2021 年春季学期/数理统计/第二周/课后作业解答

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更新：2021 年 3 月 29 日

4 **证明.** 样本容量为 $n+1$ 时的样本均值 $\bar{x}_{n+1}$：

$$\begin{aligned}\bar{x}_{n+1} =& \frac{1}{n+1}\sum_{i=1}^{n+1} x_i = \frac{1}{n+1}\left(\sum_{i=1}^{n} x_i + x_{n+1}\right) = \frac{1}{n+1}\left(n\bar{x}_n + x_{n+1}\right) \\ =& \frac{1}{n+1}\left[(n+1)\bar{x}_n - \bar{x}_n + x_{n+1}\right] = \bar{x}_n + \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)\end{aligned}$$

样本容量为 $n+1$ 时的样本方差 s_{n+1}^2：

$$\begin{aligned}s_{n+1}^2 =& \frac{1}{n}\sum_{i=1}^{n+1}\left(x_i - \bar{x}_{n+1}\right)^2 \\ =& \frac{1}{n}\sum_{i=1}^{n}\left\{x_i - \left[\bar{x}_n + \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)\right]\right\}^2 \\ =& \frac{1}{n}\sum_{i=1}^{n+1}\left[\left(x_i - \bar{x}_n\right) - \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)\right]^2 \\ =& \frac{1}{n}\sum_{i=1}^{n+1}\left\{\left(x_i - \bar{x}_n\right)^2 - \frac{2}{n+1}\left(x_i - \bar{x}_n\right)\left(x_{n+1} - \bar{x}_n\right) + \frac{1}{(n+1)^2}\left(x_{n+1} - \bar{x}_n\right)^2\right\} \\ =& \frac{1}{n}\left[\sum_{i=1}^{n}\left(x_i - \bar{x}_n\right)^2\right] + \frac{1}{n}\left(x_{n+1} - \bar{x}_n\right)^2 - \frac{2}{n(n+1)}\left(x_{n+1} - \bar{x}_n\right)\left[\sum_{i=1}^{n}\left(x_i - \bar{x}_n\right)\right] \\ & - \frac{2}{n(n+1)}\left(x_{n+1} - \bar{x}_n\right)^2 + \frac{1}{n(n+1)}\left(x_{n+1} - \bar{x}_n\right)^2 \\ =& \frac{n-1}{n}s_n^2 + \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)^2\end{aligned}$$

（另一种思路）

$$
\begin{aligned}
s_{n+1}^2 =& \frac{1}{n}\sum_{i=1}^{n+1}(x_i-\bar{x}_{n+1})^2 = \frac{1}{n}\sum_{i=1}^{n+1}[(x_i-\bar{x}_n)+(\bar{x}_n-\bar{x}_{n+1})]^2 \\
=& \frac{1}{n}\left[\sum_{i=1}^{n+1}(x_i-\bar{x}_n)^2\right] + \frac{2}{n}(\bar{x}_n-\bar{x}_{n+1})\sum_{i=1}^{n+1}(x_i-\bar{x}_n) + \frac{1}{n}(\bar{x}_n-\bar{x}_{n+1})^2 \\
=& \frac{1}{n}\left[\sum_{i=1}^{n+1}(x_i-\bar{x}_n)^2\right] - \frac{1}{n}(\bar{x}_n-\bar{x}_{n+1})^2 \\
=& \frac{1}{n}\left[\sum_{i=1}^{n+1}(x_i-\bar{x}_n)^2\right] - \frac{n+1}{n}(\bar{x}_n-\bar{x}_{n+1})^2 \\
=& \frac{1}{n}\left[\sum_{i=1}^{n}(x_i-\bar{x}_n)^2 + (x_{n+1}-\bar{x}_n)^2\right] - \frac{n+1}{n}\frac{1}{(n+1)^2}(x_{n+1}-\bar{x}_n)^2 \\
=& \frac{1}{n}\left[(n-1)\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x}_n)^2 + \frac{n}{n+1}(x_{n+1}-\bar{x}_n)^2\right] \\
=& \frac{n-1}{n}s_n^2 + \frac{1}{n+1}(x_{n+1}-\bar{x}_n)^2
\end{aligned}
$$

