Commutative Algebra

Sumanth N R

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Contents

Chapter 1

Rings and Ideals

Exercise 1.1

Let x be a nilpotent element of a ring A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Suppose $x^n = 0$ for some $n \in \mathbb{N}$.
Clearly, $(1+x)\left(1-x+x^2 \ldots + (-1)^{n-1} x^{n-1}\right) = 1 - x^n = 1$.
Hence, $1+x$ is a unit.
Consider $u+x$ for some unit u. Write $u+x = u\left(1+xu^{-1}\right)$.
Now, we know that xu^{-1} is nilpotent with the same n and hence, $\left(1+xu^{-1}\right)$ is a unit.
Since product of units is a unit, we can conclude $u+x$ is a unit.

Exercise 1.2

Let A be a ring and $A[x]$ be the ring of polynomials in an intermediate x with coefficients in A. Let $f = a_0 + a_1 x + \ldots + a_n x^n \in A[x]$. Prove that

1. f is a unit $\iff$ a_0 is a unit in A and $a_1, \ldots, a_n$ are nilpotent.
2. f is nilpotent $\iff$ $a_0, a_1, \ldots, a_n$ are nilpotent.
3. f is a zero-divisor $\iff$ there exists $a \neq 0$ in A such that $af = 0$.
4. f is said to be primitive if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\iff$ f and g are primitive.

1. We prove the result using induction on the degree of the polynomial. Base case $n = 0$ is trivially correct since a_0 is a unit in $A[x]$ $\iff$ a_0 is a unit in A. Let us assume the result for all polynomials of degree $n-1$ or lower.

   $\implies$
   Suppose the inverse of f be $g(x) = b_0 + b_1 x + \ldots + b_m x^m$.
   Observe that $a_n^{r+1} b_{m-r} = 0 \; \forall \; r$.
   $r = 0$: Co-efficient of x^{n+m} is 0. Thus, $a_n b_m = 0 \implies a_n^{0+1} b_{m-0} = 0$.
   $r = 1$: Co-efficient of x^{n+m-1} is 0. Thus, $a_n b_{m-1} + a_{n-1} b_m = 0 \implies a_n^{1+1} b_{m-1} = 0$. after multiplying by a_n.
   ...
   $r = m : a_n^{m+1} b_0 = 0$
   Since we also know that $a_0 b_0 = 1$, we know that b_0 is a unit. Multiplying a_0 in the above equation, we can see that $a_n^{m+1} = 0$ and hence a_n is nilpotent.
   We also know that the element $a_n x^n$ is nilpotent in the polynomial ring and hence $f - a_n x^n$ is a unit using the result that the sum of a unit and a nilpotent element is a unit.
   But $f - a_n x^n$ is an $n-1$ or a lower degree polynomial and hence, the result holds, that is $a_1, \ldots, a_{n-1}$ are also nilpotent along with a_0 being a unit.

$\Longleftarrow$

Using $a_0, a_1, \ldots, a_{n-1}$, we construct $f - a_n x^n$ which is a unit using the induction hypothesis and we know that $a_n x^n$ is a nilpotent element and hence, $f - a_n x^n + a_n x^n = f$ being the sum of unit and a nilpotent element, is a unit.

2. We prove the result by induction on the degree of the polynomial. $n = 0$ is trivially correct since a_0 is a nilpotent element of A if and only if a_0 is a nilpotent element of $A[x]$.
   Assume the result for all polynomials of degree $n-1$ or lower.

   $\Longrightarrow$

   $f = a_0 + a_1 x + \ldots + a_n x^n$ is nilpotent. Thus, for some $r \in \mathbb{N}$, $f^r = 0$. Considering the rn^{th} coefficient and setting it to zero, we get $a_n^r = 0$ and thus, $a_n x^n$ is a nilpotent element of $A[x]$. Since we know that the sum of 2 nilpotent elements is nilpotent, $f - a_n x^n$ being a polynomial of degree less than n is nilpotent and we get $a_0, \ldots, a_{n-1}$ are nilpotent.

   $\Longleftarrow$

   $a_0, \ldots, a_n$ are nilpotent elements of A.
   Clearly, $a_n x^n$ is nilpotent and the polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$ is nilpotent using the induction hypothesis. Adding the above two, we get f is nilpotent.

