

# Circuits and Transforms

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**Abstract**—This manual provides a simple introduction to Transforms

## 1 DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

2. The Laplace transform of  $g(t)$  is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

## 2 LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes  $q_1 \mu C$ . Then S is switched to position Q. After a long time, the charge on the capacitor is  $q_2 \mu C$ .

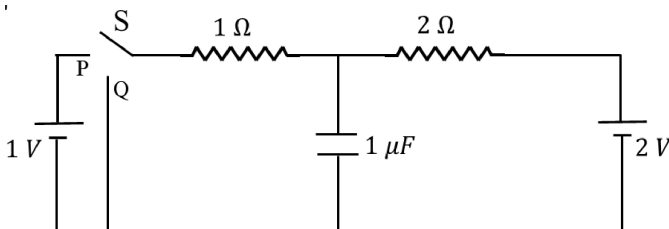
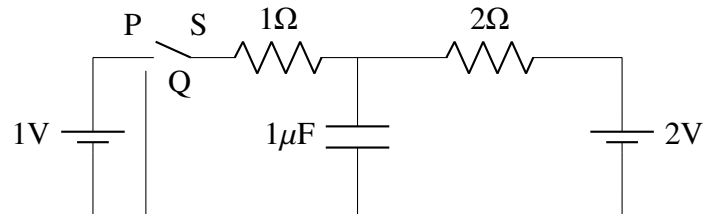


Fig. 2.1

2. Draw the circuit using latex-tikz.

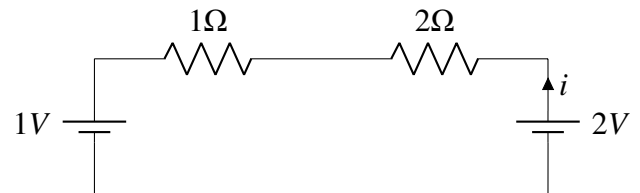
**Solution:**



3. Find  $q_1$ .

**Solution:**

Current through the capacitor is 0 at steady state. So, current only flows in the outer loop.



$$i = \frac{(2 - 1) V}{(1 + 2) \Omega}$$

$$i = \frac{1}{3} A$$

Potential difference across the capacitor,

$$V_C = 2 - 2 \times \frac{1}{3} = \frac{4}{3} V$$

$$q_1 = CV_C$$

$$q_1 = 1\mu F \times \frac{4}{3} V$$

$$q_1 = \frac{4}{3} \mu C$$

4. Show that the Laplace transform of  $u(t)$  is  $\frac{1}{s}$  and find the ROC.

**Solution:**

By the definition of Laplace transform

$$u(t) \xrightarrow{\mathcal{L}} \int_{-\infty}^{\infty} u(t)e^{-st} dt = \int_0^{\infty} e^{-st} dt$$

$$= \left. \frac{e^{-st}}{-s} \right|_0^{\infty} = \frac{1}{s}$$

$\lim_{s \rightarrow \infty} e^{-st} = 0$  only when  $Re(s) > 0$ .

The ROC is  $Re(s) > 0$ .

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{H}} L \frac{1}{s+a}, \quad a > 0 \quad (2.1)$$

and find the ROC.

**Solution:**

By the definition of Laplace transform

$$\begin{aligned} e^{-at}u(t) &\xrightarrow{\mathcal{L}} \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \\ &= \int_0^{\infty} e^{-(s+a)t}dt \\ &= \left. \frac{e^{-(s+a)t}}{-(s+a)} \right|_0^{\infty} = \frac{1}{s+a} \end{aligned}$$

$\lim_{s \rightarrow \infty} e^{-(s+a)t} = 0$  only when  $\text{Re}(s+a) > 0$ .

The ROC is  $\text{Re}(s+a) > 0$  or  $\text{Re}(s) > -a$ .

