

Circuits and Transforms

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Abstract—This manual provides a simple introduction to Transforms

1 DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

2 LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1 \mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2 \mu C$.

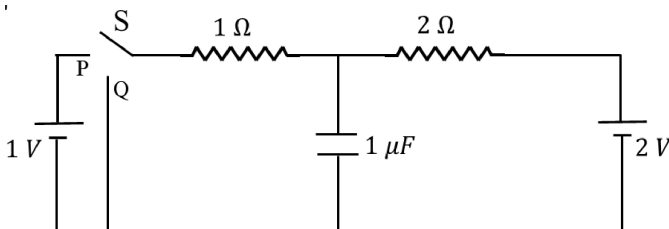
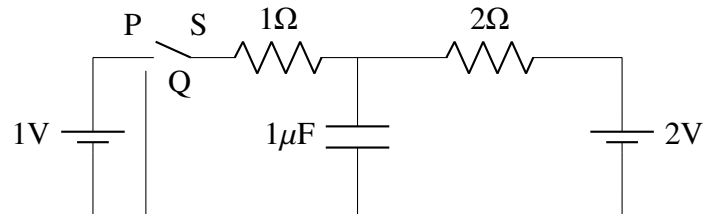


Fig. 2.1

2. Draw the circuit using latex-tikz.

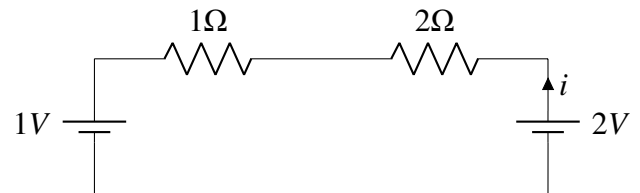
Solution:



3. Find q_1 .

Solution:

Current through the capacitor is 0 at steady state. So, current only flows in the outer loop.



$$i = \frac{(2 - 1) V}{(1 + 2) \Omega}$$

$$i = \frac{1}{3} A$$

Potential difference across the capacitor,

$$V_C = 2 - 2 \times \frac{1}{3} = \frac{4}{3} V$$

$$q_1 = CV_C$$

$$q_1 = 1\mu F \times \frac{4}{3} V$$

$$q_1 = \frac{4}{3} \mu C$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.

Solution:

By the definition of Laplace transform

$$u(t) \xrightarrow{\mathcal{L}} \int_{-\infty}^{\infty} u(t)e^{-st} dt = \int_0^{\infty} e^{-st} dt$$

$$= \frac{e^{-st}}{-s} \Big|_0^{\infty} = \frac{1}{s}$$

$\lim_{s \rightarrow \infty} e^{-st} = 0$ only when $Re(s) > 0$.

The ROC is $Re(s) > 0$.

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{H}} L \frac{1}{s+a}, \quad a > 0 \quad (2.1)$$

and find the ROC.

Solution:

By the definition of Laplace transform

$$\begin{aligned} e^{-at}u(t) &\xrightarrow{\mathcal{L}} \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \\ &= \int_0^{\infty} e^{-(s+a)t}dt \\ &= \left. \frac{e^{-(s+a)t}}{-(s+a)} \right|_0^{\infty} = \frac{1}{s+a} \end{aligned}$$

$\lim_{s \rightarrow \infty} e^{-(s+a)t} = 0$ only when $\text{Re}(s+a) > 0$.

The ROC is $\text{Re}(s+a) > 0$ or $\text{Re}(s) > -a$.

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

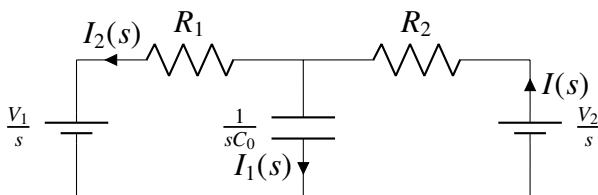


Fig. 2.2

Applying KCL at the upper junction, we have,

$$\frac{V_{C_0}(s) - V_1(s)}{R_1} + \frac{V_{C_0}(s) - V_2(s)}{R_2} + \frac{V_{C_0}(s)}{\frac{1}{sC_0}} = 0$$

$$V_{C_0}(s) \left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{V_1(s)}{R_1} + \frac{V_2(s)}{R_2}$$

$$V_{C_0}(s) = \frac{V_1(s)R_2 + V_2(s)R_1}{R_1 + R_2 + sC_0R_1R_2}$$

$$V_{C_0}(s) = \frac{R_2 + 2R_1}{s(R_1 + R_2 + sC_0R_1R_2)}$$

after substitution.