3. The only if part is trivial since $a \in A[x]$. Proof for the $\Longrightarrow$ part.
   Suppose f is a zero-divisor. Then, there exists $g \in A[x]$ with the least degree such that $fg = 0$. We have $a_n b_m = 0$ since the coefficient of x^{n+m} is 0.
   This implies $a_n g = 0$ since $a_n g f = 0$ and if $a_n g \neq 0$, it is a polynomial of degree less than n and annihilates f which contradicts the minimality of the degree of g.

   We show by induction that $a_{n-r} g = 0$ for $0 \leqslant r \leqslant n$. The base case being $r = 0, a_n g = 0$ is shown above.
   Let us just go 1 level for better underdstanding.
   Since we proved $a_n g = 0$, we showed that $a_n b_i = 0$ for all i.
   Since $fg = 0$, considering the coefficient of x^{n+m-1}, we have $a_n b_{m-1} + a_{n-1} b_m = 0$ but $a_n b_{m-1} = 0$.
   Thus, $a_{n-1} b_m = 0$ and a_{n-1} annihilates the leading co-efficient of g.
   Using the same argument as before, we can see that $a_{n-1} g = 0$.

   Now, we can use the induction hypothesis to show that $a_{n-i} g = 0$ for $0 \leqslant i \leqslant n$ and hence, $a_0 g = 0$.
   Thus, $a_i b_j = 0$ for all $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant m$.
   Also, $b_m \neq 0$ and $a_i b_m = 0$ for all i, we can conclude that $b_m f = 0$.

4. Suppose $f = a_0 + a_1 x + \ldots + a_n x^n$
   $g = b_0 + b_1 x + \ldots + b_m x^m$ and
   $fg = c_0 + c_1 x + \ldots + c_{n+m} x^{n+m}$ where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$ for $0 \leqslant i \leqslant n+m$.

   $\Longrightarrow$

   Suppose fg is primitive.
   Then, $\exists\ \gamma_1, \gamma_2, \ldots, \gamma_n \in A$ such that $\gamma_1 c_1 + \gamma_2 c_2 + \ldots + \gamma_{m+n} c_{m+n} = 1$.
   Thus,
   $$\sum_{i=0}^{n} \sum_{j=0}^{m} a_i b_j \gamma_{i+j} = 1$$
   Putting $\alpha_i := \sum_{j=0}^{m} b_j \gamma_{i+j}$, we get $\sum_{i=1}^{n} a_i \alpha_i = 1$ and hence, f is primitive.
   Similarly, we get g is primitive.

   $\Longleftarrow$

   We prove the result using the contrapositive.
   Suppose fg is not primitive.
   Then, $(c_0, c_1, \ldots, c_{m+n})$ is a non-trivial ideal of A, and is contained in some maximal ideal, say $\mathfrak{c}$ and $1 \notin \mathfrak{c}$.
   Clearly, $A/_{\mathfrak{c}}$ is a field.
   We have $fg \equiv 0 \pmod{\mathfrak{c}}$. Since $A/_{\mathfrak{c}}$ is a field, this implies $f \equiv 0 \pmod{\mathfrak{c}}$ or $g \equiv 0 \pmod{\mathfrak{c}}$.
   This means, either f or g is contained in $\mathfrak{c}$ which implies 1 does not belong to it and hence is not primitive.

Exercise 1.3
Generalize the results of Exercise 2 to a polynomial ring $A[x_1, \dots, x_r]$ in several indeterminates.

Suppose we consider the polynomial ring $A[x_1, \dots, x_r]$. An element of the above ring looks like,

$$f = \sum_{i_1+i_2+\dots+i_r \leqslant n} a_{i_1,i_2,\dots,i_r} x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}$$

1. f is a unit in $A[x_1, \dots, x_r] \iff a_{0,0,\dots,0}$ is a unit in A and every other co-efficient is nilpotent.
2. f is a nilpotent element in $A[x_1, \dots, x_r] \iff a_{i_1,i_2,\dots,i_r}$ is a nilpotent element in A for any combination of $(i_1, i_2, \dots, i_r)$.
3. f is a zero-divisor in $A[x_1, \dots, x_r] \iff \exists\ a \neq 0 \in A$ such that $af = 0$.
4. Suppose $f, g \in A[x_1, \dots, x_r]$.
   fg is primitive in $A[x_1, \dots, x_r] \iff f$ and g are primitive in $A[x_1, \dots, x_r]$.

Exercise 1.4
In the ring $A[x]$, the Jacobson radical is equal to nilradical.

