

# Pingala Series

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**Abstract**—This manual provides a simple introduction to Transforms

## 1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \quad (1.1)$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \quad (1.2)$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^n a_k = a_{n+2} - 1, \quad n \geq 1 \quad (1.3)$$

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \quad (1.4)$$

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \quad (1.5)$$

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \quad (1.6)$$

**Solution:** The following code verifies all the equations

```
wget https://raw.githubusercontent.com/Sigma1084/EE3900/master/pingala/code/Ex1_verify.py
```

## 2 PINGALA SERIES

2.1 The *one sided* Z-transform of  $x(n)$  is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \quad (2.1)$$

2.2 The *Pingala* series is generated using the difference equation

$$\begin{aligned} x(n+2) &= x(n+1) + x(n) \\ x(0) &= x(1) = 1, n \geq 0 \end{aligned} \quad (2.2)$$

Generate a stem plot for  $x(n)$ .

**Solution:** The following code generates the plot

```
wget https://raw.githubusercontent.com/Sigma1084/EE3900/master/pingala/code/Ex2_plot_xn.py
```

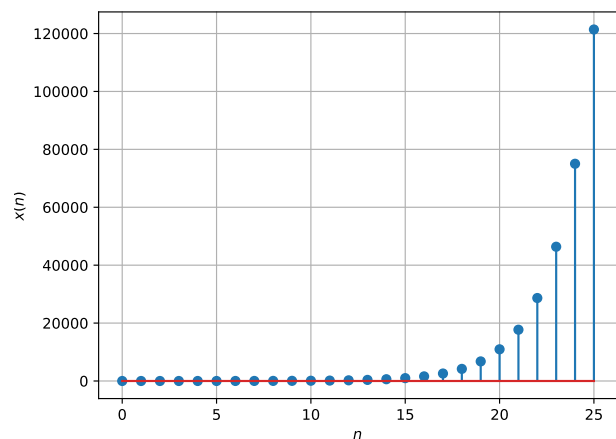


Fig. 2.2: Plot of  $x(n)$

### 2.3 Find $X^+(z)$ .

**Solution:** Taking the *one-sided* Z-transform on both sides of (2.2)

$$\mathcal{Z}^+[x(n+2)] = \mathcal{Z}^+[x(n+1)] + \mathcal{Z}^+[x(n)]$$

$$\Rightarrow z^2 X^+(z) - z^2 - z = z X^+(z) - z + X^+(z)$$

$$\Rightarrow (z^2 - z - 1) X^+(z) = z^2$$

$$\Rightarrow X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}}$$

$$X^+(z) = \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})}, \quad |z| > |\alpha| \quad (2.3)$$

Here  $\alpha$  and  $\beta$  are the roots of the characteristic equation of (2.2) and  $|z| > |\alpha|$  is the region of convergence of  $X^+(z)$ .

(w.l.o.g,  $|\alpha| > |\beta|$  is assumed)

### 2.4 Find $x(n)$ .

**Solution:** Expanding  $X^+(z)$  in (2.3) using partial fractions, we get

$$\begin{aligned} X^+(z) &= \frac{1}{(\alpha - \beta) z^{-1}} \left[ \frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}} \right] \\ &= \frac{1}{(\alpha - \beta)} \sum_{n=0}^{\infty} (\alpha^n - \beta^n) z^{-n} \\ &= \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} z^{-n+1} \\ &= \sum_{n=0}^{\infty} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} z^{-n} \\ x(n) &= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) = a_{n+1} u(n) \quad (2.4) \end{aligned}$$

### 2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \quad (2.5)$$

**Solution:** The following code generates the plot

```
wget https://raw.githubusercontent.com/
Sigma1084/EE3900/master/pingala/code/
Ex2_plot_yn.py
```

