Statistical Inference

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Spring 2024

HW 3: Part I

- Since your midterm is coming up, we didn't want to keep you busy with computer assignments and topics that are not covered in your exam, so we broke this homework into two parts. Part I is for exam preparation, and Part II will cover coding-related problems and the rest of hypothesis testing.
- Please note that this is the bare minimum number of questions you need to solve in preparation for your mideterm.
- If you have any questions about the homework, don't hesitate to use the class group to ask questions or drop an email to the HW Authors.
- Please consult the course page for important information on submission guidelines and delay policies to ensure your homework is turned in correctly and on time.
- This course aims to equip you with the skills to tackle all problems in this domain and encourages you to engage in independent research. Utilize your learnings to extend beyond the classroom teachings where necessary.

Problem 1: MoM (Optional)

An urn contains B black balls and N - B white balls. A sample of n balls is to be selected without replacement. Let Y denote the number of black balls in the sample. Show that (N/n)Y is the method-of-moments (MOM) estimator of B.

Problem 2: Cramér-Rao

Let $X_1, \ldots, X_n$ be i.i.d. random variables from an exponential distribution with parameter $\lambda > 0$, which has the following probability density function:

$$f(x; \lambda) = \lambda e^{-\lambda x}, \text{ for } x > 0$$

Consider two unbiased estimators of λ:

- $\hat{\lambda}_1 = 1/\bar{X}$, where $\bar{X}$ is the sample mean.
- $\hat{\lambda}_2 = 2/(X_1 + X_n)$, which uses only the first and last observations.

1. Find the variances of $\hat{\lambda}_1$ and $\hat{\lambda}_2$.
2. Determine the Cramér-Rao lower bound for unbiased estimators of λ.
3. Which estimator is more efficient, $\hat{\lambda}_1$ or $\hat{\lambda}_2$? Justify your answer using the concept of efficiency and the Cramér-Rao inequality.

Problem 3: GLRT

Let $X_1, \ldots, X_n$ be i.i.d. from an exponential distribution with unknown mean θ. Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

1. Derive the generalized likelihood ratio test for this setup.
2. Find the asymptotic distribution of $-2 \log \Lambda$ under H_0, where Λ is the likelihood ratio.

Problem 4: Playing around with the hypothesis

Suppose that a single observation X is taken from a uniform density on $[0, \theta]$, and consider testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

1. Find a test that has significance level $\alpha = 0$. What is its power?
2. For $0 < \alpha < 1$, consider the test that rejects when $X \in [0, \alpha]$. What is its significance level and power?
3. What is the significance level and power of the test that is rejected when $X \in [1 - \alpha, 1]$?
4. Find another test that has the same significance level and power as the previous one.
5. Does the likelihood ratio test determine a unique rejection region?
6. What happens if the null and alternative hypotheses are interchanged-$H_0 : \theta = 2$ versus $H_1 : \theta = 1$?

Problem 5: Largest Possible Power

Consider two probability density functions on $[0, 1]$: $f_0(x) = 1$, and $f_1(x) = 2x$. Among all tests of the null hypothesis $H_0 : X \sim f_0(x)$ versus the alternative $X \sim f_1(x)$, with significance level $\alpha = 0.10$, how large can the power possibly be?

Problem 6: Hypothesis about Distribution + more (Optional

1. For data $X_1, \ldots, X_n \in \mathbb{R}$ and two fixed and known values $\sigma_0^2 < \sigma_1^2$, consider the following testing problem:

$$H_0 : X_1, \ldots, X_n \sim N(0, \sigma_0^2)$$
$$H_1 : X_1, \ldots, X_n \overset{IID}{\sim} N(0, \sigma_1^2)$$

   What is the most powerful test for testing H_0 versus H_1 at level α? Letting $\chi_n^2(\alpha)$ denote the $1 - \alpha$ quantile of the χ_n^2 distribution, describe explicitly both the test statistic T and the rejection region for this test.

