2021 年春季学期/数理统计/第四周/课后作业解答

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5 **证明.** 样本的联合密度函数为

$$p\left(x_1, x_2, \ldots, x_n; \theta\right) = \theta^n \left(x_1 x_2 \ldots x_n\right)^{\theta-1} = \prod_{i=1}^{n} \theta x_i^{\theta-1}$$

令

$$T = \prod_{i=1}^{n} x_i, \quad g\left(t; \theta\right) = t^{\theta-1}\theta^n, \quad h\left(x_1, x_2, \ldots, x_n\right) = 1.$$

由因子分解定理有，$T = \prod_{i=1}^{n} x_i$ 为 θ 的充分统计量。

6 **证明.** 样本的联合密度函数为

$$\begin{aligned} p\left(x_1, x_2, \ldots, x_n; \theta\right) =& \prod_{i=1}^{n} m x_i^{m-1} \theta^{-m} \mathrm{e}^{-\left(x_i/\theta\right)^m} \\ =& m^n \left(x_1 x_2 \ldots x_n\right)^{m-1} \theta^{-mn} \mathrm{e}^{-\sum_{i=1}^{n}\left(x_i/\theta\right)^m} \\ =& = \theta^{-mn} \mathrm{e}^{-\frac{\sum_{i=1}^{n} x_i^m}{\theta^m}} \cdot m^n \left(\prod_{i=1}^{n} x_i\right)^{m-1} \end{aligned}$$

令

$$T = \sum_{i=1}^{n} x_i^m, \quad g\left(t; \theta\right) = \theta^{-mn} \mathrm{e}^{-\frac{t}{\theta^m}}, \quad h\left(x_1, x_2, \ldots, x_n\right) = m^n \left(\prod_{i=1}^{n} x_i\right)^{m-1}.$$

由因子分解定理有，$T = \sum_{i=1}^{n} x_i^m$ 为 θ 的充分统计量。

8 **证明.** 样本的联合密度函数为

$$p\left(x_1, x_2, \ldots, x_n; \mu\right) = \prod_{i=1}^{n} \frac{1}{2\theta} \mathrm{e}^{-\frac{|x_i|}{\theta}} = \frac{1}{(2\theta)^n} \mathrm{e}^{-\frac{1}{\theta}\sum_{i=1}^{n}|x_i|}$$

令

$$T = \sum_{i=1}^{n} \left|X_i\right|, \quad g\left(t; \theta\right) = \frac{1}{(2\theta)^n} \mathrm{e}^{-\frac{1}{\theta}t}, \quad h\left(x_1, x_2, \ldots, x_n\right) = 1.$$

由因子分解定理有，$T = \sum_{i=1}^{n} \left|X_i\right|$ 为 θ 的充分统计量。

10 **证明.** (a). 在 μ 已知时，样本联合密度函数为

$$\begin{aligned} p_1\left(x_1, x_2, \ldots, x_n; \sigma^2\right) =& \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{\left(x_i - \mu\right)^2}{2\sigma^2}\right) \\ =& \frac{1}{(\sqrt{2\pi}\sigma)^n} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(x_i - \mu\right)^2\right) \end{aligned}$$

令

$$T=\sum_{i=1}^{n}(x_i-\mu)^2,\quad g\left(t;\sigma^2\right)=\frac{1}{(\sqrt{2\pi}\sigma)^n}\exp\left(-\frac{t}{2\sigma^2}\right),\quad h\left(x_1,x_2,\ldots,x_n\right)=1.$$

由因子分解定理有，$T=\sum_{i=1}^{n}(x_i-\mu)^2$ 为 σ^2 的充分统计量。

(b). 在 σ^2 已知时，样本联合密度函数为

$$\begin{aligned}p_1\left(x_1,x_2,\ldots,x_n;\mu\right)&=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(\frac{(x_i-\mu)^2}{2\sigma^2}\right)\\&=\frac{1}{(\sqrt{2\pi}\sigma)^n}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(x_i^2-2\mu x_i+\mu^2\right)\right)\\&=\frac{1}{(\sqrt{2\pi}\sigma)^n}\exp\left(-\frac{\sum_{i=1}^{n}x_i^2}{2\sigma^2}\right)\cdot\exp\left[-\frac{1}{2\sigma^2}\left(n\mu^2-2\mu\sum_{i=1}^{n}x_i\right)\right]\end{aligned}$$

令

$$T=\sum_{i=1}^{n}x_i,\quad g\left(t;\mu\right)=\exp\left[-\frac{1}{2\sigma^2}\left(n\mu^2-2\mu t\right)\right],$$
$$h\left(x_1,x_2,\ldots,x_n\right)=\frac{1}{(\sqrt{2\pi}\sigma)^n}\exp\left(-\frac{\sum_{i=1}^{n}x_i^2}{2\sigma^2}\right).$$

由因子分解定理有，$T=\sum_{i=1}^{n}x_i$ 为 μ 的充分统计量。

12 **证明.** 样本的联合密度函数为

$$\begin{aligned}p\left(x_1,x_2,\ldots,x_n;\theta\right)&=\prod_{i=1}^{n}\frac{1}{\theta}I_{\{\theta<x_i<2\theta\}}\\&=\frac{1}{\theta^n}I_{\{\theta<x_1,x_2,\ldots,x_n<2\theta\}}\\&=\frac{1}{\theta^n}I_{\{\theta<x_{(1)}\le x_{(n)}<2\theta\}}\end{aligned}$$

