Circuits and Transforms

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

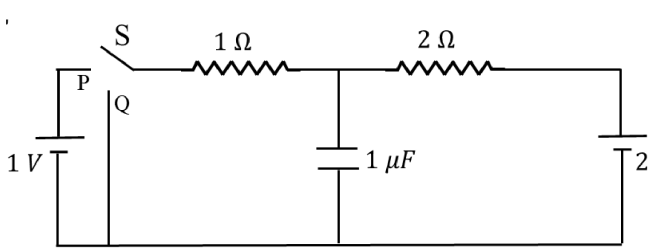
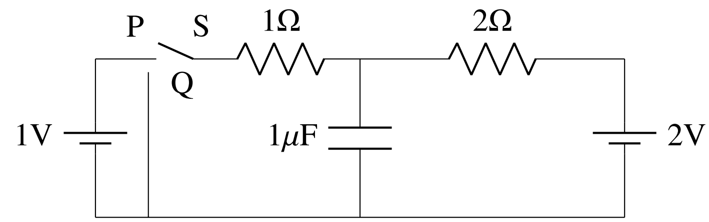
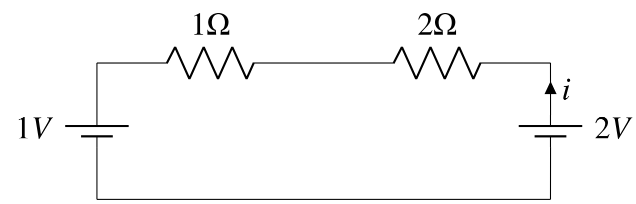


Fig. 2.1

2. Draw the circuit using latex-tikz.
Solution:



3. Find q_1.
Solution:
Current through the capacitor is 0 at steady state. So, current only flows in the outer loop.



$$\begin{aligned} i &= \frac{(2-1)\ V}{(1+2)\ \Omega} \\ i &= \frac{1}{3}A \end{aligned}$$

Potential difference across the capacitor,

$$\begin{aligned} V_C &= 2 - 2 \times \frac{1}{3} = \frac{4}{3}V \\ q_1 &= CV_C \\ q_1 &= 1\mu F \times \frac{4}{3}V \\ q_1 &= \frac{4}{3}\mu C \end{aligned}$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution:
By the definition of Laplace transform

$$\begin{aligned} u(t) \xrightarrow{\mathcal{L}} \int_{-\infty}^{\infty} u(t)e^{-st}dt &= \int_{0}^{\infty} e^{-st}dt \\ &= \left.\frac{e^{-st}}{-s}\right|_{0}^{\infty} = \frac{1}{s} \end{aligned}$$

$\lim_{s\to\infty} e^{-st} = 0$ only when $Re(s) > 0$.
The ROC is $Re(s) > 0$.

5. Show that

$$e^{-at}u(t) \overset{\mathcal{H}}{\longleftrightarrow} L\frac{1}{s+a}, \quad a > 0 \tag{2.1}$$

and find the ROC.
Solution:
By the definition of Laplace transform

$$\begin{aligned} e^{-at}u(t) &\xrightarrow{\mathcal{L}} \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \\ &= \int_{0}^{\infty} e^{-(s+a)t}dt \\ &= \left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_0^{\infty} = \frac{1}{s+a} \end{aligned}$$

$\lim_{s\to\infty} e^{-(s+a)t} = 0$ only when $Re(s+a) > 0$.
The ROC is $Re(s+a) > 0$ or $Re(s) > -a$.

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

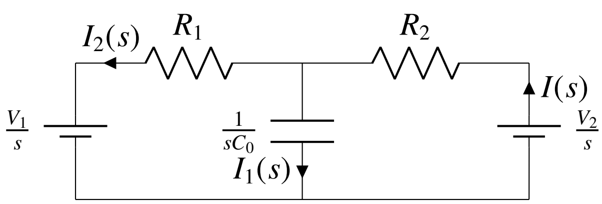


Fig. 2.2

$$u(t) \overset{\mathcal{H}}{\longleftrightarrow} LV_1(s) \tag{2.2}$$

$$2u(t) \overset{\mathcal{H}}{\longleftrightarrow} LV_2(s) \tag{2.3}$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution:
Using the Transform from time domain to s-domain

$$V_1(s) = \mathcal{L}(u(t)) = \frac{V_1}{s}$$

$$V_2(s) = \mathcal{L}(2u(t)) = \frac{V_2}{s}$$

Potential across the capacitor is $V_{C_0}(s)$ and the assuming bottom is grounded.

