

Analysis of CIR and Heston Model

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Introduction

Stochastic Differential Equations(SDEs) and Ito Doebelin Formula

Stochastic Differential Equation

An SDE is a differential equation that contains a stochastic term

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

where μ is called the **drift term** and σ is called the **diffusion term**.

Theorem (Ito Doebelin Formula)

Let $f(t, X)$ be a continuous function in time t and X where X is a Random Variable that is continuous in t . Then, the Ito Doebelin Formula is given by

$$df(t, X) = \left(f_t + \frac{1}{2} f_{XX} \right) dt + f_X dX$$

Ito's Integral

Ito's Integral is defined as follows

$$I_t := I(t) = \int_0^t \Delta(s) dW_s$$

where $\Delta(s)$ is an adapted process and W_s is a Brownian motion. It has the following properties

- ➊ $I(t)$ is a continuous process.
- ➋ $I(t)$ is a martingale.
- ➌ Zero Mean Property $\mathbb{E} \left[\int_0^t \Delta(s) dW_s \right] = 0$
- ➍ Ito Isometry $\mathbb{E} \left[\left(\int_0^t \Delta(s) dW_s \right)^2 \right] = \mathbb{E} \left[\int_0^t \Delta(s)^2 ds \right]$

CIR Model

Rate Models

Introduced by Oldrich Vasicek (1977) to model interest rates.

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t$$

where κ is the **mean reversion rate**, θ is the **long term mean** and σ is the **volatility** of the interest rate.

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where κ is the **mean reversion rate**, θ is the **long term mean** and σ is the **volatility** of the interest rate.

The model can be easily solved and the closed form solution is given by

$$r_t \sim \mathcal{N} \left(e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t}), \frac{\sigma^2}{2\kappa} (1 - e^{-2\kappa t}) \right)$$

The main disadvantage is that the interest rate can become negative.

CIR Model

The **CIR** model was introduced by John Carrington **Cox**, Jonathan Edwards **Ingersoll** and Stephen Alan **Ross** (1985) and follows the following SDE.

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

where

- r_t is the **interest rate** at time t
- κ is the **speed of mean reversion**
- θ is the **long term mean**
- σ is the **volatility coefficient**
- W_t is a **Brownian Motion**

with initial rate r_0 given.

Proposition

The exact solution of CIR is given by

$$r_t = e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t}) + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s$$

Proof.



Proposition

The exact solution of CIR is given by

$$r_t = e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t}) + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s$$

Proof.

$$\begin{aligned} dr_t &= \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t \\ e^{\kappa t} dr_t + \kappa e^{\kappa t} r_t dt &= \kappa\theta e^{\kappa t} dt + \sigma e^{\kappa t} \sqrt{r_t} dW_t \end{aligned}$$



Proposition

The exact solution of CIR is given by

$$r_t = e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t}) + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s$$

Proof.

$$\begin{aligned} dr_t &= \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t \\ e^{\kappa t}dr_t + \kappa e^{\kappa t}r_tdt &= \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}\sqrt{r_t}dW_t \\ \implies d(e^{\kappa t}r_t) &= \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}\sqrt{r_t}dW_t \quad (\text{Ito Doebelin Formula}) \\ \implies \int_0^t d(e^{\kappa s}r_s) &= \int_0^t \theta\kappa e^{\kappa s}ds + \int_0^t \sigma e^{\kappa s}\sqrt{r_s}dW_s \end{aligned}$$



Proposition

The exact solution of CIR is given by

$$r_t = e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t}) + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s$$

Proof.

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

$$e^{\kappa t} dr_t + \kappa e^{\kappa t} r_t dt = \kappa \theta e^{\kappa t} dt + \sigma e^{\kappa t} \sqrt{r_t} dW_t$$

$$\implies d(e^{\kappa t} r_t) = \kappa \theta e^{\kappa t} dt + \sigma e^{\kappa t} \sqrt{r_t} dW_t \quad (\text{Ito Doebelin Formula})$$

$$\implies \int_0^t d(e^{\kappa s} r_s) = \int_0^t \kappa \theta e^{\kappa s} ds + \int_0^t \sigma e^{\kappa s} \sqrt{r_s} dW_s$$

$$\therefore r_t = e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t}) + \sigma \int_0^t e^{\kappa s} \sqrt{r_s} dW_s$$



Proposition

The expectation of r_t is given by

$$\mathbb{E}[r_t] = e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t})$$

Proof.

$$\begin{aligned}\mathbb{E}[r_t] &= \mathbb{E}\left[e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t}) + \sigma \int_0^t e^{\kappa s} \sqrt{r_s} dW_s\right] \\ &= e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t}) + \mathbb{E}\left[\sigma \int_0^t e^{\kappa s} \sqrt{r_s} dW_s\right] \\ \therefore \mathbb{E}[r_t] &= e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t})\end{aligned}$$



