2022 年春季学期/数理统计/第十周/课后作业解答

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5 **证明.** 令

$$S_\theta = \frac{\partial \ln p(x;\theta)}{\partial \theta}$$

则，

$$\begin{aligned}
E(S_\theta) &= \int_{-\infty}^{+\infty} \frac{1}{p(x;\theta)} \cdot \frac{\partial p(x;\theta)}{\partial \theta} \cdot p(x;\theta)\ \mathrm{d}x \\
&= \int_{-\infty}^{+\infty} \frac{\partial}{\partial \theta} p(x;\theta)\ \mathrm{d}x = \frac{\partial}{\partial \theta} \int_{-\infty}^{+\infty} p(x;\theta)\ \mathrm{d}x = 0
\end{aligned}$$

所以，

$$\frac{\partial}{\partial \theta} E(S_\theta) = 0$$

同时，

$$\begin{aligned}
\frac{\partial E(S_\theta)}{\partial \theta} &= \frac{\partial}{\partial \theta} \int_{-\infty}^{+\infty} S_\theta \cdot p(x;\theta)\ \mathrm{d}x = \int_{-\infty}^{+\infty} \frac{\partial}{\partial \theta} \left[S_\theta \cdot p(x;\theta)\right]\ \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left[\frac{\partial S_\theta}{\partial \theta} \cdot p(x;\theta) + S_\theta \cdot \frac{\partial p(x;\theta)}{\partial \theta}\right]\ \mathrm{d}x \\
&= \int_{-\infty}^{+\infty} \frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2} \cdot p(x;\theta)\ \mathrm{d}x + \int_{-\infty}^{+\infty} \left[\frac{\partial \ln p(x;\theta)}{\partial \theta}\right]^2 \cdot p(x;\theta)\ \mathrm{d}x \\
&= E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2}\right] + E\left(S_\theta^2\right) \\
&= E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2}\right] + I(\theta) = 0
\end{aligned}$$

故，

$$I(\theta) = -E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2}\right]$$

6 **证明.** (a). 样本 $x_1, x_2, \ldots, x_n$ 的似然函数为

$$L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$$

对数似然函数为

$$\ln L(\theta) = n \ln \theta + (\theta - 1) \sum_{i=1}^{n} \ln x_i = -n \ln g(\theta) + \left[\frac{1}{g(\theta)} - 1\right] \sum_{i=1}^{n} \ln x_i$$

令

$$\frac{\partial \ln L(\theta)}{\partial g(\theta)} = -\frac{n}{g(\theta)} - \frac{1}{g^2(\theta)} \sum_{i=1}^{n} \ln x_i = 0$$

所以，$g(\theta)$ 的极大似然估计为

$$\hat{g}(\theta) = -\frac{1}{n} \sum_{i=1}^{n} \ln x_i$$

(b). 令 $Y = -\ln X$，则

$$P(Y < y) = P(-\ln X < y) = P\left(X > \mathrm{e}^{-y}\right) = \int_{e^{-y}}^{1} \theta x^{\theta - 1}\,\mathrm{d}x = 1 - \mathrm{e}^{-\theta y}$$

因此，

$$Y \sim \mathrm{Exp}(\theta), \quad \hat{g}(\theta) = \frac{1}{n} \sum_{i=1}^{n} Y \sim \mathrm{Ga}(n, n\theta)$$

于是，

$$E(\hat{g}) = \frac{n}{n\theta} = \frac{1}{\theta} = g(\theta), \quad \mathrm{Var}(\hat{g}) = \frac{n}{(n\theta)^2} = \frac{1}{n\theta^2}$$

$$\frac{\partial p(x;\theta)}{\partial \theta} = \frac{1}{\theta} + \ln x, \quad \frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2} = -\frac{1}{\theta^2}$$

因此，θ 的费舍尔信息量为

$$I(\theta) = -E\left[\frac{\partial^2}{\partial \theta^2} \ln p(x;\theta)\right] = \frac{1}{\theta^2}$$

故，$g(\theta)$ 的任一无偏估计的 C-R 下界为

$$\frac{[g'(\theta)]^2}{nI(\theta)} = \frac{1}{n\theta^2}$$

所以，$\hat{g}(\theta) = -\frac{1}{n}\sum_{i=1}^{n} \ln x_i$ 是 $g(\theta)$ 的有效估计。

7 **证明.** 对数密度函数为

$$\ln p(x;\theta) = \ln 2 + \ln \theta - 3\ln x - \frac{\theta}{x^2}$$

于是，

$$\frac{\partial \ln p(x;\theta)}{\partial \theta} = \frac{1}{\theta} - \frac{1}{x^2}, \quad \frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2} = -\frac{1}{\theta^2}$$

因此，θ 的费舍尔信息量为

$$I(\theta) = -E\left[\frac{\partial^2}{\partial \theta^2} \ln p(x;\theta)\right] = \frac{1}{\theta^2}$$