Analysis of CIR and Heston Model

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Introduction

Stochastic Differential Equations(SDEs) and Ito Doeblin Formula

Stochastic Differental Equation

An SDE is a differential equation that contains a stochastic term

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

where μ is called the **drift term** and σ is called the **diffusion term**.

Theorem (Ito Doeblin Formula)

Let $f(t, X)$ be a continuous function in time t and X where X is a Random Variable that is continuous in t. Then, the Ito Doeblin Formula is given by

$$df(t, X) = \left(f_t + \frac{1}{2} f_{XX}\right) dt + f_X dX$$

Ito's Integral

Ito's Integral is defined as follows

$$I_t := I(t) = \int_0^t \Delta(s) dW_s$$

where $\Delta(s)$ is an adopted process and W_s is a Brownian motion.
It has the following properties

1. $I(t)$ is a continuous process.
2. $I(t)$ is a martingale.
3. Zero Mean Property $\mathbb{E}\left[\int_0^t \Delta(s) dW_s\right] = 0$
4. Ito Isometry $\mathbb{E}\left[\left(\int_0^t \Delta(s) dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t \Delta(s)^2 ds\right]$

CIR Model

Rate Models

Introduced by Oldrich Vasicek (1977) to model interest rates.

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t$$

where κ is the **mean reversion rate**, θ is the **long term mean** and σ is the **volatility** of the interest rate.
The model can be easily solved and the closed form solution is given by

$$r_t \sim \mathcal{N}\left(e^{-\kappa t} r_0 + \theta\left(1 - e^{-\kappa t}\right), \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right)\right)$$

The main disadvantage is that the interest rate can become negative.

CIR Model

The **CIR** model was introduced by John Carrington **Cox**, Jonathan Edwards **Ingersoll** and Stephen Alan **Ross** (1985) and follows the following SDE.

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

where

- r_t is the **interest rate** at time t
- κ is the **speed of mean reversion**
- θ is the **long term mean**
- σ is the **volatility coefficient**
- W_t is a **Brownian Motion**

with initial rate r_0 given.

Proposition

The exact solution of CIR is given by

$$r_t = e^{-\kappa t} r_0 + \theta \left(1 - e^{-\kappa t}\right) + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_t} dW_s$$

Proof.

$$\begin{aligned}
dr_t &= \kappa(\theta - r_t)dt + \sigma \sqrt{r_t} dW_t \\
e^{\kappa t} dr_t + \kappa e^{\kappa t} r_t dt &= \kappa \theta e^{\kappa t} dt + \sigma e^{\kappa t} \sqrt{r_t} dW_t \\
\implies d\left(e^{\kappa t} r_t\right) &= \kappa \theta e^{\kappa t} dt + \sigma e^{\kappa t} \sqrt{r_t} dW_t \quad \text{(Ito Doeblin Formula)} \\
\implies \int_0^t d\left(e^{\kappa s} r_s\right) &= \int_0^t \theta \kappa e^{\kappa s} ds + \int_0^t \sigma e^{\kappa s} \sqrt{r_s} dW_s \\
\therefore \ r_t &= e^{-\kappa t} r_0 + \theta \left(1 - e^{-\kappa t}\right) + \sigma \int_0^t e^{\kappa s} \sqrt{r_s} dW_s
\end{aligned}$$

$\square$

Proposition

The expectation of r_t is given by

$$\mathbb{E}\left[r_t\right] = e^{-\kappa t} r_0 + \theta\left(1 - e^{-\kappa t}\right)$$

Proof.

$$\begin{aligned}
\mathbb{E}\left[r_t\right] &= \mathbb{E}\left[e^{-\kappa t} r_0 + \theta\left(1 - e^{-\kappa t}\right) + \sigma \int_0^t e^{\kappa s}\sqrt{r_s}\, dW_s\right] \\
&= e^{-\kappa t} r_0 + \theta\left(1 - e^{-\kappa t}\right) + \mathbb{E}\left[\sigma \int_0^t e^{\kappa s}\sqrt{r_s}\, dW_s\right] \\
\therefore \mathbb{E}\left[r_t\right] &= e^{-\kappa t} r_0 + \theta\left(1 - e^{-\kappa t}\right)
\end{aligned}$$

$\square$

Note that $\lim_{t\to\infty} \mathbb{E}\left[r_t\right] = \theta$ (**long term mean**).

