Fourier Series

Sumanth N R

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***Abstract*—This manual provides a simple introduction to Fourier Series**

1 Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution:
The following code yields the graph

```
wget https://raw.githubusercontent.com/
    Sigma1084/EE3900/master/charger/codes/
    Ex1_1_plotxt.py
```

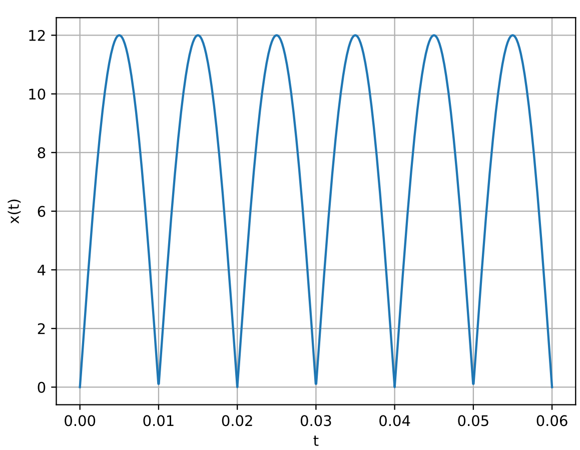

Fig. 1.1: Plot of $x(t)$

1.2 Show that $x(t)$ is periodic and find its period.
Solution:
We know that $|\sin(x)|$ is periodic with fundamental period of π.
$\implies$ Fundamental period of $A|\sin(ax)$ is $\frac{\pi}{a}$
Fundamental period of $A_0|\sin(2\pi f_0 t)|$ is $\frac{\pi}{2\pi f_0}$
$\implies$ Fundamental period of $x(t)$ is $\frac{1}{2f_0}$

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.2}$$

Solution:
Consider for some $n \in \mathbb{Z}$,

$$x(t)e^{j2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-n) f_0 t}$$

We know using the periodicity of $e^{j2\pi k f_0 t}$,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi k f_0 t} dt = \begin{cases} \frac{1}{f_0} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Now,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{j2\pi n f_0 t} dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-n) f_0 t}$$
$$= \frac{c_n}{f_0}$$

$$\implies c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{j2\pi n f_0 t} dt$$

2.2 Find c_k for (1.1)
Solution:

$$
\begin{aligned}
c_k &= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{j2\pi k f_0 t} dt \\
&= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| e^{j2\pi k f_0 t} dt \\
&= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| \cos(2\pi k f_0 t) dt \\
&\quad + j f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| \sin(2\pi k f_0 t) dt \\
&= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| \cos(2\pi k f_0 t) dt + 0 \\
&= 2 f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 t\right) \cos(2\pi k f_0 t) dt \\
&= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \sin\left(2\pi \left(n+1\right) f_0 t\right) dt \\
&\quad - f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \sin\left(2\pi \left(n-1\right) f_0 t\right) dt \\
&= f_0 A_0 \frac{1+(-1)^n}{2\pi f_0} \left(\frac{1}{n+1} - \frac{1}{n-1}\right) \\
&= \begin{cases} \frac{2A_0}{\pi\left(1-n^2\right)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}
\end{aligned}
\tag{2.3}
$$

2.3 Verify (1.1) and (2.1) using python.
Solution: The following code yields the graph

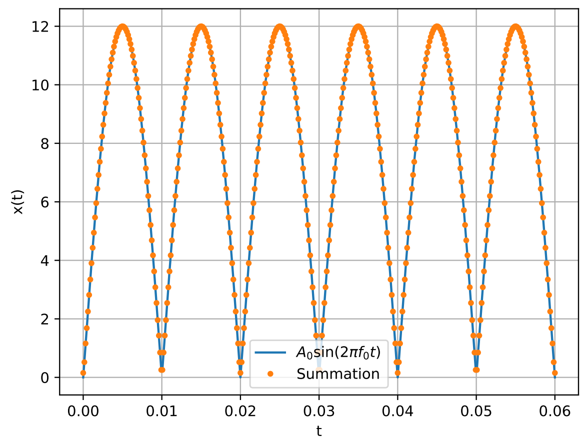

Fig. 2.3: Verification of (2.1).

```
wget https://raw.githubusercontent.com/
    Sigma1084/EE3900/master/charger/codes/
    Ex2_3_verify_xt.py
```