6 **证明.** 样本 B 的均值 $\bar{y}_B$：

$$
\begin{aligned}
\bar{y}_B =& \frac{1}{n}\sum_{i=1}^{n}y_i = \frac{1}{n}\sum_{i=1}^{n}(ax_i+b) = \frac{1}{n}\left(a\sum_{i=1}^{n}x_i+nb\right) \\
=& a\cdot\frac{1}{n}\sum_{i=1}^{n}x_i + b = a\bar{x}_A + b
\end{aligned}
$$

样本 B 的标准差 s_B：

$$
\begin{aligned}
s_B =& \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(y_i-\bar{y}_B)^2} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(ax_i+b-a\bar{x}_A-b)^2} \\
=& |a|\cdot\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x}_A)^2} = |a|s_A
\end{aligned}
$$

样本 B 的极差 R_B：

$$
R_B = y_{(n)} - y_{(1)} = ax_{(n)} + b - ax_{(1)} - b = a\left[x_{(n)} - x_{(1)}\right] = aR_A
$$

样本 B 的中位数 $m_{0.5B}$：

(a). 当 n 为偶数时，

$$
m_{0.5B} = y_{\left(\frac{n+1}{2}\right)} = ax_{\left(\frac{n+1}{2}\right)} + b = am_{0.5A} + b
$$

(b). 当 n 为奇数时，

$$
\begin{aligned}
m_{0.5B} =& \frac{1}{2}\left[y_{\left(\frac{n}{2}\right)} + y_{\left(\frac{n}{2}+1\right)}\right] = \frac{1}{2}\left[ax_{\left(\frac{n}{2}\right)} + b + ax_{\left(\frac{n}{2}+1\right)} + b\right] \\
=& \frac{a}{2}\left[x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n}{2}+1\right)}\right] + b = am_{0.5A} + b
\end{aligned}
$$

因此，样本 B 的中位数 $m_{0.5B}$ 为

$$m_{0.5B} = am_{0.5A} + b.$$

8 **证明.** 由定理 5.3.2 有，

$$E(\bar{X}) = E(X_i) = \mu = 0, \quad \operatorname{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{1}{3n}.$$

9 **证明.** 因为样本 $X_i, i = 1, 2, \ldots, n$ 是相互独立的，所以

$$\operatorname{Cov}(X_i, X_j) = 0, \quad (i \neq j).$$

因此，

$$\operatorname{Cov}\left(X_i, \bar{X}\right) = \operatorname{Cov}\left(X_i, \frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\operatorname{Cov}(X_i, X_i) = \frac{\sigma^2}{n},$$

$$\operatorname{Cov}(\bar{X}, \bar{X}) = \operatorname{Var}\left(\bar{X}\right) = \frac{1}{n}\sum_{i=1}^{n}\operatorname{Cov}\left(X_i, \bar{X}\right) = \frac{\sigma^2}{n}.$$

$$\begin{aligned}\operatorname{Cov}\left(X_i - \bar{X}, X_j - \bar{X}\right) =& \operatorname{Cov}(X_i, X_j) - \operatorname{Cov}\left(X_i, \bar{X}\right) - \operatorname{Cov}\left(X_j, \bar{X}\right) + \operatorname{Cov}(\bar{X}, \bar{X}) \\ =& 0 - \frac{1}{n}\sigma^2 - \frac{1}{n}\sigma^2 + \frac{1}{n}\sigma^2 = -\frac{1}{n}\sigma^2\end{aligned}$$

$$\begin{aligned}\operatorname{Var}\left(X_i - \bar{X}\right) =& \operatorname{Var}(X_i) + \operatorname{Var}(\bar{X}) - 2\operatorname{Cov}\left(X_i, \bar{X}\right) \\ =& \sigma^2 + \frac{1}{n}\sigma^2 - \frac{2}{n}\sigma^2 = \frac{n-1}{n}\sigma^2\end{aligned}$$