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

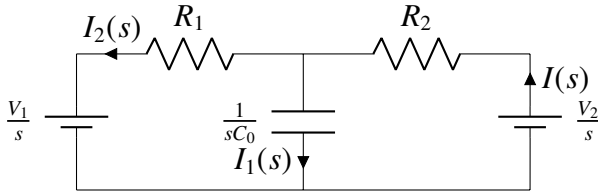


Fig. 2.2

Applying KCL at the upper junction, we have,

$$\frac{V_{C_0}(s) - V_1(s)}{R_1} + \frac{V_{C_0}(s) - V_2(s)}{R_2} + \frac{V_{C_0}(s)}{\frac{1}{sC_0}} = 0$$

$$V_{C_0}(s) \left( \frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{V_1(s)}{R_1} + \frac{V_2(s)}{R_2}$$

$$V_{C_0}(s) = \frac{V_1(s)R_2 + V_2(s)R_1}{R_1 + R_2 + sC_0R_1R_2}$$

$$V_{C_0}(s) = \frac{R_2 + 2R_1}{s(R_1 + R_2 + sC_0R_1R_2)}$$

after substitution.

7. Find  $v_{C_0}(t)$ . Plot using python.

**Solution:**

Factoring  $V_{C_0}(s)$ , we have

$$\begin{aligned} V_{C_0}(s) &= \left( \frac{1}{R_1 + R_2} \right) \left( \frac{(R_2 + 2R_1)}{s} - \frac{R_1R_2C_0(R_2 + 2R_1)}{R_1 + R_2 + sC_0R_1R_2} \right) \\ &= \left( \frac{R_2 + 2R_1}{R_1 + R_2} \right) \left( \frac{1}{s} - \frac{R_1R_2C_0}{R_1 + R_2 + sC_0R_1R_2} \right) \\ &= \left( \frac{R_2 + 2R_1}{R_1 + R_2} \right) \left( \frac{1}{s} - \frac{1}{\frac{1}{R_2C_0} + \frac{1}{R_1C_0} + s} \right) \end{aligned}$$

Applying inverse Laplace transform,

$$v_{C_0}(t) = \frac{R_2 + 2R_1}{R_1 + R_2} \left( u(t) - e^{-\left(\frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right)t} u(t) \right) \quad (2.4)$$

Using the values  $R_1 = 1 \Omega$ ,  $R_2 = 2 \Omega$ ,  $C_0 = 1 \mu F$ , we get

$$v_{C_0}(t) = \frac{4}{3}u(t) \left( 1 - e^{-1.5 \times 10^6 t} \right)$$

The following code yields  $v_{C_0}(t)$ .

```
wget https://github.com/Sigma1084/EE3900/
blob/master/cktsig/code/Ex2_plotVt.py
```

$$u(t) \xleftrightarrow{\mathcal{H}} LV_1(s) \quad (2.2)$$

$$2u(t) \xleftrightarrow{\mathcal{H}} LV_2(s) \quad (2.3)$$

Find the voltage across the capacitor  $V_{C_0}(s)$ .

**Solution:**

Using the Transform from time domain to s-domain

$$\begin{aligned} V_1(s) &= \mathcal{L}(u(t)) = \frac{V_1}{s} \\ V_2(s) &= \mathcal{L}(2u(t)) = \frac{V_2}{s} \end{aligned}$$

Potential across the capacitor is  $V_{C_0}(s)$  and the assuming bottom is grounded.

