Workshop 6: General equilibrium

FIE463: Numerical Methods in Macroeconomics and Finance using Python

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February 20, 2025

See GitHub repository for notebooks and data:

https://github.com/richardfoltyn/FIE463-V25

Exercise 1: Labor supply without capital

Recall the consumption & labor choice problem studied in the lecture. In this exercise, we revisit this setting but assume that there is no capital in the economy.

Household problem

Households choose c and h to maximize utility

$$u(c,h) = \frac{c^{1-\gamma}-1}{1-\gamma} - \psi \frac{h^{1+1/\theta}}{1+1/\theta}$$

subject to the budget constraint

$$c = w \cdot h + \pi$$

where π are firm profits which are distributed to all households equally. Since all households are identical, we assume that the economy is populated by a *single* representative household.

Firm problem

We assume that firms have the decreasing-returns-to-scale production function

$$Y = zL^{1-\alpha}$$

where z is productivity (TFP) and labor L is the only input factor. Firms maximize profits Π,

$$\max_L \Pi = zL^{1-\alpha} - wL$$

which gives rise to the first-order condition

$$\frac{\partial \Pi}{\partial L} = (1-\alpha)zL^{-\alpha} - w = 0$$

We can solve for L to obtain the firm's optimal labor demand for given w:

$$L = \left(\frac{(1-\alpha)z}{w}\right)^{\frac{1}{\alpha}} \tag{1.1}$$

For simplicity, we assume there is a *single* firm which takes wages and the price of output as given, where the latter is normalized to one.

Equilibrium

The general equilibrium in this economy is a set of quantities (L, Y, Π, c, h, π) and the wage rate w which solve the household's and firm's problem, and the following conditions are satisfied:

- Labor market: $L = h$ (hours h supplied by households equal labor L demanded by firms).
- Goods market: $Y = c$ (the amount of goods c consumed by households equals aggregate output).
- Profits: $\Pi = \pi$ (profits distributed by firms equal profits received by households).

Analytical solution

By combining the household and firm first-order conditions, the problem can be reduced to a single equation in a single unknown, L (or h):

$$h = L = \left(\frac{(1-\alpha)z^{1-\gamma}}{\psi}\right)^{\frac{1}{1/\theta+\alpha+\gamma(1-\alpha)}} \tag{1.2}$$

We will use this expression later to compare the numerical to this exact solution.

Numerical solution

In the following, you are asked to adapt the code from the lecture to solve this problem. You should use the template file `workshop06_ex1.py` provided for this exercise to implement your solution.

1. Adapt the `Parameters` data class

   ```
   @dataclass
   class Parameters:
       pass
   ```

   so that it contains the following parameters as attributes: $\alpha = 0.36$, $z = 1$, $\gamma = 2$, $\psi = 1$, $\theta = 0.5$.

2. Write the function `solve_hh(w, pi, par)` to solve the household problem for a given w and π. This function should return the household choices, in particular the **labor supply** h.

   Use the utility function `util(c, h, par)` defined in the template file for this purpose (this is the same function we used in the lecture).

3. Write the function `solve_firm(w, par)` which returns the firm's **labor demand** L given by (1.1), output Y, and profits Π for a given wage w.

4. Write the function `compute_labor_ex_demand(w, par)` which returns the **excess labor demand** for a given wage w.

5. Write the function `compute_equilibrium(par)` which uses a root-finder to locate the equilibrium, computes the equilibrium quantities (L, Y, Π, c, h, π) and prices (w, r) and stores these using an instance of the `Equilibrium` data class defined in `workshop06_ex1.py`.

6. Compute the equilibrium using the function you just implemented and print the quantities and prices using `print_equilibrium()` implemented in `workshop06_ex1.py` (you don't need to write this function yourself).

7. Compare your numerical solution to the analytical solution for the equilibrium L returned by `compute_analytical_solution()` implemented in `workshop06_ex1.py`.

Note: Include the following cell magic to automatically reload any changes you make to the template file:

[1]:
```
%load_ext autoreload
%autoreload 2
```

Exercise 2: Unequal distribution of profits

We now extend the setting from Exercise 1 and assume that a fraction of households solely live on their labor income (type 1), while profits are only distributed to a subset of households (type 2). We can think of these households as workers and entrepreneurs, respectively. We assume the economy is populated by N_1 households of type 1 and N_2 households of type 2.

Household problem

All households have identical preferences which are unchanged from the previous exercise, but their budget constraints differ. For type-1 households, it is given by

$$c_1 = w \cdot h_1$$

whereas for type-2 households it's

$$c_2 = w \cdot h_2 + \pi_2$$

The subscripts in c_1, c_2, h_1, h_2, and π_2 index the household type since different households will choose different levels of consumption and labor supply.