2.4 Show that

$$
x(t) = \sum_{k=0}^{\infty} \left(a_k \cos j2\pi k f_0 t + b_k \sin j2\pi k f_0 t\right) \tag{2.4}
$$

and obtain the formulae for a_k and b_k.
Solution: From (2.1),

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \\
&= c_0 + \sum_{k=1}^{\infty} \left(c_k e^{j2\pi k f_0 t} + c_{-k} e^{-j2\pi k f_0 t}\right) \\
&= c_0 + \sum_{k=1}^{\infty} \left(c_k + c_{-k}\right) \cos\left(2\pi k f_0 t\right) \\
&\quad + j \sum_{k=0}^{\infty} \left(c_k - c_{-k}\right) \sin\left(2\pi k f_0 t\right)
\end{aligned}
$$

Hence, for $k \geq 0$,

$$
a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.5}
$$

$$
b_k = j\left(c_k - c_{-k}\right) \tag{2.6}
$$

2.5 Find a_k and b_k for (1.1)
Solution:
Using the expression for c_k, from (2.3) and using (2.5) and (2.6), we have,

$$
\begin{aligned}
a_k &= \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi\left(1-k^2\right)} & k \text{ even} \\ 0 & k \text{ odd} \end{cases} \\
b_k &= 0
\end{aligned}
$$

2.6 Verify (2.4) using python
Solution:
The following code block yields the graph

```
wget https://raw.githubusercontent.com/
    Sigma1084/EE3900/master/charger/codes/
    Ex2_3_verify_xt_real.py
```

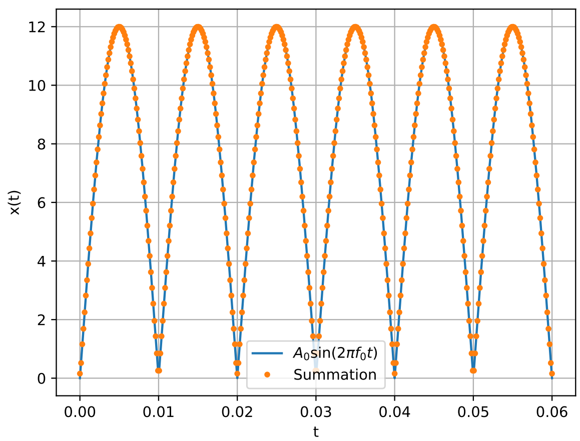

Fig. 2.6: Verification of (2.4).

3 Fourier Transform

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \tag{3.3}$$

3.3 Show that

$$g(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} G(f) e^{-j2\pi f t_0} \tag{3.4}$$

Solution:

$$\begin{aligned} \mathcal{F}\left(g\left(t-t_0\right)\right) &= \int_{-\infty}^{\infty} g\left(t-t_0\right) e^{-j2\pi f t} dt \\ &= \int_{-\infty}^{\infty} g\left(t\right) e^{-j2\pi f (t+t_0)} dt \\ &= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} g\left(t\right) e^{-j2\pi f t} dt \\ &= G(f) e^{-j2\pi f t_0} \end{aligned}$$

$$\implies g\left(t-t_0\right) \overset{\mathcal{F}}{\longleftrightarrow} G(f) e^{-j2\pi f t_0}$$

Hence proved

3.4 Show that

$$G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.5}$$

Solution:
Using the definition of inverse Fourier Transform,

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df$$

Now, putting $-f := t,\ t := f \implies df = dt$,

$$g(-f) = \int_{-\infty}^{\infty} G(t) e^{-j2\pi f t} dt$$

$$\implies G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.6}$$

3.5 $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} ?$
Solution:

$$\begin{aligned} \delta(t) &\overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt \\ &= e^{-j2\pi f(0)} \int_{-\infty}^{\infty} \delta(t) dt \end{aligned}$$

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1 \tag{3.7}$$

3.6 $e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} ?$
Solution: Suppose $g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$

$$\begin{aligned} g(t) e^{-j2\pi f_0 t} &\overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t) e^{-j2\pi (f_0+f) t} dt \\ &= G(f+f_0) \end{aligned}$$

Now, using (3.7) and (3.6) , we can get,

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1 \overset{\mathcal{F}}{\longleftrightarrow} \delta(-t)$$

Putting $g(t) = 1$ and hence, $G(f) = \delta(-f) = \delta(f)$, we get $G(f+f_0) = \delta(f+f_0)$. Hence,

$$e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(f+f_0) \tag{3.8}$$

3.7 $\cos(2\pi f_0 t) \overset{\mathcal{F}}{\longleftrightarrow} ?$
Solution: We can write

$$\cos(2\pi f_0 t) = \frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$$

Using (3.8) and the linearity of Fourier Transform, we get,

$$\cos(2\pi f_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\Big(\delta(f-f_0) + \delta(f+f_0)\Big)$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.
Solution: Using (2.1), we have,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$$