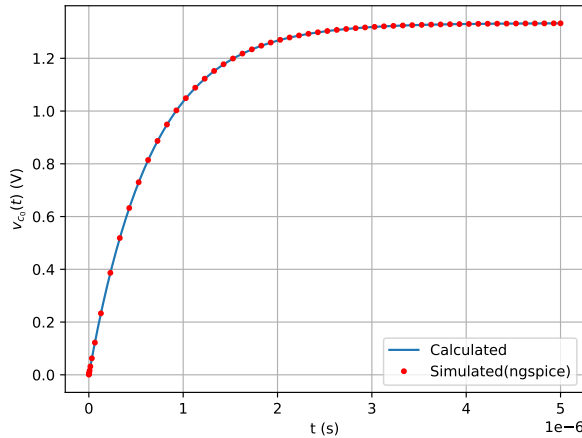


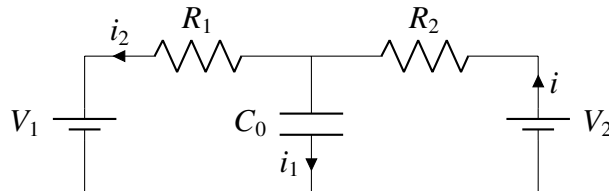
Fig. 2.3

8. Verify your result using ngspice. The following code yields  $v_{C_0}(t)$ . using ngspice

```
wget https://github.com/Sigma1084/EE3900/
blob/master/cktsig/code/Ex2.spice
```

9. Obtain Fig. 2.2 using the equivalent differential equation.

**Solution:**



Using KVL on the left and right loops, we get

$$i_2(t) R_1 + V_1 - \int_0^t \frac{i_1(t)}{C_0} dt = 0 \quad (2.5)$$

$$-i(t) R_2 + V_2 - \int_0^t \frac{i_1(t)}{C_0} dt = 0 \quad (2.6)$$

Taking the Laplace Transform after multiplying  $u(t)$  in both equations,

$$\mathcal{L}\left(R_1 i_2(t)u(t) + V_1 u(t) - u(t) \int_0^t \frac{i_1}{C_0} dt\right) = 0$$

$$\mathcal{L}\left(-R_2 i(t)u(t) + V_2 u(t) - u(t) \int_0^t \frac{i_1}{C_0} dt\right) = 0$$

Suppose  $i(t)u(t) \xrightarrow{\mathcal{L}} I(s)$  and  $Vu(t) \xrightarrow{\mathcal{L}} \frac{V}{s}$

$$\Rightarrow \mathcal{L}\left(u(t) \int_0^t \frac{i_1}{C_0} dt\right) = \frac{1}{sC_0} \mathcal{L}(u(t)i_1) = \frac{I_1(s)}{sC_0}$$

(Using the properties of Laplace Transform)

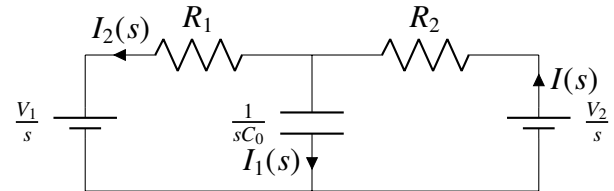
The above equations after transform

$$R_1 I_2(s) + \frac{V_1}{s} - \frac{I_1(s)}{sC_0} = 0 \quad (2.7)$$

$$-R_2 I(s) + \frac{V_2}{s} - \frac{I_1(s)}{sC_0} = 0 \quad (2.8)$$

Now after applying the transformation, from time domain to s-domain, we replace the capacitor with an equivalent resistor of resistance  $\frac{1}{sC_0}$  and replace  $I$  by  $I(s)$ ,  $V$  by  $\frac{V}{s}$

The equivalent s-domain circuit is now,



### 3 INITIAL CONDITIONS

1. Find  $q_2$  in Fig. 2.1

**Solution:**

Current through the capacitor is 0 in steady state. Since the current flows only in the outer loop, let the current in the outer loop be  $I$ .

$$I = \frac{2V}{(1+2)\Omega} = \frac{2}{3}A$$

So, the potential difference across the capacitor is

$$V_{C_0} = 2 - \left(\frac{2}{3} \times 2\right) = \frac{2}{3}V$$

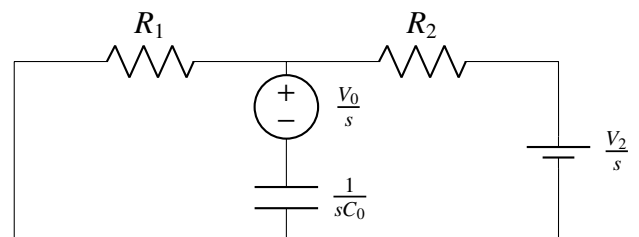
$$q_2 = CV_{C_0}$$

$$q_2 = 1\mu C \times \frac{2}{3}V$$

$$q_2 = \frac{2}{3}\mu C$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables  $R_1, R_2, C_0$  for the passive elements. Use latex-tikz.

