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Software simulation of a lock-in amplifier with application to the evaluation of uncertainties in real measuring systems

P Clarkson[1], T J Esward[1], P M Harris[1], A A Smith[2] and I M Smith[1]

[1] National Physical Laboratory, Teddington, Middlesex, TW11 0LW, UK
[2] The University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK

E-mail: trevor.esward@npl.co.uk

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Abstract
Lock-in amplifiers are widely used in metrology to extract a sinusoidal component of a known frequency from a signal that is noisy, returning estimates of the amplitude and phase of the component. A number of questions arise regarding the use by metrologists of the results returned by a lock-in amplifier. For example, what uncertainties should be associated with the estimates returned? How do these uncertainties vary as the signal-to-noise ratio changes? How do instrument errors affect the estimates and the associated uncertainties? In this paper a software simulation tool is described that may be used to process simulated or real (measured) signals, with the user allowed to associate uncertainties with the values of parameters that define the simulated signal and with those that define the operation of the instrument. The results of applying the software simulation tool to both simulated signals and real signals from infrared radiation and nanoindentation experiments are described.

Keywords: simulation, uncertainty evaluation, Monte Carlo method, lock-in amplifier

1. Introduction

A lock-in amplifier, also known as a phase-sensitive detector, is an instrument widely used in metrology to extract features of a sinusoidal component of a known frequency from a signal that is noisy. Given a measured signal and a frequency of interest, the instrument returns estimates of the amplitude and phase of the sinusoidal component of that frequency.

A number of questions arise regarding the use by metrologists of the results returned by a lock-in amplifier. For example, what uncertainties should be associated with the estimates of amplitude and phase? How do these uncertainties vary as the signal-to-noise ratio changes? How do instrument errors affect the estimates and the associated uncertainties? While software implementations of a lock-in amplifier are available [1, 2], they do not address the problem of uncertainty evaluation for such an instrument. The primary aim of this paper is to overcome this limitation by describing software that allows users of lock-in amplifiers to investigate systematically the effect of uncertainties associated with key features of these instruments.

A software simulation tool [3] to investigate the behaviour of a lock-in amplifier and to help answer the above questions has been developed at NPL. The simulation tool allows the user to enter either (a) values for the parameters that define a simulated signal, or (b) the name of a data file containing a sequence of measured signal values, along with values for the parameters that characterize the operation of the lock-in amplifier. The user may also associate uncertainties with many of the parameter values, and a Monte Carlo method [4, 5] is applied to show how uncertainties propagate through a mathematical model of the instrument.

The simulation tool, which was developed in LabVIEW 8.5 [6], is provided as a stand-alone executable that can run under the Windows XP and Vista operating systems. The simulation tool may be downloaded from the NPL website [3].

The paper is organized as follows. Section 2 provides a motivation for the development of the software simulation tool. Section 3 discusses the operation of a lock-in amplifier and section 4 the application of a Monte Carlo method for uncertainty evaluation. Section 5 describes the software

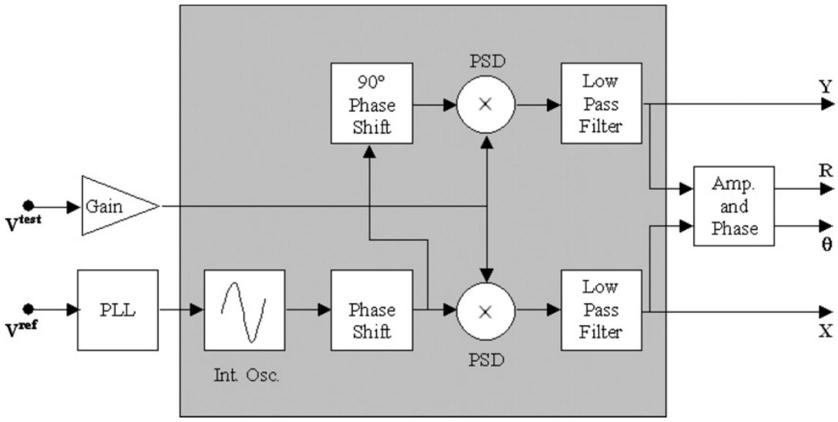

Figure 1. Schematic view of the operation of a lock-in amplifier.

simulation tool. Section 6 shows how the simulation tool can be used to study the behaviour of ideal and imperfect instruments, and how it can be applied to two metrology problems, one from applied infrared radiometry and a second from dynamic nanoindentation measurement. Concluding remarks and suggestions for future work are given in section 7.

2. Motivation for the approach

The main aims of the software simulation tool are as follows.

- To allow processing of both simulated signals, i.e. signals having well-defined properties, and real, i.e. measured, signals. The ability to process simulated signals provides a means of gaining insight into the behaviour of lock-in amplifiers, while the properties of a real signal can be investigated in a faster and more repeatable way than is permitted in the laboratory using a real instrument.
- To allow investigation into how uncertainties propagate through a mathematical model of the lock-in amplifier.
- To allow examination of how well sinusoidal components can be recovered from noisy signals. The signal-to-noise ratio can be varied and, for example, a comparison of the estimate of the amplitude and the 'true' amplitude, or the dispersion of estimates of phase, can be obtained.
- To provide an 'instrument-like' interface that allows parameter values to be easily examined and modified, and results to be displayed in a clear manner.
- To implement a generalizable approach that can be applied to other measuring systems in a straightforward way. The approach is also easily extendible, for example, to allow for a range of simulated signals and real signals stored in different data formats.

It is hoped that the simulation tool will prove to be useful to metrologists in gaining a fuller understanding of results obtained using a lock-in amplifier in the laboratory.

3. Operation of a lock-in amplifier

The operation of the lock-in amplifier is illustrated schematically in figure 1. It can be considered to consist of four stages.

- Gain—amplification of the test signal V^{test}.[3]
- Reference circuit—generation of two orthogonal phase-shifted signals from a reference signal V^{ref}, the frequency of which is that of the sinusoidal component to be recovered.
- Demodulation—multiplication of the amplified test signal by the (phase-shifted) reference signals.
- Low-pass filtering—removal of high-frequency components from the two demodulated signals.

An internal oscillator in the lock-in amplifier uses what is known as a phase-locked loop (PLL) to generate a digitally synthesized sine wave with the same phase and rising zero-crossing as an external reference signal. This signal may take the form of a sine or square wave and is generally used as the excitation signal for the experiment providing the test signal. The PLL introduces phase noise to the internal oscillator sine wave, meaning that the instantaneous phase shift between the sine wave and the external reference signal is not zero, but the average of the shifts can be assumed to be zero. In the absence of an external reference signal, the internal oscillator may itself be used to provide the excitation signal for the experiment (and the PLL is then not used).