Applying KCL at the upper junction, we have,

$$\frac{V_{C_0}(s) - V_1(s)}{R_1} + \frac{V_{C_0}(s) - V_2(s)}{R_2} + \frac{V_{C_0}(s)}{\frac{1}{sC_0}} = 0$$

$$V_{C_0}(s)\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_1(s)}{R_1} + \frac{V_2(s)}{R_2}$$

$$V_{C_0}(s) = \frac{V_1(s)R_2 + V_2(s)R_1}{R_1 + R_2 + sC_0R_1R_2}$$

$$V_{C_0}(s) = \frac{R_2 + 2R_1}{s(R_1 + R_2 + sC_0R_1R_2)}$$

after substitution.

7. Find $v_{C_0}(t)$. Plot using python.
Solution:
Factoring $V_{C_0}(s)$, we have

$$\begin{aligned} V_{C_0}(s) &= \left(\frac{1}{R_1 + R_2}\right)\left(\frac{(R_2 + 2R_1)}{s} - \frac{R_1R_2C_0\,(R_2 + 2R_1)}{R_1 + R_2 + sC_0R_1R_2}\right) \\ &= \left(\frac{R_2 + 2R_1}{R_1 + R_2}\right)\left(\frac{1}{s} - \frac{R_1R_2C_0}{R_1 + R_2 + sC_0R_1R_2}\right) \\ &= \left(\frac{R_2 + 2R_1}{R_1 + R_2}\right)\left(\frac{1}{s} - \frac{1}{\frac{1}{R_2C_0} + \frac{1}{R_1C_0} + s}\right) \end{aligned}$$

Applying inverse Laplace transform,

$$v_{C_0}(t) = \frac{R_2 + 2R_1}{R_1 + R_2}\left(u(t) - e^{-\left(\frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right)t}u(t)\right) \tag{2.4}$$

Using the values $R_1 = 1\,\Omega$, $R_2 = 2\,\Omega$, $C_0 = 1\,\mu F$, we get

$$v_{C_0}(t) = \frac{4}{3}u(t)\left(1 - e^{-1.5\times10^6\,t}\right)$$

The following code yields $v_{C_0}(t)$.

```
wget https://github.com/Sigma1084/EE3900/
    blob/master/cktsig/code/Ex2_plotVt.py
```

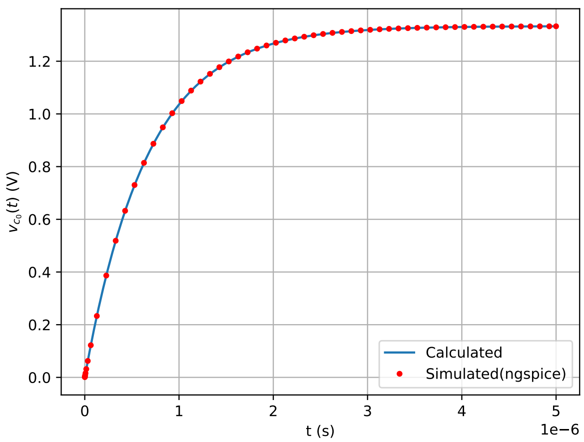
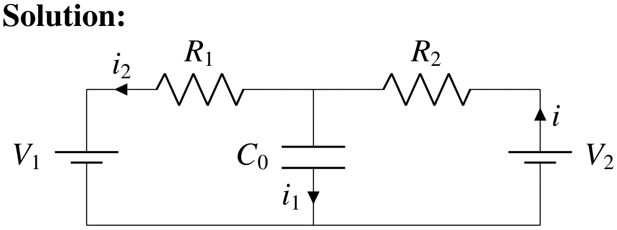


Fig. 2.3

8. Verify your result using ngspice. The following code yields $v_{C_0}(t)$. using ngspice

```
wget https://github.com/Sigma1084/EE3900/
    blob/master/cktsig/code/Ex2.spice
```