**Solution:**



3.  $V_{C_0}(s) = ?$

**Solution:**

Applying KCL at the upper junction and assuming the bottom is grounded, we have,

$$\begin{aligned} \frac{V_{C_0}(s)}{R_1} + \frac{V_{C_0}(s) - \frac{V_2}{s}}{R_2} + \frac{V_{C_0}(s) - \frac{V_0}{s}}{\frac{1}{sC_0}} &= 0 \\ V_{C_0}(s) \left( \frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) &= \frac{V_2}{sR_2} + V_0C_0 \\ V_{C_0}(s) &= \frac{\frac{V_2}{sR_2} + V_0C_0}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \\ V_{C_0}(s) &= V_0 \left( \frac{1}{\frac{1}{C_0} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \\ &+ \frac{V_2}{R_2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \left( \frac{1}{s} - \frac{1}{\frac{1}{C_0} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \end{aligned}$$

4.  $v_{C_0}(t) = ?$  Plot using python.

**Solution:**

Applying inverse Laplace transform to  $V_{C_0}(s)$ , we have,

$$\begin{aligned} v_{C_0}(t) &= V_0 e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} u(t) \\ &+ \frac{V_2}{R_2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \left( 1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} \right) u(t) \end{aligned}$$

Substituting the values  $R_1 = 1, R_2 = 2, C_0 = 1\mu F, V_0 = \frac{4}{3}V, V_2 = 2V$

$$v_{C_0}(t) = \frac{2}{3} \left( 1 + e^{-(1.5 \times 10^6)t} \right) u(t) \quad (3.1)$$

The following code yields  $v_{C_0}(t)$ .

```
wget https://github.com/Sigma1084/EE3900/
blob/master/cktsig/code/Ex3_plotVt.py
```

5. Verify your result using ngspice. The following code yields  $v_{C_0}(t)$ . using ngspice

```
wget https://github.com/Sigma1084/EE3900/
blob/master/cktsig/code/Ex3.spice
```

6. Find  $v_{C_0}(0-), v_{C_0}(0+)$  and  $v_{C_0}(\infty)$ .

**Solution:**

Using the initial conditions, we have,

$$v_{C_0}(0-) = \frac{q_1}{C_0} = \frac{4}{3}V$$

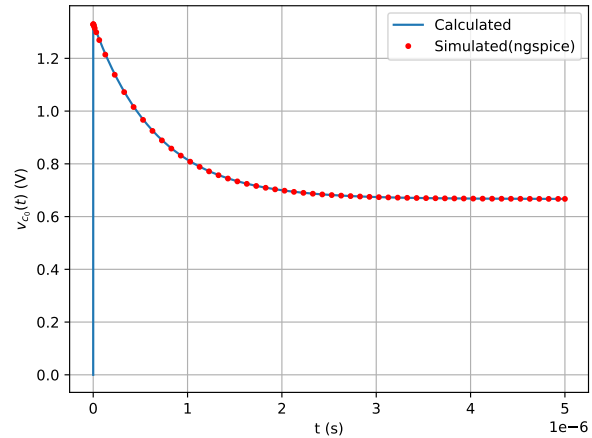


Fig. 3.1

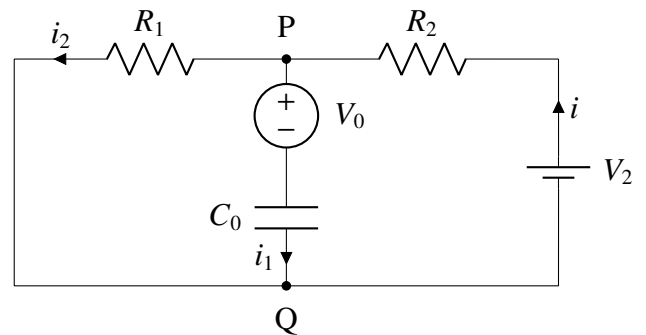
Using (3.1), we have,

$$\begin{aligned} v_{C_0}(0+) &= \frac{2}{3} \left( 1 + e^{-(1.5 \times 10^6)0} \right) u(0+) = \frac{4}{3}V \\ v_{C_0}(\infty) &= \frac{2}{3} \left( 1 + e^{-(1.5 \times 10^6)\infty} \right) u(\infty) = \frac{2}{3}V \end{aligned}$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.