A second sine wave, phase-shifted from the internal oscillator sine wave, provides the reference input to the first (X) phase-sensitive detector (PSD). A third sine wave, phase-shifted by 90° from the second sine wave, provides the reference input to the second (Y) PSD. The PSDs multiply the amplified test signal by their reference inputs.

The resulting (phase-shifted) reference signals are passed through a low-pass filter to remove high-frequency components, and the filtered signals are combined in such a way as to obtain values for the amplitude and phase of the sinusoidal component of frequency equal to that of the reference signal.

The mathematics underlying the operation of the lock-in amplifier is described in the appendix.

[3] In the context of the software simulation tool described, the term 'test signal' is used to mean either a simulated signal or a real (measured) signal.

4. A Monte Carlo method for uncertainty evaluation

The basis for the evaluation of measurement uncertainty in the simulation tool is the propagation of (probability) distributions. In order to apply the propagation of distributions, a measurement function of the generic form $Y = f(\boldsymbol{X})$ relating input quantities $\boldsymbol{X} = (X_1, \ldots, X_N)^\top$, about which information is available, and an output quantity Y, about which information is required, is formulated. Additionally, information concerning $\boldsymbol{X}$ is encoded as probability distributions, such as rectangular (uniform) and Gaussian (normal). The information can take a variety of forms, including a series of measured signal values, 'data' (estimates and associated uncertainties) from the specification of an instrument, and the expert knowledge of the user. An implementation of the propagation of distributions provides a probability distribution for Y, from which can be obtained an estimate y of Y and the standard uncertainty $u(y)$ associated with the estimate. Particular implementations of the propagation of distributions are the GUM uncertainty framework [7] and a Monte Carlo method [4, 5].

4.1. The GUM uncertainty framework

The '*Guide to the Expression of Uncertainty in Measurement*' (GUM) [7] presents a framework for uncertainty evaluation based on the law of propagation of uncertainty and the central limit theorem. The estimate y of the output quantity Y is given in terms of the estimates $\boldsymbol{x} = (x_1, \ldots, x_N)^\top$ of the input quantities by $y = f(\boldsymbol{x})$. The uncertainty associated with this estimate is given, using the law of propagation of uncertainty, by propagating the uncertainties and covariances associated with the estimates of the input quantities through the measurement function linearized about $\boldsymbol{X} = \boldsymbol{x}$. A Gaussian distribution (or t-distribution with appropriate degrees of freedom) is then assigned to Y and can be used, for example, to provide a coverage interval for Y corresponding to a prescribed coverage probability.

While the GUM uncertainty framework can be expected to work well in many circumstances, there are issues associated with its application.

- It is difficult to quantify the effects of the approximations introduced, including linearization of the measurement function, the evaluation of effective degrees of freedom using the Welch–Satterthwaite formula (taken as the degrees of freedom of a t-distribution) and the assumption that the output quantity can be described adequately by a Gaussian or t-distribution.
- It requires the evaluation of sensitivity coefficients (the values of the partial derivatives of the measurement function with respect to the input quantities evaluated at the estimates of the input quantities) as the basis of the linearization of the measurement function. Such evaluation is not always straightforward, for example, when the measurement function is algebraically complicated or symbolizes an algorithm.

4.2. A Monte Carlo method

A Monte Carlo method for uncertainty evaluation can be implemented in the following manner. For each input quantity, a random draw is made from the probability distribution assigned to that quantity, and the measurement function is evaluated for these values[4]. This process is repeated a large number of times to obtain M, say, values y_r, $r = 1, \ldots, M$, of the output quantity Y. The values y_r are then used to provide an approximation to the probability distribution for Y.

The method has a number of advantages over the GUM uncertainty framework.

- It is applicable regardless of the nature of the measurement function, i.e. whether it is linear or nonlinear.
- It does not require evaluation of effective degrees of freedom.
- It makes no assumption about the distribution for the output quantity.
- It does not require calculation of sensitivity coefficients, instead requiring only evaluations of the measurement function.

There are also some practical issues associated with the application of a Monte Carlo method. The number M of Monte Carlo trials required to achieve a sufficiently accurate approximation to the probability distribution for the output quantity may be large, for example, of the order of 10^5 or 10^6. For cases where evaluation of the measurement function takes a significant amount of time, implementing a large number of trials may be impractical. The method also requires the ability to make random draws from the probability distributions for the input quantities. Robust algorithms for psuedo-random number generation are therefore required [8].

For the lock-in amplifier application, a Monte Carlo method is used in preference to the GUM uncertainty framework. The complexity of the measurement function (section 4.3.4) means that the calculation of sensitivity coefficients is (algebraically) complicated, and it is difficult to test that the assumptions of the GUM uncertainty framework (section 4.1) are valid.

Since there are two output quantities, namely the amplitude and phase (difference) of the required sinusoidal component, the measurement function for the lock-in amplifier is bivariate [9] and has the form $\boldsymbol{Y} = \boldsymbol{f}(\boldsymbol{X})$, where $\boldsymbol{Y} = (Y_1, Y_2)^\top$. Information about the output quantities $\boldsymbol{Y}$ takes the form of a bivariate (joint) probability distribution. In general the output quantities of a bivariate model are correlated as they depend on the same input quantities. To mimic the operation of a real instrument, in both this paper and the software simulation tool, information about the correlation associated with the estimates of the output quantities is not provided. However, applications that make use of the estimates of both amplitude and phase (difference) may need this information.

[4] For the case where input quantities are correlated, random draws are made from the appropriate joint probability distribution.

4.3. Application of a Monte Carlo method to the lock-in amplifier

The number of input quantities in the measurement function depends on both the type of signal being processed and the type of low-pass filter applied. In this section we describe the application of a Monte Carlo method for the case of a simulated sine wave signal and where a simple passive RC (resistor–capacitor) filter is used. The necessary modifications for a simulated test signal that takes a different form, such as a square wave, are straightforward. For a real signal and/or where a fast Fourier transform-based (FFT) filter is used, there are fewer input quantities. The number of output quantities does not depend on the type of signal being processed and is always two.

Throughout this section, the units of all parameters and (input and output) quantities are given in parentheses. The choice of volts (V) as the unit of parameters and quantities representing amplitude is arbitrary.

4.3.1. Sampling and analogue-to-digital conversion parameters. (Integer) values are assigned to the following sampling and analogue-to-digital conversion parameters.

(i) The number N_b of bits for quantization.
(ii) The sampling frequency f_s (Hz).
(iii) The duration T of the signal (s).