9. Obtain Fig. 2.2 using the equivalent differential equation.
Solution:



Using KVL on the left and right loops, we get

$$i_2(t)\ R_1 + V_1 - \int_0^t \frac{i_1(t)}{C_0} dt = 0 \tag{2.5}$$

$$-i(t)\ R_2 + V_2 - \int_0^t \frac{i_1(t)}{C_0} dt = 0 \tag{2.6}$$

Taking the Laplace Transform after multiplying u(t) in both equations,

$$\mathcal{L}\left(R_1 i_2(t)u(t) + V_1 u(t) - u(t)\int_0^t \frac{i_1}{C_0}dt\right) = 0$$

$$\mathcal{L}\left(-R_2 i(t)u(t) + V_2 u(t) - u(t)\int_0^t \frac{i_1}{C_0}dt\right) = 0$$

Suppose $i(t)u(t) \xrightarrow{\mathcal{L}} I(s)$ and $Vu(t) \xrightarrow{\mathcal{L}} \frac{V}{s}$

$$\implies \mathcal{L}\left(u(t)\int_0^t \frac{i_1}{C_0}dt\right) = \frac{1}{sC_0}\mathcal{L}\left(u(t)i_1\right) = \frac{I_1(s)}{sC_0}$$

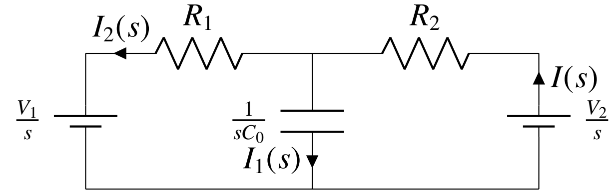
(Using the properties of Laplace Transform)

The above equations after transform

$$R_1 I_2(s) + \frac{V_1}{s} - \frac{I_1(s)}{sC_0} = 0 \tag{2.7}$$

$$-R_2 I(s) + \frac{V_2}{s} - \frac{I_1(s)}{sC_0} = 0 \tag{2.8}$$

Now after applying the transformation, from time domain to s-domain, we replace the capacitor with an equivalent resistor of resistence $\frac{1}{sC_0}$ and replace I by $I(s)$, V by $\frac{V}{s}$
The equivalent s-domain circuit is now,



3 Initial Conditions

1. Find q_2 in Fig. 2.1
Solution:
Current through the capacitor is 0 in steady state. Since the current flows only in the outer loop, let the current in the outer loop be I.

$$I = \frac{2V}{(1+2)\Omega} = \frac{2}{3}A$$

So, the potential difference across the capacitor is

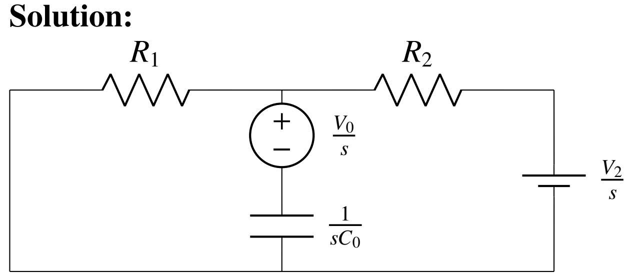
$$V_{C_0} = 2 - \left(\frac{2}{3}\times 2\right) = \frac{2}{3}V$$

$$q_2 = CV_{C_0}$$

$$q_2 = 1\mu C \times \frac{2}{3}V$$

$$q_2 = \frac{2}{3}\mu C$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.
Solution:



3. $V_{C_0}(s) = ?$
Solution:

Applying KCL at the upper junction and assuming the bottom is grounded, we have,

$$\frac{V_{C_0}(s)}{R_1} + \frac{V_{C_0}(s) - \frac{V_2}{s}}{R_2} + \frac{V_{C_0}(s) - \frac{V_0}{s}}{\frac{1}{sC_0}} = 0$$

$$V_{C_0}(s)\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_2}{sR_2} + V_0C_0$$

$$V_{C_0}(s) = \frac{\frac{V_2}{sR_2} + V_0C_0}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0}$$

$$V_{C_0}(s) = V_0\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) + \frac{V_2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right)$$

