Pingala Series

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1
$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

1.2
$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

1.3
$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

1.4
$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution: The following code verifies all the equations

```
wget https://raw.githubusercontent.com/Sigma1084
    /EE3900/master/pingala/code/Ex1_verify.py
```

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$\begin{aligned} x(n+2) &= x(n+1) + x(n) \\ x(0) &= x(1) = 1, n \geq 0 \end{aligned} \tag{2.2}$$

Generate a stem plot for $x(n)$.
Solution: The following code generates the plot

```
wget https://raw.githubusercontent.com/
    Sigma1084/EE3900/master/pingala/code/
    Ex2_pingala.py
```

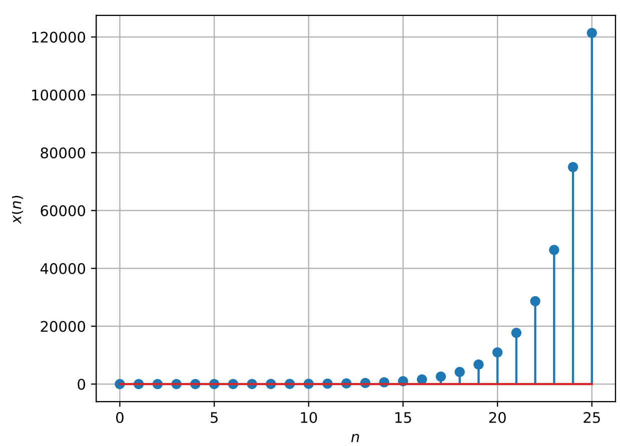

Fig. 2.2: Plot of $x(n)$

2.3 Find $X^+(z)$.
Solution: Taking the *one-sided* Z-transform on both sides of (2.2)

$$\mathcal{Z}^+\left[x(n+2)\right] = \mathcal{Z}^+\left[x(n+1)\right] + \mathcal{Z}^+\left[x(n)\right]$$

$$\implies z^2X^+(z) - z^2 - z = zX^+(z) - z + X^+(z)$$
$$\implies \left(z^2 - z - 1\right)X^+(z) = z^2$$
$$\implies X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}}$$

$$X^+(z) = \frac{1}{(1-\alpha z^{-1})(1-\beta z^{-1})}, \quad |z| > |\alpha| \tag{2.3}$$

Here α and β are the roots of the characteristic equation of (2.2) and $|z| > |\alpha|$ is the region of convergence of $X^+(z)$.
(w.l.o.g, $|\alpha| > |\beta|$ is assumed)

2.4 Find $x(n)$.
Solution: Expanding $X^+(z)$ in (2.3) using partial fractions, we get

$$X^+(z) = \frac{1}{(\alpha-\beta)z^{-1}}\left[\frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\beta z^{-1}}\right]$$
$$= \frac{1}{(\alpha-\beta)}\sum_{n=0}^{\infty}(\alpha^n - \beta^n)z^{-n}$$
$$= \sum_{n=1}^{\infty}\frac{\alpha^n - \beta^n}{\alpha-\beta}z^{-n+1}$$
$$= \sum_{n=0}^{\infty}\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha-\beta}z^{-n}$$

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}u(n) = a_{n+1}u(n) \tag{2.4}$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \tag{2.5}$$

Solution: The following code generates the plot

```
wget https://raw.githubusercontent.com/
    Sigma1084/EE3900/master/pingala/code/
    Ex2_pingala.py
```

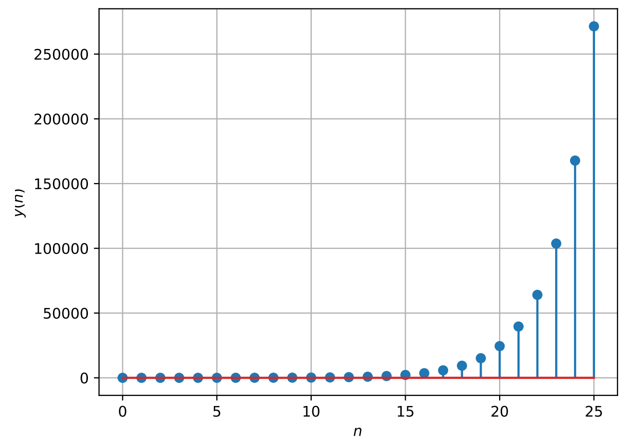

Fig. 2.5: Plot of $y(n)$

2.6 Find $Y^+(z)$.
Solution: Taking the one-sided Z-transform on both sides of (2.5),

$$\mathcal{Z}^+\left[y(n)\right] = \mathcal{Z}^+\left[x(n+1)\right] + \mathcal{Z}^+\left[x(n-1)\right]$$

$$Y^+(z) = zX^+(z) - zx(0) + z^{-1}X^+(z) + zx(-1)$$
$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z$$
$$Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > |\alpha| \tag{2.6}$$

($x(-1) = 0$ since $x(n) = 0 \ \forall \ n < 0$)

