4 **证明.** 样本容量为 $n+1$ 时的样本均值 $\bar{x}_{n+1}$：

$$\begin{aligned}\bar{x}_{n+1} =& \frac{1}{n+1}\sum_{i=1}^{n+1} x_i = \frac{1}{n+1}\left(\sum_{i=1}^{n} x_i + x_{n+1}\right) = \frac{1}{n+1}\left(n\bar{x}_n + x_{n+1}\right) \\ =& \frac{1}{n+1}\left[\left(n+1\right)\bar{x}_n - \bar{x}_n + x_{n+1}\right] = \bar{x}_n + \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)\end{aligned}$$

样本容量为 $n+1$ 时的样本方差 s^2_{n+1}：

$$\begin{aligned}s^2_{n+1} =& \frac{1}{n}\sum_{i=1}^{n+1}\left(x_i - \bar{x}_{n+1}\right)^2 \\ =& \frac{1}{n}\sum_{i=1}^{n}\left\{x_i - \left[\bar{x}_n + \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)\right]\right\}^2 \\ =& \frac{1}{n}\sum_{i=1}^{n+1}\left[\left(x_i - \bar{x}_n\right) - \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)\right]^2 \\ =& \frac{1}{n}\sum_{i=1}^{n+1}\left\{\left(x_i - \bar{x}_n\right)^2 - \frac{2}{n+1}\left(x_i - \bar{x}_n\right)\left(x_{n+1} - \bar{x}_n\right) + \frac{1}{(n+1)^2}\left(x_{n+1} - \bar{x}_n\right)^2\right\} \\ =& \frac{1}{n}\left[\sum_{i=1}^{n}\left(x_i - \bar{x}_n\right)^2\right] + \frac{1}{n}\left(x_{n+1} - \bar{x}_n\right)^2 - \frac{2}{n\left(n+1\right)}\left(x_{n+1} - \bar{x}_n\right)\left[\sum_{i=1}^{n}\left(x_i - \bar{x}_n\right)\right] \\ &- \frac{2}{n\left(n+1\right)}\left(x_{n+1} - \bar{x}_n\right)^2 + \frac{1}{n\left(n+1\right)}\left(x_{n+1} - \bar{x}_n\right)^2 \\ =& \frac{n-1}{n}s^2_n + \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)^2\end{aligned}$$

（另一种思路）

$$\begin{aligned}s^2_{n+1} =& \frac{1}{n}\sum_{i=1}^{n+1}\left(x_i - \bar{x}_{n+1}\right)^2 = \frac{1}{n}\sum_{i=1}^{n+1}\left[\left(x_i - \bar{x}_n\right) + \left(\bar{x}_n - \bar{x}_{n+1}\right)\right]^2 \\ =& \frac{1}{n}\left[\sum_{i=1}^{n+1}\left(x_i - \bar{x}_n\right)^2\right] + \frac{2}{n}\left(\bar{x}_n - \bar{x}_{n+1}\right)\sum_{i=1}^{n+1}\left(x_i - \bar{x}_n\right) + \frac{1}{n}\sum_{i=1}^{n+1}\left(\bar{x}_n - \bar{x}_{n+1}\right)^2 \\ =& \frac{1}{n}\left[\sum_{i=1}^{n+1}\left(x_i - \bar{x}_n\right)^2\right] - \frac{n+1}{n}\left(\bar{x}_n - \bar{x}_{n+1}\right)^2 \\ =& \frac{1}{n}\left[\sum_{i=1}^{n}\left(x_i - \bar{x}_n\right)^2 + \left(x_{n+1} - \bar{x}_n\right)^2\right] - \frac{n+1}{n}\frac{1}{(n+1)^2}\left(x_{n+1} - \bar{x}_n\right)^2 \\ =& \frac{1}{n}\left[\left(n-1\right)\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}_n\right)^2 + \frac{n}{n+1}\left(x_{n+1} - \bar{x}_n\right)^2\right] \\ =& \frac{n-1}{n}s^2_n + \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)^2\end{aligned}$$

6 **证明.** 样本 B 的均值 $\bar{y}_B$：

$$\begin{aligned}\bar{y}_B =& \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\sum_{i=1}^{n}\left(ax_i + b\right) = \frac{1}{n}\left(a\sum_{i=1}^{n} x_i + nb\right) \\ =& a\cdot\frac{1}{n}\sum_{i=1}^{n} x_i + b = a\bar{x}_A + b\end{aligned}$$

样本 B 的标准差 s_B：

$$\begin{aligned}
s_B =&\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(y_i-\bar{y}_B\right)^2}=\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(ax_i+b-a\bar{x}_A-b\right)^2}\\
=&|a|\cdot\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-\bar{x}_A\right)^2}=|a|s_A
\end{aligned}$$

样本 B 的极差 R_B：

$$R_B=y_{(n)}-y_{(1)}=ax_{(n)}+b-ax_{(1)}-b=a\left[x_{(n)}-x_{(1)}\right]=aR_A$$

样本 B 的中位数 $m_{0.5B}$：

(a). 当 n 为偶数时，

$$m_{0.5B}=y_{\left(\frac{n+1}{2}\right)}=ax_{\left(\frac{n+1}{2}\right)}+b=am_{0.5A}+b$$