Proposition

The variance of r_t is given by

$$Var\left[r_t\right] = \frac{\sigma^2}{\kappa} r_0 \left(e^{-\kappa t} - e^{-2\kappa t}\right) + \frac{\theta\sigma^2}{2\kappa}\left(1 - e^{-\kappa t}\right)^2$$

Proof.

$$\begin{aligned}
\mathrm{Var}\left[r_t\right] &= \mathbb{E}\left[r_t^2\right] - \mathbb{E}\left[r_t\right]^2 \\
&= \mathbb{E}\left[\left(e^{-\kappa t} r_0 + \theta\left(1 - e^{-\kappa t}\right) + \sigma e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{r_s}dW_s\right)^2\right] - \mathbb{E}\left[r_t\right]^2 \\
&= \mathbb{E}\left[\left(\mathbb{E}\left[r_t\right] + \sigma e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{r_s}dW_s\right)^2\right] - \mathbb{E}\left[r_t\right]^2 \\
&= 2\mathbb{E}\left[\mathbb{E}\left[r_t\right]\sigma e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{r_s}dW_s\right] + \mathbb{E}\left[\left(\sigma e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{r_s}dW_s\right)^2\right]
\end{aligned}$$

Proof.

$$
\begin{aligned}
&= 2\,\mathbb{E}\left[r_t\right] e^{-\kappa t} \cdot 0 + \mathbb{E}\left[\left(\sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s\right)^2\right] \\
&= \sigma^2 e^{-2\kappa t}\ \mathbb{E}\left[\int_0^t e^{2\kappa s} r_s ds\right] \quad \text{(Iso Isometry)} \\
&= \sigma^2 e^{-2\kappa t} \int_0^t e^{2\kappa s} \mathbb{E}\left[r_s\right] ds \\
&= \sigma^2 e^{-2\kappa t} \int_0^t e^{2\kappa s} \left(e^{-\kappa s} r_0 + \theta\left(1 - e^{-\kappa s}\right)\right) ds \\
&= \frac{\sigma^2}{\kappa} e^{-2\kappa t} r_0 \left(e^{\kappa s} - 1\right) + \theta \sigma^2 e^{-2\kappa t} \int_0^t \left(e^{2\kappa s} - e^{\kappa s}\right) ds \\
&= \frac{\sigma^2}{\kappa} r_0 \left(e^{-\kappa t} - e^{-2\kappa t}\right) + \frac{\theta \sigma^2}{2\kappa}\left(1 - e^{-\kappa t}\right)^2
\end{aligned}
$$

□

Method of Moments of Ito Integral

$$I_t = \int_0^t \Delta_s dW_s \quad \text{and} \quad \mathbb{E}[I_t] = 0 \quad \text{(Zero Mean Property)}$$

$$\mathbb{E}\left[I_t^2\right] = \mathbb{E}\left[\int_0^t \Delta^2(s) ds\right] \quad \text{(Ito Isometry)}$$

Proposition

$$\mathbb{E}\left[I_t^3\right] = 0$$

Proof.

$$X_t^n = n \int_0^t X_s^{n-1} dX_s + \frac{n(n-1)}{2} \int_0^t X_s^{n-2} ds \quad \text{(Ito Doeblin Formula)}$$

$$\mathbb{E}\left[\int_0^t I_s^2 \Delta(t) dW_t\right] = 0 \implies \mathbb{E}\left[I_t^3\right] = 0 + 3 \int_0^t \mathbb{E}[I_s] ds = 0$$

□

3^{rd} Moment of r_t

$$\mathbb{E}\left[r_t^3\right] = \mathbb{E}\left[\left(A_t + B_t I_t\right)^3\right]$$

where $A_t = e^{-\kappa t} r_0 + \theta\left(1 - e^{-\kappa t}\right) \qquad B_t = \sigma e^{-\kappa t}$ and
$I_t = \int_0^t e^{\kappa s} \sqrt{r_s} dW_s$

$$\begin{aligned}
\mathbb{E}\left[r_t^3\right] &= \mathbb{E}\left[\left(A_t + B_t I_t\right)^3\right] \\
&= A_t^3 + 3A_t^2 B_t \mathbb{E}\left[I_t\right] + 3A_t B_t^2 \mathbb{E}\left[I_t^2\right] + B_t^3 \mathbb{E}\left[I_t^3\right] \\
&= A_t^3 + 3A_t B_t^2 \mathbb{E}\left[I_t^2\right] \\
&= A_t\left(A_t^2 + B_t^2 \int_0^t e^{2\kappa s} \mathbb{E}\left[r_s\right] ds\right) \\
&= \mathbb{E}\left[r_t\right]\left(\mathbb{E}\left[r_t\right]^2 + \mathrm{Var}\left[r_t\right]\right)
\end{aligned}$$

Heston Model

Introduction to Heston Model

Improvement to Black Scholes Model.