7. Find $v_{C_0}(t)$. Plot using python.

Solution:

Factoring $V_{C_0}(s)$, we have

$$\begin{aligned} V_{C_0}(s) &= \left(\frac{1}{R_1 + R_2} \right) \left(\frac{(R_2 + 2R_1)}{s} - \frac{R_1R_2C_0(R_2 + 2R_1)}{R_1 + R_2 + sC_0R_1R_2} \right) \\ &= \left(\frac{R_2 + 2R_1}{R_1 + R_2} \right) \left(\frac{1}{s} - \frac{R_1R_2C_0}{R_1 + R_2 + sC_0R_1R_2} \right) \\ &= \left(\frac{R_2 + 2R_1}{R_1 + R_2} \right) \left(\frac{1}{s} - \frac{1}{\frac{1}{R_2C_0} + \frac{1}{R_1C_0} + s} \right) \end{aligned}$$

Applying inverse Laplace transform,

$$v_{C_0}(t) = \frac{R_2 + 2R_1}{R_1 + R_2} \left(u(t) - e^{-\left(\frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right)t} u(t) \right) \quad (2.4)$$

Using the values $R_1 = 1 \Omega$, $R_2 = 2 \Omega$, $C_0 = 1 \mu F$, we get

$$v_{C_0}(t) = \frac{4}{3}u(t) \left(1 - e^{-1.5 \times 10^6 t} \right)$$

The following code yields $v_{C_0}(t)$.

```
wget https://github.com/Sigma1084/EE3900/
blob/master/cktsig/code/Ex2_plotVt.py
```

$$u(t) \xleftrightarrow{\mathcal{H}} LV_1(s) \quad (2.2)$$

$$2u(t) \xleftrightarrow{\mathcal{H}} LV_2(s) \quad (2.3)$$

Find the voltage across the capacitor $V_{C_0}(s)$.

Solution:

Using the Transform from time domain to s-domain

$$\begin{aligned} V_1(s) &= \mathcal{L}(u(t)) = \frac{V_1}{s} \\ V_2(s) &= \mathcal{L}(2u(t)) = \frac{V_2}{s} \end{aligned}$$

Potential across the capacitor is $V_{C_0}(s)$ and the assuming bottom is grounded.

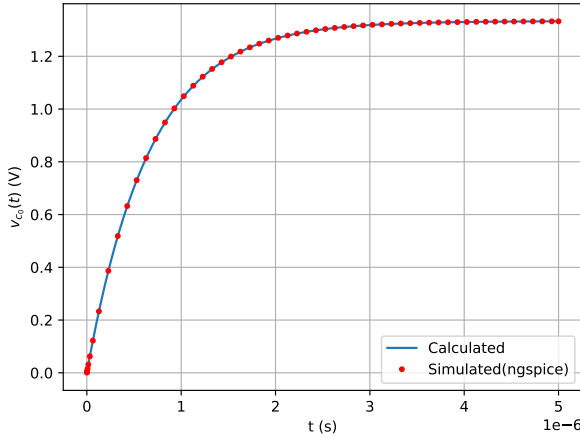


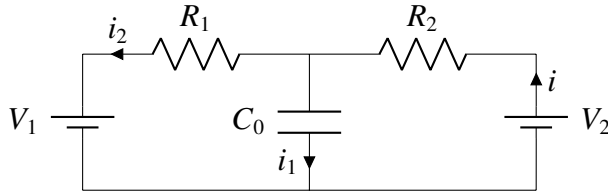
Fig. 2.3

8. Verify your result using ngspice. The following code yields $v_{C_0}(t)$. using ngspice

```
wget https://github.com/Sigma1084/EE3900/
blob/master/cktsig/code/Ex2.spice
```

9. Obtain Fig. 2.2 using the equivalent differential equation.

Solution:



Using KVL on the left and right loops, we get

$$i_2(t) R_1 + V_1 - \int_0^t \frac{i_1(t)}{C_0} dt = 0 \quad (2.5)$$

$$-i(t) R_2 + V_2 - \int_0^t \frac{i_1(t)}{C_0} dt = 0 \quad (2.6)$$

Taking the Laplace Transform after multiplying $u(t)$ in both equations,

$$\mathcal{L}\left(R_1 i_2(t)u(t) + V_1 u(t) - u(t) \int_0^t \frac{i_1}{C_0} dt\right) = 0$$

$$\mathcal{L}\left(-R_2 i(t)u(t) + V_2 u(t) - u(t) \int_0^t \frac{i_1}{C_0} dt\right) = 0$$