We know that every maximal ideal is a prime ideal and hence, the nilradical is contained in the Jacobson radical. We need to prove that Jacobson radical is contained in nilradical.
Let $f(x) \in \mathfrak{R} \implies 1 - fg$ is a unit in $A[x]\ \forall\ g \in A[x]$. Suppose

$$f(x) = a_0 + a_1 x \cdots a_n x^n$$

We use the fact that if $h(x) = h_0 + \cdots + h_k x^k$ is unit, h_0 is a unit and $h_1, \cdots, h_k$ are nilpotent.
Putting $g = x$, we see that $1 - fx$ is a unit and hence $a_0, a_1, \cdots a_n$ are nilpotent.
Since coefficients of f are nilpotent, we can conclude that f is nilpotent.

$$f \in \mathfrak{R} \implies f \in \mathfrak{N}$$

Hence, Jacobson radical is contained in nilradical and we proved the result.

Exercise 1.5
Let A be a ring and let $A[[x]]$ be a ring of formal power series $f = \sum_{i=0}^{\infty} a_n x^n$ with coefficients in A. Show that

1. f is a unit in $A[[x]] \iff a_o$ is a unit in A.
2. f is a nilpotent element in $A[[x]] \implies a_n$ is nilpotent for any $n \geqslant 0$. Is the converse true?
3. f belongs to the Jacobson radical of $A[[x]] \iff a_0$ belongs to the Jacobson radical of A.
4. The contraction of a maximal ideal $\mathfrak{m}$ of $A[[x]]$ is a maximal ideal of A and $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.
5. Every prime ideal of A is a contraction of a prime ideal of $A[[x]]$.

1. $(\implies)$
   Now, f is a unit in $A[[x]]$
   $\implies \exists\ g = \sum_{i=0}^{\infty} b_i x^i \in A[[x]]$ such that $fg = 1$
   $\implies a_0 b_0 = 1 \implies a_0$ is a unit.

   $(\impliedby)$
   Suppose a_0 is a unit in A.

Let us try to construct $g = \sum_{i \geqslant 0} b_i$ such that $fg = 1$.
Since a_0 is a unit, there is a b_0 such that $a_0 b_0 = 1$. Since the coefficient of x^n is $0\ \forall\ n \in \mathbb{N}$ in fg, we have

$$0 = \sum_{i=0}^{n} a_{n-i} b_i \implies a_0 b_n = -\sum_{i=0}^{n-1} a_{n-i} b_i \implies b_n = -b_0 \sum_{i=0}^{n-1} a_{n-i} b_i$$

We just expressed b_n in terms of $b_0, \cdots, b_{n-1}$ such that coefficient of x^n in $fg = 0$.
$\exists\ g \in A[[x]]$ such that $fg = 1$.

2. Suppose f be a nilpotent element in $A[[x]]$. Then, $f^n = 0$ for some $n \in \mathbb{N}$.
Coefficient of 1 in f^n is $a_0^n = 0 \implies a_0$ is nilpotent.
Now, consider the element $f_1 = f - a_0$. Since $f' = a_1 x + a_2 x^2 + \cdots$, being the sum of nilpotent elements, is nilpotent, and $f' = x f_1$. Thus, f_1 is nilpotent.
a_1 being the coefficient of 1 in f_1, is nilpotent and using the same argument, we can show that a_2 is nilpotent.
Thus, a_n is nilpotent for any $n \geqslant 0$.

3. $(\implies)$
f belongs to the Jacobson radical of $A[[x]] \implies 1 - fg$ is a unit $\forall\ g \in A[[x]]$.
Putting $g = y$ for some $y \in A$, we get $1 - a_0 y$ is a unit in $A\ \forall\ y \in A$ using the previous proposition.
This means $a_0 \in \mathfrak{R}_A$.

$(\impliedby)$
Suppose $a_0 \in \mathfrak{R} \implies 1 - a_0 y$ is a unit in $A\ \forall\ y \in A$.
Cosider some $g = \sum_{i \geqslant 0} b_i x^i \in A[[x]]$.
$1 - a_0 b_0$ is a unit in $A \implies 1 - fg$ is a unit in $A[[x]]$ for any $g \in A[[x]]$.
This proves the claim.

Exercise 1.6

A ring A is such that every ideal not contained in the Nilradical contains a non-zero idempotent (that is, an element e such that $e^2 = e$ and $e \neq 0$). Prove that the Nilradical and the Jacobson radical of A are equal.