4.3.2. Input quantities. Probability distributions, defined in terms of estimates and associated standard uncertainties, are assigned to the following input quantities.

(i) Test signal amplitude A_{test} (V).
(ii) Test signal phase θ_{test} (°).
(iii) Test signal frequency f_{test} (Hz).
(iv) Noise associated with the amplitude values (amplitude noise) $\delta V_i, i = 1, \ldots, m := f_s T$ (V).
(v) Noise associated with the sampled time values (jitter noise) $\delta t_i, i = 1, \ldots, m$ (s).
(vi) Reference signal amplitude A_{ref} (V).
(vii) Reference signal phase θ_{ref} (°).
(viii) Reference signal frequency f_{ref} (Hz).
(ix) Reference signal orthogonality angle θ_{orth} (°).
(x) Phase noise $\delta\theta_i, i = 1, \ldots, m$ (°).
(xi) Time constant τ for the RC filter (s).

The input quantities can be categorized into those ((i)–(v)) that describe the properties of the simulated test signal, and those ((vi)–(xi)) that describe the operation of the instrument. The notation

$$\boldsymbol{X} = (A_{\text{test}}, \theta_{\text{test}}, f_{\text{test}}, \boldsymbol{\Delta V}^\top, \boldsymbol{\Delta t}^\top, A_{\text{ref}}, \theta_{\text{ref}}, f_{\text{ref}}, \theta_{\text{orth}}, \boldsymbol{\Delta\theta}^\top, \tau)^\top,$$

where $\boldsymbol{\Delta V} = (\delta V_1, \ldots, \delta V_m)^\top$, $\boldsymbol{\Delta t} = (\delta t_1, \ldots, \delta t_m)^\top$ and $\boldsymbol{\Delta\theta} = (\delta\theta_1, \ldots, \delta\theta_m)^\top$, is used as a compact means of representing the complete set of $8 + 3m$ input quantities. In the examples in section 6, the estimates of the amplitude noise are all zero, i.e. the estimate of the vector $\boldsymbol{\Delta V}$ is the vector $\boldsymbol{0} = (0, \ldots, 0)^\top$ of dimension $m \times 1$, and, within a particular example, the estimates have the same associated standard uncertainty. A similar statement applies to the input quantities describing jitter noise and phase noise. When listing the values $\boldsymbol{x}$ of the input quantities, units are omitted to ease reading but are as given in this section.

4.3.3. Output quantities. There are two output quantities.

(i) Estimate R of the amplitude of the sinusoidal component of frequency f_{ref} (V).
(ii) Estimate θ of the difference between the phase of the sinusoidal component of frequency f_{ref} and that of the reference signal (°).

4.3.4. Measurement function. The measurement function here symbolizes an algorithm. It can be considered to consist of a number of sub-functions, and is based on a quantization of the underlying mathematics of the lock-in amplifier described in the appendix. Note that the gain stage described in section 3 is not included in the measurement function.

Output quantities R and θ are obtained as follows.

(i) Calculate $m = f_s T$, $t_i = i/f_s$, $i = 1, \ldots, m$.

(ii) Generate the test signal:

$$V_i^{\text{test}} = Q\left[A_{\text{test}} \sin\left(2\pi f_{\text{test}} \tilde{t}_i + \theta_{\text{test}}^{\text{r}}\right) + \delta V_i\right], \quad i = 1, \ldots, m,$$

where $\tilde{t}_i = t_i + \delta t_i$ and Q denotes the operation of quantization (depending on the number N_b of bits) applied to a sampled analogue signal. The superscript 'r' notation is used here (and elsewhere) to denote an angle expressed in radians rather than degrees (°), e.g. $\theta_{\text{test}}^{\text{r}} = (\pi/180)\,\theta_{\text{test}}$, etc.

(iii) Generate the orthogonal phase-shifted signals:

$$V_i^{\text{X,ref}} = Q\left[A_{\text{ref}} \sin\left(2\pi f_{\text{ref}} t_i + \theta_{\text{ref}}^{\text{r}} + \delta\theta_i^{\text{r}}\right)\right], \quad i = 1, \ldots, m,$$

and

$$V_i^{\text{Y,ref}} = Q\left[A_{\text{ref}} \sin\left(2\pi f_{\text{ref}} t_i + \theta_{\text{ref}}^{\text{r}} + \delta\theta_i^{\text{r}} + \theta_{\text{orth}}^{\text{r}}\right)\right], \quad i = 1, \ldots, m.$$

The same number N_b of bits is used for the quantization of the (phase-shifted) reference signals as for the test signal.

(iv) Apply phase-sensitive detectors to the test signal and the (phase-shifted) reference signals to obtain product signals:

$$V_i^{\text{X}} = V_i^{\text{test}} V_i^{\text{X,ref}}, \qquad i = 1, \ldots, m,$$

and

$$V_i^{\text{Y}} = V_i^{\text{test}} V_i^{\text{Y,ref}}, \qquad i = 1, \ldots, m.$$

(v) Apply a low-pass filter based on an RC filter with time constant τ to the product signals:

$$X_1 = \alpha V_1^{\text{X}},$$
$$X_i = \alpha V_i^{\text{X}} + (1 - \alpha) X_{i-1}, \qquad i = 2, \ldots, m,$$
$$Y_1 = \alpha V_1^{\text{Y}},$$
$$Y_i = \alpha V_i^{\text{Y}} + (1 - \alpha) Y_{i-1}, \qquad i = 2, \ldots, m,$$

where $\alpha = \Delta t/(\Delta t + \tau)$, $\Delta t = 1/f_s$.

