Circuits and Transforms

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CONTENTS

***Abstract*—This manual provides a simple introduction to Transforms**

1 DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t) e^{-st}\, dt \tag{1.2}$$

2 LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

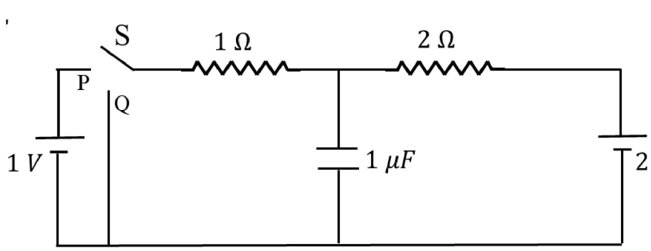
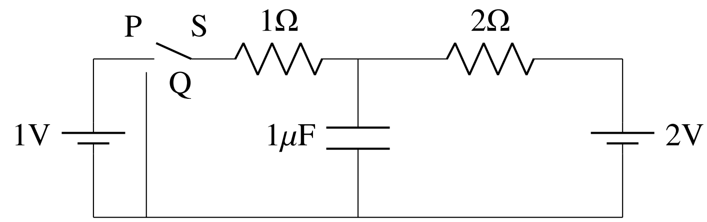
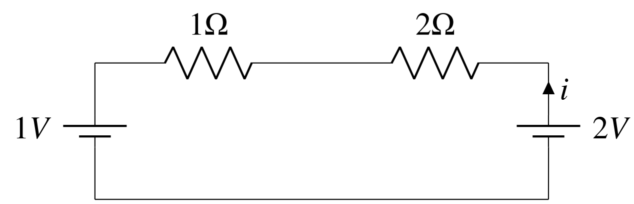


Fig. 2.1

2. Draw the circuit using latex-tikz.
   Solution:



3. Find q_1.
   Solution:
   Current through the capacitor is 0 at steady state. So, current only flows in the outer loop.



$$i = \frac{(2-1)\ V}{(1+2)\ \Omega}$$
$$i = \frac{1}{3}A$$

Potential difference across the capacitor,

$$V_C = 2 - 2 \times \frac{1}{3} = \frac{4}{3}V$$
$$q_1 = CV_C$$
$$q_1 = 1\mu F \times \frac{4}{3}V$$
$$q_1 = \frac{4}{3}\mu C$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
   Solution:
   By the definition of Laplace transform

$$u(t) \xrightarrow{\mathcal{L}} \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt$$
$$= \left.\frac{e^{-st}}{-s}\right|_0^{\infty} = \frac{1}{s}$$

$\lim_{s\to\infty} e^{-st} = 0$ only when $Re(s) > 0$.
The ROC is $Re(s) > 0$.

5. Show that

$$e^{-at}u(t) \overset{\mathcal{H}}{\longleftrightarrow} L\frac{1}{s+a}, \quad a > 0 \tag{2.1}$$

and find the ROC.
Solution:
By the definition of Laplace transform

$$\begin{aligned} e^{-at}u(t) &\overset{\mathcal{L}}{\longrightarrow} \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \\ &= \int_{0}^{\infty} e^{-(s+a)t}dt \\ &= \left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_0^\infty = \frac{1}{s+a} \end{aligned}$$

$\lim_{s\to\infty} e^{-(s+a)t} = 0$ only when $Re(s+a) > 0$. The ROC is $Re(s+a) > 0$ or $Re(s) > -a$.

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

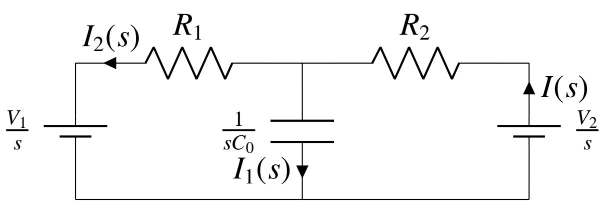


Fig. 2.2

$$u(t) \overset{\mathcal{H}}{\longleftrightarrow} LV_1(s) \tag{2.2}$$

$$2u(t) \overset{\mathcal{H}}{\longleftrightarrow} LV_2(s) \tag{2.3}$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution:
Using the Transform from time domain to s-domain

$$V_1(s) = \mathcal{L}(u(t)) = \frac{V_1}{s}$$

$$V_2(s) = \mathcal{L}(2u(t)) = \frac{V_2}{s}$$

Potential across the capacitor is $V_{C_0}(s)$ and the assuming bottom is grounded.