4. $v_{C_0}(t) = ?$ Plot using python.
Solution:
Applying inverse Laplace transform to $V_{C_0}(s)$, we have,

$$v_{C_0}(t) = V_0 e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}} u(t) + \frac{V_2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right) u(t)$$

Substituting the values $R_1 = 1, R_2 = 2, C_0 = 1\mu F, V_0 = \frac{4}{3}V, V_2 = 2V$

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-\left(1.5\times10^6\right)t}\right) u(t) \tag{3.1}$$

The following code yields $v_{C_0}(t)$.

```
wget https://github.com/Sigma1084/EE3900/
    blob/master/cktsig/code/Ex3_plotVt.py
```

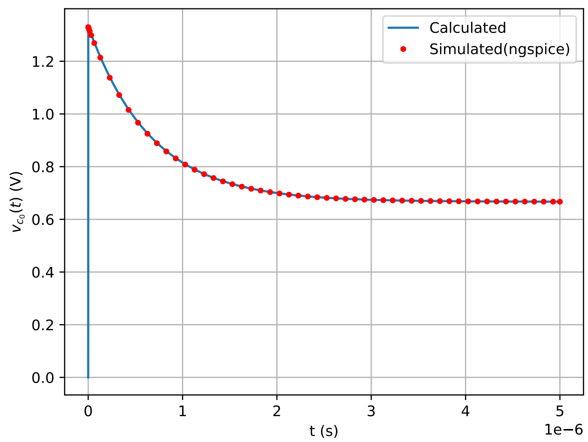


Fig. 3.1

5. Verify your result using ngspice. The following code yields $v_{C_0}(t)$. using ngspice

```
wget https://github.com/Sigma1084/EE3900/
    blob/master/cktsig/code/Ex3.spice
```

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution:
Using the initial conditions, we have,

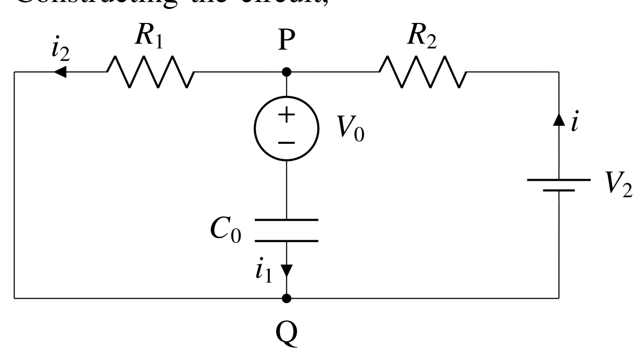
$$v_{C_0}(0-) = \frac{q_1}{C_0} = \frac{4}{3}V$$

Using (3.1), we have,

$$v_{C_0}(0+) = \frac{2}{3}\left(1 + e^{-\left(1.5\times10^6\right)0}\right) u(0^+) = \frac{4}{3}V$$

$$v_{C_0}(\infty) = \frac{2}{3}\left(1 + e^{-\left(1.5\times10^6\right)\infty}\right) u(\infty) = \frac{2}{3}V$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.
Solution:
Constructing the circuit,



Using KVL on both the left and outer loops, we have

$$-i_2(t)\ R_1 + V_0 + \int_0^t \frac{i_1(t)}{C_0} dt = 0$$

$$-i(t)\ R_2 + V_2 - i_2(t)\ R_1 = 0$$

Taking the Laplace Transform after multiplying u(t) in both equations,

$$\mathcal{L}\left(-R_1 i_2(t)u(t) + V_0 u(t) + u(t)\int_0^t \frac{i_1}{C_0} dt\right) = 0$$

$$\mathcal{L}\left(-R_2 i(t)u(t) + V_2 u(t) - R_1 i_2(t)u(t)\right) = 0$$

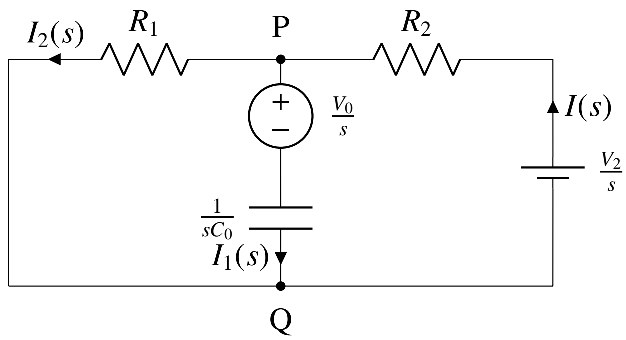
Suppose $i(t)u(t) \xrightarrow{\mathcal{L}} I(s)$ and $Vu(t) \xrightarrow{\mathcal{L}} \frac{V}{s}$

$$\implies \mathcal{L}\left(u(t)\int_0^t \frac{i_1}{C_0}dt\right) = \frac{1}{sC_0}\mathcal{L}\left(u(t)i_1\right) = \frac{I_1(s)}{sC_0}$$