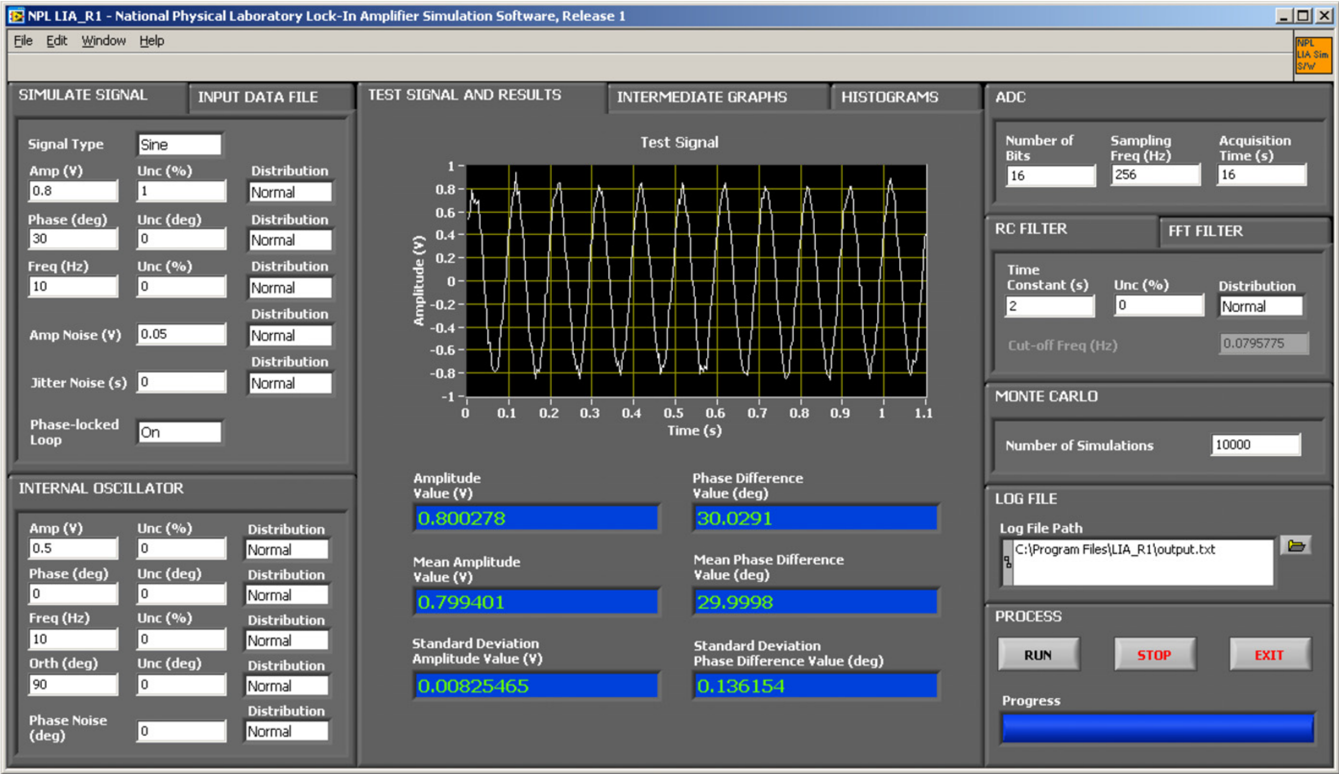

Figure 2. Screenshot of the front panel of the lock-in amplifier software simulation tool. (This figure is in colour only in the electronic version)

(vi) Evaluate amplitude and phase difference values:

$$R = \text{average}\left\{\frac{2}{A_{\text{ref}}}\sqrt{X_i^2 + Y_i^2}, i = n+1, \ldots, m\right\},$$

and

$$\theta = \text{average}\left\{\tan^{-1}\left(\frac{Y_i}{X_i}\right), i = n+1, \ldots, m\right\},$$

where n is chosen so that $t_i \leqslant T - \tau$ for all $i \leqslant n$. Since the outputs of the low-pass filter require some time to stabilize, the estimates R and θ are calculated using a subset of the values corresponding to times after which saturation is expected to have occurred. In practice, the choice of value of T should depend on that of τ, e.g. in the LabVIEW-based software simulation tool, T must be greater than 5τ (see section 6.1).

5. Software simulation

The software simulation tool has been developed in LabVIEW, based on the Stanford Research Systems Model SR830 [10] lock-in amplifier, and compiled into a stand-alone Windows executable.

The software simulation tool allows both simulated (sine or square) signals and signals read from data files to be processed. The operation of the lock-in amplifier is determined by assigning probability distributions to the parameters defining the (phase-shifted) reference signals and low-pass filter (when an RC filter is used). A Monte Carlo calculation is used to investigate the influence of the different input quantities (section 4.3.2) on the evaluation of the output quantities (section 4.3.3), and the performance and limitations of ideal and imperfect instruments. Results, comprising estimates of the output quantities and the associated standard uncertainties and histograms illustrating the probability distributions for the output quantities, are provided.

Figure 2 shows a screenshot of the front panel of the software simulation tool.

Documentation for the software is provided in the form of HTML files generated from MATLAB [11] implementations. These implementations, which may only be run by a user who has access to MATLAB, form the basis of specifying and testing the LabVIEW-based software and may be accessed using the MATLAB *grabcode* command. For the user who does not have access to MATLAB, the LabVIEW-based version of the software is provided. The MATLAB and LabVIEW-based versions of the software differ only in the following ways.

- There is a single LabVIEW-based executable while there are two main MATLAB routines—one for processing simulated signals, the other for processing signals read in from file.
- The LabVIEW-based version provides an intuitive instrument-like front panel for ease of input whereas the MATLAB implementations require the user to edit the underlying MATLAB code.
- Restrictions on the permitted values for parameters that are applied in the LabVIEW-based software are generally not applied in the MATLAB code.

- The user has access to more figures of accuracy in the results in the MATLAB implementations.

In order to present results to a sufficient level of numerical accuracy in this paper[5], the MATLAB version of the software has been used in the examples in section 6.

5.1. User inputs

The user assigns the number M of Monte Carlo trials and assigns values to parameters (i)–(iii) of section 4.3. A probability distribution is assigned to each of the input quantities (i)–(xi) of section 4.3.2 given an estimate, associated standard uncertainty and probability distribution type (normal or rectangular).

5.2. Monte Carlo method

Step (i) of section 4.3.4 is first carried out. Then, for each Monte Carlo trial $r = 1, \ldots, M$, random draws are made from the distributions used to characterize the input quantities and steps (ii) to (vii) are performed. Values R_r and $\theta_r, r = 1, \ldots, M$, of, respectively, R and θ are thus obtained.

5.3. Outputs

The software outputs the following.

(i) For each Monte Carlo trial, a graph showing an initial portion of the test signal (for a signal read in from file, the graph is the same for each Monte Carlo trial).
(ii) For each Monte Carlo trial, a graph showing the signals provided by the low-pass filter applied to the (in-phase and quadrature) outputs of the phase-sensitive detectors.
(iii) A graph showing frequency distributions (in the form of scaled histograms) of, respectively, the values R_r and θ_r obtained from all M Monte Carlo trials.
(iv) Numerical values for the average and standard deviation of the values $R_r, r = 1, \ldots, M$, taken, respectively, as an estimate $\widehat{R}$ of the amplitude of the sinusoidal component of interest and the associated standard uncertainty $u(\widehat{R})$.
(v) Numerical values for the average and standard deviation of the values $\theta_r, r = 1, \ldots, M$, taken, respectively, as an estimate $\widehat{\theta}$ of the phase difference for the sinusoidal component of interest and the associated standard uncertainty $u(\widehat{\theta})$.