(b). 当 n 为奇数时，

$$\begin{aligned}
m_{0.5B}=&\frac{1}{2}\left[y_{\left(\frac{n}{2}\right)}+y_{\left(\frac{n}{2}+1\right)}\right]=\frac{1}{2}\left[ax_{\left(\frac{n}{2}\right)}+b+ax_{\left(\frac{n}{2}+1\right)}+b\right]\\
=&\frac{a}{2}\left[x_{\left(\frac{n}{2}\right)}+x_{\left(\frac{n}{2}+1\right)}\right]+b=am_{0.5A}+b
\end{aligned}$$

因此，样本 B 的中位数 $m_{0.5B}$ 为

$$m_{0.5B}=am_{0.5A}+b.$$

23 **证明.**（可参考例 3.3.4 与例 3.3.5）

$$P\left(X\leq k\right)=\sum_{i=1}^{k}pq^{i-1}=\frac{p\left(1-q^k\right)}{1-q}=1-q^k,\quad k=1,2,\ldots,$$

对于 $X_{(n)}$，有

$$\begin{aligned}
P\left(X_{(n)}=k\right)=&P\left(X_{(n)}\leq k\right)-P\left(X_{(n)}\leq k-1\right)\\
=&\prod_{i=1}^{n}P\left(X_i\leq k\right)-\prod_{i=1}^{n}P\left(X_i\leq k-1\right)\\
=&\left(1-q^k\right)^n-\left(1-q^{k-1}\right)^n
\end{aligned}$$

对于 $X_{(1)}$，有

$$\begin{aligned}
P\left(X_{(1)}=k\right)=&P\left(X_{(1)}\leq k\right)-P\left(X_{(1)}\leq k-1\right)\\
=&1-P\left(X_{(1)}>k\right)-\left[1-P\left(X_{(1)}>k-1\right)\right]\\
=&\prod_{i=1}^{n}P\left(X_i>k-1\right)-\prod_{i=1}^{n}P\left(X_i>k\right)\\
=&q^{n(k-1)}-q^{nk}=q^{n(k-1)}\left(1-q^n\right)
\end{aligned}$$

32 **证明.** 总体 X 的密度函数和分布函数分别为

$$p(x)=\begin{cases}3x^2, & 0<x<1;\\ 0, & \text{其他}.\end{cases}\qquad F(x)=\begin{cases}0, & x<0;\\ x^3, & 0\le x<1;\\ 1, & x\ge 1.\end{cases}$$

所以，根据 P242 (5.3.16)，$\left(X_{(2)},X_{(4)}\right)$ 的联合密度函数为

$$\begin{aligned}p\left(x,y\right)&=\frac{5!}{1!\cdot 1!\cdot 1!}\left[F\left(x\right)\right]^{2-1}\left[F\left(y\right)-F\left(x\right)\right]^{4-2-1}\left[1-F\left(y\right)\right]^{5-4}p\left(x\right)p\left(y\right)\\&=120x^3\left(y^3-x^3\right)\left(1-y^3\right)\cdot 3x^2\cdot 3y^2\\&=1080x^5y^2\left(y^3-x^3\right)\left(1-y^3\right),\quad 0<x<y<1\end{aligned}$$

为求 $\left(\frac{X_{(2)}}{X_{(4)}},X_{(4)}\right)$ 的联合密度，令

$$\begin{cases}\mu=\frac{x}{y}\\ \nu=y\end{cases}\implies\begin{cases}x=\mu\nu\\ y=\nu\end{cases}$$

则，其雅可比行列式为

$$J=\frac{\partial\left(x,y\right)}{\partial\left(\mu,\nu\right)}=\begin{vmatrix}\nu & \mu\\ 0 & 1\end{vmatrix}=\nu$$

由 $0<x<y<1$ 可得 $0<\mu<1,0<\nu<1$。于是，$\left(\frac{X_{(2)}}{X_{(4)}},X_{(4)}\right)$ 的联合密度函数为

$$\begin{aligned}p\left(\mu,\nu\right)=p\left(\mu\nu,\nu\right)\cdot|J|&=1080\left(\mu\nu\right)^5\nu^2\left[\nu^3-\left(\mu\nu\right)^3\right]\left(1-\nu^3\right)\\&=1080\mu^5\left(1-\mu^3\right)\cdot\nu^{11}\left(1-\nu^3\right)\end{aligned}$$

可求得其边际密度函数为，

$$\begin{aligned}U=\frac{X_{(2)}}{X_{(4)}}\sim p(\mu)&=\int_0^1 p(\mu,\nu)\,\mathrm{d}\nu=1080\mu^5\left(1-\mu^3\right)\int_0^1\nu^{11}\left(1-\nu^3\right)\mathrm{d}\nu\\&=18\mu^5\left(1-\mu^3\right),\quad 0<\mu<1\end{aligned}$$

类似可得

$$V=X_{(4)}\sim p(\nu)=60\nu^{11}\left(1-\nu^3\right),\quad 0<\nu<1$$

因此，可以证明

$$p\left(\mu,\nu\right)=p(\mu)\cdot p(\nu)$$

即，$\frac{X_{(2)}}{X_{(4)}}$ 与 $X_{(4)}$ 独立。

评论 也可利用变量独立的一个推论，判断 $p\left(\mu,\nu\right)$ 可分离变量，省去求解边际密度函数的步骤。

推论 令 $(X,Y)\sim p(x,y)$, 则 X 与 Y 相互独立的充要条件是, $p(x,y)$ 可分离变量, 即存在非负函数 $g(x)$，$h(y)$ 使得 $p(x,y)=g(x)\cdot h(y)$。