(Using the properties of Laplace Transform)
The above equations after transform

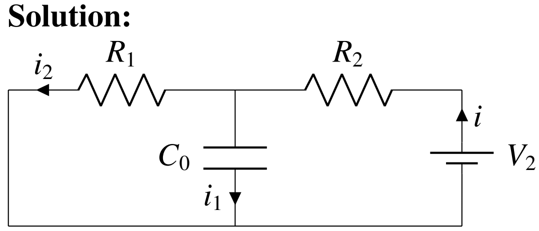
$$R_1 I_2(s) + \frac{V_0}{s} + \frac{I_1(s)}{sC_0} = 0$$
$$-R_2 I(s) + \frac{V_2}{s} - R_1 I_2(s) = 0$$

Now after applying the transformation, from time domain to s-domain, we replace the capacitor with an equivalent resistor of resistence $\frac{1}{sC_0}$ and replace I by $I(s)$, V by $\frac{V}{s}$, and hence we get,



4 Bilinear Transform

1. In Fig. 2.1, Consider the case when S is switched to Q right in the beginning. Formulate the differential equation.
Solution:



Special Case of (2.5), putting $V_1 = 0$, we get

$$i_2(t)\ R_1 - \int_0^t \frac{i_1(t)}{C_0}dt = 0 \tag{4.1}$$
$$V_2 - i(t)\ R_2 - i_2(t)\ R_1 = 0 \tag{4.2}$$
$$-i(t)\ R_2 + V_2 - \int_0^t \frac{i_1(t)}{C_0}dt = 0 \tag{4.3}$$

After transformation to s-domain, we get

$$R_1 I_2(s) - \frac{I_1(s)}{sC_0} = 0$$
$$-R_2 I(s) + V_2(s) - \frac{I_1(s)}{sC_0} = 0$$

Differentiating w.r.t. time, we get,

$$R_1\frac{di_2(t)}{dt} - \frac{i_1(t)}{C_0} = 0$$
$$R_2\frac{di(t)}{dt} - \frac{i_1(t)}{C_0} = 0$$

We need voltages in the final equation.

$$V_{C_0} = \frac{1}{C_0}q \implies \frac{dV_{C_0}}{dt} = \frac{1}{C_0}i_1$$
$$i(t) = i_1(t) + i_2(t) = C_0\frac{dV_{C_0}}{dt} + \frac{1}{R_1}V_{C_0}$$

And this implies, equation (4.3) corresponds to

$$-\left(C_0\frac{dV_{C_0}}{dt} + \frac{1}{R_1}V_{C_0}\right)R_2 + V_2 - V_{C_0} = 0$$
$$\frac{dV_{C_0}}{dt} + \frac{V_{C_0}}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V_2}{C_0 R_2}$$
$$\frac{dV_{C_0}}{dt} + \frac{V_{C_0}}{\frac{C_0 R_1 R_2}{R_1+R_2}} = \frac{V_2}{C_0 R_2}$$
$$\implies \frac{dV_{C_0}}{dt} + \frac{V_{C_0}}{\tau} = \frac{V_2}{C_0 R_2} \tag{4.4}$$

Here, $\tau = \frac{C_0 R_1 R_2}{R_1+R_2}$ is the time constant of the circuit.

