Option Pricing of NIFTY 50 Index options using Dual Heston model

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I. Terminology

- **Options** are financial derivatives that give buyers the right, but not the obligation, to buy or sell an underlying asset at an agreed-upon price and date.
- A **call option** gives the holder the right, but not the obligation, to buy the underlying security at the strike price on or before expiration.
- A **put option** gives the holder the right, but not the obligation, to sell the underlying stock at the strike price on or before expiration.
- **American options** can be exercised at any time between the date of purchase and the expiration date.
- **European options** can only be exercised at the end of their lives on their expiration date.
- **Volatility** represents how large an asset's prices swing around the mean price—it is a statistical measure of its dispersion of returns. Volatile assets are often considered riskier than less volatile assets because the price is expected to be less predictable.
- **Market risk** is the possibility that an individual or other entity will experience losses due to factors that affect the overall performance of investments. Market risk may arise due to changes to interest rates, exchange rates, geopolitical events, or recessions.
- **Mean reversion** is a theory used in finance that suggests that asset price volatility and historical returns eventually will revert to the long-run mean or average level of the entire dataset.
  Credit : Investopedia

II. Introduction

A. Background

In 1973, Black and Scholes studied and obtained the famous Black–Scholes (B-S) pricing formula and established the classical parametric option pricing model. The B-S formula can lead all investors to a risk-neutral world with risk-free interest rate as the rate of return, and it can predict the price of options better, regardless of their preferences.

However, the formula has made many assumptions in advance, for example, the volatility and interest rate of options are assumed to be a constant; the underlying asset follow geometric Brownian motion, etc., is not completely consistent with the actual market situation; thus, the option price calculated by the formula is far from the real world situation.

B. Topic of Interest

Many Researchers proposed improved models to obtain option price values more closer to actual values, two of such models are **Heston** model, which uses stochastic volatility to overcome the constraints of constant volatility and **CIR** model, which uses stochastic interest rate to overcome the constraints of constant interest rate in B-S model.

C. Aim of the Project

The aim of this project is to study Heston, stochastic volatility model and CIR, stochastic interest rate model, and implement them on real world data.

III. CIR Model

The Cox-Ingersoll-Ross model (CIR) is a mathematical formula used to model interest rate movements. It was introduced by John Carrington Cox, Jonathan Edwards Ingersoll and Stephen Alan Ross in 1985. An interest rate model is, essentially, a probabilistic description of how interest rates can change over time.

Interest rates have properties such as positivity, boundedness, and return to equilibrium which make them behave differently from stock and asset prices. CIR model describes interest movements as driven by a sole source of market risk. It is used as a method to forecast interest rates and is based on a stochastic differential equation.

CIR is a mean reverting process, the square root element($\sqrt{r_t}$) does not allow for negative rates and the model assumes mean reversion towards a long-term normal interest rate level.

A. SDE

The CIR model describes the dynamics of the interest rate r(t) as a solution of the following stochastic differential equation (SDE):

$$\begin{cases} dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t) \\ r(0) = r_0 \end{cases} \tag{1}$$

where,

1) κ mean reversion coefficient
2) θ long term mean
3) σ volatility coefficient

In this model, the stochastic term $\sigma\sqrt{r(t)}dW(t)$ has a standard deviation proportion to the square root of the current rate. This implies that as the rate increase, its standard deviation increase and as it falls and approach zero, the stochastic term also approach zero.

B. Assumptions

There are three assumptions in this model,

1) The change in the interest rate over time is described by a single state variable r.
2) The means and variances of the rates of return in the processes are proportional to r, which means that they won't dominate the portfolio decision for large values of r.
3) The development of the state variable r follows the equation 1.

C. Modelling the formula

For modelling the CIR formula, we need to find the parameters of the formula θ, κ etc. based on the real world data, for this we use Method of Moments technique for parameter estimation.

This technique has been thoroughly explained in the following Heston model Section, intuitively one may understand that this technique is based on equating the n^{th} theoretical moment of a random variable with corresponding sample moment, and solving these equations to find the required parameters.

So, in the following subsections we derive some results towards finding a general equation for the theoretical moments of r.

