Analysis of CIR and Heston Model

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Stochastic Differential Equations(SDEs) and Ito Doeblin Formula

Stochastic Differential Equation

An SDE is a differential equation that contains a stochastic term

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

where μ is called the **drift term** and σ is called the **diffusion term**.

Theorem (Ito Doeblin Formula)

Let $f(t, X)$ be a continuous function in time t and X where X is a Random Variable that is continuous in t. Then, the Ito Doeblin Formula is given by

$$df(t, X) = \left(f_t + \frac{1}{2} f_{XX} \right) dt + f_X \, dX$$

Rate Models

Introduced by Oldrich Vasicek (1977) to model interest rates.

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t$$

where κ is the **mean reversion rate**, θ is the **long term mean** and σ is the **volatility** of the interest rate.

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The model can be easily solved and the closed form solution is given by

$$r_t \sim \mathcal{N}\left(e^{-\kappa t}r_0 + \theta\left(1 - e^{-\kappa t}\right), \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right)\right)$$

The main disadvantage is that the interest rate can become negative.

CIR Model

The **CIR** model was introduced by John Carrington **Cox**, Jonathan Edwards **Ingersoll** and Stephen Alan **Ross** (1985) and follows the following SDE.

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

where

- κ is the **speed of mean reversion**
- θ is the **long term mean**
- σ is the **volatility coefficient**

with initial rate r_0 given.

Proposition

The exact solution of CIR is given by

$$r_t = e^{-\kappa t}r_0 + \theta\left(1 - e^{-\kappa t}\right) + \sigma e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{r_t}dW_s$$

Proof.

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

$$e^{\kappa t}dr_t + \kappa e^{\kappa t}r_t dt = \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}\sqrt{r_t}dW_t$$

$$\implies d\left(e^{\kappa t}r_t\right) = \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}\sqrt{r_t}dW_t \quad \text{(Ito Doeblin Formula)}$$

$$\implies \int_0^t d\left(e^{\kappa s}r_s\right) = \int_0^t \theta\kappa e^{\kappa s}ds + \int_0^t \sigma e^{\kappa s}\sqrt{r_s}dW_s$$

$$\therefore\ r_t = e^{-\kappa t}r_0 + \theta\left(1 - e^{-\kappa t}\right) + \sigma\int_0^t e^{\kappa s}\sqrt{r_s}dW_s$$

Proposition

The expectation of r_t is given by

$$\mathbb{E}\left[r_t\right] = e^{-\kappa t}r_0 + \theta\left(1 - e^{-\kappa t}\right)$$

Proof.

$$\mathbb{E}\left[r_t\right] = \mathbb{E}\left[e^{-\kappa t}r_0 + \theta\left(1 - e^{-\kappa t}\right) + \sigma\int_0^t e^{\kappa s}\sqrt{r_s}dW_s\right]$$

$$= e^{-\kappa t}r_0 + \theta\left(1 - e^{-\kappa t}\right) + \mathbb{E}\left[\sigma\int_0^t e^{\kappa s}\sqrt{r_s}dW_s\right]$$

$$\therefore \mathbb{E}\left[r_t\right] = e^{-\kappa t}r_0 + \theta\left(1 - e^{-\kappa t}\right)$$

□

Note that $\lim_{t\to\infty}\mathbb{E}\left[r_t\right] = \theta$ (**long term mean**).

Proposition

The variance of r_t is given by

$$Var[r_t] = \frac{\sigma^2}{\kappa} r_0 \left(e^{-\kappa t} - e^{-2\kappa t}\right) + \frac{\theta \sigma^2}{2\kappa} \left(1 - e^{-\kappa t}\right)^2$$

Proof.

$$\begin{aligned}
\text{Var}[r_t] &= \mathbb{E}\left[r_t^2\right] - \mathbb{E}[r_t]^2 \\
&= \mathbb{E}\left[\left(e^{-\kappa t} r_0 + \theta\left(1 - e^{-\kappa t}\right) + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s\right)^2\right] - \mathbb{E}[r_t]^2 \\
&= \mathbb{E}\left[\left(\mathbb{E}[r_t] + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s\right)^2\right] - \mathbb{E}[r_t]^2 \\
&= 2\mathbb{E}\left[\mathbb{E}[r_t] \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s\right] + \mathbb{E}\left[\left(\sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s\right)^2\right]
\end{aligned}$$

Proof.

$$= 2 \, \mathbb{E}\left[r_t\right] e^{-\kappa t} \cdot 0 + \mathbb{E}\left[\left(\sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s \right)^2 \right]$$

$$= \mathbb{E}\left[\sigma^2 e^{-2\kappa t} \int_0^t e^{2\kappa s} r_s ds \right] \quad \text{(Iso Isometry)}$$

$$= \sigma^2 e^{-2\kappa t} \int_0^t e^{2\kappa s} \mathbb{E}\left[r_s\right] ds$$

$$= \sigma^2 e^{-2\kappa t} \int_0^t e^{2\kappa s} \left(e^{-\kappa s} r_0 + \theta \left(1 - e^{-\kappa s} \right) \right) ds$$

$$= \frac{\sigma^2}{\kappa} e^{-2\kappa t} r_0 \left(e^{\kappa s} - 1 \right) + \theta \sigma^2 e^{-2\kappa t} \int_0^t \left(e^{2\kappa s} - e^{\kappa s} \right) ds$$

$$= \frac{\sigma^2}{\kappa} r_0 \left(e^{-\kappa t} - e^{-2\kappa t} \right) + \frac{\theta \sigma^2}{2\kappa} \left(1 - e^{-\kappa t} \right)^2$$