2. What is the distribution of this test statistic T under the alternative hypothesis H_1? Using this result, and letting F denote the CDF of the χ_n^2 distribution, provide a formula for the power of this test against H_1 in terms of $\chi_n^2(\alpha)$, σ_0^2, σ_1^2, and F. Keeping σ_0^2 fixed, what happens to the power of the test as σ_1^2 increases to ∞?

Problem 7: Sufficient Evidence

The average adult male height in a certain country is 170 cm. We suspect that the men in a certain city in that country might have a different average height due to some environmental factors. We pick a random sample of size 9 from the adult males in the city and obtain the following values for their heights (in cm):
176.2, 157.9, 160.1, 180.9, 165.1, 167.2, 162.9, 155.7, 166.2
Assume that the height distribution in this population is normally distributed. Here, we need to decide between

$$H_0 : \mu = 170$$

$$H_1 : \mu \neq 170$$

Based on the observed data, is there enough evidence to reject H_0 at significance level $\alpha = 0.05$?

Problem 8: Alternate Hypothesis

- Let $X_1, X_2, \cdots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution, where μ is unknown and σ is known. Design a level α test to choose between

$$H_0 : \mu \leq \mu_0,$$

$$H_1 : \mu > \mu_0.$$

- Let X_1, X_2, X_3, X_4 be a random sample from a $N(\mu, \sigma^2)$ distribution, where μ and σ are unknown. Suppose that we have observed the following values 3.58, 10.03, 4.77, 14.66 .We would like to decide between

$$H_0 :\geq 10$$

$$H_1 : \mu < 10$$

Assuming $\alpha = 0.05$, Do you accept H_0 or $H1$?

Problem 9: Comparing Means

The manager of a lemonade bottling plant is interested in comparing the performance of two production lines, one of which has only recently been installed. For each line she selects 10 one hour periods at random and records the number of crates completed in each hour. The table below gives the results. From past experience

Production line	**Number of crates completed per hour**									
1 (new)	78	87	79	82	87	81	85	80	82	83
2 (old)	74	77	78	70	87	83	76	78	81	76

Table 1: Production output comparison between new and old production lines

with this kind of equipment it is known that the variance in these figures will be 10 for Line 1 and 25 for Line 2. Assuming that these samples came from normal populations with these variances, test the hypothesis that the two populations have the same mean.

Problem 10: Significant Difference

The temperature of the earth may be measured either by thermometers on the ground (x), which is an accurate but tedious method, or by sensors mounted in space satellites (y), which is a less accurate method and may be

Site	Ground therm, x	Satellite sensors, y
1	4.6	4.7
2	17.3	19.5
3	12.2	12.5
4	3.6	4.2
5	6.2	6.0
6	14.8	15.4
7	11.4	14.9
8	14.9	17.8
9	9.3	9.7
10	10.4	10.5
11	7.2	7.4

Table 2: Comparison of ground and satellite sensor measurements

biased. The following table gives readings taken by both methods at eleven sites.
Given that all readings are normally distributed, investigate the hypothesis that satellite sensors give, on average, significantly higher readings than the ground thermometers.

Problem 11: Testing Independance

Suppose that 300 persons are selected at random from a large population, and each person in the sample is classified according to blood type, O, A, B, or AB, and also according to Rh, positive or negative. The observed numbers are given in Table below. Test the hypothesis that the two classifications of blood types are

Table 3: Data for Question 5

	O	A	B	AB
Rh positive	82	89	54	19
Rh negative	13	27	7	9

independent.

Problem 12: Tests Comparing Variance

Two college instructors are interested in whether or not there is any variation in the way they grade math exams. They each grade the same set of 30 exams. The first instructor's grades have a variance of 52.3. The second instructor's grades have a variance of 89.9. Test the claim that the first instructor's variance is smaller. The level of significance is 10%. (In most colleges, it is desirable for the variances of exam grades to be nearly the same among instructors.)