Firm problem

The firm problem remains unchanged from the previous exercise. For convenience, we repeat the central equations:

$$\begin{aligned} \text{Labor demand:} \quad & L = \left(\frac{(1-\alpha)z}{w}\right)^{\frac{1}{\alpha}} \\ \text{Output:} \quad & Y = zL^{1-\alpha} \\ \text{Profits:} \quad & \Pi = zL^{1-\alpha} - wL \end{aligned}$$

Equilibrium

The general equilibrium in this economy is a set of quantities $(L, Y, \Pi, c_1, c_2, h_1, h_2, \pi_2)$ and the wage rate w which solve the household's and firm's problem, and the following conditions are satisfied:

- Labor market clearing: $L = N_1 h_1 + N_2 h_2$ (hours supplied by households equal labor L demanded by firms).
- Goods market clearing: $Y = N_1 c_1 + N_2 h_2$ (the amount of goods consumed by households equals aggregate output).
- Profits: $\Pi = N_2 \pi_2$ (profits distributed by firms equal profits received by type-2 households).

Numerical solution

In the following, you are asked to adapt the code you wrote for exercise 1 to solve the modified problem. The new solution only requires changes at a few selected points to take into account the unequal distribution of profits. You should use the template file `workshop06_ex2.py` provided for this exercise.

1. Adapt the `Parameters` class to include the two new parameters `N1` and `N2` which represent the number of type-1 and type-2 households, respectively. Set $N_1 = 5$ and $N_2 = 5$.

   For the remaining parameters, use the same values as in exercise 1.

2. Write the function `compute_labor_ex_demand(w, par)` which returns the excess labor demand for given w. Use the function `solve_hh()` and `solve_firm()` you wrote for exercise 1 to solve this task.

   Hint: Don't copy the implementations for `solve_hh()` and `solve_firm()` but directly import them from the module which contains the solution for exercise 1:

   ```
   from workshop01_ex1 import solve_firm, solve_hh
   ```

3. Write the function `compute_equilibrium(par)` which uses a root-finder to locate the equilibrium, computes the equilibrium quantities $(L, Y, \Pi, c_1, h_1, c_2, h_2, \pi_2)$ and the wage rate w, and stores these using an instance of the `Equilibrium` data class defined in `workshop06_ex2.py`.

4. Compute the equilibrium using the function you just implemented and print the quantities and prices using `print_equilibrium()` defined in `workshop06_ex2.py`.

   How does the unequal distribution of profits affect consumption and labor supply of type-1 vs type-2 households?

5. You are interested to see if and how the allocation and prices in the economy change as we vary the number of type-1 and type-2 households. Assume that there are a total of $N = N_1 + N_2 = 10$ households in the economy

   - Using the function `compute_equilibrium()` you wrote earlier, compute the equilibrium when N_1 takes on the integer values from $0, \dots, 9$ and $N_2 = N - N_1$.
   - Create a graph with four panels (2×2) which shows the aggregates Y, L, Π, and w as a function of N_1.
   - Create a graph with three columns which shows (c_1, c_2) in the first, (h_1, h_2) in the second, and π_2 in the third column. Use different colors and line styles to distinguish household types and include a legend.

     What do you conclude about the effects of inequality on the equilibrium allocation and prices?

Bonus: using analytical results and root-finding

Unlike the previous exercise, this economy no longer has a closed-form solution for the equilibrium quantities. From the households' first-order conditions, we can derive that the equilibrium is characterized by the two non-linear equations

$$(wh_1)^{-\gamma} = \psi \frac{h_1^{1/\theta}}{w}$$
$$(wh_1 + \pi_2)^{-\gamma} = \psi \frac{h_2^{1/\theta}}{w}$$

Note that w and $\pi_2 = \frac{\Pi}{N_2}$ itself are functions of (h_1, h_2) via the labor market clearing $L = N_1 h_1 + N_2 h_2$:

$$w = (1-\alpha) z L^{-\alpha} = (1-\alpha) z \big(N_1 h_1 + N_2 h_2\big)^{-\alpha}$$
$$\Pi = \alpha z L^{1-\alpha} = \alpha z \big(N_1 h_1 + N_2 h_2\big)^{1-\alpha}$$

We can substitute these two equations into the non-linear equation system above and numerically find a solution (h_1, h_2) that satisfies these conditions.

1. Use the multivariate root-finder `root()` from `scipy.optimize` with `method='hybr'` to solve the above equation system.

   To do this, you need to write a function which takes as argument a vector `x` which contains the values (h_1, h_2), and return the errors in the two first-order conditions, i.e., a vector that contains the left-hand minus the right-hand side for each of the two equations.

2. Make sure the results obtained from this approach are the same as in the main exercise.