

# Fourier Series

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**Abstract**—This manual provides a simple introduction to Fourier Series

## 1 PERIODIC FUNCTION

Let

$$x(t) = A_0 |\sin(2\pi f_0 t)| \quad (1.1)$$

1.1 Plot  $x(t)$ .

**Solution:**

The following code yields the graph

```
wget https://raw.githubusercontent.com/
Sigma1084/EE3900/master/charger/codes/
Ex1_1_plotxt.py
```

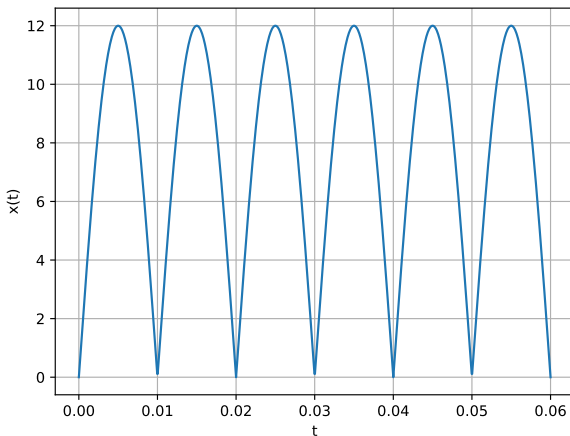


Fig. 1.1: Plot of  $x(t)$

1.2 Show that  $x(t)$  is periodic and find its period.

**Solution:**

We know that  $|\sin(x)|$  is periodic with fundamental period of  $\pi$ .

$\Rightarrow$  Fundamental period of  $A|\sin(ax)$  is  $\frac{\pi}{a}$   
 Fundamental period of  $A_0|\sin(2\pi f_0 t)|$  is  $\frac{\pi}{2\pi f_0}$

$\Rightarrow$  Fundamental period of  $x(t)$  is  $\frac{1}{f_0}$

## 2 FOURIER SERIES

Consider  $A_0 = 12$  and  $f_0 = 50$  for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.1)$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t} dt \quad (2.2)$$

**Solution:**

Consider for some  $n \in \mathbb{Z}$ ,

$$x(t) e^{j2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi (k-n) f_0 t}$$

We know using the periodicity of  $e^{j2\pi k f_0 t}$ ,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi k f_0 t} dt = \begin{cases} \frac{1}{f_0} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Now,

$$\begin{aligned} \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{j2\pi n f_0 t} dt &= \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_{k=-\infty}^{\infty} c_k e^{j2\pi (k-n) f_0 t} dt \\ &= \frac{c_n}{f_0} \end{aligned}$$

$$\Rightarrow c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{j2\pi n f_0 t} dt$$

## 2.2 Find $c_k$ for (1.1)

**Solution:**

$$\begin{aligned}
 c_k &= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{j2\pi k f_0 t} dt \\
 &= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 |\sin(2\pi f_0 t)| e^{j2\pi k f_0 t} dt \\
 &= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 |\sin(2\pi f_0 t)| \cos(2\pi k f_0 t) dt \\
 &\quad + j f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 |\sin(2\pi f_0 t)| \sin(2\pi k f_0 t) dt \\
 &= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 |\sin(2\pi f_0 t)| \cos(2\pi k f_0 t) dt + 0 \\
 &= 2f_0 \int_0^{\frac{1}{2f_0}} A_0 \sin(2\pi f_0 t) \cos(2\pi k f_0 t) dt \\
 &= f_0 A_0 \int_0^{\frac{1}{2f_0}} \sin(2\pi(n+1)f_0 t) dt \\
 &\quad - f_0 A_0 \int_0^{\frac{1}{2f_0}} \sin(2\pi(n-1)f_0 t) dt \\
 &= f_0 A_0 \frac{1 + (-1)^n}{2\pi f_0} \left( \frac{1}{n+1} - \frac{1}{n-1} \right) \\
 &= \begin{cases} \frac{2A_0}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \quad (2.3)
 \end{aligned}$$

## 2.3 Verify (1.1) and (2.1) using python.

**Solution:** The following code yields the graph

```
wget https://raw.githubusercontent.com/
Sigma1084/EE3900/master/charger/codes/
Ex2_3_verify_xt.py
```

## 2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} (a_k \cos j2\pi k f_0 t + b_k \sin j2\pi k f_0 t) \quad (2.4)$$

and obtain the formulae for  $a_k$  and  $b_k$ .

