A CONSTRUCTING DATASETS FOR THE ACCURACY EXPERIMENT

The goal of this section is to show that given an upper-triangular matrix $\mathbf{R}_{\text{fixed}}$ we can devise matrices $\mathbf{S}$ and $\mathbf{T}$ with arbitrary dimensions such that $\mathbf{S} \times \mathbf{T} = \mathbf{QR}$ for an upper-triangular $\mathbf{R} = \begin{bmatrix} \mathbf{R}_{\text{fixed}} & \mathbf{V} \\ \mathbf{0} & \mathbf{W} \end{bmatrix}$, for some matrices $\mathbf{V}, \mathbf{W}$.

We denote by $\mathbf{1}_{m\times n}$ the $m \times n$ matrix that consists entirely of 1s and by $\mathbf{0}_{m\times n}$ the $m \times n$ matrix that consists entirely of 0s.

We use the following two observations.

Lemma A.1 ([1]). *For arbitrary real matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *with respective QR decompositions* $\mathbf{A} = \mathbf{Q}_A\mathbf{R}_A$ *and* $\mathbf{B} = \mathbf{Q}_B\mathbf{R}_B$ *it holds*

$$\mathbf{A} \otimes \mathbf{B} = (\mathbf{Q}_A \otimes \mathbf{Q}_B)(\mathbf{R}_A \otimes \mathbf{R}_B).$$

Lemma A.2. *Let* $\mathbf{A} \in \mathbb{R}^{m\times n_1}$ *and* $\mathbf{B} \in \mathbb{R}^{m\times n_2}$ *be arbitrary and let* $\mathbf{A} = \mathbf{Q}_A\mathbf{R}_A$ *be the QR decomposition of* $\mathbf{A}$*, where* $\mathbf{Q}_A \in \mathbb{R}^{m\times n_1}, \mathbf{R}_A \in \mathbb{R}^{n_1\times n_1}$*. There is an orthogonal matrix* $\mathbf{Q}' \in \mathbb{R}^{m\times n_2}$ *and matrices* $\mathbf{V} \in \mathbb{R}^{n_1\times n_2}, \mathbf{W} \in \mathbb{R}^{n_2\times n_2}$ *such that*

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_A & \mathbf{Q}' \end{bmatrix} \begin{bmatrix} \mathbf{R}_A & \mathbf{V} \\ \mathbf{0}_{n_2\times n_1} & \mathbf{W} \end{bmatrix}.$$

We revisit the notion of Kronecker products. For an $m \times n$ matrix $\mathbf{A}$ and a $p \times q$ matrix $\mathbf{B}$, the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is the $mp \times nq$ matrix

$$\begin{bmatrix} \mathbf{A}[1:1]\mathbf{B} & \cdots & A[1:n]\mathbf{B} \\ \vdots & \ddots & \vdots \\ \mathbf{A}[m:1]\mathbf{B} & \cdots & A[m:n]\mathbf{B} \end{bmatrix},$$

where each $A[i:j]\mathbf{B}$ is the matrix $\mathbf{B}$ multiplied by the scalar $A[i:j]$.

Now observe that we can express a Cartesian products in terms of Kronecker products: for $\mathbf{S} \in \mathbb{R}^{m_1\times n_1}$ and $\mathbf{T} \in \mathbb{R}^{m_2\times n_2}$, we have

$$\mathbf{S} \times \mathbf{T} = \begin{bmatrix} \mathbf{S} \otimes \mathbf{1}_{m_2\times 1} & \mathbf{1}_{m_1\times 1} \otimes \mathbf{T} \end{bmatrix}.$$

From Lemma A.1 it follows that for a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ with QR decomposition $\mathbf{A} = \mathbf{QR}$ we have

$$\mathbf{A} \otimes \mathbf{1}_{m\times 1} = (\mathbf{Q} \otimes \mathbf{Q}_1)(\mathbf{R}\sqrt{m})$$
$$\mathbf{1}_{m\times 1} \otimes \mathbf{A} = (\mathbf{Q}_1 \otimes \mathbf{Q})(\mathbf{R}\sqrt{m}),$$

where $\mathbf{1}_{m\times 1} = \mathbf{Q}_1 \begin{bmatrix} \sqrt{m} \end{bmatrix}$ is the QR decomposition of $\mathbf{1}_{m\times 1}$, $\begin{bmatrix} \sqrt{m} \end{bmatrix}$ is a 1×1 matrix with only entry $\sqrt{m}$, and $\mathbf{R}\sqrt{m}$ is the multiplication $\mathbf{R}$ with the scalar $\sqrt{m}$.