Applying KCL at the upper junction, we have,

$$\frac{V_{C_0}(s) - V_1(s)}{R_1} + \frac{V_{C_0}(s) - V_2(s)}{R_2} + \frac{V_{C_0}(s)}{\frac{1}{sC_0}} = 0$$

$$V_{C_0}(s)\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_1(s)}{R_1} + \frac{V_2(s)}{R_2}$$

$$V_{C_0}(s) = \frac{V_1(s)R_2 + V_2(s)R_1}{R_1 + R_2 + sC_0R_1R_2}$$

$$V_{C_0}(s) = \frac{R_2 + 2R_1}{s(R_1 + R_2 + sC_0R_1R_2)}$$

after substitution.

7. Find $v_{C_0}(t)$. Plot using python.
Solution:
Factoring $V_{C_0}(s)$, we have

$$\begin{aligned} V_{C_0}(s) &= \left(\frac{1}{R_1+R_2}\right)\left(\frac{(R_2+2R_1)}{s} - \frac{R_1R_2C_0\,(R_2+2R_1)}{R_1+R_2+sC_0R_1R_2}\right) \\ &= \left(\frac{R_2+2R_1}{R_1+R_2}\right)\left(\frac{1}{s} - \frac{R_1R_2C_0}{R_1+R_2+sC_0R_1R_2}\right) \\ &= \left(\frac{R_2+2R_1}{R_1+R_2}\right)\left(\frac{1}{s} - \frac{1}{\frac{1}{R_2C_0}+\frac{1}{R_1C_0}+s}\right) \end{aligned}$$

Applying inverse Laplace transform,

$$v_{C_0}(t) = \frac{R_2+2R_1}{R_1+R_2}\left(u(t) - e^{-\left(\frac{1}{R_1C_0}+\frac{1}{R_2C_0}\right)t}u(t)\right) \tag{2.4}$$

Using the values $R_1 = 1\,\Omega$, $R_2 = 2\,\Omega$, $C_0 = 1\,\mu F$, we get

$$v_{C_0}(t) = \frac{4}{3}u(t)\left(1 - e^{-1.5\times10^6\,t}\right)$$

The following code yields $v_{C_0}(t)$.

```
wget https://github.com/Sigma1084/EE3900/
    blob/master/cktsig/code/Ex2_plotVt.py
```

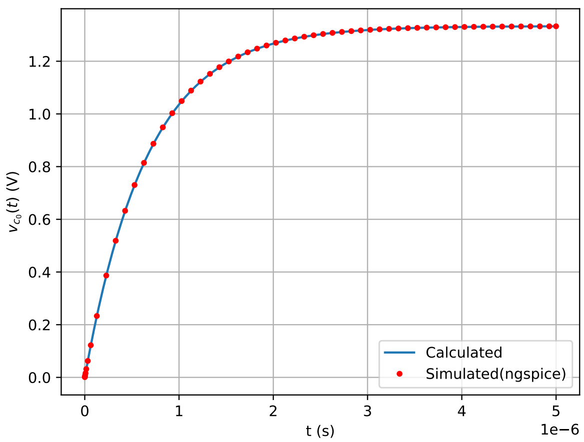
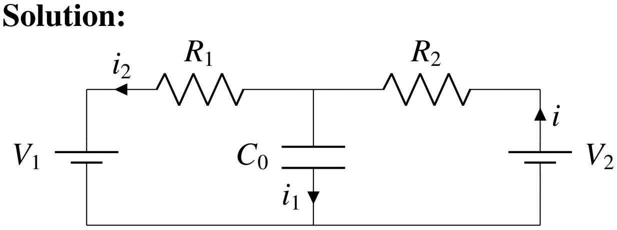