**Solution:**

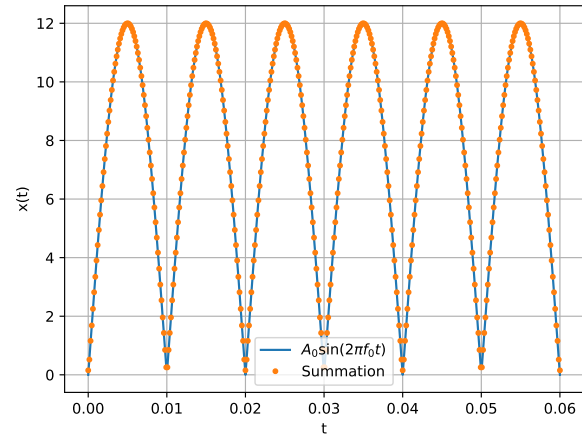


Fig. 2.3: Verification of (2.1).

**Solution:** From (2.1),

$$\begin{aligned}
 x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \\
 &= c_0 + \sum_{k=1}^{\infty} (c_k e^{j2\pi k f_0 t} + c_{-k} e^{-j2\pi k f_0 t}) \\
 &= c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k}) \cos(2\pi k f_0 t) \\
 &\quad + j \sum_{k=0}^{\infty} (c_k - c_{-k}) \sin(2\pi k f_0 t)
 \end{aligned}$$

Hence, for  $k \geq 0$ ,

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \quad (2.5)$$

$$b_k = j(c_k - c_{-k}) \quad (2.6)$$

## 2.5 Find $a_k$ and $b_k$ for (1.1)

**Solution:**

Using the expression for  $c_k$ , from (2.3) and using (2.5) and (2.6), we have,

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi(1-k^2)} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

$$b_k = 0$$

## 2.6 Verify (2.4) using python

### Solution:

The following code block yields the graph

```
wget https://raw.githubusercontent.com/
Sigma1084/EE3900/master/charger/codes/
Ex2_3_verify_xt_real.py
```

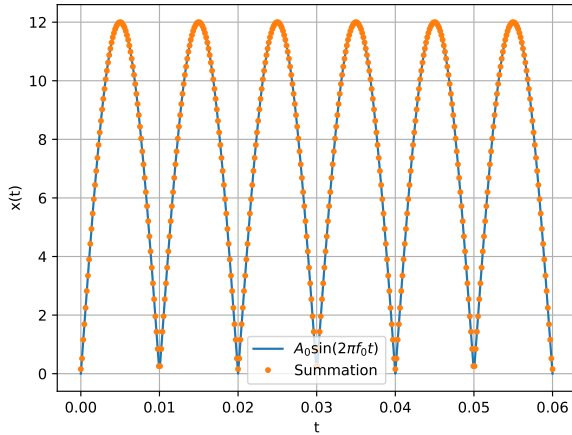


Fig. 2.6: Verification of (2.4).

## 3 FOURIER TRANSFORM

### 3.1

$$\delta(t) = 0, \quad t \neq 0 \quad (3.1)$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (3.2)$$

### 3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt \quad (3.3)$$

### 3.3 Show that

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} G(f) e^{-j2\pi f t_0} \quad (3.4)$$

### Solution:

$$\begin{aligned} \mathcal{F}(g(t - t_0)) &= \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} g(t) e^{-j2\pi f(t+t_0)} dt \\ &= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt \\ &= G(f) e^{-j2\pi f t_0} \end{aligned}$$

$$\Rightarrow g(t - t_0) \xleftrightarrow{\mathcal{F}} G(f) e^{-j2\pi f t_0}$$

Hence proved

### 3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \quad (3.5)$$

### Solution:

Using the definition of inverse Fourier Transform,

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df$$

Now, putting  $-f := t$ ,  $t := f \Rightarrow df = dt$ ,

$$\begin{aligned} g(-f) &= \int_{-\infty}^{\infty} G(t) e^{-j2\pi f t} dt \\ \Rightarrow G(t) &\xleftrightarrow{\mathcal{F}} g(-f) \end{aligned} \quad (3.6)$$

### 3.5 $\delta(t) \xleftrightarrow{\mathcal{F}} ?$

### Solution:

$$\begin{aligned} \delta(t) &\xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt \\ &= e^{-j2\pi f(0)} \int_{-\infty}^{\infty} \delta(t) dt \\ \delta(t) &\xleftrightarrow{\mathcal{F}} 1 \end{aligned} \quad (3.7)$$