### 2.6 Find $Y^+(z)$ .

**Solution:** Taking the one-sided Z-transform on both sides of (2.5),

$$\mathcal{Z}^+[y(n)] = \mathcal{Z}^+[x(n+1)] + \mathcal{Z}^+[x(n-1)]$$

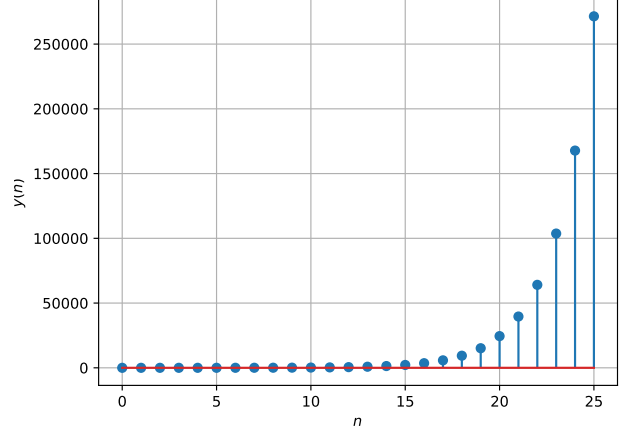


Fig. 2.5: Plot of  $y(n)$

$$\begin{aligned} Y^+(z) &= z X^+(z) - z x(0) + z^{-1} X^+(z) + z x(-1) \\ &= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \end{aligned}$$

$$Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > |\alpha| \quad (2.6)$$

( $x(-1) = 0$  since  $x(n) = 0 \forall n < 0$ )

### 2.7 Find $y(n)$ .

**Solution:** Using (2.6),

$$\begin{aligned} Y^+(z) &= (1 + 2z^{-1}) X^+(z) \\ &= (1 + 2z^{-1}) \sum_{n=0}^{\infty} x(n) z^{-n} \\ &= \sum_{n=0}^{\infty} x(n) z^{-n} + \sum_{n=1}^{\infty} 2x(n-1) z^{-n} \\ &= x(0) + \sum_{n=1}^{\infty} (x(n) + 2x(n-1)) z^{-n} \\ &= x(0) + \sum_{n=1}^{\infty} (x(n+1) + x(n-1)) z^{-n} \end{aligned}$$

$$\Rightarrow y(0) = x(0) = 1$$

$$\begin{aligned} \forall n > 0 \quad y(n) &= x(n+1) + x(n-1) \\ &= a_{n+2} + a_n = b_{n+1} \end{aligned}$$

$\alpha$  and  $\beta$  are the roots of the characteristic equation of (2.2),  $z^2 - z - 1 = 0$

(w.l.o.g,  $|\alpha| > |\beta|$  is assumed)

$$\Rightarrow \alpha\beta = -1 \quad \text{and} \quad \alpha + \beta = 1$$

$$\begin{aligned}
y(n) &= \frac{(\alpha^{n+2} - \beta^{n+2}) + (\alpha^n + \beta^n)}{\alpha - \beta} \\
&= \frac{(\alpha^{n+2} - \beta^{n+2}) - \alpha\beta(\alpha^n + \beta^n)}{\alpha - \beta} \\
&= \frac{(\alpha - \beta)(\alpha^{n+1} + \beta^{n+1})}{\alpha - \beta} \\
y(n) &= \alpha^{n+1} + \beta^{n+1} \quad (2.7)
\end{aligned}$$

Thus,  $y(n) = \alpha^{n+1} + \beta^{n+1}$  for  $n \geq 0$  as  $\alpha + \beta = 1$ . Comparing (2.7) with the definition of  $b_n$ , we see that  $y(n) = b_{n+1}$ .

$$\Rightarrow b_{n+1} = y(n) = \alpha^{n+1} + \beta^{n+1} \quad \forall n > 0 \quad (2.8)$$

### 3 POWER OF THE Z TRANSFORM

3.1 Show that

$$\sum_{k=1}^n a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \quad (3.1)$$

