Statistical Inference

HW Author: Ali Mollahosseini, Pouya Gohari, Parham Sazdar
Instructor: Mohammadreza A. Dehaqani

Spring 2024

HW 3: Part I Solutions

Problem 1: MoM

An urn contains B black balls and N - B white balls. A sample of n balls is to be selected without replacement. Let Y denote the number of black balls in the sample. Show that (N/n)Y is the method-of-moments (MOM) estimator of B.

Solution Considering the Hypergeometric distribution,

$$\mu = \frac{\theta n}{N}$$

thus

$$\theta = \frac{N}{n}\mu$$

thus the MOM estimator of θ is

$$\hat{\theta}_{MM} = \frac{N}{n}\bar{Y}_n$$

because it has been obtained substituting the first moment with the first sample moment.

Problem 2: Cramér-Rao

Let $X_1, \ldots, X_n$ be i.i.d. random variables from an exponential distribution with parameter $\lambda > 0$, which has the following probability density function:

$$f(x; \lambda) = \lambda e^{-\lambda x}, \text{ for } x > 0$$

Consider two unbiased estimators of λ:

- $\hat{\lambda}_1 = 1/\bar{X}$, where $\bar{X}$ is the sample mean.
- $\hat{\lambda}_2 = 2/(X_1 + X_n)$, which uses only the first and last observations.

1. Find the variances of $\hat{\lambda}_1$ and $\hat{\lambda}_2$.
2. Determine the Cramér-Rao lower bound for unbiased estimators of λ.
3. Which estimator is more efficient, $\hat{\lambda}_1$ or $\hat{\lambda}_2$? Justify your answer using the concept of efficiency and the Cramér-Rao inequality.

Solution:

1. To find the variances of $\hat{\lambda}_1$ and $\hat{\lambda}_2$, we first need to find their expected values and then use the variance formula. Copy codeFor $\hat{\lambda}_1 = 1/\bar{X}$: $\mathbb{E}[\bar{X}] = 1/\lambda$, and $\text{Var}(\bar{X}) = 1/(n\lambda^2)$ Using the delta method, $\text{Var}(\hat{\lambda}_1) \approx (1/\mathbb{E}[\bar{X}]^2)^2 * \text{Var}(\bar{X}) = \lambda^2 * 1/(n\lambda^2) = 1/(n\lambda^2)$

   For $\hat{\lambda}_2 = 2/(X_1 + X_n)$: $\mathbb{E}[X_1 + X_n] = 2/\lambda$, and $\text{Var}(X_1 + X_n) = 2/\lambda^2$ Using the delta method, $\text{Var}(\hat{\lambda}_2) \approx (2/\mathbb{E}[X_1 + X_n]^2)^2 * \text{Var}(X_1 + X_n) = (\lambda/2)^2 * 2/\lambda^2 = 1/(2\lambda^2)$

2. The Cramér-Rao lower bound for unbiased estimators of λ is given by: $\text{CRLB}(\lambda) = 1/(nI(\lambda))$, where $I(\lambda)$ is the Fisher information.

   For the exponential distribution, $I(\lambda) = 1/\lambda^2$. Therefore, $\text{CRLB}(\lambda) = 1/(n\lambda^2)$

3. To determine which estimator is more efficient, we compare their variances to the Cramér-Rao lower bound.

   $\text{Var}(\hat{\lambda}_1) = 1/(n\lambda^2)$, which is equal to the CRLB. Therefore, $\hat{\lambda}_1$ is efficient. $\text{Var}(\hat{\lambda}_2) = 1/(2\lambda^2)$, which is greater than the CRLB for any $n > 2$. Therefore, $\hat{\lambda}_2$ is not efficient.

   According to the Cramér-Rao inequality, the variance of any unbiased estimator is greater than or equal to the CRLB. Since $\hat{\lambda}_1$ achieves the CRLB, it is the more efficient estimator compared to $\hat{\lambda}_2$.

   In conclusion, the sample mean-based estimator $\hat{\lambda}_1$ is more efficient than the estimator $\hat{\lambda}_2$, which uses only the first and last observations.

Problem 3: GLRT

Let $X_1, \ldots, X_n$ be i.i.d. from an exponential distribution with unknown mean θ. Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

1. Derive the generalized likelihood ratio test for this setup.
2. Find the asymptotic distribution of $-2 \log \Lambda$ under H_0, where Λ is the likelihood ratio.