Method of Moments of Ito Integral

$$I_t = \int_0^t \Delta_s dW_s \quad \text{and} \quad \mathbb{E}[I_t] = 0 \quad \text{(Zero Mean Property)}$$

$$\mathbb{E}\left[I_t^2\right] = \mathbb{E}\left[\int_0^t \Delta^2(s)ds\right] \quad \text{(Ito Isometry)}$$

Proposition

$$\mathbb{E}\left[I_t^3\right] = 0$$

Proof.

$$X_t^n = n\int_0^t X_s^{n-1}dX_s + \frac{n(n-1)}{2}\int_0^t X_s^{n-2}ds \quad \text{(Ito Doeblin Formula)}$$

$$\mathbb{E}\left[\int_0^t I_s^2 \Delta(t)dW_t\right] = 0 \implies \mathbb{E}\left[I_t^3\right] = 0 + 3\int_0^t \mathbb{E}[I_s]\,ds = 0$$

3rd Moment of r_t

$$\mathbb{E}\left[r_t^3\right] = \mathbb{E}\left[(A_t + B_t I_t)^3\right]$$

where $A_t = e^{-\kappa t} r_0 + \theta\left(1 - e^{-\kappa t}\right)$ $B_t = \sigma e^{-\kappa t}$ and
$I_t = \displaystyle\int_0^t e^{\kappa s}\sqrt{r_s}\,dW_s$

$$
\begin{aligned}
\mathbb{E}\left[r_t^3\right] &= \mathbb{E}\left[(A_t + B_t I_t)^3\right] \\
&= A_t^3 + 3A_t^2 B_t \mathbb{E}\left[I_t\right] + 3A_t B_t^2 \mathbb{E}\left[I_t^2\right] + B_t^3 \mathbb{E}\left[I_t^3\right] \\
&= A_t^3 + 3A_t B_t^2 \mathbb{E}\left[I_t^2\right] \\
&= A_t\left(A_t^2 + B_t^2 \int_0^t e^{2\kappa s}\mathbb{E}\left[r_s\right]ds\right) \\
&= \mathbb{E}\left[r_t\right]\left(\mathbb{E}\left[r_t\right]^2 + \mathsf{Var}\left[r_t\right]\right)
\end{aligned}
$$

Introduction to Heston Model

Developed by Steven Heston in 1993 with the following features

- The stock price follows a general Brownian motion
- The stochastic process of the volatility is a CIR process

Stochastic Differential Equation

Heston SDE

The Heston model is given by

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^x \qquad\qquad S_0 > 0$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v \qquad\qquad v_0 > 0$$

where W_t^x and W_t^v are Brownian motions with correlation ρ where

$$dW_t^x dW_t^v = \rho dt \quad |\rho| \leq 1$$

- μ is the **drift**
- κ is the **speed of mean reversion**
- θ is the **long term mean**
- σ is the **volatility coefficient**
- ρ is the **correlation coefficient**

Garsinov Theorem

Theorem (Garsinov Theorem)

Euler-Maruyama Method

Numerical Method to solve Stochastic Differential Equations.
Divide the interval $[0, T]$ into N equal sub-intervals of length $\Delta t = \frac{T}{N}$

$$0 = t_0 < t_1 < \cdots < t_N = T \qquad \Delta t = \frac{T}{N} = t_{n+1} - t_n$$

An SDE

$$dX_t = r(t, X_t)dt + \sigma(t, W_t)dW_t \quad X_0 = 0$$

can be approximated by Euler-Maruyama method, given by

$$X_{n+1} = X_n + r(t_n, X_n)\Delta t + \sigma(t_n, X_n)\Delta W_n$$

where $\Delta W_n \sim \mathcal{N}\left(0, \sqrt{\Delta t}\right)$

Discretized Heston Model

Discretized Heston Model

The Heston model is given by

$$S_{t+1} = S_t + rS_t\Delta t + \sqrt{v_t}Z_x\sqrt{\Delta t}$$
$$v_{t+1} = v_t + \kappa(\theta - v_t)\Delta t + \sigma Z_v\sqrt{\Delta t}$$

where $Z_x, Z_v \sim \mathcal{N}(0,1)$ with correlation ρ So, we can write Z_x, Z_v as

$$Z_x = \rho Z_1 + \sqrt{1-\rho^2}Z_2$$
$$Z_v = Z_1$$

where $Z_1, Z_2 \sim \mathcal{N}(0,1)$ are independent.

Method of Moments

We define

$$Q_{t+1} := \frac{S_{t+1}}{S_t} = 1 + r + \sqrt{v_t} Z_s$$

Sample Moments of Q_{t+1} are given by

$$\mathbb{E}\left[Q_{t+1}\right] = 1 + r$$
$$\mathbb{E}\left[Q_{t+1}^2\right] = (1+r)^2 + \mathbb{E}\left[v_t\right]$$
$$\mathbb{E}\left[Q_{t+1}^3\right] = (1+r)^3 + 3(1+r)\mathbb{E}\left[v_t\right]$$
$$\mathbb{E}\left[Q_{t+1}^4\right] = (1+r)^4 + 6(1+r)^2\mathbb{E}\left[v_t\right] + 3\mathbb{E}\left[v_t^2\right]$$
$$\mathbb{E}\left[Q_{t+1}^5\right] = (1+r)^5 +$$