

2021 年春季学期/数理统计/第二周/课后作业解答

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4 证明. 样本容量为 $n+1$ 时的样本均值 \bar{x}_{n+1} :

$$\begin{aligned}\bar{x}_{n+1} &= \frac{1}{n+1} \sum_{i=1}^{n+1} x_i = \frac{1}{n+1} \left(\sum_{i=1}^n x_i + x_{n+1} \right) = \frac{1}{n+1} (n\bar{x}_n + x_{n+1}) \\ &= \frac{1}{n+1} [(n+1)\bar{x}_n - \bar{x}_n + x_{n+1}] = \bar{x}_n + \frac{1}{n+1} (x_{n+1} - \bar{x}_n)\end{aligned}$$

样本容量为 $n+1$ 时的样本方差 s_{n+1}^2 :

$$\begin{aligned}s_{n+1}^2 &= \frac{1}{n} \sum_{i=1}^{n+1} (x_i - \bar{x}_{n+1})^2 \\ &= \frac{1}{n} \sum_{i=1}^n \left\{ x_i - \left[\bar{x}_n + \frac{1}{n+1} (x_{n+1} - \bar{x}_n) \right] \right\}^2 \\ &= \frac{1}{n} \sum_{i=1}^{n+1} \left[(x_i - \bar{x}_n) - \frac{1}{n+1} (x_{n+1} - \bar{x}_n) \right]^2 \\ &= \frac{1}{n} \sum_{i=1}^{n+1} \left\{ (x_i - \bar{x}_n)^2 - \frac{2}{n+1} (x_i - \bar{x}_n) (x_{n+1} - \bar{x}_n) + \frac{1}{(n+1)^2} (x_{n+1} - \bar{x}_n)^2 \right\} \\ &= \frac{1}{n} \left[\sum_{i=1}^n (x_i - \bar{x}_n)^2 \right] + \frac{1}{n} (x_{n+1} - \bar{x}_n)^2 - \frac{2}{n(n+1)} (x_{n+1} - \bar{x}_n) \left[\sum_{i=1}^n (x_i - \bar{x}_n) \right] \\ &\quad - \frac{2}{n(n+1)} (x_{n+1} - \bar{x}_n)^2 + \frac{1}{n(n+1)} (x_{n+1} - \bar{x}_n)^2 \\ &= \frac{n-1}{n} s_n^2 + \frac{1}{n+1} (x_{n+1} - \bar{x}_n)^2\end{aligned}$$

(另一种思路)

$$\begin{aligned}
s_{n+1}^2 &= \frac{1}{n} \sum_{i=1}^{n+1} (x_i - \bar{x}_{n+1})^2 = \frac{1}{n} \sum_{i=1}^{n+1} [(x_i - \bar{x}_n) + (\bar{x}_n - \bar{x}_{n+1})]^2 \\
&= \frac{1}{n} \left[\sum_{i=1}^{n+1} (x_i - \bar{x}_n)^2 \right] + \frac{2}{n} (\bar{x}_n - \bar{x}_{n+1}) \sum_{i=1}^{n+1} (x_i - \bar{x}_n) + \frac{1}{n} \sum_{i=1}^{n+1} (\bar{x}_n - \bar{x}_{n+1})^2 \\
&= \frac{1}{n} \left[\sum_{i=1}^{n+1} (x_i - \bar{x}_n)^2 \right] - \frac{n+1}{n} (\bar{x}_n - \bar{x}_{n+1})^2 \\
&= \frac{1}{n} \left[\sum_{i=1}^n (x_i - \bar{x}_n)^2 + (x_{n+1} - \bar{x}_n)^2 \right] - \frac{n+1}{n} \frac{1}{(n+1)^2} (x_{n+1} - \bar{x}_n)^2 \\
&= \frac{1}{n} \left[(n-1) \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x}_n)^2 + \frac{n}{n+1} (x_{n+1} - \bar{x}_n)^2 \right] \\
&= \frac{n-1}{n} s_n^2 + \frac{1}{n+1} (x_{n+1} - \bar{x}_n)^2
\end{aligned}$$

6 证明. 样本 B 的均值 \bar{y}_B :

$$\begin{aligned}
\bar{y}_B &= \frac{1}{n} \sum_{i=1}^n y_i = \frac{1}{n} \sum_{i=1}^n (ax_i + b) = \frac{1}{n} \left(a \sum_{i=1}^n x_i + nb \right) \\
&= a \cdot \frac{1}{n} \sum_{i=1}^n x_i + b = a\bar{x}_A + b
\end{aligned}$$

样本 B 的标准差 s_B :

$$\begin{aligned}
s_B &= \sqrt{\frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y}_B)^2} = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (ax_i + b - a\bar{x}_A - b)^2} \\
&= |a| \cdot \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x}_A)^2} = |a|s_A
\end{aligned}$$

样本 B 的极差 R_B :

$$R_B = y_{(n)} - y_{(1)} = ax_{(n)} + b - ax_{(1)} - b = a[x_{(n)} - x_{(1)}] = aR_A$$

样本 B 的中位数 $m_{0.5B}$:

(a). 当 n 为偶数时,

$$m_{0.5B} = y_{(\frac{n+1}{2})} = ax_{(\frac{n+1}{2})} + b = am_{0.5A} + b$$

(b). 当 n 为奇数时,

$$\begin{aligned}
m_{0.5B} &= \frac{1}{2} \left[y_{(\frac{n}{2})} + y_{(\frac{n}{2}+1)} \right] = \frac{1}{2} \left[ax_{(\frac{n}{2})} + b + ax_{(\frac{n}{2}+1)} + b \right] \\
&= \frac{a}{2} \left[x_{(\frac{n}{2})} + x_{(\frac{n}{2}+1)} \right] + b = am_{0.5A} + b
\end{aligned}$$