Suppose $i(t)u(t) \xrightarrow{\mathcal{L}} I(s)$ and $Vu(t) \xrightarrow{\mathcal{L}} \frac{V}{s}$

$$\Rightarrow \mathcal{L}\left(u(t) \int_0^t \frac{i_1}{C_0} dt\right) = \frac{1}{sC_0} \mathcal{L}(u(t)i_1) = \frac{I_1(s)}{sC_0}$$

(Using the properties of Laplace Transform)

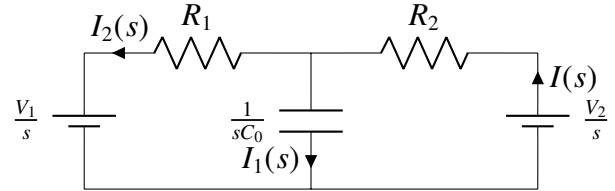
The above equations after transform

$$R_1 I_2(s) + \frac{V_1}{s} - \frac{I_1(s)}{sC_0} = 0 \quad (2.7)$$

$$-R_2 I(s) + \frac{V_2}{s} - \frac{I_1(s)}{sC_0} = 0 \quad (2.8)$$

Now after applying the transformation, from time domain to s-domain, we replace the capacitor with an equivalent resistor of resistance $\frac{1}{sC_0}$ and replace I by $I(s)$, V by $\frac{V}{s}$

The equivalent s-domain circuit is now,



3 INITIAL CONDITIONS

1. Find q_2 in Fig. 2.1

Solution:

Current through the capacitor is 0 in steady state. Since the current flows only in the outer loop, let the current in the outer loop be I .

$$I = \frac{2V}{(1+2)\Omega} = \frac{2}{3}A$$

So, the potential difference across the capacitor is

$$V_{C_0} = 2 - \left(\frac{2}{3} \times 2\right) = \frac{2}{3}V$$

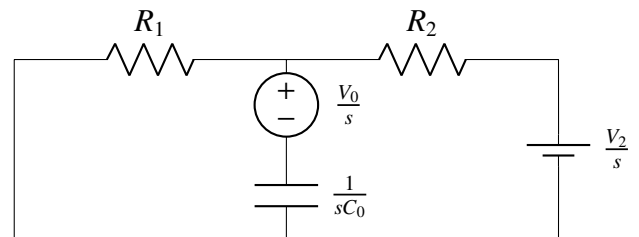
$$q_2 = CV_{C_0}$$

$$q_2 = 1\mu C \times \frac{2}{3}V$$

$$q_2 = \frac{2}{3}\mu C$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.

Solution:



3. $V_{C_0}(s) = ?$

Solution:

Applying KCL at the upper junction and assuming the bottom is grounded, we have,

$$\frac{V_{C_0}(s)}{R_1} + \frac{V_{C_0}(s) - \frac{V_2}{s}}{R_2} + \frac{V_{C_0}(s) - \frac{V_0}{s}}{\frac{1}{sC_0}} = 0$$

$$V_{C_0}(s) \left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{V_2}{sR_2} + V_0C_0$$

$$V_{C_0}(s) = \frac{\frac{V_2}{sR_2} + V_0C_0}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0}$$

$$V_{C_0}(s) = V_0 \left(\frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) + \frac{V_2}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \left(\frac{1}{s} - \frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right)$$

4. $v_{C_0}(t) = ?$ Plot using python.

Solution:

Applying inverse Laplace transform to $V_{C_0}(s)$, we have,

$$v_{C_0}(t) = V_0 e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} u(t) + \frac{V_2}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} \right) u(t)$$

Substituting the values $R_1 = 1, R_2 = 2, C_0 = 1\mu F, V_0 = \frac{4}{3}V, V_2 = 2V$

$$v_{C_0}(t) = \frac{2}{3} \left(1 + e^{-(1.5 \times 10^6)t} \right) u(t) \quad (3.1)$$

The following code yields $v_{C_0}(t)$.

```
wget https://github.com/Sigma1084/EE3900/
blob/master/cktsig/code/Ex3_plotVt.py
```

5. Verify your result using ngspice. The following code yields $v_{C_0}(t)$. using ngspice

```
wget https://github.com/Sigma1084/EE3900/
blob/master/cktsig/code/Ex3.spice
```

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.