Proposition

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$$\mathbb{E}[r_t] = e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t})$$

Proof.

$$\begin{aligned}\mathbb{E}[r_t] &= \mathbb{E}\left[e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t}) + \sigma \int_0^t e^{\kappa s} \sqrt{r_s} dW_s\right] \\ &= e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t}) + \mathbb{E}\left[\sigma \int_0^t e^{\kappa s} \sqrt{r_s} dW_s\right] \\ \therefore \mathbb{E}[r_t] &= e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t})\end{aligned}$$



Note that $\lim_{t \rightarrow \infty} \mathbb{E}[r_t] = \theta$ (**long term mean**).

Proposition

The variance of r_t is given by

$$\text{Var}[r_t] = \frac{\sigma^2}{\kappa} r_0 (e^{-\kappa t} - e^{-2\kappa t}) + \frac{\theta \sigma^2}{2\kappa} (1 - e^{-\kappa t})^2$$

Proof.

$$\begin{aligned} \text{Var}[r_t] &= \mathbb{E}[r_t^2] - \mathbb{E}[r_t]^2 \\ &= \mathbb{E} \left[\left(e^{-\kappa t} r_0 + \theta (1 - e^{-\kappa t}) + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s \right)^2 \right] - \mathbb{E}[r_t]^2 \\ &= \mathbb{E} \left[\left(\mathbb{E}[r_t] + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s \right)^2 \right] - \mathbb{E}[r_t]^2 \\ &= 2\mathbb{E} \left[\mathbb{E}[r_t] \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s \right] + \mathbb{E} \left[\left(\sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s \right)^2 \right] \end{aligned}$$

Proof.

$$\begin{aligned} &= 2 \mathbb{E}[r_t] e^{-\kappa t} \cdot 0 + \mathbb{E} \left[\left(\sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s \right)^2 \right] \\ &= \sigma^2 e^{-2\kappa t} \mathbb{E} \left[\int_0^t e^{2\kappa s} r_s ds \right] \quad (\text{Iso Isometry}) \\ &= \sigma^2 e^{-2\kappa t} \int_0^t e^{2\kappa s} \mathbb{E}[r_s] ds \end{aligned}$$



Proof.

$$\begin{aligned}
 &= 2 \mathbb{E}[r_t] e^{-\kappa t} \cdot 0 + \mathbb{E} \left[\left(\sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s \right)^2 \right] \\
 &= \sigma^2 e^{-2\kappa t} \mathbb{E} \left[\int_0^t e^{2\kappa s} r_s ds \right] \quad (\text{Iso Isometry}) \\
 &= \sigma^2 e^{-2\kappa t} \int_0^t e^{2\kappa s} \mathbb{E}[r_s] ds \\
 &= \sigma^2 e^{-2\kappa t} \int_0^t e^{2\kappa s} (e^{-\kappa s} r_0 + \theta (1 - e^{-\kappa s})) ds \\
 &= \frac{\sigma^2}{\kappa} e^{-2\kappa t} r_0 (e^{\kappa t} - 1) + \theta \sigma^2 e^{-2\kappa t} \int_0^t (e^{2\kappa s} - e^{\kappa s}) ds \\
 &= \frac{\sigma^2}{\kappa} r_0 (e^{-\kappa t} - e^{-2\kappa t}) + \frac{\theta \sigma^2}{2\kappa} (1 - e^{-\kappa t})^2
 \end{aligned}$$



Method of Moments of Ito Integral

$$I_t = \int_0^t \Delta_s dW_s \quad \text{and} \quad \mathbb{E}[I_t] = 0 \quad (\text{Zero Mean Property})$$

$$\mathbb{E}[I_t^2] = \mathbb{E}\left[\int_0^t \Delta^2(s) ds\right] \quad (\text{Ito Isometry})$$

Proposition

$$\mathbb{E}[I_t^3] = 0$$

Proof.

$$X_t^n = n \int_0^t X_s^{n-1} dX_s + \frac{n(n-1)}{2} \int_0^t X_s^{n-2} ds \quad (\text{Ito Doebelin Formula})$$

$$\mathbb{E}\left[\int_0^t I_s^2 \Delta(s) dW_s\right] = 0 \implies \mathbb{E}[I_t^3] = 0 + 3 \int_0^t \mathbb{E}[I_s] ds = 0$$