Using (3.8), we get,

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_0)$$
$$= \frac{2A_0}{\pi} \sum_{k=-\infty}^{\infty} \frac{\delta(f - 2kf_0)}{1 - 4k^2}$$

```
wget https://raw.githubusercontent.com/
    Sigma1084/EE3900/master/charger/codes/
    Ex3_08_x-fourier.py
```

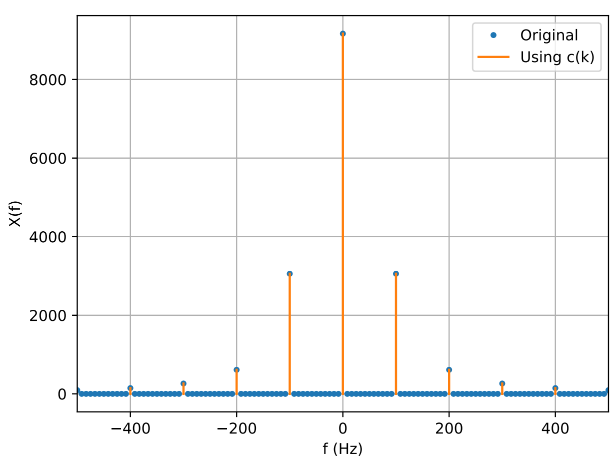

Fig. 3.8: Fourier Transform of $x(t)$

3.9 Show that

$$\mathrm{rect}\,(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}\,(t) \tag{3.9}$$

Verify using python.
Solution:

$$\mathrm{rect}\,(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \mathrm{rect}\,(t)\, e^{-j2\pi f t} dt$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi f t} dt$$
$$= \frac{1}{-j2\pi f}\left(e^{-j\pi f} - e^{j\pi f}\right)$$
$$= \frac{\sin(\pi f)}{\pi f} = \mathrm{sinc}\,(f)$$

```
wget https://raw.githubusercontent.com/
    Sigma1084/EE3900/master/charger/codes/
    Ex3_09_rect-fourier.py
```

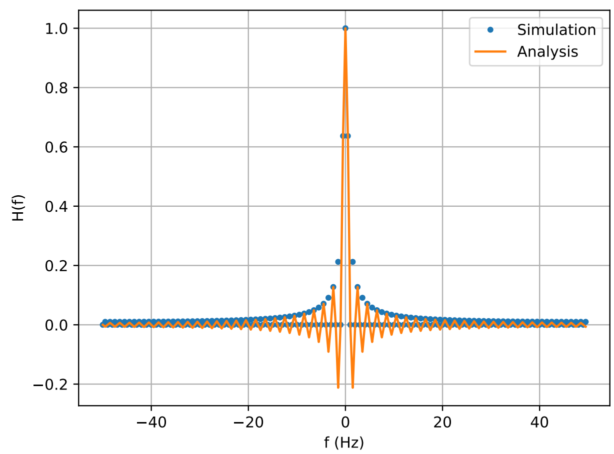

Fig. 3.9: Fourier Transform of $x(t)$

3.10 $\mathrm{sinc}\,(t) \overset{\mathcal{F}}{\longleftrightarrow}$? Verify using python.
Solution:
Using (3.9), and (3.6), we get,

$$\mathrm{sinc}\,(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}\,(-f) = \mathrm{rect}\,(f) \tag{3.10}$$

```
wget https://raw.githubusercontent.com/
    Sigma1084/EE3900/master/charger/codes/
    Ex3_10_sinc-fourier.py
```

4 Filter

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
Solution: The function $H(f)$ is a low pass filter which filters out even harmonics and leaves the zero frequency component behind. The rectangular function represents an ideal