2. Find $H(s)$ considering the output voltage at the capacitor.
Solution:
Recall that $H(s) = \frac{V_{C_0}(s)}{V_2(s)}$
Using KCL at point P, after s-domain trans-

form, we have,

$$\frac{V_{C_0}(s)}{R_1} + \frac{V_{C_0}(s)}{\frac{1}{sC_0}} + \frac{V_{C_0}(s) - V_2(s)}{R_2} = 0$$
$$V_{C_0}(s)\left(\frac{1}{R_1} + sC_0 + \frac{1}{R_2}\right) = \frac{V_2(s)}{R_2}$$
$$\frac{V_{C_0}(s)}{V_2(s)} = H(s) = \frac{1}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right)}$$
$$H(s) = \frac{1}{C_0R_2\left(\frac{1}{\tau} + s\right)} \tag{4.5}$$

3. Plot $H(s)$. What kind of filter is it?
   Solution: The following code yields the graph Putting $s = j\omega$ in the above equation, we can see that
   $|H(j\omega)$ is decreasing with increasing frequency. Clearly, H(s) is a low pass filter

4. Using trapezoidal rule for integration, formulate the difference equation by considering
   $$y(n) = y(t)|_{t=n} \tag{4.6}$$
   Solution:
   From equation (4.4), putting $V = V_{C_0}$ we have
   $$\frac{dV}{dt} + \frac{V}{\tau} = \frac{V_2}{C_0R_2}$$
   where $\tau = \frac{C_0R_1R_2}{R_1+R_2}$ is the time constant.

   Trapezoidal rule applied to $y' = f(t, y)$ gives
   $$y_{n+1} - y_n = \frac{1}{2}h\Big(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\Big)$$
   where $h = t_{n+1} - t_n$ is the step size.

   Using the trapezoidal rule on (4.4), we get
   $$\frac{v_{n+1} - v_n}{t_{n+1} - t_n} = \frac{1}{2}\left[\frac{V_2\Big(u(n) + u(n+1)\Big)}{C_0R_2} - \frac{v_n + v_{n+1}}{\tau}\right]$$
   Now, putting $t_n = n \ \forall \ n$, we get the difference equation
   $$v_{n+1} - v_n + \frac{v_n + v_{n+1}}{2\tau} = \frac{V_2}{2C_0R_2}\Big(u(n) + u(n+1)\Big)$$
   $$v_{n+1}\left(1 + \frac{1}{2\tau}\right) - v_n\left(1 - \frac{1}{2\tau}\right) = \frac{V_2}{2C_0R_2}\Big(u(n) + u(n+1)\Big) \tag{4.7}$$
   where $v_0 = 0$ is the initial condition.
   We get $u(n)$ since $V_2 = 0$ when $t < 0$.

5. Find $H(z)$.
   Solution:
   Taking $\mathcal{Z}$-Transform on (4.7), we get
   $$zV(z)\left(1 + \frac{1}{2\tau}\right) - V(z)\left(1 - \frac{1}{2\tau}\right) = \frac{V_2}{2C_0R_2}\frac{1 + z}{1 - z^{-1}}$$
   Now since V_2 is constant at $t > 0$, we have $\mathcal{Z}(V_2) = \mathcal{Z}(V_2u(n)) = \frac{V_2}{1-z^{-1}}$
   We can substitute V_2 in the above equation and since we know $H(z) = \frac{V(z)}{V_2(z)}$, we get
   $$H(z)\left(\frac{2\tau - 1}{2\tau}\right)\left[\frac{2\tau + 1}{2\tau - 1} - z^{-1}\right] = \frac{1 + z^{-1}}{2C_0R_2}$$
   $$H(z) = \frac{1}{C_0R_2}\left(\frac{\tau}{2\tau - 1}\right)\frac{1 + z^{-1}}{\frac{2\tau+1}{2\tau-1} - z^{-1}}$$
   $$H(z) = \left(\frac{\tau}{C_0R_2}\right)\frac{1 + z^{-1}}{(2\tau + 1) - (2\tau - 1)z^{-1}} \tag{4.8}$$
   $|z| > 1$ and $|z| > \frac{2\tau-1}{2\tau+1} \implies$ ROC is $|z| > 1$