6. Example applications

In this section, the application of the software simulation tool to simulated and real test signals to answer some of the questions posed in section 1 is described. In the following, a test signal may be 'ideal', i.e. none of the parameters defining the signal is subject to uncertainty, or 'imperfect', i.e. the uncertainty associated with the estimate of at least one of the parameters is non-zero. Similarly, the operation of the instrument may be 'ideal', i.e. the reference signal is ideal and the uncertainty associated with the time constant for the RC filter is zero or, otherwise, 'imperfect'. Throughout this section, where a non-zero uncertainty is associated with the estimate of an input quantity, a normal probability distribution is assigned to that quantity.

[5] Uncertainties are always quoted to two significant figures, and the estimates with which the uncertainties are associated are quoted to a corresponding number of decimal digits.

Table 1. Results returned for the examples of sections 6.1 and 6.2.

$u(A_{\text{test}})$ (V)	$\widehat{R}$ (V)	$u(\widehat{R})$ (V)	$\widehat{\theta}$ (°)	$u(\widehat{\theta})$ (°)
0	0.799 5614		30.000 09	
0.000 0080	0.799 5621	0.000 0080	30.000 08	0.000 13
0.000 080	0.799 562	0.000 080	30.000 05	0.000 11
0.000 80	0.799 56	0.000 80	30.000 03	0.000 11
0.008 0	0.799 6	0.008 0	30.000 03	0.000 11

6.1. Ideal test signal, ideal instrument

For this example, the test signal type is a sine wave and the estimates of the input quantities are

$$\boldsymbol{x} = (0.8, 30, 10, \mathbf{0}^\top, \mathbf{0}^\top, 0.5, 0, 10, 90, \mathbf{0}^\top, 2)^\top;$$

all associated uncertainties are set to zero and the test signal is processed only once, i.e. $M = 1$. The analogue-to-digital conversion parameters are $N_\mathrm{b} = 16$, $f_\mathrm{s} = 256$ Hz and $T = 16$ s. The results returned are $\widehat{R} \equiv R_1 = 0.799\,5614$ V and $\widehat{\theta} \equiv \theta_1 = 30.000\,09°$ (row 1 of table 1).

The estimate $\widehat{R}$ is close to, but not equal to, the amplitude 0.8 V of the test signal and, similarly, the estimate $\widehat{\theta}$ is close to the phase difference 30° between the test and first (phase-shifted) reference signals. The difference can be explained by (a) the effect of quantization (in this case, 16-bit) of the test and (phase-shifted) reference signals, and (b) the effect of the RC filter. The filter allows a small amount of leakage of the sum component at 20 Hz in the products of the test and (phase-shifted) reference signals (see equations (A.5) and (A.6) in the appendix). Additionally, the outputs of the filter may not have completely saturated. Figure 3 shows the in-phase (X) and quadrature (Y) outputs of the RC filter and illustrates the time taken for saturation of both outputs. The estimates $\widehat{R}$ and $\widehat{\theta}$ are calculated using those values of the outputs corresponding to times greater than $T - \tau = 14$ s. The 20 Hz oscillations in both outputs can also be clearly identified in a magnified version of figure 3.

This use of the software simulation tool (with an ideal instrument applied to an ideal test signal) permits the dependence of the accuracy of the results on the parameters N_b, f_s and T of the instrument to be investigated. Furthermore, it informs how values of these parameters may be chosen to give results that are fit for a particular purpose.

6.2. Imperfect test signal, ideal instrument I

For this example, the test signal type is a sine wave, estimates of the input quantities are assigned the same values as in section 6.1 and all uncertainties are set to zero excepting $u(A_{\text{test}})$ associated with the estimate of the test signal amplitude. The analogue-to-digital conversion parameters are $N_\mathrm{b} = 16$, $f_\mathrm{s} = 256$ Hz and $T = 16$ s, as before.

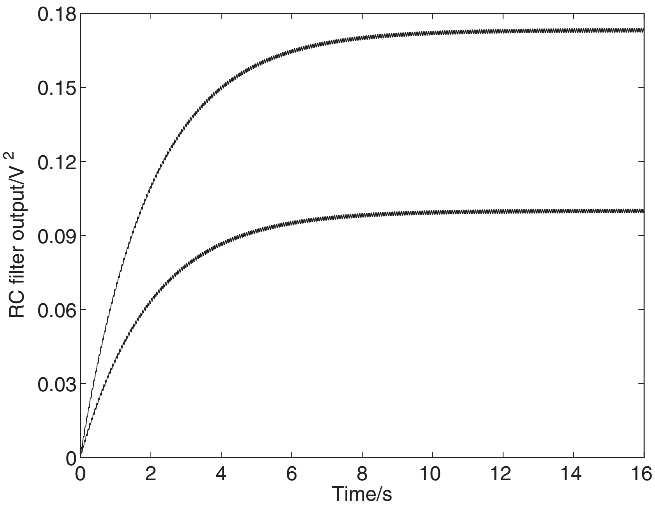

Figure 3. In-phase (upper) and quadrature (lower) outputs of the RC filter obtained for the ideal test signal, ideal instrument example of section 6.1.

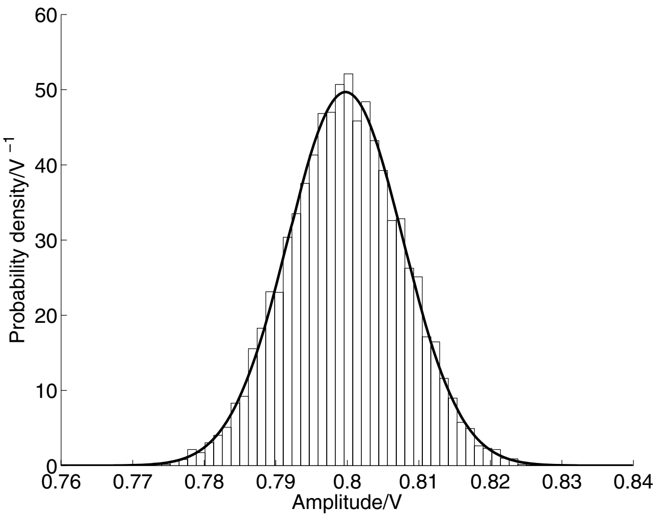

Figure 4. Scaled histogram of values of the output quantity R obtained for the case $u(A_{\text{test}}) = 0.008\,0$ V in the imperfect test signal, ideal instrument example of section 6.2.