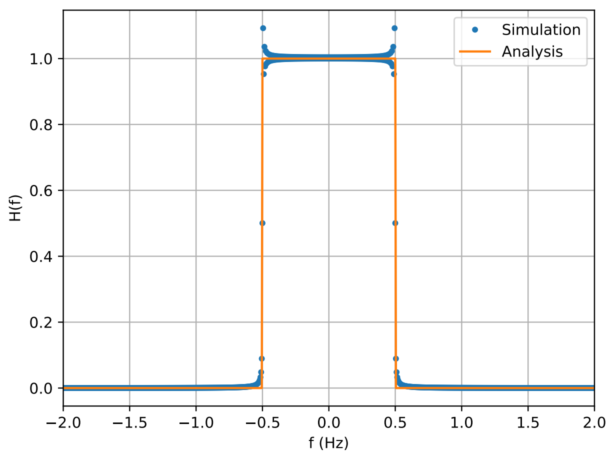

Fig. 3.10: Fourier Transform of $x(t)$

low pass filter. Suppose the cutoff frequency is $f_c = 50$ Hz, then

$$H(f) = \text{rect}\left(\frac{f}{2f_c}\right) = \begin{cases} 1 & |f| < f_c \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Multiplying by a scaling factor to get DC 5V,

$$H(f) = \frac{\pi V_0}{2A_0}\text{rect}\left(\frac{f}{2f_c}\right)$$

where $V_0 = 5$ V.

4.2 Find $h(t)$.

Solution: Suppose $g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$. Then, for some nonzero $a \in \mathbb{R}$

$$\begin{aligned} g(at) &\overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(at)e^{-j2\pi ft}\,dt \\ &= \frac{1}{a}\int_{-\infty}^{\infty} g(u)e^{\left(-j2\pi\frac{f}{a}t\right)}\,dt \\ &= \frac{1}{a}G\left(\frac{f}{a}\right) \end{aligned}$$

where we have substituted $u := at$.

$$\therefore h(t) = \frac{2\pi V_0}{A_0}f_c\text{sinc}\left(2f_c t\right)$$

4.3 Verify your result using through convolution.

Solution: Fourier transform of $x(t)$ and $h(t)$ respectively is

$$X(f) = \sum_{k=-\infty}^{\infty}\frac{2A_0}{\pi}\frac{\delta\left(f+2kf_0\right)}{1-4k^2} \tag{4.2}$$

$$H(f) = \frac{\pi V_0}{2A_0}\text{rect}\left(\left(\frac{f}{2f_c}\right)\right) \tag{4.3}$$

$$X(f)\times H(f) = \sum_{k=-\infty}^{\infty} V_0\frac{\delta\left(f+2kf_0\right)}{1-4k^2}\times\text{rect}\left(\left(\frac{f}{2f_c}\right)\right) \tag{4.4}$$

$$X(f)\times H(f) = \sum_{k=0}^{0} V_0\frac{\delta\left(f+2kf_0\right)}{1-4k^2} \tag{4.5}$$

Hence,

$$X(f)\times H(f) = V_0\frac{\delta\left(f\right)}{1-4\times 0} \tag{4.6}$$

$$X(f)\times H(f) = V_0\delta\left(f\right) \tag{4.7}$$

Since $1 \overset{\mathcal{F}}{\longleftrightarrow} \delta\left(0\right)$, Hence,

$$V_0\delta\left(t\right) \overset{\mathcal{F}}{\longleftrightarrow} V_0\times 1 \tag{4.8}$$

$$\implies H(t)\circledast x(t) = V_0 \tag{4.9}$$

Hence verified. The following python yields

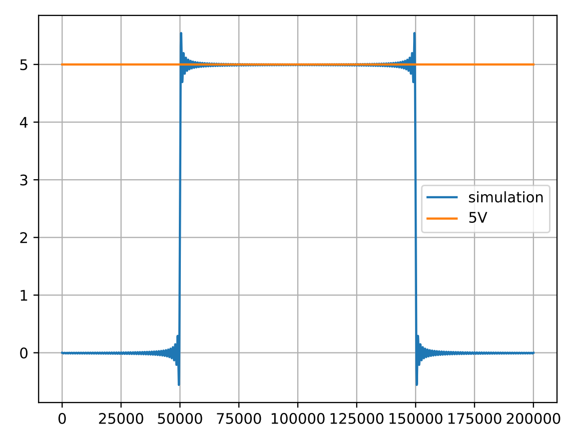

Fig. 4.3: Convolution of the $x(t)$ and $h(t)$.

the plot 4.3.

```
wget https://raw.githubusercontent.com/
    Sigma1084/EE3900/master/charger/codes/
    Ex4_convolution.py
```