6. How can you obtain $H(z)$ from $H(s)$?
   Solution:
   Using bilear transform on (4.5),
   setting $s := \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$, where T is the step size.
   $$H(s) = \frac{1}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}C_0\right)}$$
   $$H(s) = \frac{1}{R_2C_0\left(\frac{1}{\tau} + \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)} \quad \left(\text{Since } \tau = \frac{C_0R_1R_2}{R_1 + R_2}\right)$$
   $$H(s) = \left(\frac{\tau}{C_0R_2}\right)\frac{T(1 + z^{-1})}{T + Tz^{-1} + 2\tau - 2\tau z^{-1}}$$
   $$H(s) = \left(\frac{\tau}{C_0R_2}\right)\frac{T(1 + z^{-1})}{(2\tau + T) - (2\tau - T)z^{-1}}$$
   Putting step size $T = 1$, we get $H(z)$

Observe that

$$H(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = H(z)$$

7. Find $v(n)$. Verify using ngspice and the difference equation.
Solution:
We know that $V(z) = H(z)V_2(z)$ and $V_2(z) = \frac{V_2}{1-z^{-1}}$
Using (4.8), we get

$$V(z) = \frac{1}{C_0R_2\left(\frac{1}{\tau}+\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)}\frac{V_2}{1-z^{-1}}$$

We can use Residue Theorem to find the inverse $\mathcal{Z}$ transform

$$x(n) = \left[\sum_{\text{Pole } z_i}\text{Residue of } X(z)z^{n-1} \text{ at } z_i\right]$$

$$\therefore v(n) = \sum_{\text{Pole } z_i}\left[\frac{V_2}{C_0R_2}\frac{(z-z_i)z^{n-1}}{\left(\frac{1}{\tau}+\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)}\frac{1}{1-z^{-1}}\right]_{z=z_i}$$

The only poles are $z = 1$ and $z = \frac{2\tau-T}{2\tau+T}$ At $z = 1$, we get,

$$\frac{V_2}{C_0R_2}\frac{(z-1)z^{n-1}}{\left(\frac{1}{\tau}+\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)}\frac{1}{1-z^{-1}}\Bigg|_{z=1}$$

$$= \frac{V_2}{C_0R_2}\frac{z^n}{\left(\frac{1}{\tau}+\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)}\Bigg|_{z=1}$$

$$= \frac{V_2\tau}{C_0R_2}$$

At $z = \frac{2\tau-T}{2\tau+T}$, we get,

$$\frac{V_2}{C_0R_2}\frac{\left(z-\frac{2\tau-T}{2\tau+T}\right)z^{n-1}}{\left(\frac{1}{\tau}+\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)}\frac{1}{1-z^{-1}}\Bigg|_{z=\frac{2\tau-T}{2\tau+T}}$$

$$= \frac{V_2T\tau}{C_0R_2}\frac{\left(z-\frac{2\tau-T}{2\tau+T}\right)}{(2\tau+T-(2\tau-T)z^{-1})}\frac{z^{n-1}(1+z^{-1})}{1-z^{-1}}\Bigg|_{z=\frac{2\tau-T}{2\tau+T}}$$

$$= \frac{V_2}{C_0R_2}\cdot\frac{\tau T z^n}{2\tau+T}\cdot\frac{1+z^{-1}}{1-z^{-1}}\Bigg|_{z=\frac{2\tau-T}{2\tau+T}}$$

$$= \frac{V_2}{C_0R_2}\cdot\frac{\tau T z^n}{2\tau+T}\cdot\frac{\frac{2\tau-T}{2\tau+T}+1}{\frac{2\tau-T}{2\tau+T}-1}$$

$$= \frac{V_2}{C_0R_2}\cdot\frac{\tau T z^n}{2\tau+T}\cdot\frac{4\tau}{-2T}$$

$$= -\frac{V_2\tau}{C_0R_2}\cdot\frac{2\tau}{2\tau+T}\cdot z^n$$

$$= -\frac{V_2\tau}{C_0R_2}\cdot\frac{1}{1+\frac{T}{2\tau}}\cdot\left(\frac{2\tau-T}{2\tau+T}\right)^n$$

Hence, we get

$$v(n) = \frac{V_2\tau}{C_0R_2}\left[1-\frac{1}{1+\frac{T}{2\tau}}\cdot\left(\frac{2\tau-T}{2\tau+T}\right)^n\right]u(n)$$

ROC is $z > 1$
Taking the sampling frequency, T to be low and plugging in values, we get

$$y(n) \approx \frac{2}{3}\left(1-\frac{1-7.5\times10^5 n}{1+7.5\times10^5 n}\right)u(n)$$