Table 1 (rows 2 to 5) gives the results returned by the simulation tool implementing a Monte Carlo calculation with 10^5 trials[6] for different values of $u(A_{\text{test}})$. In table 1 (and subsequent tables), uncertainties are reported to two significant figures and corresponding estimates to a number of decimal figures that is consistent with the way the associated uncertainties are reported. The results suggest that R is proportional to A_{test} with a sensitivity coefficient equal to unity, whereas θ is insensitive to A_{test} and its associated uncertainty.

Figures 4 and 5 show scaled histograms for, respectively, R and θ. Each figure also shows the normal probability distribution function for a quantity having mean and standard deviation given by, respectively, the estimate of the output

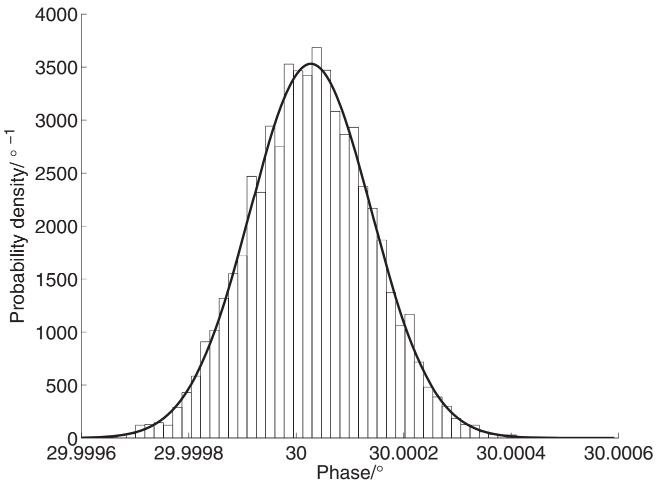

Figure 5. As figure 4, but for the output quantity θ.

quantity and its associated uncertainty. The fact that, in each figure, the Monte Carlo and normal distributions are very similar indicates that, for this example, R and θ may be approximated by normally distributed random variables.

6.3. Imperfect test signal, ideal instrument II

For this example, the test signal type is a sine wave and the estimates of the input quantities are

$$\boldsymbol{x} = (10^{-4}, 30, 10, \mathbf{0}^\top, \mathbf{0}^\top, 0.5, 0, 10, 90, \mathbf{0}^\top, 2)^\top,$$

and all associated uncertainties are set to zero apart from that $u(\delta V)$ associated with the estimates δV_i of the amplitude noise. The analogue-to-digital conversion parameters are $N_b = 24$,[7] $f_s = 256$ Hz and $T = 16$ s.

The choice of a value for $u(\delta V)$, together with the estimate of the test signal amplitude, defines a signal-to-noise ratio. For example, $u(\delta V) = 10^{-7}$ V represents a signal-to-noise ratio of $10^{-4}/10^{-7} = 10^3$, and $u(\delta V) = 10^{-4}$ V represents a ratio of unity. Table 2 gives the results returned by the simulation tool implementing a Monte Carlo calculation with 10^5 trials for different values of $u(\delta V)$.

The results show that, for signal-to-noise ratios greater than about 0.1, the estimates returned are close to the 'true' values. The uncertainties associated with the estimates are inversely proportional to the signal to noise ratio. For signal-to-noise ratios less than about 0.1, the uncertainties associated with the estimates are large compared with the estimates, and the information provided by the instrument about the test signal is consequently poor.

It is possible to achieve improved estimates of the output quantities and lower associated uncertainties by increasing the time constant τ of the RC filter (requiring a corresponding increase in the acquisition time T). Improvement in the results is therefore achieved at a cost of increased processing time.

[6] For this simulated test signal, on a personal computer with a 'standard specification', a Monte Carlo calculation with 10^5 trials takes approximately 30 min to run using the MATLAB version of the software and approximately 15 min using the LabVIEW version.

[7] Since the estimate of the test signal amplitude is lower than in the previous two examples, a higher value of the number of bits is used to reduce the effects of quantization.

Table 2. Results returned for the imperfect test signal, ideal instrument example of section 6.3.

$u(\delta V)$ (V)	Signal-to-noise ratio	$\widehat{R}$ (V)	$u(\widehat{R})$ (V)	$\widehat{\theta}$ (°)	$u(\widehat{\theta})$ (°)
10^{-7}	1000	0.000 099 9422	0.000 000 0040	30.0000	0.0023
10^{-6}	100	0.000 099 942	0.000 000 038	30.000	0.022
10^{-5}	10	0.000 099 94	0.000 000 38	30.00	0.22
10^{-4}	1	0.000 100 0	0.000 003 8	30.0	2.2
10^{-3}	0.1	0.000 108	0.000 036	30	25
10^{-2}	0.01	0.000 48	0.000 25	8	94
10^{-1}	0.001	0.004 7	0.002 5	1	103

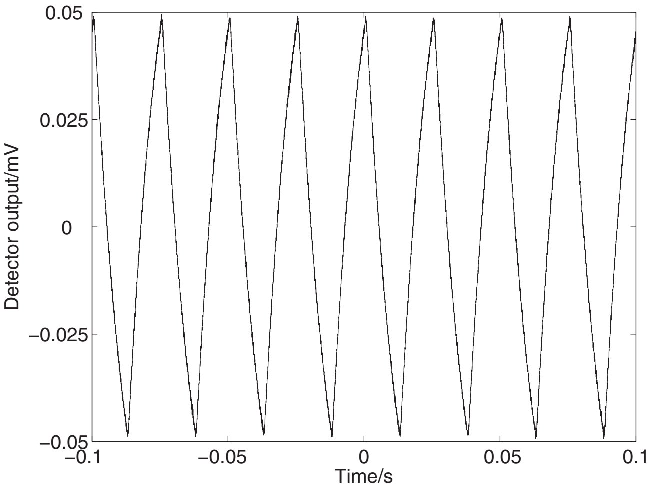

Figure 6. Distorted triangular wave output from the low-bandwidth detector.

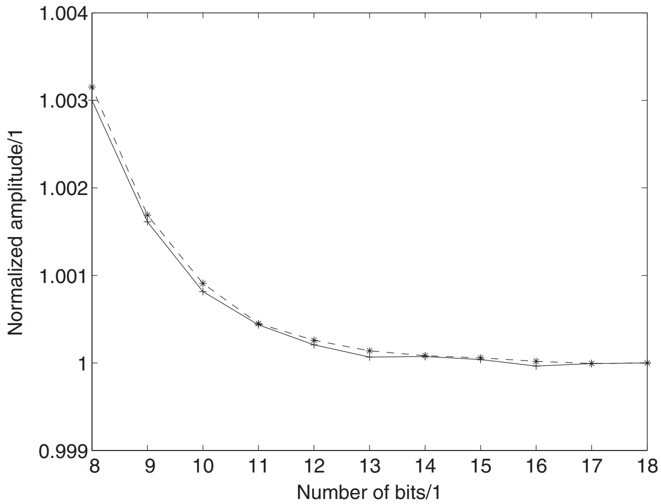

Figure 7. Normalized amplitude as a function of number of bits of quantization for frequencies 40 Hz (solid) and 200 Hz (dashed).