Solution:

Using the initial conditions, we have,

$$v_{C_0}(0-) = \frac{q_1}{C_0} = \frac{4}{3}V$$

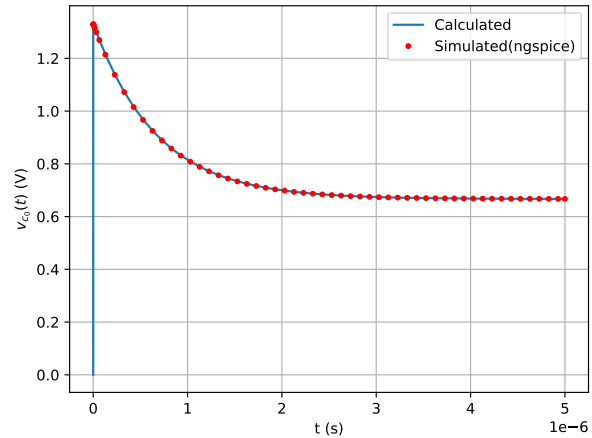


Fig. 3.1

Using (3.1), we have,

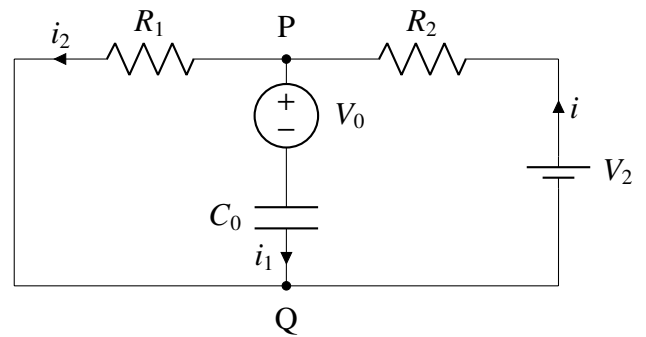
$$v_{C_0}(0+) = \frac{2}{3} \left(1 + e^{-(1.5 \times 10^6)0} \right) u(0+) = \frac{4}{3}V$$

$$v_{C_0}(\infty) = \frac{2}{3} \left(1 + e^{-(1.5 \times 10^6)\infty} \right) u(\infty) = \frac{2}{3}V$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.

Solution:

Constructing the circuit,



Using KVL on both the left and outer loops, we have

$$-i_2(t) R_1 + V_0 + \int_0^t \frac{i_1(t)}{C_0} dt = 0$$

$$-i(t) R_2 + V_2 - i_2(t) R_1 = 0$$

Taking the Laplace Transform after multiplying $u(t)$ in both equations,

$$\mathcal{L} \left(-R_1 i_2(t) u(t) + V_0 u(t) + u(t) \int_0^t \frac{i_1}{C_0} dt \right) = 0$$

$$\mathcal{L} (-R_2 i(t) u(t) + V_2 u(t) - R_1 i_2(t) u(t)) = 0$$

Suppose $i(t)u(t) \xrightarrow{\mathcal{L}} I(s)$ and $Vu(t) \xrightarrow{\mathcal{L}} \frac{V}{s}$

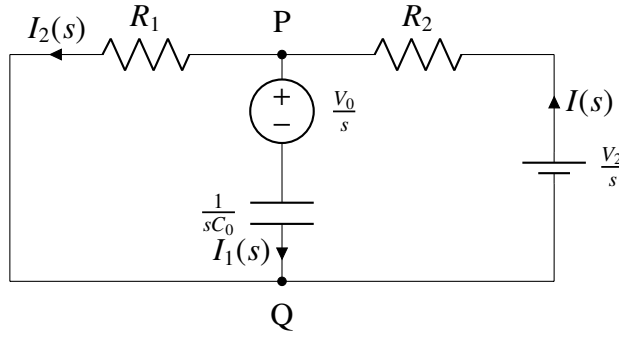
$$\Rightarrow \mathcal{L}\left(u(t) \int_0^t \frac{i_1}{C_0} dt\right) = \frac{1}{sC_0} \mathcal{L}(u(t)i_1) = \frac{I_1(s)}{sC_0}$$

(Using the properties of Laplace Transform)