Now, considering the Laplace Transform

$$V(s) = H(s)\cdot V_2(s) = \frac{5\times10^5}{s+1.5\times10^6}\frac{2}{s}$$

Taking partial fractions, we get

$$V(s) = \frac{10^6}{1.5\times10^6}\left(\frac{1}{s}-\frac{1}{s+1.5\times10^6}\right)$$

Taking the inverse Laplace Transform, we get

$$y(t) = \frac{2}{3}\left(1-e^{-1.5\times10^6 t}\right)u(t)$$

with ROC $s > 0$

Now, approximating,

$$
\begin{aligned}
e^{-1.5\times 10^6 t} &= \frac{e^{-0.75\times 10^6 t}}{e^{0.75\times 10^6 t}} \\
&\approx \frac{1 - 7.5\times 10^5 t}{1 + 7.5\times 10^5 t} \quad \text{when } t \ll 10^{-6}
\end{aligned}
$$

$$
\implies y(t) \approx \frac{2}{3}\left(1 - \frac{1 - 7.5\times 10^5 t}{1 + 7.5\times 10^5 t}\right)u(t)
$$

We just proved that $y(n) = y(t)|_{t=n}$ And hence, verified the correctness of Bilinear Transform. The following code yields the graph

```
wget https://github.com/Sigma1084/EE3900/
    blob/master/cktsig/code/Ex4_plot.py
```

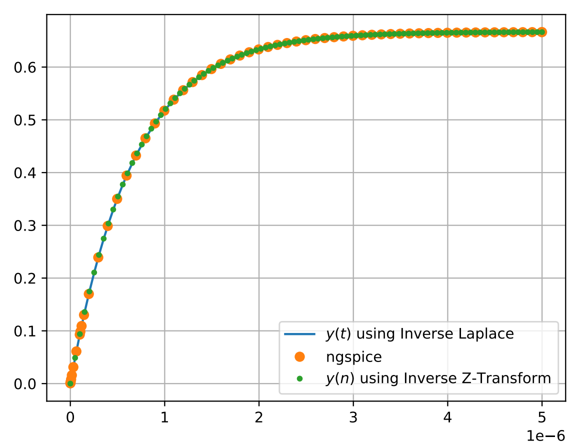


Fig. 4.1

8. Find $y(n)$ using the difference equation and verify
Solution:
Using the equation (4.7), we get

$$
v_{n+1}\left(1 + \frac{1}{2\tau}\right) - v_n\left(1 - \frac{1}{2\tau}\right) = \frac{V_2}{2C_0R_2}\Big(u(n) + u(n+1)\Big)
$$

And hence, we have

$$
v_{n+1} = v_n\left(\frac{2\tau - 1}{2\tau + 1}\right) + \left(\frac{2\tau}{2\tau + 1}\right)\frac{V_2}{2C_0R_2}\Big(u(n)+u(n+1)\Big)
$$

Since $v_0 = 0$ and $\forall n \geq 0$, $u(n) + u(n+1) = 2$, we can solve the linear difference equation.

For all $n \geq 0$

$$
v_{n+1} = -\frac{7.5\times 10^5 - 1}{7.5\times 10^5 + 1}v_n + \frac{10^6}{7.5\times 10^5 + 1}
$$

Write $v_{n+1} = av_n + b$

$$
\begin{aligned}
v_1 &= av_0 + b = b \\
v_2 &= av_1 + b = ab + b = b(a+1) \\
v_3 &= av_2 + b = a^2b + ab + b = b(a^2 + 2a + 1) \\
&\vdots \\
v_n &= a^n v_0 + b\sum_{k=0}^{n-1} a^k = b\left(\frac{a^n - 1}{a - 1}\right)
\end{aligned}
$$

$$
\implies v_n = \frac{10^6}{2\times 7.5\times 10^5} \times \left(1 - \left(-\frac{7.5\times 10^5 - 1}{7.5\times 10^5 + 1}\right)^n\right)
$$

$$
\implies v_n = \frac{2}{3}\left(1 - \left(\frac{1 - 7.5\times 10^5}{1 + 7.5\times 10^5}\right)^n\right)u(n)
$$

Notice that this is the same equation we got from earlier