D. Exact Solution

The Integral form of the above stochastic differential equation 1 is:

$$r(t) = \theta + (r_0 - \theta)e^{-\kappa t} + \sigma e^{\kappa t}\int_0^t e^{\kappa s}\sqrt{r(s)}dW(s) \quad (2)$$

Proof :

The equation 1 changes to the following form

$$dr(t) + \kappa r(t)dt = \kappa\theta dt + \sigma\sqrt{r(t)}dW(t) \quad (3)$$

Multiplying both sides of the relation 3 by $e^{\kappa t}$ results in

$$e^{\kappa t}dr(t) + \kappa e^{\kappa t}r(t)dt = \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}\sqrt{r(t)}dW(t) \quad (4)$$

$$d\left(e^{\kappa t}r(t)\right) = \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}\sqrt{r(t)}dW(t) \quad (5)$$

Now, integrating both sides of the relation 5 on $[0,t]$ gives us

$$e^{\kappa t}r(t) - r_0 = \kappa\theta\int_0^t e^{\kappa s}ds + \sigma\int_0^t e^{\kappa s}\sqrt{r(s)}dW(s) \quad (6)$$

Consequently, the relation 6 is derived.

According to the above result, the CIR model has no general explicit solution. However, its mean and variance can be calculated explicitly.

E. Expectation and Variance

The expectation and variance of $r_t := r(t)$ are given by

$$\begin{cases}\mathbb{E}\left[r_t\right] = e^{-\kappa t}r_0 + \theta\left(1 - e^{-\kappa t}\right) \\ \mathrm{Var}\left[r_t\right] = \frac{\sigma^2}{\kappa}r_0\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \frac{\theta\sigma^2}{2\kappa}\left(1 - e^{-\kappa t}\right)^2\end{cases} \quad (7)$$

Proof.

$$\begin{aligned}\mathbb{E}\left[r_t\right] &= \mathbb{E}\left[e^{-\kappa t}r_0 + \theta\left(1 - e^{-\kappa t}\right) + \sigma\int_0^t e^{\kappa s}\sqrt{r_s}dW_s\right] \\ &= e^{-\kappa t}r_0 + \theta\left(1 - e^{-\kappa t}\right) + \mathbb{E}\left[\sigma\int_0^t e^{\kappa s}\sqrt{r_s}dW_s\right] \\ \therefore \mathbb{E}\left[r_t\right] &= e^{-\kappa t}r_0 + \theta\left(1 - e^{-\kappa t}\right) \quad (8)\end{aligned}$$

$$\begin{aligned}\mathrm{Var}\left[r_t\right] &= \mathbb{E}\left[r_t^2\right] - \mathbb{E}\left[r_t\right]^2 \\ &= \mathbb{E}\left[\left(e^{-\kappa t}r_0 + \theta\left(1 - e^{-\kappa t}\right) + \sigma e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{r_s}dW_s\right)^2\right] \\ &\quad - \mathbb{E}\left[r_t\right]^2 \\ &= \mathbb{E}\left[\left(\mathbb{E}\left[r_t\right] + \sigma e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{r_s}dW_s\right)^2\right] - \mathbb{E}\left[r_t\right]^2 \\ &= 2\mathbb{E}\left[\mathbb{E}\left[r_t\right]\sigma e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{r_s}dW_s\right] \\ &\quad + \mathbb{E}\left[\left(\sigma e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{r_s}dW_s\right)^2\right] \\ &= 2\ \mathbb{E}\left[r_t\right]e^{-\kappa t}\cdot 0 + \mathbb{E}\left[\left(\sigma e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{r_s}dW_s\right)^2\right] \\ &= \mathbb{E}\left[\sigma^2 e^{-2\kappa t}\int_0^t e^{2\kappa s}r_s ds\right] \quad \text{(Iso Isometry)} \\ &= \sigma^2 e^{-2\kappa t}\int_0^t e^{2\kappa s}\mathbb{E}\left[r_s\right]ds \\ &= \sigma^2 e^{-2\kappa t}\int_0^t e^{2\kappa s}\left(e^{-\kappa s}r_0 + \theta\left(1 - e^{-\kappa s}\right)\right)ds \\ &= \frac{\sigma^2}{\kappa}e^{-2\kappa t}r_0\left(e^{\kappa s} - 1\right) + \theta\sigma^2 e^{-2\kappa t}\int_0^t\left(e^{2\kappa s} - e^{\kappa s}\right)ds \\ &= \frac{\sigma^2}{\kappa}r_0\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \frac{\theta\sigma^2}{2\kappa}\left(1 - e^{-\kappa t}\right)^2 \quad (9)\end{aligned}$$