The above equations after transform

$$\begin{aligned} R_1 I_2(s) + \frac{V_0}{s} + \frac{I_1(s)}{sC_0} &= 0 \\ -R_2 I(s) + \frac{V_2}{s} - R_1 I_2(s) &= 0 \end{aligned}$$

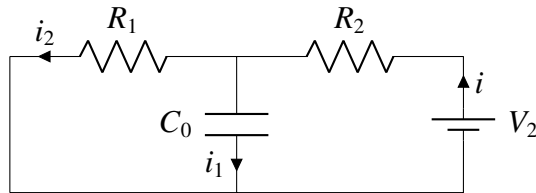
Now after applying the transformation, from time domain to s-domain, we replace the capacitor with an equivalent resistor of resistance $\frac{1}{sC_0}$ and replace I by $I(s)$, V by $\frac{V}{s}$, and hence we get,



4 BILINEAR TRANSFORM

1. In Fig. 2.1, Consider the case when S is switched to Q right in the beginning. Formulate the differential equation.

Solution:



Special Case of (2.5), putting $V_1 = 0$, we get

$$i_2(t) R_1 - \int_0^t \frac{i_1(t)}{C_0} dt = 0 \quad (4.1)$$

$$V_2 - i(t) R_2 - i_2(t) R_1 = 0 \quad (4.2)$$

$$-i(t) R_2 + V_2 - \int_0^t \frac{i_1(t)}{C_0} dt = 0 \quad (4.3)$$

After transformation to s-domain, we get

$$\begin{aligned} R_1 I_2(s) - \frac{I_1(s)}{sC_0} &= 0 \\ -R_2 I(s) + \frac{V_2(s)}{s} - \frac{I_1(s)}{sC_0} &= 0 \end{aligned}$$

Differentiating w.r.t. time, we get,

$$\begin{aligned} R_1 \frac{di_2(t)}{dt} - \frac{i_1(t)}{C_0} &= 0 \\ R_2 \frac{di(t)}{dt} - \frac{i_1(t)}{C_0} &= 0 \end{aligned}$$

We need voltages in the final equation.

$$V_{C_0} = \frac{1}{C_0} q \Rightarrow \frac{dV_{C_0}}{dt} = \frac{1}{C_0} i_1$$

$$i(t) = i_1(t) + i_2(t) = C_0 \frac{dV_{C_0}}{dt} + \frac{1}{R_1} V_{C_0}$$

And this implies, equation (4.3) corresponds to

$$\begin{aligned} -\left(C_0 \frac{dV_{C_0}}{dt} + \frac{1}{R_1} V_{C_0}\right) R_2 + V_2 - V_{C_0} &= 0 \\ \frac{dV_{C_0}}{dt} + \frac{V_{C_0}}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2}\right) &= \frac{V_2}{C_0 R_2} \\ \frac{dV_{C_0}}{dt} + \frac{V_{C_0}}{\frac{C_0 R_1 R_2}{R_1 + R_2}} &= \frac{V_2}{C_0 R_2} \\ \Rightarrow \frac{dV_{C_0}}{dt} + \frac{V_{C_0}}{\tau} &= \frac{V_2}{C_0 R_2} \quad (4.4) \end{aligned}$$

Here, $\tau = \frac{C_0 R_1 R_2}{R_1 + R_2}$ is the time constant of the circuit.

2. Find $H(s)$ considering the output voltage at the capacitor.

Solution:

Recall that $H(s) = \frac{V_{C_0}(s)}{V_2(s)}$

Using KCL at point P, after s-domain transform, we have,

$$\frac{V_{C_0}(s)}{R_1} + \frac{V_{C_0}(s)}{\frac{1}{sC_0}} + \frac{V_{C_0}(s) - V_2(s)}{R_2} = 0$$

$$V_{C_0}(s) \left(\frac{1}{R_1} + sC_0 + \frac{1}{R_2} \right) = \frac{V_2(s)}{R_2}$$

$$\frac{V_{C_0}(s)}{V_2(s)} = H(s) = \frac{1}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right)}$$

$$H(s) = \frac{1}{C_0 R_2 \left(\frac{1}{\tau} + s \right)} \quad (4.5)$$

3. Plot $H(s)$. What kind of filter is it?

Solution: The following code yields the graph
Putting $s = j\omega$ in the above equation, we can see that

$|H(j\omega)|$ is decreasing with increasing frequency.
Clearly, $H(s)$ is a low pass filter

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \quad (4.6)$$

Solution:

From equation (4.4), putting $V = V_{C_0}$ we have

$$\frac{dV}{dt} + \frac{V}{\tau} = \frac{V_2}{C_0 R_2}$$

where $\tau = \frac{C_0 R_1 R_2}{R_1 + R_2}$ is the time constant.