3rd Moment of r_t

$$\mathbb{E}[r_t^3] = \mathbb{E}[(A_t + B_t I_t)^3]$$

where $A_t = e^{-\kappa t} r_0 + \theta(1 - e^{-\kappa t})$ $B_t = \sigma e^{-\kappa t}$ and

$$I_t = \int_0^t e^{\kappa s} \sqrt{r_s} dW_s$$

$$\begin{aligned} \mathbb{E}[r_t^3] &= \mathbb{E}[(A_t + B_t I_t)^3] \\ &= A_t^3 + 3A_t^2 B_t \mathbb{E}[I_t] + 3A_t B_t^2 \mathbb{E}[I_t^2] + B_t^3 \mathbb{E}[I_t^3] \\ &= A_t^3 + 3A_t B_t^2 \mathbb{E}[I_t^2] \\ &= A_t \left(A_t^2 + B_t^2 \int_0^t e^{2\kappa s} \mathbb{E}[r_s] ds \right) \\ &= \mathbb{E}[r_t] \left(\mathbb{E}[r_t]^2 + \text{Var}[r_t] \right) \end{aligned}$$

Heston Model

Introduction to Heston Model

Improvement to Black Scholes Model.

Developed by Steven Heston in 1993.

- The stock price follows a general Brownian motion
- The stochastic process of the volatility is a CIR process

Some important assumptions

- Interest Rate is constant
- No dividends
- European Style Options
- Frictionless Market

Stochastic Differential Equation

Heston SDE

The Heston model is given by

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^x \quad S_0 > 0$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^v \quad v_0 > 0$$

where W_t^x and W_t^v are Brownian motions with correlation ρ where

$$dW_t^x dW_t^v = \rho dt \quad |\rho| \leq 1$$

- μ is the **drift**
- κ is the **speed of mean reversion**
- θ is the **long term mean**
- σ is the **volatility coefficient**
- ρ is the **correlation coefficient**

Garsinov Theorem

The stock price and variance are under the historical measure \mathbb{P} .

By applying the Girsanov theorem one can find a probability measure \mathbb{Q} such that we can represent the stock price in the form

$$\begin{aligned}dS_t &= rS_t dt + \sqrt{v_t} S_t d\tilde{W}_t^x, S_0 > 0, \\dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^v, v_0 > 0\end{aligned}$$

where,

$$\tilde{W}_t^x = \left(W_t^x + \frac{\mu - r}{\sqrt{v_t}} \right)$$

Euler-Maruyama Method

Numerical Method to solve Stochastic Differential Equations.

Divide the interval $[0, T]$ into N equal sub-intervals of length $\Delta t = \frac{T}{N}$

$$0 = t_0 < t_1 < \dots < t_N = T \quad \Delta t = \frac{T}{N} = t_{n+1} - t_n$$

An SDE

$$dX_t = r(t, X_t)dt + \sigma(t, W_t)dW_t \quad X_0 = 0$$

can be approximated by Euler-Maruyama method, given by

$$X_{n+1} = X_n + r(t_n, X_n)\Delta t + \sigma(t_n, X_n)\Delta W_n$$

where $\Delta W_n \sim \mathcal{N}(0, \sqrt{\Delta t})$

Discretized Heston Model

Discretized Heston Model

The Heston model is given by

$$\begin{aligned}S_{t+1} &= S_t + rS_t\Delta t + \sqrt{v_t}S_tZ_x\sqrt{\Delta t} \\ v_{t+1} &= v_t + \kappa(\theta - v_t)\Delta t + \sigma Z_v\sqrt{\Delta t}\end{aligned}$$

where $Z_x, Z_v \sim \mathcal{N}(0, 1)$ with correlation ρ So, we can write Z_x, Z_v as

$$\begin{aligned}Z_x &= \rho Z_1 + \sqrt{1 - \rho^2} Z_2 \\ Z_v &= Z_1\end{aligned}$$

where $Z_1, Z_2 \sim \mathcal{N}(0, 1)$ are independent.

Method of Moments

We define

$$Q_{t+1} := \frac{S_{t+1}}{S_t} = 1 + r + \sqrt{v_t} Z_s$$

Sample Moments of Q_{t+1} are given by

$$\mathbb{E}[Q_{t+1}] = (1 + r)$$

$$\mathbb{E}[Q_{t+1}^2] = (1 + r)^2 + \mathbb{E}[v_t]$$

$$\mathbb{E}[Q_{t+1}^3] = (1 + r)^3 + 3(1 + r)\mathbb{E}[v_t]$$

$$\mathbb{E}[Q_{t+1}^4] = (1 + r)^4 + 6(1 + r)^2\mathbb{E}[v_t] + 3\mathbb{E}[v_t^2]$$

$$\mathbb{E}[Q_{t+1}^5] = (1 + r)^5 + 10(1 + r)^3\mathbb{E}[v_t] + 15(1 + r)\mathbb{E}[v_t^2]$$

Improvement of Heston Model

Rate of interest changes once every 3 months.

An improvement of the Heston model is given by a piecewise constant interest rate model.

Here, we assume that the interest rate is constant for 3 months and use CIR Model to estimate the new rate.