F. 3^{rd} Moment of r_t

$$\mathbb{E}\left[r_t^3\right] = \mathbb{E}\left[\left(A_t + B_t I_t\right)^3\right]$$

where $A_t = e^{-\kappa t}r_0 + \theta\left(1 - e^{-\kappa t}\right) \quad B_t = \sigma e^{-\kappa t}$ and

$$I_t = \int_0^t e^{\kappa s}\sqrt{r_s}dW_s$$

$$\begin{aligned}\mathbb{E}\left[r_t^3\right] &= \mathbb{E}\left[(A_t + B_t I_t)^3\right] \\ &= A_t^3 + 3A_t^2 B_t \mathbb{E}\left[I_t\right] + 3A_t B_t^2 \mathbb{E}\left[I_t^2\right] \\ &\quad + B_t^3 \mathbb{E}\left[I_t^3\right] \\ &= A_t^3 + 3A_t B_t^2 \mathbb{E}\left[I_t^2\right] \\ &= A_t\left(A_t^2 + B_t^2 \int_0^t e^{2\kappa s}\mathbb{E}\left[r_s\right] ds\right) \\ &= \mathbb{E}\left[r_t\right]\left(\mathbb{E}\left[r_t\right]^2 + \text{Var}\left[r_t\right]\right) \end{aligned} \tag{10}$$

G. Parameter Estimation

We equate the moments obtained from method of moments and sample moments to estimate the parameters κ,θ and σ.

H. CIR-path

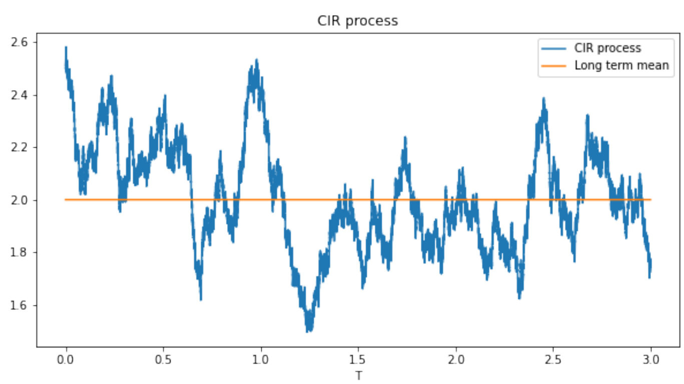

Fig. 1. Caption

IV. Heston Model

The *Heston Model* uses statistical methods to calculate and forecast option pricing with the assumption that volatility is arbitrary. The assumption that volatility is arbitrary, rather than constant, is the key factor that makes *stochastic volatility* models unique.

Key features of Heston Model:

- It considers the possible correlation between a stock's price and its volatility.
- It does not require the stock prices to follow a lognormal probability distribution.(In the Black Scholes model we assume that stock prices follow a lognormal probability distribution).

A. Application of Heston Model

Developed by mathematician Steven Heston in 1993, the Heston model was created to price options, which are a type of financial derivative. Unlike other financial assets such as equities, the value of an option is not based on the value of an asset but rather the change in an underlying asset's price.

In the Heston model, volatility is a mean reverting process. Mean reversion means that the process does not wander off to infinity but oscillates around a well-defined long term average value.

Each option is a contract between a buyer and seller, which gives the holder of the option the right to buy or sell the underlying asset at a specific price. All options have a specific expiration date, at which point the contract must be executed at the previously set price or risk expiring.

However, the volatility of options depends on the price and maturity. Therefore, the Heston model was designed to price an option while accounting for these variations in market volatility.

There are two categories of options: calls and puts. Calls allow the holder to buy at a specific price, and puts allow the holder to sell at a specific price.

Once a call or put option has been purchased, the date at which the holder can buy or sell depends on whether it is an American or European option. American options allow the holder to execute the option anytime before the expiry date, while European options only allow the holder to execute the option on the expiry date. It's important to note that the Heston model is only capable of pricing European options.

B. SDE

Under the risk-neutral measure the Heston stochastic volatility model, which is our point-of-departure, is specified by the following system of SDEs:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^x, S_0 > 0 \tag{11}$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^v, v_0 > 0 \tag{12}$$

where,

1) μ drift of the stock process(risk-free interest rate under risk-neutral measure)
2) κ mean reversion coefficient of the variance process
3) θ long term mean of the variance process
4) σ volatility coefficient of the variance process
5) ρ correlation between W_t^x and dW_t^v i.e.,

$$dW_t^x dW_t^v = \rho dt \tag{13}$$

with $|\rho_{x,\sigma}| < 1$.

The variance process, v_t, of the stock S_t is a mean reverting square root process, in which $\kappa > 0$ determines the speed of adjustment of the volatility towards its theoretical mean, $\theta > 0$, and $\sigma > 0$ is the second-order volatility, i.e., the variance of the volatility.