Trapezoidal rule applied to $y' = f(t, y)$ gives

$$y_{n+1} - y_n = \frac{1}{2}h(f(t_n, y_n) + f(t_{n+1}, y_{n+1}))$$

where $h = t_{n+1} - t_n$ is the step size.

Using the trapezoidal rule on (4.4), we get

$$\frac{v_{n+1} - v_n}{t_{n+1} - t_n} = \frac{1}{2} \left[\frac{V_2(u(n) + u(n+1))}{C_0 R_2} - \frac{v_n + v_{n+1}}{\tau} \right]$$

Now, putting $t_n = n \forall n$, we get the difference equation

$$\begin{aligned} v_{n+1} - v_n + \frac{v_n + v_{n+1}}{2\tau} &= \frac{V_2}{2C_0 R_2} (u(n) + u(n+1)) \\ v_{n+1} \left(1 + \frac{1}{2\tau} \right) - v_n \left(1 - \frac{1}{2\tau} \right) &= \frac{V_2}{2C_0 R_2} (u(n) + u(n+1)) \end{aligned} \quad (4.7)$$

where $v_0 = 0$ is the initial condition.

We get $u(n)$ since $V_2 = 0$ when $t < 0$.

5. Find $H(z)$.

Solution:

Taking \mathcal{Z} -Transform on (4.7), we get

$$zV(z) \left(1 + \frac{1}{2\tau} \right) - V(z) \left(1 - \frac{1}{2\tau} \right) = \frac{V_2}{2C_0 R_2} \frac{1+z}{1-z^{-1}}$$

Now since V_2 is constant at $t > 0$, we have

$$\mathcal{Z}(V_2) = \mathcal{Z}(V_2 u(n)) = \frac{V_2}{1-z^{-1}}$$

We can substitute V_2 in the above equation and since we know $H(z) = \frac{V(z)}{V_2(z)}$, we get

$$H(z) \left(\frac{2\tau - 1}{2\tau} \right) \left[\frac{2\tau + 1}{2\tau - 1} - z^{-1} \right] = \frac{1 + z^{-1}}{2C_0 R_2}$$

$$H(z) = \frac{1}{C_0 R_2} \left(\frac{\tau}{2\tau - 1} \right) \frac{1 + z^{-1}}{\frac{2\tau + 1}{2\tau - 1} - z^{-1}}$$

$$H(z) = \left(\frac{\tau}{C_0 R_2} \right) \frac{1 + z^{-1}}{(2\tau + 1) - (2\tau - 1)z^{-1}} \quad (4.8)$$

$$|z| > 1 \text{ and } |z| > \frac{2\tau - 1}{2\tau + 1} \implies \text{ROC is } |z| > 1$$

6. How can you obtain $H(z)$ from $H(s)$?

Solution:

Using bilinear transform on (4.5),

setting $s := \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$, where T is the step size.

$$H(s) = \frac{1}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} C_0 \right)}$$

$$H(s) = \frac{1}{R_2 C_0 \left(\frac{1}{\tau} + \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \right)} \quad \left(\text{Since } \tau = \frac{C_0 R_1 R_2}{R_1 + R_2} \right)$$

$$H(s) = \left(\frac{\tau}{C_0 R_2} \right) \frac{T(1 + z^{-1})}{T + Tz^{-1} + 2\tau - 2\tau z^{-1}}$$

$$H(s) = \left(\frac{\tau}{C_0 R_2} \right) \frac{T(1 + z^{-1})}{(2\tau + T) - (2\tau - T)z^{-1}}$$

Putting step size $T = 1$, we get $H(z)$

Observe that

$$H(s) \Big|_{s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}} = H(z)$$

7. Find $v(n)$. Verify using ngspice and the difference equation.

Solution:

We know that $V(z) = H(z)V_2(z)$ and $V_2(z) =$

$$\frac{V_2}{1 - z^{-1}}$$

Using (4.8), we get

$$V(z) = \frac{1}{C_0 R_2 \left(\frac{1}{\tau} + \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \right)} \frac{V_2}{1 - z^{-1}}$$