同理，我们有

$$\operatorname{Var}\left(X_j - \bar{X}\right) = \frac{n-1}{n}\sigma^2;$$

所以，

$$\begin{aligned}\operatorname{Corr}\left(X_i - \bar{X}, X_j - \bar{X}\right) =& \frac{\operatorname{Cov}\left(X_i - \bar{X}, X_j - \bar{X}\right)}{\sqrt{\operatorname{Var}\left(X_i - \bar{X}\right)} \cdot \sqrt{\operatorname{Var}\left(X_j - \bar{X}\right)}} \\ =& \frac{-\frac{1}{n}\sigma^2}{\sqrt{\frac{n-1}{n}\sigma^2} \cdot \sqrt{\frac{n-1}{n}\sigma^2}} = -\frac{1}{n-1}\end{aligned}$$

18 **证明.** 由定理 5.3.2 有，

$$\operatorname{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{9}{8}, \quad \sigma(\bar{X}) = \sqrt{\operatorname{Var}(\bar{X})} = \frac{3\sqrt{2}}{4}.$$

23 **证明.**（可参考例 3.3.4 与例 3.3.5）

$$P(X \leq k) = \sum_{i=1}^{k} pq^{i-1} = \frac{p\left(1 - q^k\right)}{1 - q} = 1 - q^k, \quad k = 1, 2, \ldots,$$

对于 $X_{(n)}$，有

$$
\begin{aligned}
P\left(X_{(n)}=k\right) =&P\left(X_{(n)} \leq k\right)-P\left(X_{(n)} \leq k-1\right) \\
=&\prod_{i=1}^{n} P\left(X_{i} \leq k\right)-\prod_{i=1}^{n} P\left(X_{i} \leq k-1\right) \\
=&\left(1-q^{k}\right)^{n}-\left(1-q^{k-1}\right)^{n}
\end{aligned}
$$

对于 $X_{(1)}$，有

$$
\begin{aligned}
P\left(X_{(1)}=k\right) =&P\left(X_{(1)} \leq k\right)-P\left(X_{(1)} \leq k-1\right) \\
=&1-P\left(X_{(1)}>k\right)-\left[1-P\left(X_{(1)}>k-1\right)\right] \\
=&\prod_{i=1}^{n} P\left(X_{i}>k-1\right)-\prod_{i=1}^{n} P\left(X_{i}>k\right) \\
=&q^{n(k-1)}-q^{nk}=q^{n(k-1)}\left(1-q^{n}\right)
\end{aligned}
$$

25 **证明.** 韦布尔分布的总体分布函数 $F(x)$ 为

$$
\begin{aligned}
F(x) &= \int_0^x p(t)\mathrm{d}t = \int_0^x \frac{mt^{m-1}}{\eta^m}\mathrm{e}^{-\left(\frac{t}{\eta}\right)^m}\,\mathrm{d}t = \int_0^x \mathrm{e}^{-\left(\frac{t}{\eta}\right)^m}\,\mathrm{d}\left(\frac{t}{\eta}\right)^m \\
&= -\,\mathrm{e}^{-\left(\frac{t}{\eta}\right)^m}\Big|_0^x = 1-\mathrm{e}^{-\left(\frac{x}{\eta}\right)^m}
\end{aligned}
$$

因此，

$$
\begin{aligned}
p_{(1)}(x) =&n[1-F(x)]^{n-1}p(x) = n\mathrm{e}^{-(n-1)\left(\frac{x}{\eta}\right)^m}\cdot\frac{mx^{m-1}}{\eta^m}\mathrm{e}^{-\left(\frac{x}{\eta}\right)^m} \\
=&\frac{mnx^{m-1}}{\eta^m}\mathrm{e}^{-n\left(\frac{x}{\eta}\right)^m} = \frac{mx^{m-1}}{(\eta/\sqrt[m]{n})^m}\mathrm{e}^{-\left(\frac{x}{\eta/\sqrt[m]{n}}\right)^m}
\end{aligned}
$$

所以，$X_{(1)}$ 服从参数为 $\left(m, \frac{\eta}{\sqrt[m]{n}}\right)$ 的韦布尔分布。