**Solution:** Using (2.4), and the definition of convolution,

$$\begin{aligned}
\sum_{k=1}^n a_k &= \sum_{k=0}^{n-1} a_{k+1} = \sum_{k=-\infty}^{n-1} a_{k+1} u(k) \\
&= \sum_{k=-\infty}^{n-1} x(k) \\
&= \sum_{k=-\infty}^{\infty} x(k) u(n-1-k) \\
&= u(n-1) * x(n) \\
\Rightarrow \sum_{k=1}^n a_k &= x(n) * u(n-1)
\end{aligned}$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \quad (3.2)$$

can be expressed as

$$[x(n+1) - 1] u(n) \quad (3.3)$$

**Solution:** Using (2.4), and the definition of

$$u(n)$$

$$\begin{aligned}
&a_{n+2} - 1 \quad n \geq 1 \\
&= [x(n+1) - 1] \quad n \geq 0 \\
&= [x(n+1) - 1] u(n)
\end{aligned}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \quad (3.4)$$

**Solution:** Using the definitions of  $x(n)$  (2.4) and  $X^+(z)$  (2.3),

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{a_k}{10^k} &= \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \\
&= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \\
&= \frac{1}{10} X^+(10) \\
&= \frac{1}{10} \times \frac{1}{1 - \frac{1}{10} - \frac{1}{100}} \\
\Rightarrow \sum_{k=1}^{\infty} \frac{a_k}{10^k} &= \frac{10}{89} \quad (3.5)
\end{aligned}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \quad (3.6)$$

can be expressed as

$$w(n) = (\alpha^{n+1} + \beta^{n+1}) u(n) \quad (3.7)$$

and find  $W(z)$ .

**Solution:** Putting  $n = n + 1$  in (3.6), we get

$$\begin{aligned}
&\alpha^n + \beta^n \quad n \geq 1 \\
&= \alpha^{n+1} + \beta^{n+1} \quad n \geq 0 \\
w(n) &:= (\alpha^{n+1} + \beta^{n+1}) u(n) = y(n)
\end{aligned}$$

Since we know  $w(n) = y(n)$ , we can use (2.6) to find  $W(z)$

$$\begin{aligned}
W(z) &= \sum_{n=-\infty}^{\infty} w(n) = \sum_{n=0}^{\infty} y(n) \\
&= Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > |\alpha|
\end{aligned}$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \quad (3.8)$$

**Solution:** Using the definitions of  $y(n)$  (2.7) and  $Y^+(z)$  (2.6),

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{b_k}{10^k} &= \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \\ &= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \\ &= \frac{1}{10} Y^+(10) \\ &= \frac{1}{10} \times \frac{1 + 2\frac{1}{10}}{1 - \frac{1}{10} - \frac{1}{100}} \\ \Rightarrow \sum_{k=1}^{\infty} \frac{a_k}{10^k} &= \frac{12}{89} \end{aligned} \quad (3.9)$$

3.6 Solve the JEE 2019 problem.

**Solution:** We know that

$$\sum_{k=1}^n a_k = x(n) * u(n-1) \quad (3.10)$$

But

$$\begin{aligned} x(n) * u(n-1) &\stackrel{Z}{\rightleftharpoons} X(z)z^{-1}U(z) \\ &= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \\ &= z \left[ \frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right] \\ &\stackrel{Z}{\rightleftharpoons} z \sum_{n=0}^{\infty} (x(n) - 1) z^{-n} \\ &= \sum_{n=0}^{\infty} (x(n) - 1) z^{-n+1} \\ &= \sum_{n=0}^{\infty} (x(n+1) - 1) z^{-n} \end{aligned}$$

From (2.4), we get

$$\sum_{k=1}^n a_k = a_{n+2} - 1 \quad (3.11)$$

Using (3.11), (3.5), (2.8) and (3.9), we can

conclude that Options 1, 2, 3 are right and Option 4 is wrong.