C. Girsanov Theorem

The stock price and variance in 12 are under the historical measure $\mathbb{P}$. By applying the Girsanov theorem one can find a probability measure $\mathbb{Q}$ such that we can represent the stock price in the form

$$dS_t = rS_t dt + \sqrt{v_t} S_t d\tilde{W}_t^x, S_0 > 0, \tag{14}$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v, v_0 > 0 \tag{15}$$

where,

$$\tilde{W}_t^x = \left(W_t^x + \frac{\mu - r}{\sqrt{v_t}} \right) \tag{16}$$

D. Euler - Maruyama Method

Let us divide the time interval $[0, T]$ into n parts

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = T$$

We can choose equally spaced points t_i such that

$$\Delta t = t_{i+1} - t_i = \frac{T}{N} \quad \forall\, 1 \leq i \leq N$$

An SDE:

$$\begin{aligned} dX_t &= r(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 &= 0 \end{aligned}$$

can be approximated by:

$$\begin{aligned} X_{i+1} &= X_i + r(t_i, X_i)\Delta t + \sigma(t_i, X_i)\Delta W_i \\ X_0 &= 0 \end{aligned}$$

where $X(t_i) = X_i$, $W(t_i) = W_i$ and $\Delta W_t \sim \mathcal{N}\left(0, \sqrt{\Delta t}\right)$

E. Discretized Heston Model

The Heston process can be discretized using the Euler-Maruyama method. Applying Euler-Maruyama method to Heston SDE 15 we obtain:

$$S_{t+1} = S_t + rS_t\Delta t + \sqrt{v_t} S_t Z_s \sqrt{\Delta t} \tag{17}$$
$$v_{t+1} = v_t + \kappa(\theta - v_t)\Delta t + \sigma\sqrt{v_t} Z_v \sqrt{\Delta t} \tag{18}$$

where Z_s, Z_v are standard normal random variables with correlation ρ. So, we can write Z_s, Z_v as,

$$Z_s = \rho Z_1 + \sqrt{1 - \rho^2} Z_2 \tag{19}$$
$$Z_v = Z_1 \tag{20}$$

F. Parameter Estimation using MOM

In this method we equate the theoretical moments with the sample moments from the data to obtain the parameters.

The j^{th} moment of the random variable Q_t is defined as $E(Q_t^j)$. We use M_j to denoted the j^{th} sample moment obtained from the data.

The process to find the parameters is:

1) Write m moments in terms of the m parameters that we are trying to estimate.
2) Obtain sample moments from the data set. The j^{th} sample moment denoted by $\hat{M}_j$ is obtained by raising each observation to the power of j and taking the average of those terms. Symbolically,

$$\hat{M}_j = \frac{1}{n} \sum_{t=1}^{n} Q_{t+1}^j \tag{21}$$

where,

$$Q_{t+1} = \frac{S_{t+1}}{S_t} \tag{22}$$
$$= 1 + r + \sqrt{v_t} Z_s \tag{23}$$

from 18
3) Substitute the j^{th} sample moment for the j^{th} moment in each of the m equations. That is, let $M_j = \hat{M}_j$. Now we have a system of m equations in m unknowns.
4) Solve for each of the m parameters. The resulting parameter values are the method of moments estimates. We denote the method of moments estimate of a parameter α as $\hat{\alpha}_{MOM}$.

When working with a data set of stock values, we are given values of S_t rather than values of Q_{t+1}. We can easily transform the data set into values of Q_{t+1} by solving for $\frac{S_{t+1}}{S_t}$ for each value of t. We wish to write five moments of Q_{t+1} in terms of the five parameters $r, \kappa, \theta, \sigma$, and ρ.

Letting M_j represent the j^{th} moment of Q_{t+1}, we express formulas for the first moment, the second moment, the fourth moment, and the fifth moment. We have excluded the third moment because it does not add any information to the system beyond the information available from the first two moments.

G. Calculating Theoretical Moments

We know that

$$Q_{t+1} = 1 + r + \sqrt{v_t}\left(\rho Z_1 + \sqrt{1 - \rho^2} Z_2\right)$$

We need to find the moments of Q_{t+1}^n for $1 \leq n \leq 5$.

$$\begin{aligned} \mathbb{E}\left[Q_{t+1}\right] &= 1 + r + \mathbb{E}\left[\sqrt{v_t}\left(\rho Z_1 + \sqrt{1 - \rho^2} Z_2\right)\right] \\ &= 1 + r + \rho\mathbb{E}\left[\sqrt{v_t}\right]\mathbb{E}\left[Z_1\right] + \sqrt{1 - \rho^2}\mathbb{E}\left[\sqrt{v_t}\right]\mathbb{E}\left[Z_2\right] \\ &= 1 + r + \rho\mathbb{E}\left[\sqrt{v_t}\right] \cdot 0 + \sqrt{1 - \rho^2}\mathbb{E}\left[\sqrt{v_t}\right] \cdot 0 \\ &= 1 + r \end{aligned} \tag{24}$$

