Fourier Series

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Contents

***Abstract*—This manual provides a simple introduction to Fourier Series**

1 Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution:
The following code yields the graph

```
wget https://raw.githubusercontent.com/
    Sigma1084/EE3900/master/charger/codes/
    Ex1_1_plotxt.py
```

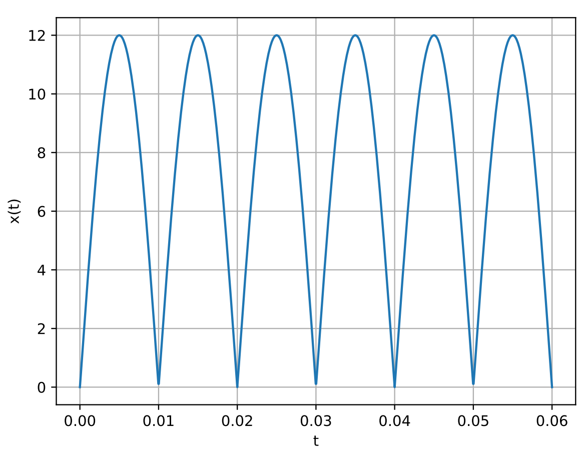

Fig. 1.1: Plot of $x(t)$

1.2 Show that $x(t)$ is periodic and find its period.
Solution:
We know that $|\sin(x)|$ is periodic with fundamental period of π.
$\implies$ Fundamental period of $A|\sin(ax)$ is $\frac{\pi}{a}$
Fundamental period of $A_0|\sin(2\pi f_0 t)|$ is $\frac{\pi}{2\pi f_0}$
$\implies$ Fundamental period of $x(t)$ is $\frac{1}{2f_0}$

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.2}$$

Solution:
Consider for some $n \in \mathbb{Z}$,

$$x(t)e^{j2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-n) f_0 t}$$

We know using the periodicity of $e^{j2\pi k f_0 t}$,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi k f_0 t} dt = \begin{cases} \frac{1}{f_0} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Now,

$$\begin{aligned} \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{j2\pi n f_0 t} dt &= \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-n) f_0 t} \\ &= \frac{c_n}{f_0} \end{aligned}$$

$$\implies c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{j2\pi n f_0 t} dt$$

2.2 Find c_k for (1.1)
Solution:

$$
\begin{aligned}
c_k &= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{j2\pi k f_0 t} dt \\
&= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| e^{j2\pi k f_0 t} dt \\
&= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| \cos(2\pi k f_0 t) dt \\
&\quad + j f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| \sin(2\pi k f_0 t) dt \\
&= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| \cos(2\pi k f_0 t) dt + 0 \\
&= 2 f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 t\right) \cos(2\pi k f_0 t) dt \\
&= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \sin\left(2\pi \left(n+1\right) f_0 t\right) dt \\
&\quad - f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \sin\left(2\pi \left(n-1\right) f_0 t\right) dt \\
&= f_0 A_0 \frac{1+(-1)^n}{2\pi f_0} \left(\frac{1}{n+1} - \frac{1}{n-1}\right) \\
&= \begin{cases} \frac{2A_0}{\pi\left(1-n^2\right)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}
\end{aligned} \tag{2.3}
$$

2.3 Verify (1.1) using python.
Solution:
The following code block yields the graph

```
wget https://raw.githubusercontent.com/
    Sigma1084/EE3900/master/charger/codes/
    Ex1_1_plotxt.py
```

2.4 Show that

$$
x(t) = \sum_{k=0}^{\infty} \left(a_k \cos j2\pi k f_0 t + b_k \sin j2\pi k f_0 t\right) \tag{2.4}
$$

and obtain the formulae for a_k and b_k.
Solution: From (2.1),

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \\
&= c_0 + \sum_{k=1}^{\infty} \left(c_k e^{j2\pi k f_0 t} + c_{-k} e^{-j2\pi k f_0 t}\right) \\
&= c_0 + \sum_{k=1}^{\infty} \left(c_k + c_{-k}\right) \cos\left(2\pi k f_0 t\right) \\
&\quad + \sum_{k=0}^{\infty} \left(c_k - c_{-k}\right) \sin\left(2\pi k f_0 t\right)
\end{aligned}
$$