6.4. Real signal, imperfect instrument I

NPL's Applied Radiometry Group uses lock-in amplifiers in the measurement of radiation from infrared light sources. A collimated circular beam of radiation is directed at a rotating beam chopper system. The chopper disc has a series of radial rectangular slots cut in it through which the beam passes. The collimated beam is narrow enough to be able to pass completely through a slot. The chopper control system allows the experimentalist to choose the frequency at which the system 'chops' the beam. After passing through the slot the beam meets the detector, typically a thermopile, bolometer or pyrometer. The amplified detector output is displayed on a digital oscilloscope and is also connected to a lock-in amplifier. The chopper controller is connected to the lock-in amplifier to provide the reference signal.

In the case of an ideal detector the output waveform should appear trapezoidal. The edge of the slot meets the collimated beam, the slot traverses the circular beam and the opposite edge of the slot then intercepts the beam. However, if the response of the detector is too slow, a trapezoid signal is not observed, but the detector output appears as a distorted triangular wave. This type of signal is still of use to the experimentalist if the aim is to make relative, rather than absolute, measurements of detector performance. Figure 6 shows an example of a distorted triangular wave detector output.

Study of the form of the signal from this detector shows that it can be simulated very accurately by modelling it as a triangular wave in which each component of the Fourier series that describes the wave is shifted in phase by 22.5°, together with a small additional random noise component. A simulation of the distorted triangular wave was provided as a real signal for the simulation tool. Monte Carlo calculations were performed to study the effect of quantization in the lock-in amplifier on the simulation results. All sources of uncertainty in the internal oscillator were set to zero, except for the phases of the internal oscillator signal and the orthogonal signal, with which a relative standard uncertainty of 0.5% was associated in both cases. Eleven simulation runs were performed to estimate the amplitude of the sinusoidal component of the simulated signal at a frequency of 40 Hz, the fundamental frequency of the triangular wave, for quantization levels of 8 to 18 bits. The same process was then repeated for the fifth harmonic frequency, 200 Hz. In each case 10^4 Monte Carlo trials were employed[8]. Figure 7 shows the results obtained, where the estimates have been normalized to that obtained for the case of 18 bit quantization for both the 40 Hz and 200 Hz results. The figure clearly shows systematic

[8] In both this example and that in section 6.5, the number of test signal values being processed makes the implementation of 10^5 Monte Carlo trials impractical. For the purpose of evaluating estimates and associated standard uncertainties, the use of 10^4 trials can be expected to be adequate.

Table 3. Results obtained in the laboratory and using the simulation tool for the real signal, imperfect instrument example of section 6.5.

Nominal frequency (Hz)	Laboratory		Simulation tool	
	$\widehat{R}$ (V)	$u(\widehat{R})$ (V)	$\widehat{R}$ (V)	$u(\widehat{R})$ (V)
10	0.074 874	0.000 038	0.075 093 8	0.000 002 4
20	0.040 3380	0.000 0080	0.040 417 30	0.000 000 56
30	0.027 180	0.000 012	0.027 546 20	0.000 000 23
60	0.006 860	0.000 020	0.006 971 23	0.000 000 20
80	0.002 544	0.000 013	0.002 487 94	0.000 000 17
100	0.003 607	0.000 024	0.003 524 910	0.000 000 099

overestimation of the amplitude at both 40 Hz and 200 Hz as fewer bits of quantization are employed. These results can be used to evaluate contributions to the experimental uncertainty budget arising from specific choices of lock-in amplifier quantization.

6.5. Real signal, imperfect instrument II

Instrumented nanoindentation is used to obtain a measure of the mechanical properties of materials. Traditionally these measures have included hardness and modulus of elasticity. More recently, it has become possible using dynamic nanoindentation to measure the properties of viscoelastic materials such as polymers. The technique consists of indenting the test material with a constant applied force (typically several mN), superimposing a small oscillatory force (typically tens of μN) and then measuring the amplitude and the phase of the oscillation of the indenter with respect to the applied force. Highly viscous materials will dampen the oscillation and so decrease the amplitude.

The amplitude of oscillation is typically a few nm, smaller than the noise level on the detected signal. One way to measure the amplitude and phase of such a small signal is to use a lock-in amplifier. A reference (nominally sine wave) signal from the driving oscillator is fed into the lock-in amplifier, and the output signal from the nanoindenter which monitors the displacement of the indenter is connected to the amplifier. The lock-in amplifier returns values of the amplitude and phase, with the phase relating a square wave signal used as a trigger for the experiment to the sine wave component within the output signal.

A single 'experiment' involves three signals.

- The square wave signal used as the trigger for the experiment.
- The sine wave signal used as a reference signal for the experiment.
- The output signal from the nanoindenter (a sine wave signal buried in noise).

All three signals have the same nominal frequency. Figure 8 shows the output signal from the nanoindenter returned using an excitation sine wave of nominal frequency 100 Hz.

For a number of experiments involving frequencies from 10 to 100 Hz, the three signals described above have been captured in data files (10 s of data sampled at 20 kHz) for processing by the simulation tool. For a given experiment, in order to obtain appropriate inputs for the 'internal oscillator' section of the simulation tool, the sine wave signal is divided into 20 segments, each of 0.5 s duration, and sine wave fitting is implemented returning estimates of the amplitude and frequency for each segment. For these two parameters, the mean and standard deviation of the 20 values provide estimates of, respectively, the parameters and their associated standard uncertainties and are used to define the simulated (phase-shifted) reference signals when using the simulation tool. 10^4 Monte Carlo trials were employed.