Fig. 2.3

8. Verify your result using ngspice. The following code yields $v_{C_0}(t)$. using ngspice

```
wget https://github.com/Sigma1084/EE3900/
    blob/master/cktsig/code/Ex2.spice
```

9. Obtain Fig. 2.2 using the equivalent differential equation.
Solution:



Using KVL on the left and right loops, we get

$$i_2(t)\ R_1 + V_1 - \int_0^t \frac{i_1(t)}{C_0} dt = 0 \tag{2.5}$$

$$-i(t)\ R_2 + V_2 - \int_0^t \frac{i_1(t)}{C_0} dt = 0 \tag{2.6}$$

Taking the Laplace Transform after multiplying u(t) in both equations,

$$\mathcal{L}\left(R_1 i_2(t)u(t) + V_1 u(t) - u(t)\int_0^t \frac{i_1}{C_0} dt\right) = 0$$

$$\mathcal{L}\left(-R_2 i(t)u(t) + V_2 u(t) - u(t)\int_0^t \frac{i_1}{C_0} dt\right) = 0$$

Suppose $i(t)u(t) \xrightarrow{\mathcal{L}} I(s)$ and $Vu(t) \xrightarrow{\mathcal{L}} \frac{V}{s}$

$$\implies \mathcal{L}\left(u(t)\int_0^t \frac{i_1}{C_0} dt\right) = \frac{1}{sC_0}\mathcal{L}(u(t)i_1) = \frac{I_1(s)}{sC_0}$$

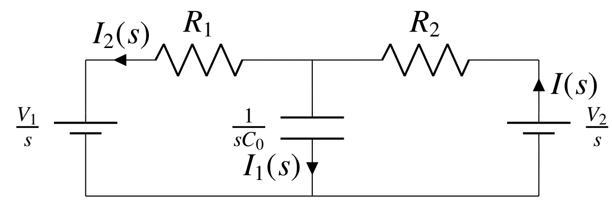
(Using the properties of Laplace Transform)

The above equations after transform

$$R_1 I_2(s) + \frac{V_1}{s} - \frac{I_1(s)}{sC_0} = 0 \tag{2.7}$$

$$-R_2 I(s) + \frac{V_2}{s} - \frac{I_1(s)}{sC_0} = 0 \tag{2.8}$$

Now after applying the transformation, from time domain to s-domain, we replace the capacitor with an equivalent resistor of resistence $\frac{1}{sC_0}$ and replace I by $I(s)$, V by $\frac{V}{s}$
The equivalent s-domain circuit is now,



3 Initial Conditions

1. Find q_2 in Fig. 2.1
Solution:
Current through the capacitor is 0 in steady state. Since the current flows only in the outer loop, let the current in the outer loop be I.

$$I = \frac{2V}{(1+2)\Omega} = \frac{2}{3}A$$

So, the potential difference across the capacitor is

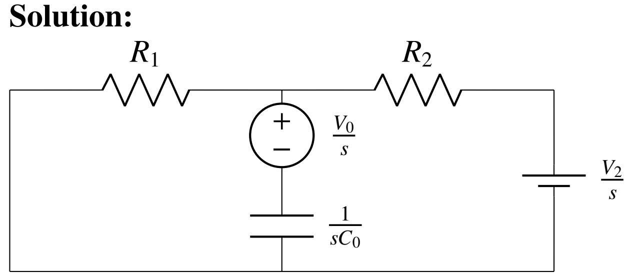
$$V_{C_0} = 2 - \left(\frac{2}{3} \times 2\right) = \frac{2}{3}V$$

$$q_2 = CV_{C_0}$$

$$q_2 = 1\mu C \times \frac{2}{3}V$$

$$q_2 = \frac{2}{3}\mu C$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.
Solution:



3. $V_{C_0}(s) = ?$
Solution:

Applying KCL at the upper junction and assuming the bottom is grounded, we have,

$$\frac{V_{C_0}(s)}{R_1} + \frac{V_{C_0}(s) - \frac{V_2}{s}}{R_2} + \frac{V_{C_0}(s) - \frac{V_0}{s}}{\frac{1}{sC_0}} = 0$$

$$V_{C_0}(s)\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_2}{sR_2} + V_0C_0$$

$$V_{C_0}(s) = \frac{\frac{V_2}{sR_2} + V_0C_0}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0}$$

$$V_{C_0}(s) = V_0\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right)$$

$$+ \frac{V_2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right)$$

4. $v_{C_0}(t) = ?$ Plot using python.
Solution:
Applying inverse Laplace transform to $V_{C_0}(s)$, we have,

$$v_{C_0}(t) = V_0 e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}} u(t)$$

$$+ \frac{V_2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right) u(t)$$

Substituting the values $R_1 = 1, R_2 = 2, C_0 = 1\mu F, V_0 = \frac{4}{3}V, V_2 = 2V$

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-\left(1.5\times 10^6\right)t}\right) u(t) \tag{3.1}$$

The following code yields $v_{C_0}(t)$.

```
wget https://github.com/Sigma1084/EE3900/
    blob/master/cktsig/code/Ex3_plotVt.py
```

5. Verify your result using ngspice. The following code yields $v_{C_0}(t)$. using ngspice

```
wget https://github.com/Sigma1084/EE3900/
    blob/master/cktsig/code/Ex3.spice
```

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution:
Using the initial conditions, we have,

$$v_{C_0}(0-) = \frac{q_1}{C_0} = \frac{4}{3}V$$

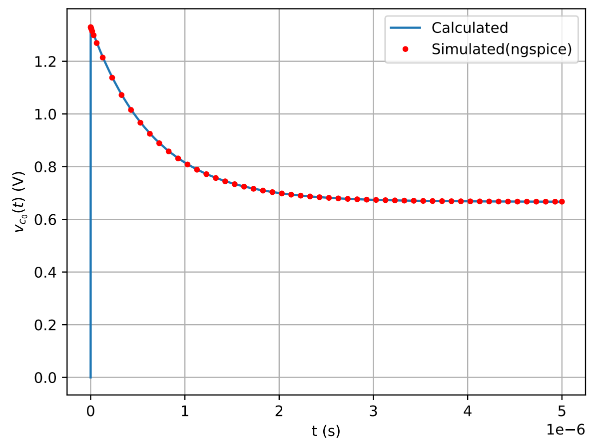


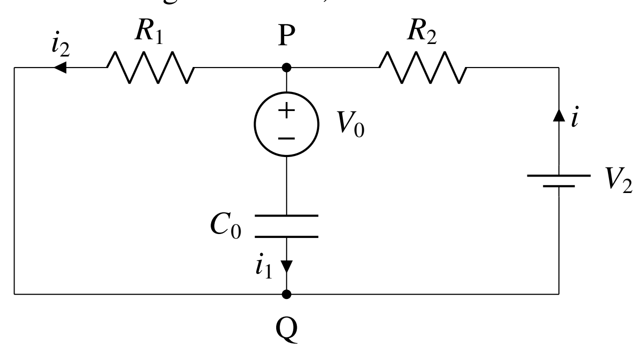
Fig. 3.1

Using (3.1), we have,

$$v_{C_0}(0+) = \frac{2}{3}\left(1 + e^{-\left(1.5\times 10^6\right)0}\right) u(0^+) = \frac{4}{3}V$$

$$v_{C_0}(\infty) = \frac{2}{3}\left(1 + e^{-\left(1.5\times 10^6\right)\infty}\right) u(\infty) = \frac{2}{3}V$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.
Solution:
Constructing the circuit,