Developed by Steven Heston in 1993.

- The stock price follows a general Brownian motion
- The stochastic process of the volatility is a CIR process

Some important assumptions

- Interest Rate is constant
- No dividends
- European Style Options
- Frictionless Market

Stochastic Differential Equation

Heston SDE

The Heston model is given by

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^x \qquad S_0 > 0$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v \qquad v_0 > 0$$

where W_t^x and W_t^v are Brownian motions with correlation ρ where

$$dW_t^x dW_t^v = \rho dt \quad |\rho| \leq 1$$

- μ is the **drift**
- κ is the **speed of mean reversion**
- θ is the **long term mean**
- σ is the **volatility coefficient**
- ρ is the **correlation coefficient**

Garsinov Theorem

The stock price and variance are under the historical measure $\mathbb{P}$.

By applying the Girsanov theorem one can find a probability measure $\mathbb{Q}$ such that we can represent the stock price in the form

$$dS_t = rS_t dt + \sqrt{v_t} S_t d\tilde{W}_t^x, S_0 > 0,$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v, v_0 > 0$$

where,

$$\tilde{W}_t^x = \left(W_t^x + \frac{\mu - r}{\sqrt{v_t}} \right)$$

Euler-Maruyama Method

Numerical Method to solve Stochastic Differential Equations.
Divide the interval $[0, T]$ into N equal sub-intervals of length $\Delta t = \frac{T}{N}$

$$0 = t_0 < t_1 < \cdots < t_N = T \qquad \Delta t = \frac{T}{N} = t_{n+1} - t_n$$

An SDE

$$dX_t = r(t, X_t)dt + \sigma(t, W_t)dW_t \quad X_0 = 0$$

can be approximated by Euler-Maruyama method, given by

$$X_{n+1} = X_n + r(t_n, X_n)\Delta t + \sigma(t_n, X_n)\Delta W_n$$

where $\Delta W_n \sim \mathcal{N}\left(0, \sqrt{\Delta t}\right)$

Discretized Heston Model

Discretized Heston Model

The Heston model is given by

$$S_{t+1} = S_t + rS_t\Delta t + \sqrt{v_t} Z_x \sqrt{\Delta t}$$
$$v_{t+1} = v_t + \kappa(\theta - v_t)\Delta t + \sigma Z_v \sqrt{\Delta t}$$

where $Z_x, Z_v \sim \mathcal{N}(0,1)$ with correlation ρ So, we can write Z_x, Z_v as

$$Z_x = \rho Z_1 + \sqrt{1-\rho^2} Z_2$$
$$Z_v = Z_1$$

where $Z_1, Z_2 \sim \mathcal{N}(0,1)$ are independent.

Method of Moments

We define

$$Q_{t+1} := \frac{S_{t+1}}{S_t} = 1 + r + \sqrt{v_t} Z_s$$

Sample Moments of Q_{t+1} are given by

$$\begin{aligned}
\mathbb{E}\left[Q_{t+1}\right] &= (1+r) \\
\mathbb{E}\left[Q_{t+1}^2\right] &= (1+r)^2 + \mathbb{E}\left[v_t\right] \\
\mathbb{E}\left[Q_{t+1}^3\right] &= (1+r)^3 + 3(1+r)\mathbb{E}\left[v_t\right] \\
\mathbb{E}\left[Q_{t+1}^4\right] &= (1+r)^4 + 6(1+r)^2\mathbb{E}\left[v_t\right] + 3\mathbb{E}\left[v_t^2\right] \\
\mathbb{E}\left[Q_{t+1}^5\right] &= (1+r)^5 + 10(1+r)^3\mathbb{E}\left[v_t\right] + 15(1+r)\mathbb{E}\left[v_t^2\right]
\end{aligned}$$

Improvement of Heston Model

Rate of interest changes once every 3 months.

An improvement of the Heston model is given by a piecewise constant interest rate model.

Here, we assume that the interest rate is constant for 3 months and use CIR Model to estimate the new rate.