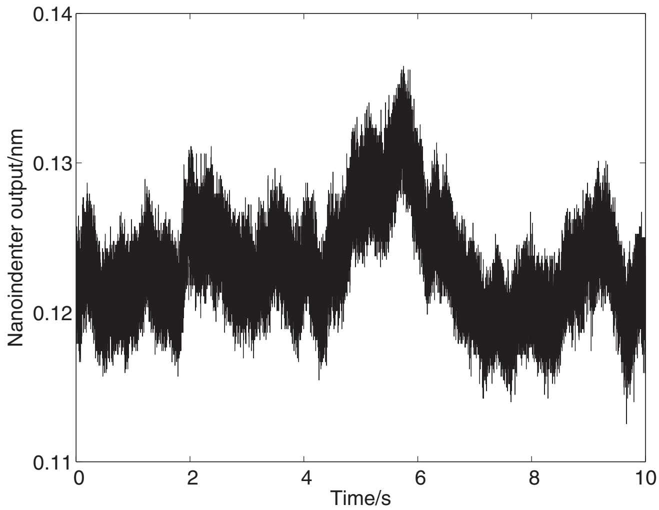

Figure 8. Example output signal from the nanoindenter using excitation sine wave of nominal frequency 100 Hz.

Laboratory results, namely the mean and standard deviation of the amplitude values returned from 20 repetitions of the experiment, are taken, respectively, as an estimate of the amplitude and its associated standard uncertainty.

Table 3 shows the results obtained for a number of nominal frequencies.

The standard uncertainties $u(\widehat{R})$ obtained in the laboratory are consistently greater (by more than an order of magnitude) than those obtained using the simulation tool, indicating that uncertainties associated with the (phase-shifted) reference signals are not the major contributors to uncertainty in the experiment. Therefore, when considering an uncertainty budget related to the experiment, consideration should be given to possible additional sources of uncertainty. Differences in the estimates $\widehat{R}$ may reflect the fact that some time elapsed between the end of the experiments and the generation of the data sets for processing.

7. Conclusions and future work

In this paper the issue of uncertainty evaluation associated with the results returned by a lock-in amplifier has been discussed. In particular, a number of questions concerning the interpretation of the results have been addressed. A software simulation tool that helps provide answers to these questions has been described, and its application illustrated using both simulated and real signals.

Application of the software simulation tool to simulated signals allows the user to understand better the uncertainties resulting from the use of an imperfect instrument, for example, identifying the parameters defining the operation of the lock-in amplifier that contribute most to the uncertainties associated with estimates of the output quantities, namely amplitude and phase difference.

A new version of the software simulation tool is to be developed. Proposed additional functionality includes allowing for more types of simulated signal, e.g. triangle and trapezoid, and noise structures other than white noise.

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The National Measurement Office of the UK's Department for Business, Innovation and Skills supported this work as part of its Software Support for Metrology (SS*f*M) programme. John Mountford and John Nunn from NPL took part in helpful discussions on their use of lock-in amplifiers and provided the authors with real signals for processing by the software simulation tool.

Appendix. Mathematics underlying the lock-in amplifier

The lock-in amplifier relies only on the following basic trigonometric identities for its operation:

$$2\sin(a)\sin(b) = \cos(a-b) - \cos(a+b), \tag{A.1}$$

$$2\sin(a)\cos(b) = \sin(a+b) + \sin(a-b). \tag{A.2}$$

Consider a continuous test signal made up of components of the form

$$V^{\mathrm{sig}} = A_{\mathrm{sig}} \sin\left(\omega_{\mathrm{sig}} t + \theta_{\mathrm{sig}}\right), \qquad \omega_{\mathrm{sig}} = 2\pi f_{\mathrm{sig}}.$$

Let $V^{\mathrm{X,ref}}$ and $V^{\mathrm{Y,ref}}$ be two continuous (phase-shifted) reference signals given by

$$V^{\mathrm{X,ref}} = A_{\mathrm{ref}} \sin\left(\omega_{\mathrm{ref}} t + \theta_{\mathrm{ref}}\right), \qquad \omega_{\mathrm{ref}} = 2\pi f_{\mathrm{ref}},$$

and

$$V^{\mathrm{Y,ref}} = A_{\mathrm{ref}} \sin\left(\omega_{\mathrm{ref}} t + \theta_{\mathrm{ref}} + \pi/2\right).$$

Multiply the test signal by the two (phase-shifted) reference signals to give output signals with components of the form

$$V^{\mathrm{X}} = A_{\mathrm{sig}} A_{\mathrm{ref}} \sin(\omega_{\mathrm{sig}} t + \theta_{\mathrm{sig}}) \sin(\omega_{\mathrm{ref}} t + \theta_{\mathrm{ref}}) \tag{A.3}$$

and

$$V^{\mathrm{Y}} = A_{\mathrm{sig}} A_{\mathrm{ref}} \sin(\omega_{\mathrm{sig}} t + \theta_{\mathrm{sig}}) \sin(\omega_{\mathrm{ref}} t + \theta_{\mathrm{ref}} + \pi/2). \tag{A.4}$$

Using identities (A.1) and (A.2), and the relationship $\sin(a + \pi/2) = \cos(a)$ allows components (A.3) and (A.4) to be expressed as

$$\begin{aligned} V^{\mathrm{X}} = \tfrac{1}{2} A_{\mathrm{sig}} A_{\mathrm{ref}} [&\cos((\omega_{\mathrm{sig}} - \omega_{\mathrm{ref}})t + (\theta_{\mathrm{sig}} - \theta_{\mathrm{ref}})) \\ &- \cos((\omega_{\mathrm{sig}} + \omega_{\mathrm{ref}})t + (\theta_{\mathrm{sig}} + \theta_{\mathrm{ref}}))] \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} V^{\mathrm{Y}} = \tfrac{1}{2} A_{\mathrm{sig}} A_{\mathrm{ref}} [&\sin((\omega_{\mathrm{sig}} - \omega_{\mathrm{ref}})t + (\theta_{\mathrm{sig}} - \theta_{\mathrm{ref}})) \\ &+ \sin((\omega_{\mathrm{sig}} + \omega_{\mathrm{ref}})t + (\theta_{\mathrm{sig}} + \theta_{\mathrm{ref}}))]. \end{aligned} \tag{A.6}$$

Apply a (perfect) low-pass filter to obtain the respective dc components corresponding to $\omega_{\mathrm{sig}} = \omega_{\mathrm{ref}}$:

$$X = \tfrac{1}{2} A_{\mathrm{sig}} A_{\mathrm{ref}} \cos(\theta_{\mathrm{sig}} - \theta_{\mathrm{ref}}),$$
$$Y = \tfrac{1}{2} A_{\mathrm{sig}} A_{\mathrm{ref}} \sin(\theta_{\mathrm{sig}} - \theta_{\mathrm{ref}}),$$

from which are obtained

$$A_{\mathrm{sig}} = \frac{2}{A_{\mathrm{ref}}} \sqrt{X^2 + Y^2}, \qquad \theta_{\mathrm{sig}} = \theta_{\mathrm{ref}} + \tan^{-1}\left(\frac{Y}{X}\right).$$

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