2.7 Find $y(n)$.
Solution: Using (2.6),

$$Y^+(z) = (1 + 2z^{-1})\ X^+(z)$$
$$= (1 + 2z^{-1})\sum_{n=0}^{\infty}x(n)z^{-n}$$
$$= \sum_{n=0}^{\infty}x(n)z^{-n} + \sum_{n=1}^{\infty}2x(n-1)z^{-n}$$
$$= x(0) + \sum_{n=1}^{\infty}(x(n) + 2x(n-1))z^{-n}$$
$$= x(0) + \sum_{n=1}^{\infty}(x(n+1) + x(n-1))z^{-n}$$

$$\implies \quad y(0) = x(0) = 1$$
$$\forall n > 0 \quad y(n) = x(n+1) + x(n-1)$$
$$= a_{n+2} + a_n = b_{n+1}$$

α and β are the roots of the characteristic equation of (2.2), $z^2 - z - 1 = 0$
(w.l.o.g, $|\alpha| > |\beta|$ is assumed)
$\implies \alpha\beta = -1$ and $\alpha + \beta = 1$

$$y(n) = \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) + (\alpha^n + \beta^n)}{\alpha - \beta}$$
$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) - \alpha\beta(\alpha^n + \beta^n)}{\alpha - \beta}$$
$$= \frac{(\alpha - \beta)\left(\alpha^{n+1} + \beta^{n+1}\right)}{\alpha - \beta}$$
$$y(n) = \alpha^{n+1} + \beta^{n+1} \tag{2.7}$$

Thus, $y(n) = \alpha^{n+1} + \beta^{n+1}$ for $n \geq 0$ as $\alpha + \beta = 1$. Comparing (2.7) with the definition of b_n, we see that $y(n) = b_{n+1}$.

$$\implies b_{n+1} = y(n) = \alpha^{n+1} + \beta^{n+1} \quad \forall n > 0 \tag{2.8}$$

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \tag{3.1}$$

Solution: Using (2.4), and the definition of convolution,

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1} = \sum_{k=-\infty}^{n-1} a_{k+1} \; u(k)$$
$$= \sum_{k=-\infty}^{n-1} x(k)$$
$$= \sum_{k=-\infty}^{\infty} x(k) \; u(n-1-k)$$
$$= u(n-1) * x(n)$$
$$\implies \sum_{k=1}^{n} a_k = x(n) * u(n-1)$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.2}$$

can be expressed as

$$[x(n+1) - 1]\, u(n) \tag{3.3}$$

Solution: Using (2.4), and the definition of $u(n)$

$$\begin{aligned} & a_{n+2} - 1 && n \geq 1 \\ &= [x(n+1) - 1] && n \geq 0 \\ &= [x(n+1) - 1]\, u(n) && \end{aligned}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \tag{3.4}$$

Solution: Using the definitions of $x(n)$ (2.4) and $X^+(z)$ (2.3),

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k}$$
$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k}$$
$$= \frac{1}{10} X^+(10)$$
$$= \frac{1}{10} \times \frac{1}{1 - \frac{1}{10} - \frac{1}{100}}$$
$$\implies \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{3.5}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.6}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.7}$$

and find $W(z)$.
Solution: Putting $n = n + 1$ in (3.6), we get

$$\begin{aligned} & \alpha^n + \beta^n && n \geq 1 \\ &= \alpha^{n+1} + \beta^{n+1} && n \geq 0 \end{aligned}$$
$$w(n) := \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) = y(n)$$

Since we know $w(n) = y(n)$, we can use (2.6) to find $W(z)$

$$W(z) = \sum_{n=-\infty}^{\infty} w(n) = \sum_{n=0}^{\infty} y(n)$$
$$= Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > |\alpha|$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \tag{3.8}$$

Solution: Using the definitions of $y(n)$ (2.7) and $Y^+(z)$ (2.6),

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{b_k}{10^k} &= \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \\ &= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \\ &= \frac{1}{10} Y^+(10) \\ &= \frac{1}{10} \times \frac{1 + 2\frac{1}{10}}{1 - \frac{1}{10} - \frac{1}{100}} \end{aligned}$$

$$\implies \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{12}{89} \tag{3.9}$$

3.6 Solve the JEE 2019 problem.
Solution: We know that

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \tag{3.10}$$

But

$$\begin{aligned} & x(n) * u(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} X(z) z^{-1} U(z) \\ & = \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \\ & = z \left[\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right] \\ & \overset{\mathcal{Z}}{\rightleftharpoons} z \sum_{n=0}^{\infty} (x(n) - 1) z^{-n} \\ & = \sum_{n=0}^{\infty} (x(n) - 1) z^{-n+1} \\ & = \sum_{n=0}^{\infty} (x(n+1) - 1) z^{-n} \end{aligned}$$

From (2.4), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \tag{3.11}$$

Using (3.11), (3.5), (2.8) and (3.9), we can conclude that Options 1, 2, 3 are right and Option 4 is wrong.