5 Filter Design

5.1 Design a Butterworth filter for $H(f)$

Solution: The transfer function of a Butterworth filter is given by

$$|H_n(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^{2n}}} \tag{5.1}$$

where n is the order of the filter and f_c is the cutoff frequency
Let the passband and stopband frequency thresholds be 50 Hz and 100 Hz and their corresponding attenuations be -1 dB and -5 dB respectively

$$A_p = 10\log_{10}\left|H_n(f_p)\right|^2 \tag{5.2}$$

$$= -10\log_{10}\left(1 + \left(\frac{f_p}{f_c}\right)^{2n}\right) \tag{5.3}$$

$$A_s = -10\log_{10}\left(1 + \left(\frac{f_s}{f_c}\right)^{2n}\right) \tag{5.4}$$

$$\implies n = \frac{\log\left(\frac{10^{-\frac{A_p}{10}}-1}{10^{-\frac{A_s}{10}}-1}\right)}{2\log\left(\frac{f_p}{f_s}\right)} \approx 1.53 \tag{5.5}$$

Hence, we choose a 2nd order Butterworth filter with

$$f_c = \frac{f_p}{\left(10^{-\frac{A_p}{10}} - 1\right)^{\frac{1}{2n}}} \approx 77.74\,\text{Hz} \tag{5.6}$$

5.2 Design a Chebyschev filter for $H(f)$
Solution: The transfer function of a Chebyshev filter is given by

$$|H_n(f)| = \frac{1}{\sqrt{1 + \epsilon^2 T_n^2\left(\frac{f}{f_c}\right)}} \tag{5.7}$$

where ϵ is the ripple factor, f_c is the cutoff frequency and T_n is a Chebyshev polynomial of the n^{th} order
Assuming the same parameters as before along with a ripple of 0.1 dB, we get

$$\epsilon = \sqrt{10^{\frac{\delta}{10}} - 1} \approx 0.15 \tag{5.8}$$

Also, assume that $f_c = f_p \implies \frac{f_s}{f_c} > 1$

$$A_s = -10\log_{10}\left(1 + \epsilon^2 T_n^2\left(\frac{f_s}{f_c}\right)\right) \tag{5.9}$$

$$\implies T_n\left(\frac{f_s}{f_c}\right) = \frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon} \tag{5.10}$$

$$\implies \cosh\left(n\cosh^{-1}\left(\frac{f_s}{f_c}\right)\right) = \frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon} \tag{5.11}$$

Thus

$$n = \frac{\cosh^{-1}\left(\frac{\sqrt{10^{-\frac{A_s}{10}}-1}}{\epsilon}\right)}{\cosh^{-1}\left(\frac{f_s}{f_c}\right)} \approx 2.26 \tag{5.12}$$

Hence, we choose a 3rd order Chebyshev filter

5.3 Design a circuit for your Butterworth filter
Solution: Using the table of normalized Butterworth coefficients, we can see that for a 2nd order Butterworth filter

$$C_1 = 1.4142\,\text{F} \tag{5.13}$$

$$L_2 = 1.4142\,\text{H} \tag{5.14}$$

On denormalizing these values, we get

$$C_1' = \frac{C_1}{2\pi f_c} = 2.89\,\text{mF} \tag{5.15}$$

$$L_2' = \frac{L_2}{2\pi f_c} = 2.89\,\text{mH} \tag{5.16}$$

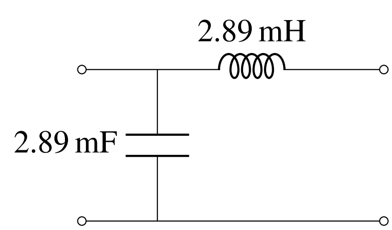

Fig. 5.3: 2nd order Butterworth filter circuit

5.4 Design a circuit for your Chebyschev filter
Solution: Using the table of normalized Chebyshev coefficients, we can see that for a 3rd order Chebyshev filter

$$C_1 = 1.4328\,\text{F} \tag{5.17}$$

$$L_2 = 1.5937\,\text{H} \tag{5.18}$$

$$C_3 = 1.4328\,\text{F} \tag{5.19}$$

On denormalizing these values, we get

$$C_1' = \frac{C_1}{2\pi f_c} = 4.56\,\text{mF} \tag{5.20}$$

$$L_2' = \frac{L_2}{2\pi f_c} = 5.07\,\text{mH} \tag{5.21}$$

$$C_3' = \frac{C_3}{2\pi f_c} = 4.56\,\text{mF} \tag{5.22}$$

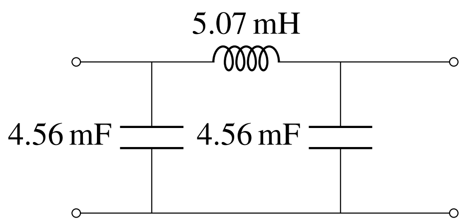

Fig. 5.4: 3rd order Chebyshev filter circuit