**Solution:**

Constructing the circuit,



Using KVL on both the left and outer loops, we have

$$\begin{aligned} -i_2(t) R_1 + V_0 + \int_0^t \frac{i_1(t)}{C_0} dt &= 0 \\ -i(t) R_2 + V_2 - i_2(t) R_1 &= 0 \end{aligned}$$

Taking the Laplace Transform after multiplying  $u(t)$  in both equations,

$$\begin{aligned} \mathcal{L} \left( -R_1 i_2(t) u(t) + V_0 u(t) + u(t) \int_0^t \frac{i_1}{C_0} dt \right) &= 0 \\ \mathcal{L} (-R_2 i(t) u(t) + V_2 u(t) - R_1 i_2(t) u(t)) &= 0 \end{aligned}$$

Suppose  $i(t)u(t) \xrightarrow{\mathcal{L}} I(s)$  and  $Vu(t) \xrightarrow{\mathcal{L}} \frac{V}{s}$

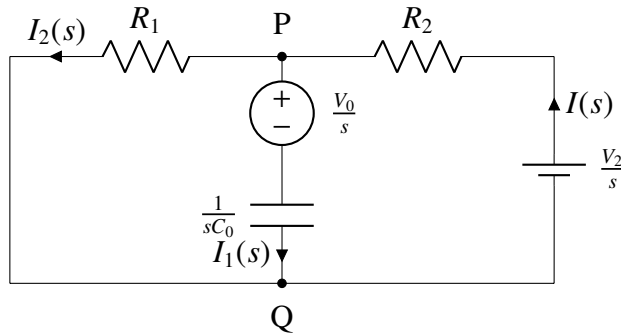
$$\Rightarrow \mathcal{L}\left(u(t) \int_0^t \frac{i_1}{C_0} dt\right) = \frac{1}{sC_0} \mathcal{L}(u(t)i_1) = \frac{I_1(s)}{sC_0}$$

(Using the properties of Laplace Transform)

The above equations after transform

$$\begin{aligned} R_1 I_2(s) + \frac{V_0}{s} + \frac{I_1(s)}{sC_0} &= 0 \\ -R_2 I(s) + \frac{V_2}{s} - R_1 I_2(s) &= 0 \end{aligned}$$

Now after applying the transformation, from time domain to s-domain, we replace the capacitor with an equivalent resistor of resistance  $\frac{1}{sC_0}$  and replace  $I$  by  $I(s)$ ,  $V$  by  $\frac{V}{s}$ , and hence we get,



#### 4 BILINEAR TRANSFORM

1. In Fig. 2.1, Consider the case when  $S$  is switched to  $Q$  right in the beginning. Formulate the differential equation.

**Solution:**

Special Case of (2.5), putting  $V_1 = 0$ , we get

$$i_2(t) R_1 - \int_0^t \frac{i_1(t)}{C_0} dt = 0 \quad (4.1)$$

$$-i(t) R_2 + V_2 - \int_0^t \frac{i_1(t)}{C_0} dt = 0 \quad (4.2)$$

The above equations after transform

$$R_1 I_2(s) - \frac{I_1(s)}{sC_0} = 0 \quad (4.3)$$

$$-R_2 I(s) + \frac{V_2}{s} - \frac{I_1(s)}{sC_0} = 0 \quad (4.4)$$

2. Find  $H(s)$  considering the output voltage at the capacitor.
3. Plot  $H(s)$ . What kind of filter is it?

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \quad (4.5)$$

5. Find  $H(z)$ .
6. How can you obtain  $H(z)$  from  $H(s)$ ?