It is trvial to note that $\mathfrak{N} \subseteq \mathfrak{R}$.
If $\mathfrak{R} \nsubseteq \mathfrak{N}$, then there exists an idempotent $e \in \mathfrak{R}$ such that $e^2 \neq e, e \neq 0$. We show that this is not possible.
Since $e \in \mathfrak{R}$, $1 - ey$ is a unit in A for every $y \in A$. Putting $y = 1$, we get $1 - e$ is a unit in A.
Since $e, 1 \in \mathfrak{m}$, we have $1 - e \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$.
Also, note that $(1 - e)e = e - e^2 = 0 \implies e = 0$ since $1 - e$ is a unit which is a contradiction.
Hence, $\mathfrak{N} = \mathfrak{R}$.

Exercise 1.7

Let A be a ring in which every element x satisfies $x^n = x$ for some $n \in \mathbb{N}$ and $n > 1$ depending on x. Show that every prime ideal of A is maximal.

Suppose $\mathfrak{p}$ be a prime ideal of A.
Consider a non zero element $x \in A/\mathfrak{p} \implies x^n \in A/\mathfrak{p}$ for some $n > 1 \in \mathbb{N}$.
Therefore, $x^n - x \in A/\mathfrak{p} \implies x\left(x^{n-1} - 1\right) \in A/\mathfrak{p}$.
Since $A/\mathfrak{p}$ is an integral domain and x is chosen to be a non zero element,
$x\left(x^{n-1} - 1\right) = 0 \implies x^{n-1} - 1 = 0 \implies x^{-1} = x^{n-2}$.
We therefore proved that any non-zero element in $A/\mathfrak{p}$ is a unit and hence $A/\mathfrak{p}$ is a field.
Thus, we conclude $\mathfrak{p}$ is a maximal ideal.

Exercise 1.8

Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

Suppose we take the set of prime ideals of A and consider the inclusion relation on it as a partial order defined by

$$\mathfrak{p}_1 < \mathfrak{p}_2 \iff \mathfrak{p}_1 \supset \mathfrak{p}_2$$

This forms a chain in the set of prime ideals of A.
Using Zorn's lemma, we can show that there exists a maximal element $\mathfrak{p}$ in the chain which is the minimal prime ideal of A.

> **Exercise 1.9**
>
> Let $\mathfrak{a} \neq (1)$ be an ideal in a ring A. Show that $\mathfrak{a} = r(\mathfrak{a}) \iff \mathfrak{a}$ is an intersection of prime ideals.

One side is trivial since $r(\mathfrak{a})$ is the intersection of all prime ideals containing $\mathfrak{a}$.

> **Exercise 1.10**
>
> Let A be a ring, $\mathfrak{N}$ its nilradical. Show that the following are equivalent.
>
> 1. A has exactly one prime ideal.
> 2. A Every element of A is a unit or nilpotent.
> 3. $A/\mathfrak{N}$ is a field.

$(1 \implies 3)$
Suppose A has exactly one prime ideal, $\mathfrak{p}$. Then, $\mathfrak{N} = \mathfrak{p}$.
Since any maximal ideal is a prime ideal, and we only have one prime ideal, the prime ideal must be a maximal ideal and since $\mathfrak{p} = \mathfrak{N}$, $\mathfrak{N}$ is a maximal ideal and A is a local ring.
We directly get the result that $A/\mathfrak{N}$ is a field.
$(3 \implies 2)$
Suppose $A/\mathfrak{N}$ is a field.
Consider some $x \in A$. If $x \in \mathfrak{N}$, it is nilpotent and if $x \notin \mathfrak{N}$,

$$\exists\, y \in A, y \notin \mathfrak{N} \text{ s.t. } (x + \mathfrak{N})(y + \mathfrak{N}) = 1 + \mathfrak{N} \qquad \implies xy - 1 \in \mathfrak{N}$$

Clearly, since the sum of a unit and a nilpotent unit is a unit, we have $xy - 1 + 1 = xy$ is a unit in A.

$$\implies \exists\, z \in A \text{ s.t. } xyz = 1$$

yz is the inverse of x and hence x is a unit in A.
Therefore, every element of A is either nilpotent or a unit.

$(2 \implies 3)$
We need to prove that $A/\mathfrak{N}$ is a field.
Suppose we consider $x \in A, x \notin \mathfrak{N}$

$$\implies \exists\, y \in A, y \notin \mathfrak{N} \text{ s.t. } xy = 1$$

Clearly,

$$\implies (x + \mathfrak{N})(y + \mathfrak{N}) = 1 + \mathfrak{N}$$

and hence, we can see that $A/\mathfrak{N}$ is a field.

