS 62.1

1. An office has four workers, and four tasks have to be performed. Workers differ in efficiency and tasks differ in their intrinsic difficulty. Time each worker would take to complete each task is given in the effectiveness matrix. How the tasks should be allocated to each worker so as to minimise the total man-hour?

		Wrokers			
		I	*II*	*III*	*IV*
	A	5	23	14	8
	B	10	25	1	23
Tasks	*C*	35	16	15	12
	D	16	23	21	7

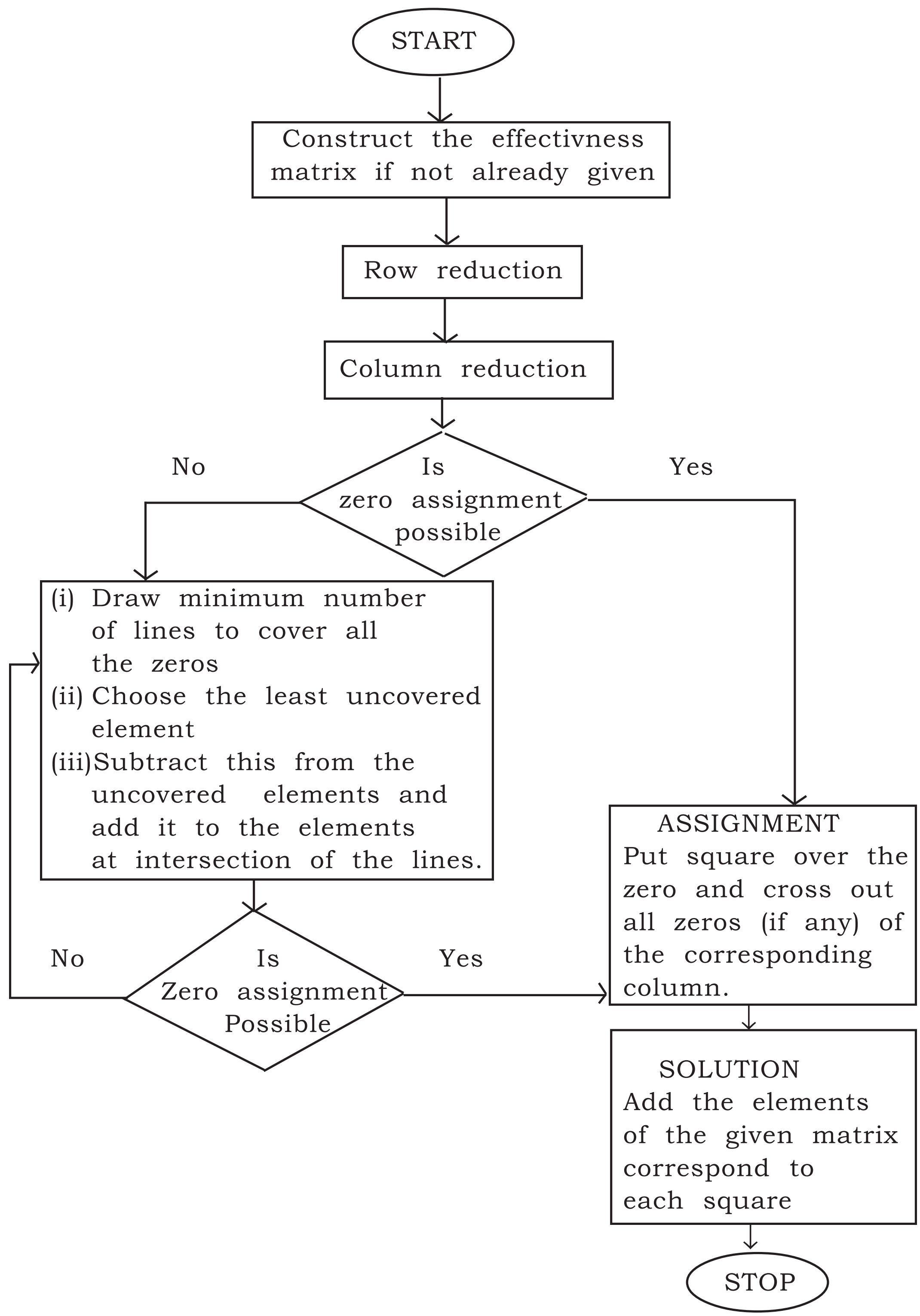
2. A taxi hire company has one taxi at each of five depots a,b,c,d and e. A customer requires a taxi in each town, namely A,B,C,D and E. Distances (in kms) between depots (origins) and towns (Destinations) are given in the following distance matrix:

	a	*b*	*c*	*d*	*e*
A	140	110	155	170	180
B	115	100	110	140	155
C	120	90	135	150	165
D	30	30	60	60	90
E	35	15	50	60	85

How should taxis be assigned to customers so as to minimize the distance travelled?

WHAT YOU HAVE LEARNT

Flow chart for Assignment problem



TERMINAL QUESTIONS

1. Solve the following assignment problems:

(a) **Workers**

Jobs

	I	II	III	IV
A	1	19	10	4
B	6	21	7	19
C	31	12	11	8
D	12	19	17	3

(b) **Persons**

Jobs

	I	II	III	IV
A	15	17	24	16
B	10	15	12	13
C	17	19	18	16
D	13	20	16	14

(c) **Persons**

Jobs

	A	B	C	D
I	8	26	17	11
II	14	29	5	27
III	40	21	20	17
IV	19	26	24	10

2. Find the proper assignment of the assignment problem whose cost matrix is given as under.

	I	*II*	*III*	*IV*	*V*
A	10	6	4	8	3
B	2	11	7	7	6
C	5	10	11	4	8
D	6	5	3	2	5
E	11	7	10	11	7

3. Solve the following assignment problem.

	I	*II*	*III*	*IV*
1	8	9	10	11
2	10	11	12	13
3	13	14	15	13
4	9	11	14	10

ANSWERS TO INTEXT QUESTIONS

62.1

1. $I \rightarrow A,\ II \rightarrow C,\ III \rightarrow B,\ IV \rightarrow D$

2. $A \rightarrow e,\ B \rightarrow c,\ C \rightarrow b,\ D \rightarrow a,\ E \rightarrow d$

   min distance (km) = 180+110+90+30+60=470 km.

ANSWERS TO TERMINAL QUESTIONS

1. (a) $A \rightarrow I,\ B \rightarrow III,\ C \rightarrow II,\ D \rightarrow IV$

   (b) A → II, B → III, C → IV, D → I

   min cost = 17+12+16+13=58

   (c) I → A, II → C, III → B, IV → D

2. $A \rightarrow V,\ B \rightarrow I,\ C \rightarrow IV,\ D \rightarrow III,\ E \rightarrow II$

3. $1 \rightarrow II,\ 2 \rightarrow III,\ 3 \rightarrow IV,\ 4 \rightarrow I$

   or

   $1 \rightarrow III,\ 2 \rightarrow II,\ 3 \rightarrow IV,\ 4 \rightarrow I$