Solution

1. The likelihood function is $L(\lambda) = \prod \lambda^{x_i} \exp(-n\lambda)/\prod x_i!$. Under H_0, the MLE of λ is $\tilde{\lambda} = \min(\bar{x}, \lambda_0)$. Under H_1, the MLE is $\hat{\lambda} = \bar{x}$. The generalized likelihood ratio is:

$$\Lambda = \frac{L(\tilde{\lambda})}{L(\hat{\lambda})} = \left(\frac{\tilde{\lambda}}{\bar{x}}\right)^{n\bar{x}} \exp(-n(\tilde{\lambda} - \bar{x}))$$

   The test rejects for large values of $\bar{x}$ or equivalently small values of Λ. An exact threshold can be determined for any given n and λ_0.

2. The signed root likelihood ratio statistic is:

$$r = \text{sign}(\bar{x} - \lambda_0)\sqrt{2\left[n\bar{x}\log\left(\frac{\bar{x}}{\tilde{\lambda}}\right) - n(\bar{x} - \tilde{\lambda})\right]}$$

   The score statistic is:

$$S = \frac{(\bar{x} - \lambda_0)}{\sqrt{\lambda_0/n}}$$

   After some algebra, we can show that $r - S = o_p(1)$. Intuitively, both statistics are comparing the sample mean $\bar{x}$ to the null value λ_0, normalized by the variance. So the generalized likelihood ratio test is asymptotically equivalent to the score test in this case.

Problem 4: Playing around with the hypothesis

Suppose that a single observation X is taken from a uniform density on $[0, \theta]$, and consider testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

1. Find a test that has significance level $\alpha = 0$. What is its power?
2. For $0 < \alpha < 1$, consider the test that rejects when $X \in [0, \alpha]$. What is its significance level and power?
3. What is the significance level and power of the test that is rejected when $X \in [1 - \alpha, 1]$?
4. Find another test that has the same significance level and power as the previous one.
5. Does the likelihood ratio test determine a unique rejection region?
6. What happens if the null and alternative hypotheses are interchanged-$H_0 : \theta = 2$ versus $H_1 : \theta = 1$?

Solution

1. A test with significance level $\alpha = 0$ would never reject the null hypothesis H_0. The power of such a test is the probability of rejecting H_0 when H_1 is true, which is also 0.
2. Let's call this test ϕ_1. Under H_0, $X \sim \text{Unif}(0, 1)$, so

   $$\mathbb{P}H_0(\phi_1 \text{ rejects}) = \mathbb{P}H_0(X \in [0, \alpha]) \qquad = \int_0^\alpha \frac{1}{1} dx = \alpha$$

   Thus, the significance level is α. Under H_1, $X \sim \text{Unif}(0, 2)$, so

   $$\mathbb{P}H_1(\phi_1 \text{ rejects}) = \mathbb{P}H_1(X \in [0, \alpha]) \qquad = \int_0^\alpha \frac{1}{2} dx = \frac{\alpha}{2}$$

   Thus, the power is $\frac{\alpha}{2}$.
3. Let's call this test ϕ_2. Under H_0,

   $$\mathbb{P}H_0(\phi_2 \text{ rejects}) = \mathbb{P}H_0(X \in [1 - \alpha, 1]) \qquad = \int_{1-\alpha}^1 \frac{1}{1} dx = \alpha$$

   So the significance level is also α. Under H_1,

   $$\mathbb{P}H_1(\phi_2 \text{ rejects}) = \mathbb{P}H_1(X \in [1 - \alpha, 1]) \qquad = \int_{1-\alpha}^1 \frac{1}{2} dx = \frac{\alpha}{2}$$

   So the power is also $\frac{\alpha}{2}$.
4. The test ϕ_3 that rejects when $X \in [1, 1 + \alpha]$ has the same significance level and power as ϕ_2. Under H_0,

   $$\mathbb{P}H_0(\phi_3 \text{ rejects}) = \mathbb{P}H_0(X \in [1, 1 + \alpha]) \qquad = \int_1^{1+\alpha} \frac{1}{1} dx = \alpha$$

   Under H_1,

   $$\mathbb{P}H_1(\phi_3 \text{ rejects}) = \mathbb{P}H_1(X \in [1, 1 + \alpha]) \qquad = \int_1^{1+\alpha} \frac{1}{2} dx = \frac{\alpha}{2}$$

5. No, the likelihood ratio test does not determine a unique rejection region. The likelihood ratio is

   $$L(x) = \frac{f_0(x)}{f_1(x)} = \frac{\frac{1}{1}}{\frac{1}{2}} = 2$$

   for all $x \in [0, 1]$. So any subset of $[0, 1]$ with measure α could be used as a rejection region.

6. If the null and alternative are interchanged, then a test ϕ has significance level α if

$$\mathbb{P}H_0(\phi \text{ rejects}) = \mathbb{P}\theta = 2(\phi \text{ rejects}) = \alpha$$

and power

$$\mathbb{P}H_1(\phi \text{ rejects}) = \mathbb{P}\theta = 1(\phi \text{ rejects})$$

The tests ϕ_1, ϕ_2, and ϕ_3 now have significance level $\frac{\alpha}{2}$ and power α. In general, tests will have higher power and lower significance level when the null and alternative are interchanged in this problem.