$(3 \implies 1)$
We can see that $\mathfrak{N}$ is a maximal ideal.
Suppose for contradiction, let there be at least 2 distinct prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ in A.
Clearly, this means that

$$\mathfrak{N} \subseteq \mathfrak{p}_1 \text{ and } \mathfrak{N} \subseteq \mathfrak{p}_2$$

We claim that

$$\mathfrak{N} \subset \mathfrak{p}_1 \text{ or } \mathfrak{N} \subset \mathfrak{p}_2$$

since if that does not hold, then we have

$$\mathfrak{N} = \mathfrak{p}_1 = \mathfrak{p}_2$$

which contradicts the fact that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are distinct and if it holds, it contradicts the maximality of $\mathfrak{N}$. And hence, we can see that there is only a single prime ideal in A.

Exercise 1.11

A ring A is called a Boolean ring if $x^2 = x$ for every $x \in A$. In a Boolean ring, show that

1. $2x = 0 \ \forall \ x \in A$
2. Every prime ideal $\mathfrak{p}$ is maximal and $A/\mathfrak{p}$ is a field with 2 elements.
3. Every finitely generated ideal in A is principal.

1. Since $x^2 = x \ \forall \ x \in A$, we have

$$\begin{aligned} (1+x)^2 &= (1+x) \\ \implies \cancel{1} + 2x + \cancel{x^2} &= \cancel{1} + \cancel{x} \\ \implies 2x &= 0 \quad \forall \ x \in A \end{aligned}$$

2. Consider some prime ideal $\mathfrak{p}$. We know that every element of $A/\mathfrak{p}$ satisfies $x^2 = x$. Since $\mathfrak{p}$ is a prime ideal, $A/\mathfrak{p}$ is an integral domain, all the elements of $A/\mathfrak{p}$ are the elements that satisfy $x = x^2$.

$$x = x^2 \implies x(1-x) = 0 \implies x = 0 \text{ or } x = 1$$

Hence, A is a field with 2 elements which implies that $\mathfrak{p}$ is a maximal ideal.

3. We need to show that any ideal generated by 2 elements can be generated by a single element. In other words, we need to show that

$$(x, y) = (z) \qquad \text{for some } z \in A \tag{1.1}$$

We claim that

$$z = x + y - xy$$

since $xz = x$ and $yz = y$ and it satisfies (1.1).

Hence, we can merge any finite number of generators into a single generator which implies every finitely generated ideal is principal.

Exercise 1.12

A local ring contains no idempotents $\neq 0, 1$.

Recall that a ring A is called a local ring if it contains a unique maximal ideal $\mathfrak{m}$.
Notice that if e is an idempotent in A, then $e(e-1) = e^2 - e = 0$ shows that e and $e-1$ are zero divisors and hence, not units. Thus, $e, 1-e \in \mathfrak{m}$ but

$$\begin{gathered} \implies e + (1-e) = 1 \in \mathfrak{m} \\ \implies\impliedby \end{gathered}$$

Chapter 2

Modules

Exercise 2.1

Show that

$$ {}^{\mathbb{Z}}\!/_{m\mathbb{Z}} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}\!/_{n\mathbb{Z}} = 0 $$

when m and n are coprime.

Let

$$ T := {}^{\mathbb{Z}}\!/_{m\mathbb{Z}} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}\!/_{n\mathbb{Z}} $$

Consider some element $x \otimes y \in T$.
Since m and n are coprime,

$$ \exists\, a, b \in \mathbb{Z} \quad \text{s.t.} \quad am + bn = 1 $$

Now,

$$ \begin{aligned} x \otimes y &= (am + bn)(x \otimes y) \\ &= (amx \otimes y) + (x \otimes bny) \\ &= (0 \otimes y) + (x \otimes 0) \\ &= 0 \end{aligned} $$

Hence, $T = 0$.

Exercise 2.2

Let A be a ring and $\mathfrak{a}$ be an ideal.
Show that

$$ {}^{A}\!/_{\mathfrak{a}} \otimes_A M \cong {}^{M}\!/_{\mathfrak{a}M} $$

Any element of ${}^{A}\!/_{\mathfrak{a}} \otimes_A M$ can be written as

$$ (r + \mathfrak{a}) \otimes m \in {}^{A}\!/_{\mathfrak{a}} \otimes_A M $$