Using KVL on both the left and outer loops, we have

$$-i_2(t)\ R_1 + V_0 + \int_0^t \frac{i_1(t)}{C_0} dt = 0$$

$$-i(t)\ R_2 + V_2 - i_2(t)\ R_1 = 0$$

Taking the Laplace Transform after multiplying u(t) in both equations,

$$\mathcal{L}\left(-R_1 i_2(t)u(t) + V_0 u(t) + u(t)\int_0^t \frac{i_1}{C_0} dt\right) = 0$$

$$\mathcal{L}\left(-R_2 i(t)u(t) + V_2 u(t) - R_1 i_2(t)u(t)\right) = 0$$

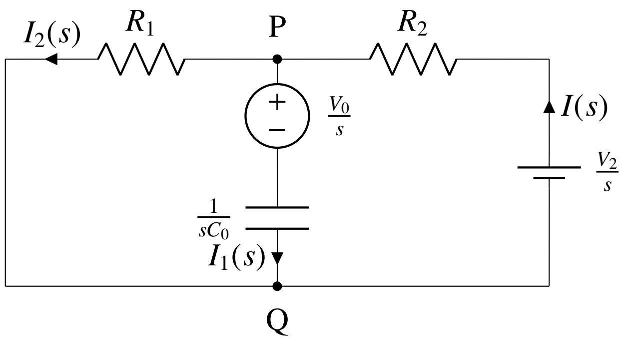
Suppose $i(t)u(t) \xrightarrow{\mathcal{L}} I(s)$ and $Vu(t) \xrightarrow{\mathcal{L}} \frac{V}{s}$

$$\implies \mathcal{L}\left(u(t)\int_0^t \frac{i_1}{C_0}dt\right) = \frac{1}{sC_0}\mathcal{L}\left(u(t)i_1\right) = \frac{I_1(s)}{sC_0}$$

(Using the properties of Laplace Transform) The above equations after transform

$$R_1 I_2(s) + \frac{V_0}{s} + \frac{I_1(s)}{sC_0} = 0$$
$$-R_2 I(s) + \frac{V_2}{s} - R_1 I_2(s) = 0$$

Now after applying the transformation, from time domain to s-domain, we replace the capacitor with an equivalent resistor of resistence $\frac{1}{sC_0}$ and replace I by $I(s)$, V by $\frac{V}{s}$, and hence we get,



4 Bilinear Transform

1. In Fig. 2.1, Consider the case when S is switched to Q right in the beginning. Formulate the differential equation.
   Solution:
   Special Case of (2.5), putting $V_1 = 0$, we get
   $$i_2(t)\ R_1 - \int_0^t \frac{i_1(t)}{C_0}dt = 0 \qquad (4.1)$$
   $$-i(t)\ R_2 + V_2 - \int_0^t \frac{i_1(t)}{C_0}dt = 0 \qquad (4.2)$$
   The above equations after transformation to s-domain
   $$R_1 I_2(s) - \frac{I_1(s)}{sC_0} = 0 \qquad (4.3)$$
   $$-R_2 I(s) + \frac{V_2}{s} - \frac{I_1(s)}{sC_0} = 0 \qquad (4.4)$$
2. Find $H(s)$ considering the ouput voltage at the capacitor.
   Solution:
   Recall that $H(s) = \frac{V_{C_0}(s)}{V_2(s)}$
   Using KCL at point P, after s-domain transform, we have,
   $$\frac{V_{C_0}(s)}{R_1} + \frac{V_{C_0}(s)}{\frac{1}{sC_0}} + \frac{V_{C_0}(s) - V_2(s)}{R_2} = 0$$
   $$V_{C_0}(s)\left(\frac{1}{R_1} + sC_0 + \frac{1}{R_2}\right) = \frac{V_2(s)}{R_2}$$
   $$\frac{V_{C_0}(s)}{V_2(s)} = H(s) = \frac{1}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right)}$$
3. Plot $H(s)$. What kind of filter is it?
   Solution: The following code yields the graph

   Clearly, H(s) is a low pass filter
4. Using trapezoidal rule for integration, formulate the difference equation by considering
   $$y(n) = y(t)|_{t=n} \qquad (4.5)$$
   Solution:
5. Find $H(z)$.
6. How can you obtain $H(z)$ from $H(s)$?