因此, 样本 B 的中位数 $m_{0.5B}$ 为

$$m_{0.5B} = am_{0.5A} + b.$$

8 证明. 由定理 5.3.2 有,

$$E(\bar{X}) = E(X_i) = \mu = 0, \quad \text{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{1}{3n}.$$

9 证明. 因为样本 $X_i, i = 1, 2, \dots, n$ 是相互独立的, 所以

$$\text{Cov}(X_i, X_j) = 0, \quad (i \neq j).$$

因此,

$$\text{Cov}(X_i, \bar{X}) = \text{Cov}\left(X_i, \frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n} \text{Cov}(X_i, X_i) = \frac{\sigma^2}{n},$$

$$\text{Cov}(\bar{X}, \bar{X}) = \text{Var}(\bar{X}) = \frac{1}{n} \sum_{i=1}^n \text{Cov}(X_i, \bar{X}) = \frac{\sigma^2}{n}.$$

$$\begin{aligned} \text{Cov}(X_i - \bar{X}, X_j - \bar{X}) &= \text{Cov}(X_i, X_j) - \text{Cov}(X_i, \bar{X}) - \text{Cov}(X_j, \bar{X}) + \text{Cov}(\bar{X}, \bar{X}) \\ &= 0 - \frac{1}{n}\sigma^2 - \frac{1}{n}\sigma^2 + \frac{1}{n}\sigma^2 = -\frac{1}{n}\sigma^2 \end{aligned}$$

$$\begin{aligned} \text{Var}(X_i - \bar{X}) &= \text{Var}(X_i) + \text{Var}(\bar{X}) - 2 \text{Cov}(X_i, \bar{X}) \\ &= \sigma^2 + \frac{1}{n}\sigma^2 - \frac{2}{n}\sigma^2 = \frac{n-1}{n}\sigma^2 \end{aligned}$$

同理, 我们有

$$\text{Var}(X_j - \bar{X}) = \frac{n-1}{n}\sigma^2;$$

所以,

$$\begin{aligned} \text{Corr}(X_i - \bar{X}, X_j - \bar{X}) &= \frac{\text{Cov}(X_i - \bar{X}, X_j - \bar{X})}{\sqrt{\text{Var}(X_i - \bar{X})} \cdot \sqrt{\text{Var}(X_j - \bar{X})}} \\ &= \frac{-\frac{1}{n}\sigma^2}{\sqrt{\frac{n-1}{n}\sigma^2} \cdot \sqrt{\frac{n-1}{n}\sigma^2}} = -\frac{1}{n-1} \end{aligned}$$

18 证明. 由定理 5.3.2 有,

$$\text{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{9}{8}, \quad \sigma(\bar{X}) = \sqrt{\text{Var}(\bar{X})} = \frac{3\sqrt{2}}{4}.$$

23 证明. (可参考例 3.3.4 与例 3.3.5)

$$P(X \leq k) = \sum_{i=1}^k pq^{i-1} = \frac{p(1-q^k)}{1-q} = 1 - q^k, \quad k = 1, 2, \dots,$$

对于 $X_{(n)}$, 有

$$\begin{aligned} P(X_{(n)} = k) &= P(X_{(n)} \leq k) - P(X_{(n)} \leq k-1) \\ &= \prod_{i=1}^n P(X_i \leq k) - \prod_{i=1}^n P(X_i \leq k-1) \\ &= (1 - q^k)^n - (1 - q^{k-1})^n \end{aligned}$$

对于 $X_{(1)}$, 有

$$\begin{aligned}
 P(X_{(1)} = k) &= P(X_{(1)} \leq k) - P(X_{(1)} \leq k-1) \\
 &= 1 - P(X_{(1)} > k) - [1 - P(X_{(1)} > k-1)] \\
 &= \prod_{i=1}^n P(X_i > k-1) - \prod_{i=1}^n P(X_i > k) \\
 &= q^{n(k-1)} - q^{nk} = q^{n(k-1)} (1 - q^n)
 \end{aligned}$$

25 证明. 韦布尔分布的总体分布函数 $F(x)$ 为

$$\begin{aligned}
 F(x) &= \int_0^x p(t) dt = \int_0^x \frac{mt^{m-1}}{\eta^m} e^{-\left(\frac{t}{\eta}\right)^m} dt = \int_0^x e^{-\left(\frac{t}{\eta}\right)^m} d\left(\frac{t}{\eta}\right)^m \\
 &= -e^{-\left(\frac{t}{\eta}\right)^m} \Big|_0^x = 1 - e^{-\left(\frac{x}{\eta}\right)^m}
 \end{aligned}$$

因此,

$$\begin{aligned}
 p_{(1)}(x) &= n[1 - F(x)]^{n-1} p(x) = ne^{-(n-1)\left(\frac{x}{\eta}\right)^m} \cdot \frac{mx^{m-1}}{\eta^m} e^{-\left(\frac{x}{\eta}\right)^m} \\
 &= \frac{mnx^{m-1}}{\eta^m} e^{-n\left(\frac{x}{\eta}\right)^m} = \frac{mx^{m-1}}{(\eta/\sqrt[n]{n})^m} e^{-\left(\frac{x}{\eta/\sqrt[n]{n}}\right)^m}
 \end{aligned}$$

所以, $X_{(1)}$ 服从参数为 $\left(m, \frac{\eta}{\sqrt[n]{n}}\right)$ 的韦布尔分布。