We can observe that we just need to calculate for even powers of $\mathbb{E}\left[\left(\sqrt{v_t}\left(\rho Z_1 + \sqrt{1 - \rho^2} Z_2\right)\right)^n\right]$ since expectations containing odd powers of Z is 0.

$$\begin{aligned} &\mathbb{E}\left[\left(\sqrt{v_t}\left(\rho Z_1 + \sqrt{1 - \rho^2} Z_2\right)\right)^2\right] \\ &= \rho^2\mathbb{E}\left[v_t\right]\mathbb{E}\left[Z_1^2\right] + \left(1 - \rho^2\right)\mathbb{E}\left[v_t\right]\mathbb{E}\left[Z_2^2\right] + 2\mathbb{E}\left[v_t\right] \cdot 0 \\ &= \mathbb{E}\left[v_t\right] + \mathbb{E}\left[v_t\right]\left(\rho^2 - \rho^2\right) \\ &= \mathbb{E}\left[v_t\right] \end{aligned} \tag{25}$$

Using the same logic,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sqrt{v_t}\left(\rho Z_1+\sqrt{1-\rho^2}Z_2\right)\right)^4\right] \\
&=\rho^4\mathbb{E}\left[v_t^2\right]\mathbb{E}\left[Z_1^4\right]+\left(1-\rho^2\right)^2\mathbb{E}\left[v_t^2\right]\mathbb{E}\left[Z_2^4\right]+ \\
&\quad+6\mathbb{E}\left[v_t^2\right]\rho^2\left(1-\rho^2\right)\mathbb{E}\left[Z_1^2\right]\mathbb{E}\left[Z_2^2\right] \\
&=\mathbb{E}\left[v_t^2\right]\left(3\rho^4+3\left(1-\rho^2\right)^2+6\rho^2\left(1-\rho^2\right)\right) \\
&=\mathbb{E}\left[v_t^2\right]\left(6\rho^4-6\rho^2+3-6\rho^4+6\rho^2\right) \\
&=3\mathbb{E}\left[v_t^2\right]
\end{aligned}
\tag{26}
$$

Now, we can use the above results to calculate the moments of Q_{t+1}.
Let's say

$$
B_n \coloneqq \mathbb{E}\left[\left(\sqrt{v_t}\left(\rho Z_1+\sqrt{1-\rho^2}Z_2\right)\right)^n\right] \tag{27}
$$

$$
\begin{aligned}
\mathbb{E}\left[Q_{t+1}^2\right]&=(1+r)^2+2(1+r)B_1+B_2 \\
&=(1+r)^2+\mathbb{E}\left[v_t\right]
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
\mathbb{E}\left[Q_{t+1}^3\right]&=(1+r)^3+3(1+r)^2B_1+3(1+r)B_2+B_3 \\
&=(1+r)^3+3(1+r)\mathbb{E}\left[v_t\right]
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
\mathbb{E}\left[Q_{t+1}^4\right]&=(1+r)^4+6(1+r)^2B_2+B_4 \\
&=(1+r)^4+6(1+r)^2\mathbb{E}\left[v_t\right]+3\mathbb{E}\left[v_t^2\right]
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
\mathbb{E}\left[Q_{t+1}^5\right]&=(1+r)^5+10(1+r)^3B_2+5(1+r)B_4 \\
&=(1+r)^5+10(1+r)^3\mathbb{E}\left[v_t\right]+15(1+r)\mathbb{E}\left[v_t^2\right]
\end{aligned}
\tag{31}
$$

V. Heston-CIR Hybrid model

A. Idea

We have assumed that interest rate $r(t)$ remains constant in Heston model. But we know that in real world it is not true. So, to we overcome this problem we are going to combine Heston and CIR model.

In this case we are taking the interest rates for every three months announced by RBI. Then we use the CIR model to find how the interest rate is evolving.

From the CIR model we will be able to find the interest rate at time t. But we know that for every three months the interest rate remains constant. So, we can divide the entire problem of finding the option price to many pieces with constant rate of interest(piece size of three months).

For every piece of problem we first use CIR model to find the interest for that piece and then use heston model for that piece with has a constant interest rate.

VI. References

- Data source credit: NSE official website
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