Together, this implies the following.

Corollary A.3. *Let* $\mathbf{S} \in \mathbb{R}^{m_1\times n_1}, \mathbf{T} \in \mathbb{R}^{m_2\times n_2}$ *be arbitrary and let* $\mathbf{S} = \mathbf{Q}_S\mathbf{R}_S$ *be the QR decomposition of* $\mathbf{S}$*, where* $\mathbf{Q}_S \in \mathbb{R}^{m_1\times n_1}, \mathbf{R}_S \in \mathbb{R}^{n_1\times n_1}$*. There is an orthogonal matrix* $\mathbf{Q}' \in \mathbb{R}^{m_1 m_2\times n_2}$ *and matrices* $\mathbf{V} \in \mathbb{R}^{n_1\times n_2}, \mathbf{W} \in \mathbb{R}^{n_2\times n_2}$ *such that*

$$\begin{aligned} \mathbf{S} \times \mathbf{T} &= \begin{bmatrix} \mathbf{S} \otimes \mathbf{1}_{m_2\times 1} & \mathbf{1}_{m_1\times 1} \otimes \mathbf{T} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Q}_S \otimes \mathbf{Q}_1 & \mathbf{Q}' \end{bmatrix} \begin{bmatrix} \mathbf{R}_S\sqrt{m_2} & \mathbf{V} \\ \mathbf{0}_{n_2\times n_1} & \mathbf{W} \end{bmatrix}. \end{aligned}$$

Let $\mathbf{R}_S \in \mathbb{R}^{n_1\times n_1}$ be an arbitrary given upper-triangular matrix. We arbitrarily choose a vector $\mathbf{v} = (v_1, \dots, v_{m_1})^\top \in \mathbb{Q}^{m_1}$ with $||\mathbf{v}||_2 = 1$ and a $m_2 \times n_2$ matrix $\mathbf{T}$ of natural numbers, where m_2 is square, so $\sqrt{m_2}$ is a natural number. We set $\mathbf{Q}_S$ to be the first n_1 columns of the orthogonal matrix of rational numbers [2]

$$\hat{\mathbf{Q}} = \begin{bmatrix} v_1 & v_2 & v_3 & \cdots & v_{n_1} \\ v_2 & \frac{v_2^2-v_1-1}{v_1+1} & \frac{v_2v_3}{v_1+1} & \cdots & \frac{v_2v_{m_1}}{v_1+1} \\ v_3 & \frac{v_3v_2}{v_1+1} & \frac{v_3^2-v_1-1}{v_1+1} & \cdots & \frac{v_3v_{m_1}}{v_1+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{m_1} & \frac{v_{m_1}v_2}{v_1+1} & \frac{v_{m_1}v_3}{v_1+1} & \cdots & \frac{v_{m_1}^2-v_1-1}{v_1+1} \end{bmatrix},$$

so $\mathbf{Q}_S = \hat{\mathbf{Q}}[:\{1,\dots,n_1\}]$. We obtain $\mathbf{S}$ as $\mathbf{S} = \mathbf{Q}_S\mathbf{R}_S$.

It follows from Corollary A.3 that there is an orthogonal matrix $\mathbf{Q}$ as well as matrices $\mathbf{V}, \mathbf{W}$ such that $\mathbf{S} \times \mathbf{T} = \mathbf{Q} \begin{bmatrix} \mathbf{R}_S\sqrt{m_2} & \mathbf{V} \\ \mathbf{0} & \mathbf{W} \end{bmatrix}$, as desired. Furthermore, if $\mathbf{R}_S$ only consists of rational numbers then so do $\mathbf{S}, \mathbf{T}$ and $\mathbf{R}_S\sqrt{m_2}$. When computing the QR decomposition of $\mathbf{S}\times\mathbf{T}$ we can compare the ground truth $\mathbf{R}_S\sqrt{m_2}$ with the corresponding part of the computed result.

REFERENCES

[1] Charles F Van Loan. 2000. The ubiquitous Kronecker product. *Journal of computational and applied mathematics* 123, 1-2 (2000), 85–100.

[2] CL Zihwei. 2006. Extending an orthonormal rational set of vectors into an orthonormal rational basis. *Unpublished online notes, available at the URL www. math. uchicago. edu/ may/VIGRE/VIGRE2006/PAPERS/Lin* (2006).