Pingala Series

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Contents

1 **JEE 2019** 1

2 **Pingala Series** 1

3 **Power of the Z transform** 3

***Abstract*—This manual provides a simple introduction to Transforms**

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1
$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

1.2
$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

1.3
$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

1.4
$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution: The following code verifies all the equations

```
wget https://raw.githubusercontent.com/Sigma1084
    /EE3900/master/pingala/code/Ex1_verify.py
```

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$\begin{aligned} x(n+2) &= x(n+1) + x(n) \\ x(0) &= x(1) = 1, n \geq 0 \end{aligned} \tag{2.2}$$

Generate a stem plot for $x(n)$.

Solution: The following code generates the plot

```
wget https://raw.githubusercontent.com/
    Sigma1084/EE3900/master/pingala/code/
    Ex2_plot_xn.py
```

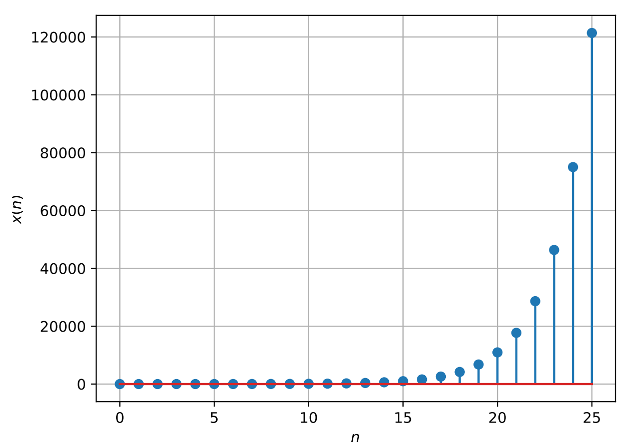

Fig. 2.2: Plot of $x(n)$

2.3 Find $X^+(z)$.
Solution: Taking the *one-sided* Z-transform on both sides of (2.2)

$$\mathcal{Z}^+\left[x(n+2)\right] = \mathcal{Z}^+\left[x(n+1)\right] + \mathcal{Z}^+\left[x(n)\right]$$

$$\implies z^2X^+(z) - z^2 - z = zX^+(z) - z + X^+(z)$$
$$\implies \left(z^2 - z - 1\right)X^+(z) = z^2$$
$$\implies X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}}$$

$$X^+(z) = \frac{1}{\left(1-\alpha z^{-1}\right)\left(1-\beta z^{-1}\right)}, \quad |z| > |\alpha| \tag{2.3}$$

Here α and β are the roots of the characteristic equation of (2.2) and $|z| > |\alpha|$ is the region of convergence of $X^+(z)$.
(w.l.o.g, $|\alpha| > |\beta|$ is assumed)

2.4 Find $x(n)$.
Solution: Expanding $X^+(z)$ in (2.3) using partial fractions, we get

$$X^+(z) = \frac{1}{(\alpha-\beta)\,z^{-1}}\left[\frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\beta z^{-1}}\right]$$
$$= \frac{1}{(\alpha-\beta)}\sum_{n=0}^{\infty}\left(\alpha^n - \beta^n\right)z^{-n}$$
$$= \sum_{n=1}^{\infty}\frac{\alpha^n-\beta^n}{\alpha-\beta}z^{-n+1}$$
$$= \sum_{n=0}^{\infty}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}z^{-n}$$

$$x(n) = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}u(n) = a_{n+1}u(n) \tag{2.4}$$

2.5 Sketch

$$y(n) = x\left(n-1\right) + x\left(n+1\right), \quad n \geq 0 \tag{2.5}$$

Solution: The following code generates the plot

```
wget https://raw.githubusercontent.com/
    Sigma1084/EE3900/master/pingala/code/
    Ex2_plot_yn.py
```

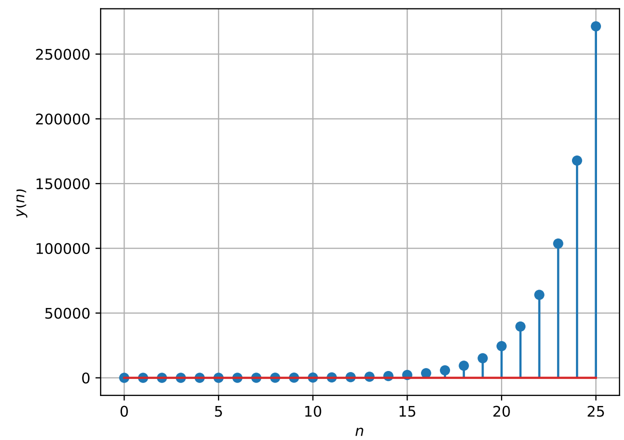

Fig. 2.5: Plot of $y(n)$

2.6 Find $Y^+(z)$.
Solution: Taking the one-sided Z-transform on both sides of (2.5),

$$\mathcal{Z}^+\left[y(n)\right] = \mathcal{Z}^+\left[x(n+1)\right] + \mathcal{Z}^+\left[x(n-1)\right]$$

$$Y^+(z) = zX^+(z) - zx(0) + z^{-1}X^+(z) + zx(-1)$$
$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z$$
$$Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > |\alpha| \tag{2.6}$$

($x(-1) = 0$ since $x(n) = 0 \ \forall \ n < 0$)

