- **Pumping Lemma for CFL** - a proof, a Language is not Context Free

If L is a Context Free Language, the, L has Pumping Length 'P' such that any string 'S', where $|S| \geq P$ may be divided into 5 pieces S = u v x y z such that the following conditions must be true:

1. $u\, v^i\, x\, y^i\, z$ is in L for every $i \geq 0$
2. $|v\, y| > 0$
3. $|v\, x\, y| \leq P$

Example:

$L = \{a^n b^n c^n \mid n \geq 0\}$

- Assume L is context Free
- Pumping length P = 4

$S = a^4b^4c^4$ = aaaabbbbcccc						
u	**v**	**x**	**y**	**z**	**i**	
∈	∈	∈	aaaa	bbbbcccc	2	aaaa aaaa bbbb cccc
∈	a	∈	aaa	bbbbcccc	2	a a aaa aaa bbbb cccc
∈	aa	∈	aa	bbbbcccc	2	aa aa aa aa bbbb cccc
∈	aaa	∈	a	bbbbcccc	2	aaa aaa a a bbbb cccc
∈	aaaa	∈	∈	bbbbcccc	2	aaaa aaaa bbbb cccc
a	∈	∈	aaaa	bbbbcccc	2	a aaaa aaaa bbbb cccc
a	a	∈	aa	bbbbcccc	2	a a a aa aa bbbb cccc
a	aa	∈	a	bbbbcccc	2	a aa aa a a bbbb cccc
a	aaa	∈	∈	bbbbcccc	2	a aaa aaa bbbb cccc
a	a	a	a	bbbbcccc	2	a a a a a a bbbb cccc
: :						
aaa	ab	bb	bc	ccc	2	aaa ab ab bb bc bc ccc

Example:

$L = \{a^n b^n | n \geq 1\}$

- Assume L is context Free
- Pumping length P = 4
- $u\, v^i\, x\, y^i\, z$ is in L for every $i \geq 0$

$S = a^4b^4$ $= aaaabbbb$						
u	**v**	**x**	**y**	**z**	**i**	
$\in$	$\in$	$\in$	aaaa	bbbb	2	aaaa aaaa bbbb
$\in$	a	$\in$	aaa	bbbb	2	a a aaa aaa bbbb
$\in$	aa	$\in$	aa	bbbb	2	aa aa aa aa bbbb
$\in$	aaa	$\in$	a	bbbb	2	aaa aaa a a bbbb
$\in$	aaaa	$\in$	$\in$	bbbb	2	aaaa aaaa bbbb
a	$\in$	$\in$	aaaa	bbbb	2	a aaaa aaaa bbbb
a	a	$\in$	aa	bbbb	2	a a a aa aa bbbb
a	aa	$\in$	a	bbbb	2	a aa aa a a bbbb
a	aaa	$\in$	$\in$	bbbb	2	a aaa aaa bbbb
a	a	a	a	bbbb	2	a a a a a a bbbb
: :						
aa	aa	$\in$	bb	bb	2	aa aa aa bb bb bb
					3	aa aa aa aa bb bb bb bb

Example:

- $L = \{0^i 1 0^i 1 0^i \mid i \geq 1\}$
- $L = \{ww \mid w \in \{0,1\}^*\}$ that is $w = 0^i 1^i$ the $ww = 0^i 1^i 0^i 1^i$

- **Properties of Context Free Language (CFL)**

A. Decision Properties

- **Emptiness: Decidable**

  - Remove all “Useless” symbols
  - If a CFG (G) holds the Starting symbol as “Useless” then the L(G) = ∅ otherwise L(G) ≠ ∅

- **Finiteness: Decidable**

  - If in the normal from of CFG(G) holds loop (cycle) then the L(G) is infinite otherwise finite

Example:

$S \rightarrow AB$
$A \rightarrow XB$
$B \rightarrow XA$
$X \rightarrow \alpha$

- **Membership: Decidable**

  - If L(G), of the normal form of CFG(G), contains the “w” then $w \in L(G)$ **(CYK Algorithm)**

- **Equivalence: Undecidable**

  - There is no such algorithm that prove the equivalency of 2 CFL’s

B. Closure Properties

- **Union:**
  - L_1 = CFL and L_2 = CFL
  - Starting symbol is S_1 of G_1 for $L_1(G_1)$
  - Starting symbol is S_2 of G_2 for $L_2(G_2)$
  - $L_1 \cup L_2$
  - $S \rightarrow S_1 \mid S_2$ is also in CFG

  $\therefore$ So, CFL is closed under Union operation

- **Concatenation:**
  - L_1 = CFL and L_2 = CFL
  - Starting symbol is S_1 of G_1 for $L_1(G_1)$
  - Starting symbol is S_2 of G_2 for $L_2(G_2)$
  - $L_1 . L_2$
  - $S \rightarrow S_1 S_2$ is also in CFG

  $\therefore$ So, CFL is closed under Concatenation operation

- **Transpose/ Reversal:**
  - $L = \{0^n1^n \mid n \geq 1\}$ is CFL
  - $S \rightarrow 0S1 \mid 01$
  - Reversal $S \rightarrow 1S0 \mid 10$
  - $L^T = \{1^n0^n \mid n \geq 1\}$ is also CFL

  $\therefore$ So, CFL is closed under Transpose operation

- **Kleene star:**
  - L = CFL
  - Starting symbol is S of G for L(G)
  - $S' \rightarrow SS' \mid \in$ [consider a new Starting symbol S']
  - $\{\in, SS', SSS', SSSS', \ldots\}$
  - $\{\in, S, SS, SSS, \ldots\}$
  - $S^* \Rightarrow L^*$ is also CFL

  $\therefore$ So, CFL is closed under Kleene star operation

- **Intersection:**
  - $L_1 = \{0^m1^n | m, n \geq 1\}$ and $L_2 = \{0^n1^n | n \geq 1\}$
  - $L_1 \cap L_2 = \{0^n1^n | n \geq 1\}$ is also CFL

  - $L_3 = \{0^m1^n2^n | m, n \geq 1\}$ and $L_4 = \{0^m1^m2^n | m, n \geq 1\}$
  - $L_3 \cap L_4 = \{0^n1^n2^n | n \geq 1\}$ is not CFL

  $\therefore$ CFL is not closed under Intersection operation

- **Complement:**
  - Assume Complement of CFL is also CFL
  - L_1 = CFL and L_2 = CFL
  - $\overline{L_1}$ = CFL and $\overline{L_2}$ = CFL
  - $\overline{L_1} \cup \overline{L_2}$ = CFL
  - $\overline{\overline{L_1} \cup \overline{L_2}}$ = CFL
  - $L_1 \cap L_2 \neq$ CFL [applying De Morgan's law]

  $\therefore$ CFL is not closed under Complement operation