Problem 5: Largest Possible Power

Consider two probability density functions on $[0, 1]$: $f_0(x) = 1$, and $f_1(x) = 2x$. Among all tests of the null hypothesis $H_0 : X \sim f_0(x)$ versus the alternative $X \sim f_1(x)$, with significance level $\alpha = 0.10$, how large can the power possibly be?

Solution: The likelihood ratio statistic is

$$L(X) = \frac{f_0(X)}{f_1(X)} = \frac{1}{2X}$$

The condition $L(X) < c$ is then equivalent to $X > \tilde{c}$, where $\tilde{c} = \frac{1}{2c}$. Under the hypothesis H_0, $X \sim \text{Uniform}(0, 1)$, so the rejection threshold $\tilde{c}$ should be $1 - 0.1 = 0.9$, i.e. the most powerful tests rejects H_0 when $X > 0.9$. Under the hypothesis H_1, $X \sim f_1(x) = 2x$. Then the type II error probability is

$$\beta = \mathbb{P}[H_1|\text{accept } H_0] = \mathbb{P}[H_1|X \leq 0.9] = \int_0^{0.9} 2x, dx = 0.81.$$

Thus the power of the test is

$$\text{Power} = 1 - \beta = 0.19$$

This is the maximum power that can be achieved: According to the Neyman-Pearson lemma, for any other test of H_0 with significance level at most 0.1, its power against H_1 is at most 0.19.

Problem 6: Hypothesis about Distribution + more

1. For data $X_1, \ldots, X_n \in \mathbb{R}$ and two fixed and known values $\sigma_0^2 < \sigma_1^2$, consider the following testing problem:

$$H_0 : X_1, \ldots, X_n \sim N(0, \sigma_0^2)$$
$$H_1 : X_1, \ldots, X_n \overset{IID}{\sim} N(0, \sigma_1^2)$$

What is the most powerful test for testing H_0 versus H_1 at level α? Letting $\chi_n^2(\alpha)$ denote the $1 - \alpha$ quantile of the χ_n^2 distribution, describe explicitly both the test statistic T and the rejection region for this test.

2. What is the distribution of this test statistic T under the alternative hypothesis H_1? Using this result, and letting F denote the CDF of the χ_n^2 distribution, provide a formula for the power of this test against H_1 in terms of $\chi_n^2(\alpha)$, σ_0^2, σ_1^2, and F. Keeping σ_0^2 fixed, what happens to the power of the test as σ_1^2 increases to ∞?

Solution:

1. The joint PDF under H_0 is

$$f_0(x_1,...,x_n) = \left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right)^n \exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma_0^2}\right)$$

The joint PDF under H_1 is

$$f_1(x_1,...,x_n) = \left(\frac{1}{\sqrt{2\pi\sigma_1^2}}\right)^n \exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma_1^2}\right)$$

So the likelihood ratio statistic is

$$L(X_1,...,X_n) = \frac{f_0(X_1,...,X_n)}{f_1(X_1,...,X_n)} = \left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\left(\frac{\sigma_0^2-\sigma_1^2}{2\sigma_0^2\sigma_1^2}\sum_{i=1}^n x_i^2\right)$$

Since $\sigma_0^2 < \sigma_1^2$, L is a decreasing function of $T := \sum_{i=1}^n X_i^2$. Then rejecting for small values of L is the same as rejecting for large values of T. Since under H_0, $\sum_{i=1}^n \left(\frac{X_i}{\sigma_0}\right)^2 \sim \chi_n^2$, we have $\frac{1}{\sigma_0^2}T \sim \chi_n^2$, so $T \sim \sigma_0^2\chi_n^2$. Then the rejection threshold should be $c = \sigma_0^2\chi_n^2(\alpha)$, and the most powerful test rejects H_0 when $T > c$.

2. Under H_1, $\sum_{i=1}^n \left(\frac{X_i}{\sigma_1}\right)^2 \sim \chi_n^2$, so $T \sim \sigma_1^2\chi_n^2$. Then the probability of type II error is

$$\beta = \mathbb{P}[H_1|\text{accept } H_0] = \mathbb{P}H_1[T \le \sigma_0^2\chi_n^2(\alpha)] \qquad = \mathbb{P}_{H_1}\left[\frac{T}{\sigma_1^2} \le \frac{\sigma_0^2}{\sigma_1^2}\chi_n^2(\alpha)\right] = F\left(\frac{\sigma_0^2}{\sigma_1^2}\chi_n^2(\alpha)\right)$$