2.7 Find $y(n)$.
Solution: Using (2.6),

$$Y^+(z) = \left(1 + 2z^{-1}\right)\ X^+(z)$$
$$= \left(1 + 2z^{-1}\right)\sum_{n=0}^{\infty}x(n)z^{-n}$$
$$= \sum_{n=0}^{\infty}x(n)z^{-n} + \sum_{n=1}^{\infty}2x(n-1)z^{-n}$$
$$= x(0) + \sum_{n=1}^{\infty}\left(x(n) + 2x(n-1)\right)z^{-n}$$
$$= x(0) + \sum_{n=1}^{\infty}\left(x(n+1) + x(n-1)\right)z^{-n}$$

$$\implies \quad y(0) = x(0) = 1$$
$$\forall n > 0 \quad y(n) = x(n+1) + x(n-1)$$
$$= a_{n+2} + a_n = b_{n+1}$$

α and β are the roots of the characteristic equation of (2.2), $z^2 - z - 1 = 0$
(w.l.o.g, $|\alpha| > |\beta|$ is assumed)
$\implies \alpha\beta = -1$ and $\alpha + \beta = 1$

$$
\begin{aligned}
y(n) &= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) + (\alpha^n + \beta^n)}{\alpha - \beta} \\
&= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) - \alpha\beta\,(\alpha^n + \beta^n)}{\alpha - \beta} \\
&= \frac{(\alpha - \beta)\left(\alpha^{n+1} + \beta^{n+1}\right)}{\alpha - \beta} \\
y(n) &= \alpha^{n+1} + \beta^{n+1} \qquad (2.7)
\end{aligned}
$$

Thus, $y(n) = \alpha^{n+1} + \beta^{n+1}$ for $n \geq 0$ as $\alpha + \beta = 1$. Comparing (2.7) with the definition of b_n, we see that $y(n) = b_{n+1}$.

$$
\implies b_{n+1} = y(n) = \alpha^{n+1} + \beta^{n+1} \quad \forall n > 0 \qquad (2.8)
$$

3 Power of the Z transform

3.1 Show that

$$
\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \qquad (3.1)
$$

Solution: Using (2.4), and the definition of convolution,

$$
\begin{aligned}
\sum_{k=1}^{n} a_k &= \sum_{k=0}^{n-1} a_{k+1} = \sum_{k=-\infty}^{n-1} a_{k+1}\; u(k) \\
&= \sum_{k=-\infty}^{n-1} x(k) \\
&= \sum_{k=-\infty}^{\infty} x(k)\; u(n-1-k) \\
&= u(n-1) * x(n) \\
\implies \sum_{k=1}^{n} a_k &= x(n) * u(n-1)
\end{aligned}
$$

3.2 Show that

$$
a_{n+2} - 1, \quad n \geq 1 \qquad (3.2)
$$

can be expressed as

$$
[x(n+1) - 1]\, u(n) \qquad (3.3)
$$

Solution: Using (2.4), and the definition of $u(n)$

$$
\begin{aligned}
& a_{n+2} - 1 && n \geq 1 \\
&= [x(n+1) - 1] && n \geq 0 \\
&= [x(n+1) - 1]\, u(n)
\end{aligned}
$$

3.3 Show that

$$
\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \qquad (3.4)
$$

Solution: Using the definitions of $x(n)$ (2.4) and $X^+(z)$ (2.3),

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{a_k}{10^k} &= \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \\
&= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \\
&= \frac{1}{10} X^+(10) \\
&= \frac{1}{10} \times \frac{1}{1 - \frac{1}{10} - \frac{1}{100}} \\
\implies \sum_{k=1}^{\infty} \frac{a_k}{10^k} &= \frac{10}{89} \qquad (3.5)
\end{aligned}
$$

3.4 Show that

$$
\alpha^n + \beta^n, \quad n \geq 1 \qquad (3.6)
$$

can be expressed as

$$
w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \qquad (3.7)
$$

and find $W(z)$.
Solution: Putting $n = n + 1$ in (3.6), we get

$$
\begin{aligned}
& \alpha^n + \beta^n && n \geq 1 \\
&= \alpha^{n+1} + \beta^{n+1} && n \geq 0 \\
w(n) &:= \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) = y(n)
\end{aligned}
$$

Since we know $w(n) = y(n)$, we can use (2.6) to find $W(z)$

$$
\begin{aligned}
W(z) &= \sum_{n=-\infty}^{\infty} w(n) = \sum_{n=0}^{\infty} y(n) \\
&= Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > |\alpha|
\end{aligned}
$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \tag{3.8}$$

Solution: Using the definitions of $y(n)$ (2.7) and $Y^+(z)$ (2.6),

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{b_k}{10^k} &= \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \\
&= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \\
&= \frac{1}{10} Y^+(10) \\
&= \frac{1}{10} \times \frac{1 + 2\frac{1}{10}}{1 - \frac{1}{10} - \frac{1}{100}}
\end{aligned}$$

$$\implies \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{12}{89} \tag{3.9}$$

3.6 Solve the JEE 2019 problem.
Solution: We know that

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \tag{3.10}$$

But

$$\begin{aligned}
&x(n) * u(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} X(z) z^{-1} U(z) \\
&= \frac{z^{-1}}{\left(1 - z^{-1} - z^{-2}\right)\left(1 - z^{-1}\right)} \\
&= z \left[\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right] \\
&\overset{\mathcal{Z}}{\rightleftharpoons} z \sum_{n=0}^{\infty} \left(x(n) - 1\right) z^{-n} \\
&= \sum_{n=0}^{\infty} \left(x(n) - 1\right) z^{-n+1} \\
&= \sum_{n=0}^{\infty} \left(x(n+1) - 1\right) z^{-n}
\end{aligned}$$

From (2.4), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \tag{3.11}$$

Using (3.11), (3.5), (2.8) and (3.9), we can conclude that Options 1, 2, 3 are right and Option 4 is wrong.