### 3.6 $e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} ?$

**Solution:** Suppose  $g(t) \xleftrightarrow{\mathcal{F}} G(f)$

$$\begin{aligned} g(t) e^{-j2\pi f_0 t} &\xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(t) e^{-j2\pi(f_0+f)t} dt \\ &= G(f + f_0) \end{aligned}$$

Now, using (3.7) and (3.6), we can get,

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1 \xleftrightarrow{\mathcal{F}} \delta(-t)$$

Putting  $g(t) = 1$  and hence,  $G(f) = \delta(-f) = \delta(f)$ , we get  $G(f + f_0) = \delta(f + f_0)$ . Hence,

$$e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \delta(f + f_0) \quad (3.8)$$

### 3.7 $\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} ?$

**Solution:** We can write

$$\cos(2\pi f_0 t) = \frac{1}{2} (e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})$$

Using (3.8) and the linearity of Fourier Transform, we get,

$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2} (\delta(f - f_0) + \delta(f + f_0))$$

3.8 Find the Fourier Transform of  $x(t)$  and plot it. Verify using python.

**Solution:** Using (2.1), we have,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$$

Using (3.8), we get,

$$\begin{aligned} x(t) &\xleftrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} c_k \delta(f - k f_0) \\ &= \frac{2A_0}{\pi} \sum_{k=-\infty}^{\infty} \frac{\delta(f - 2k f_0)}{1 - 4k^2} \end{aligned}$$

wget [https://raw.githubusercontent.com/Sigma1084/EE3900/master/charger/codes/Ex3\\_8\\_x-fourier.py](https://raw.githubusercontent.com/Sigma1084/EE3900/master/charger/codes/Ex3_8_x-fourier.py)

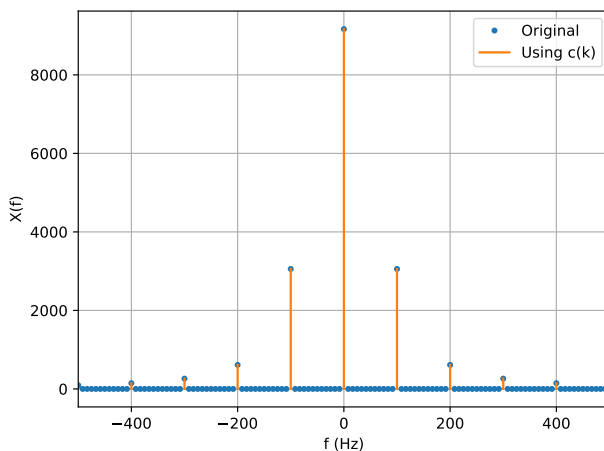


Fig. 3.8: Fourier Transform of  $x(t)$

3.9 Show that

$$\text{rect}(t) \xleftrightarrow{\mathcal{F}} \text{sinc}(t) \quad (3.9)$$

Verify using python.

**Solution:**

$$\begin{aligned} \text{rect}(t) &\xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \text{rect}(t) e^{-j2\pi f t} dt \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi f t} dt \\ &= \frac{1}{-j2\pi f} (e^{-j\pi f} - e^{j\pi f}) \\ &= \frac{\sin(\pi f)}{\pi f} = \text{sinc}(f) \end{aligned}$$

The following code yields the plot

3.10  $\text{sinc}(t) \xleftrightarrow{\mathcal{F}} ?$  Verify using python.

**Solution:**

Using (3.9), and (3.6), we get,

$$\text{sinc}(t) \xleftrightarrow{\mathcal{F}} \text{rect}(-f) = \text{rect}(f) \quad (3.10)$$

## 4 FILTER

4.1 Find  $H(f)$  which transforms  $x(t)$  to DC 5V.

**Solution:**

We need  $H(f)$  to be a low pass filter that filters out even harmonics and leaves the zero frequency behind

$\text{rect}\left(\frac{f}{2f_c}\right)$  is an ideal low pass filter with cutoff frequency  $f_c$ . Multiplying with a scalar factor  $\frac{5}{2A_0}$  gives us the required filter.

4.2 Find  $h(t)$ .

4.3 Verify your result using through convolution.

## 5 FILTER DESIGN

5.1 Design a Butterworth filter for  $H(f)$ .

5.2 Design a Chebyshev filter for  $H(f)$ .

5.3 Design a circuit for your Butterworth filter.

5.4 Design a circuit for your Chebyshev filter.