令

$$\left(T_1,T_2\right)=\left(X_{(1)},X_{(n)}\right),\quad g\left(t_1,t_2;\theta\right)=\frac{1}{\theta^n}I_{\{\theta<t_1\le t_2<2\theta\}},\quad h\left(x_1,x_2,\ldots,x_n\right)=1.$$

由因子分解定理有，$(T_1,T_2)=\left(X_{(1)},X_{(n)}\right)$ 为 θ 的充分统计量。

14 **证明.** 样本的联合密度函数为

$$\begin{aligned}p\left(x_1,x_2,\ldots,x_n;a,b\right)&=\prod_{i=1}^{n}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x_i^{a-1}\left(1-x_i\right)^{b-1}\\&=\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n\left(\prod_{i=1}^{n}x_i\right)^{a-1}\left[\prod_{i=1}^{n}\left(1-x_i\right)\right]^{b-1}\end{aligned}$$

令

$$\left(T_1,T_2\right)=\left(\prod_{i=1}^{n}x_i,\prod_{i=1}^{n}\left(1-x_i\right)\right),\quad g\left(t_1,t_2;a,b\right)=\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n t_1^{a-1}t_2^{b-1},$$
$$h\left(x_1,x_2,\ldots,x_n\right)=1.$$

由因子分解定理有，$(T_1,T_2)=\left(\prod_{i=1}^{n}x_i,\prod_{i=1}^{n}\left(1-x_i\right)\right)$ 为 (a,b) 的充分统计量。

15 **证明.** 样本的联合密度函数为

$$
\begin{aligned}
p\left(x_1, x_2, \ldots, x_n ; \theta\right)= & \prod_{j=1}^n C(\theta) \exp \left[\sum_{i=1}^k Q_i(\theta) T_i\left(x_j\right)\right] h\left(x_j\right) \\
= & C(\theta)^n \exp \left[\sum_{j=1}^n \sum_{i=1}^k Q_i(\theta) T_i\left(x_j\right)\right] \cdot \prod_{j=1}^n h\left(x_j\right) \\
= & C(\theta)^n \exp \left[\sum_{i=1}^k Q_i(\theta) \sum_{j=1}^n T_i\left(x_j\right)\right] \cdot \prod_{j=1}^n h\left(x_j\right)
\end{aligned}
$$

令

$$
T(x)=\left(\sum_{j=1}^n T_1\left(x_j\right), \ldots, \sum_{j=1}^n T_k\left(x_j\right)\right), \quad g(T(x) ; \theta)=C(\theta)^n \exp \left[\sum_{i=1}^k Q_i(\theta) t_i\right],
$$

$$
h\left(x_1, x_2, \ldots, x_n\right)=\prod_{j=1}^n h\left(x_j\right) .
$$

由因子分解定理有，$T(x)=\left(\sum_{j=1}^n T_1\left(x_j\right), \ldots, \sum_{j=1}^n T_k\left(x_j\right)\right)$ 为 θ 的充分统计量。

20 **证明.** 样本的联合密度函数为

$$
\begin{aligned}
p\left(y_1, y_2, \ldots, y_n ; \beta_0, \beta_1, \sigma^2\right)= & \prod_{i=1}^n \frac{1}{\sqrt{2 \pi} \sigma} \exp \left(-\frac{\left(y_i-\beta_0-\beta_1 x_i\right)^2}{2 \sigma^2}\right) \\
= & \frac{1}{(\sqrt{2 \pi} \sigma)^n} \exp \left(-\frac{1}{2 \sigma^2} \sum_{i=1}^n\left(y_i-\beta_0-\beta_1 x_i\right)^2\right) \\
= & \frac{1}{(\sqrt{2 \pi} \sigma)^n} \exp \left[-\frac{1}{2 \sigma^2}\left(\sum_{i=1}^n y_i^2-2 \beta_0 \sum_{i=1}^n y_i\right.\right. \\
& \left.\left.-2 \beta_1 \sum_{i=1}^n x_i y_i+\sum_{i=1}^n\left(\beta_0+\beta_1 x_i\right)^2\right)\right]
\end{aligned}
$$

令

$$
\left(T_1, T_2, T_3\right)=\left(\sum_{i=1}^n y_i, \sum_{i=1}^n x_i y_i, \sum_{i=1}^n y_i^2\right),
$$

$$
g\left(t_1, t_2, t_3 ; \beta_0, \beta_1, \sigma^2\right)=\frac{1}{(\sqrt{2 \pi} \sigma)^n} \exp \left[-\frac{1}{2 \sigma^2}\left(t_3-2 \beta_0 t_1-2 \beta_1 t_2+\sum_{i=1}^n\left(\beta_0+\beta_1 x_i\right)^2\right)\right],
$$

$$
h\left(y_1, y_2, \ldots, y_n\right)=1 .
$$

由因子分解定理有，$\left(T_1, T_2, T_3\right)=\left(\sum_{i=1}^n y_i, \sum_{i=1}^n x_i y_i, \sum_{i=1}^n y_i^2\right)$ 为 $\left(\beta_0, \beta 1, \sigma^2\right)$ 的充分统计量。