Hence, for $k \geq 0$,

$$
a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.5}
$$

$$
b_k = c_k - c_{-k} \tag{2.6}
$$

2.5 Find a_k and b_k for (1.1)
Solution:
Using the expression for c_k, from (2.3) and using (2.5) and (2.6), we have,

$$
a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi\left(1-k^2\right)} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}
$$

$$
b_k = 0
$$

2.6 Verify (2.4) using python.

3 Fourier Transform

3.1

$$
\delta(t) = 0, \quad t \neq 0 \tag{3.1}
$$

$$
\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}
$$

3.2 The Fourier Transform of $g(t)$ is

$$
G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \tag{3.3}
$$

3.3 Show that

$$
g(t - t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} G(f) e^{-j2\pi f t_0} \tag{3.4}
$$

Solution:

$$\begin{aligned}\mathcal{F}\left(g\left(t-t_0\right)\right) &= \int_{-\infty}^{\infty} g\left(t-t_0\right) e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} g\left(t\right) e^{-j2\pi f(t+t_0)} dt \\ &= e^{-j2\pi ft_0}\int_{-\infty}^{\infty} g\left(t\right) e^{-j2\pi ft} dt \\ &= G(f)e^{-j2\pi ft_0}\end{aligned}$$

$$\implies g\left(t-t_0\right) \overset{\mathcal{F}}{\longleftrightarrow} G(f)e^{-j2\pi ft_0}$$

Hence proved

3.4 Show that

$$G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.5}$$

Solution:
Using the definition of inverse Fourier Transform,

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft} df$$

Now, putting $-f := t,\ t := f \implies df = dt$,

$$g(-f) = \int_{-\infty}^{\infty} G(t)e^{-j2\pi ft} dt$$

$$\implies G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.6}$$

3.5 $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} ?$
Solution:

$$\begin{aligned}\delta(t) &\overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft} dt \\ &= e^{-j2\pi f(0)}\int_{-\infty}^{\infty} \delta(t) dt\end{aligned}$$

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1 \tag{3.7}$$

3.6 $e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} ?$
Solution: Suppose $g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$

$$\begin{aligned}g(t)e^{-j2\pi f_0 t} &\overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t)e^{-j2\pi(f_0+f)t} dt \\ &= G(f+f_0)\end{aligned}$$

Now, using (3.7) and (3.6) , we can get,

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1 \overset{\mathcal{F}}{\longleftrightarrow} \delta(-t)$$

Putting $g(t) = 1$ and hence, $G(f) = \delta(-f) = \delta(f)$, we get $G(f+f_0) = \delta(f+f_0)$. Hence,

$$e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(f+f_0) \tag{3.8}$$

3.7 $\cos(2\pi f_0 t) \overset{\mathcal{F}}{\longleftrightarrow} ?$
Solution: We can write

$$\cos(2\pi f_0 t) = \frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$$

Using (3.8) and the linearity of Fourier Transform, we get,

$$\cos(2\pi f_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\Big(\delta(f-f_0) + \delta(f+f_0)\Big)$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.
Solution: Using (2.1), we have,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$$

Using (3.8), we get,

$$\begin{aligned}x(t) &\overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} c_k \delta(f-kf_0) \\ &= \frac{2A_0}{\pi}\sum_{k=-\infty}^{\infty}\frac{\delta(f-2kf_0)}{1-4k^2}\end{aligned}$$

3.9 Show that

$$\text{rect}\,(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}\,(t) \tag{3.9}$$

Verify using python.
Solution:

$$\begin{aligned}\text{rect}\,(t) &\overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \text{rect}\,(t)\, e^{-j2\pi ft} dt \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi ft} dt \\ &= \frac{1}{-j2\pi f}\left(e^{-j\pi f} - e^{j\pi f}\right) \\ &= \frac{\sin(\pi f)}{\pi f} = \text{sinc}\,(f)\end{aligned}$$

The following code yields the plot

3.10 $\text{sinc}\,(t) \overset{\mathcal{F}}{\longleftrightarrow} ?$ Verify using python.
Solution:
Using (3.9), and (3.6), we get,

$$\text{sinc}\,(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}\,(-f) = \text{rect}\,(f) \tag{3.10}$$

4 Filter

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
Solution:

4.2 Find $h(t)$.
4.3 Verify your result using through convolution.

5 Filter Design

5.1 Design a Butterworth filter for $H(f)$.
5.2 Design a Chebyschev filter for $H(f)$.
5.3 Design a circuit for your Butterworth filter.
5.4 Design a circuit for your Chebyschev filter.