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1

1) $U(n)$

$U(12) = \{1, 5, 7, 11\}$

$\odot_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

=> From the above table, every elements belong to the set $U(12)$. Therefore the operation $\odot_{12}$ is closed under $U(12)$

=> Taking above elements for associativity

$$(1 \odot_{12} 7) \odot_{12} 11 = 1 \odot_{12} (7 \odot_{12} 11)$$

$$7 \odot_{12} 11 = 1 \odot_{12} 5$$

$$5 = 5$$

$$\therefore (1 \odot_{12} 7) \odot_{12} 11 = (1 \times 7 \times 11) \odot_{12} = 1 \odot_{12} (7 \odot_{12} 11)$$

Associativity satisfies.

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2

⇒ From the table, 1 gives back element with which operated.

∴ 1 is identity element.

⇒ And from the table, every element is inverse of itself.

∴ Inverse exists for every element.

Also, since the table is ~~commuta~~ symmetric (i.e first row = first column ...) it satisfies commutative.

∴ U(12) is an abelian group.

~~Order of group = 56~~

$$\therefore \phi(56) = 56\left(1-\frac{1}{2}\right)\left(1-\frac{1}{7}\right)$$

$$= 28\left(\frac{6}{7}\right)$$

$$= 24$$

$$\begin{array}{r|l} 2 & 56 \\ \hline 2 & 28 \\ \hline 2 & 14 \\ \hline & 7 \end{array}$$

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3

3) Given:- We know that $O(G) = O(a) = m$

and $a^m = e$

Part 1:- Suppose a^k is also a generator

then a can be written as

$$a = (a^k)^r \quad [\text{For some } r \in \mathbb{Z}]$$

$$a a^{-1} = a^{kr} \cdot a^{-1}$$

$$e = a^{kr-1}$$

$$\therefore \quad a^m = a^{kr-1}$$

By theorem, $m \mid kr-1$

and by division algorithm

$$1 = kr + ms \quad [\text{For some } s \in \mathbb{Z}]$$

$$\therefore (k, m) = 1$$

Part 2:- Suppose $(k, m) = 1$

We can write it as

$$1 = kr + ms \quad [\text{For some } r, s \in \mathbb{Z}]$$

$$\therefore \quad a = a^{kr+ms}$$

$$a = a^{kr} \cdot (a^m)^s$$

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(4)

$\Rightarrow \quad a = a^{kr} \cdot (e)^s$

$a = (a^k)^r$

$\therefore$ a^k is the generator of G.

$(Z_{18}, \oplus_{18})$, Order is 18

$\therefore$ No. of generators $= \phi(18) = 18(1-\frac{1}{2})(1-\frac{1}{3})$

$= 9(\frac{2}{3})$

$= 6$

Relatively prime no of 18

are $= \{1, 5, 7, 11, 13, 17\}$

Since here, 1 is generator

and nk is generators for additive group

List of generators $= \{1, 5, 7, 11, 13, 17\}$.

5] i] Given: $a, x \in G$

To prove:- $(xax^{-1})^n = xa^nx^{-1}$

Let $O(a) = n$ and $O(xax^{-1}) = m$

then taking LHS

$\Rightarrow (xax^{-1})^n$

$\Rightarrow x^n a^n x^{-n}$

$\Rightarrow x^n e x^{-n}$ [Given $O(a) = n$]

$\Rightarrow x^n x^{-n}$ [Inverse property]

$\Rightarrow e$ —(1)

Taking RHS

$= xa^nx^{-1}$

$= xex^{-1}$

$= xx^{-1}$ [Inverse law]

$= e$ —(2)

From (1) & (2), $(xax^{-1})^n = xa^nx^{-1}$

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(6)

Also

taking $a^n = e$

$\Rightarrow x^n a^n x^{-n} = x^n e x^{-n}$

$\Rightarrow (x a x^{-1})^n = x^n x^{-n}$

$\Rightarrow (x a x^{-1})^n = e$

$\therefore \boxed{m \leq n}$ [Since $(xax^{-1})^m = e$] — (3)

Now, $(x a x^{-1})^m = e$

$\Rightarrow x^m a^m x^{-m} = e$

$x^{-m} x^m a^m x^{-m} x^m = x^{-m} e x^m$

$a^m = x^{-m} x^m$

$a^m = e$

$\therefore \boxed{n \leq m}$ [Since $a^n = e$] — (4)

From (3) & (4), $O(a) = O(xax^{-1})$

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ii) The order of an element p of a group G is the number of least positive integer n such that

$p * p * p \dots * p$ (n times) $= e$. It is denoted by $O(p) = n$

Given:- $a, b \in G$ and $O(a) = 4$, $O(b) = 2$

$\therefore a^4 = e$ and $b^2 = e$

We know that, $a^3 b = ba$

$$(a^3 b)(ba)^3 = ba(ba)^3$$

$$(a^3 b)(ba)^3 = (ba)^4$$

$$(a^3 b)(ba)^3 = (b^2)^2 a^4$$

$$(a^3 b)(ba)^3 = e$$

$$\cancel{a^4 b^4 = e}$$

$$\cancel{(a^4)^2 (b^2)^2 = e}$$

$\therefore$ Order of $O(ab) = 3$

6] ii] Proof: Consider a group G and its subgroup H. Consider $a, b \in G$ then let Ha and Hb be its right cosets of H in G.

$\Rightarrow$ To prove, one-one

Let $f : H_a \to H_b$ i.e $f(Ha) = Hb$

Let $h_1 a = h_2 a$. By RCL $h_1 = h_2$

and $h_1 = h_2 \Rightarrow h_1 b = h_2 b$

Then $f(h_1 a) = f(h_2 a)$ [Since $h_1 b = h_2 b$]

$\therefore$ function is one-one

$\Rightarrow$ To prove, onto

There exists $h \in H$ for some $hb \in Hb$ such that $ha \in Ha$

$\therefore$ The function is onto.

$\therefore$ There is one to one correspondance between two right cosets.

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4

6) i) H is proper subgroup of Z under addition and contains 12, 30 and 54

H must contain identity element, 0

$\therefore$ H will contain inverses of 12, 30 & 54

Possibilities :- $-12, -30, -54$

~~12+42=24~~

$\therefore H = \{12, -12, 30, -30, 54, -54, 0, \dots\}$

~~2) ii) $\phi(91) = 91\left(1-\frac{1}{7}\right)\left(1-\frac{1}{13}\right)$~~

~~$= 78\left(\frac{12}{13}\right)$~~

4) ii) $a^5 = e$, $aba^{-1} = b^2$

Since we know that $O(b) = O(aba^{-1})$

0

Let $O(b) = m$ & $O(aba^{-1}) = n$

$\therefore b^m = (aba^{-1})^n$

$b^m = (b^2)^n$

$\therefore \boxed{m = 2n}$

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4] i] Let $G = \{1, -1, i, -i\}$

$H = \{1, -1\}$

$\therefore$ Left cosets = $\{1, -1\}$, $\{-1, 1\}$, $\{i, -i\}$

and $\{-i, i\}$

Distinct ~~left~~ left cosets = $\{1, -1\}$ & $\{i, -i\}$

Left cosets $\{i, -i\}$ cannot be a group because of absence of 1 [e]

$\therefore$ Every coset of group G is not a subgroup of G.