

2021 年春季学期/数理统计/第三周/课后作业解答

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2 证明. 因为 $X \sim N(\mu, 16)$, 所以 $\bar{X} \sim N(\mu, \frac{16}{n})$ 。因此,

$$\begin{aligned} P(|\bar{X} - \mu| < 1) &= P\left(\left|\frac{\bar{X} - \mu}{\sqrt{16/n}}\right| < \frac{1}{\sqrt{16/n}}\right) \\ &= \Phi\left(\frac{\sqrt{n}}{4}\right) - \Phi\left(-\frac{\sqrt{n}}{4}\right) \\ &= 2\Phi\left(\frac{\sqrt{n}}{4}\right) - 1 \geq 0.95 \end{aligned}$$

所以, $\Phi\left(\frac{\sqrt{n}}{4}\right) \geq 0.975$, $\frac{\sqrt{n}}{4} \geq 1.96$, 即 $n \geq 61.47$ 。因此, 当 n 至少为 62 时, 上述概率不等式才成立。

5 证明. 因为, $\frac{\sqrt{n}(\bar{X} - \mu)}{s} \sim t(n-1)$, 所以,

$$\begin{aligned} P(|\bar{X} - \mu| < 0.6) &= P\left(\left|\frac{\sqrt{n}(\bar{X} - \mu)}{s}\right| < \frac{\sqrt{n} \times 0.6}{s}\right) \\ &= 2t_{15}\left(\frac{4 \times 0.6}{\sqrt{5.32}}\right) - 1 \approx 2 \times 0.8427 - 1 = 0.6854 \end{aligned}$$

8 证明. 因为 $X \sim F(n, m)$, 其密度函数为

$$p_X(x) = \frac{\Gamma\left(\frac{n+m}{2}\right) \left(\frac{n}{m}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right)} x^{\frac{n}{2}-1} \left(1 + \frac{n}{m}x\right)^{-\frac{n+m}{2}}, \quad x > 0,$$

由于 $z = \frac{n}{m}x / (1 + \frac{n}{m}x)$ 在 $(0, \infty)$ 上严格单调递增, 其反函数为 $x = \frac{m}{n} \cdot \frac{z}{1-z}$, 导数为 $\frac{dx}{dz} = \frac{m}{n} \cdot \frac{1}{(1-z)^2}$, 因此, Z 的密度函数为

$$\begin{aligned} p_Z(z) &= \frac{\Gamma\left(\frac{n+m}{2}\right) \left(\frac{n}{m}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right)} \left(\frac{m}{n} \cdot \frac{z}{1-z}\right)^{\frac{n}{2}-1} \left(1 + \frac{z}{1-z}\right)^{-\frac{n+m}{2}} \cdot \frac{m}{n} \cdot \frac{1}{(1-z)^2} \\ &= \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right)} \left(\frac{z}{1-z}\right)^{\frac{n}{2}-1} \left(\frac{1}{1-z}\right)^{-\frac{n+m}{2}} \cdot \frac{1}{(1-z)^2} \\ &= \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right)} z^{\frac{n}{2}-1} (1-z)^{\frac{m}{2}-1}, \quad 0 < z < 1 \end{aligned}$$

所以, Z 服从贝塔分布 $\text{Be}\left(\frac{n}{2}, \frac{m}{2}\right)$ 。

9 证明. 因为,

$$X_1 \sim N(0, \sigma^2), \quad X_2 \sim N(0, \sigma^2),$$

$$X_1 + X_2 \sim N(0, 2\sigma^2), \quad X_1 - X_2 \sim N(0, 2\sigma^2),$$

因此, 根据卡方分布定义有

$$\left(\frac{X_1 + X_2}{\sqrt{2}\sigma}\right)^2 \sim \chi^2(1), \quad \left(\frac{X_1 - X_2}{\sqrt{2}\sigma}\right)^2 \sim \chi^2(1).$$

因为,

$$\begin{aligned} \text{Cov}(X_1 + X_2, X_1 - X_2) &= 2\text{Cov}(X_1, X_2) + \text{Var}(X_1) - \text{Var}(X_2) \\ &= 0 + \sigma^2 - \sigma^2 = 0 \end{aligned}$$

所以, 由性质 3.4.13 有, 对于二维正态分布 $(X_1 + X_2, X_1 - X_2)$, 不相关与独立是等价的。
于是, 根据 F 分布的定义有

$$Y = \left(\frac{X_1 + X_2}{X_1 - X_2}\right)^2 = \frac{\frac{(X_1 + X_2)^2}{2\sigma^2}}{\frac{(X_1 - X_2)^2}{2\sigma^2}} \sim F(1, 1)$$

13 **证明.** 假设正态总体的方差为 σ^2 , 则由定理 5.4.1 有,

$$\frac{(n_1 - 1)s_1^2}{\sigma^2} = \frac{14s_1^2}{\sigma^2} \sim \chi^2(14), \quad \frac{(n_2 - 1)s_2^2}{\sigma^2} = \frac{19s_2^2}{\sigma^2} \sim \chi^2(19),$$

由 F 分布的定义有

$$\frac{\frac{14s_1^2}{\sigma^2}/14}{\frac{19s_2^2}{\sigma^2}/19} = \frac{s_1^2}{s_2^2} \sim F(14, 19),$$

因此,

$$P\left(\frac{s_1^2}{s_2^2} > 2\right) = P(F > 2) = 1 - P(F \leq 2) \approx 0.0798.$$

18 **证明.** 对于 $F \sim F(k, m)$, 其密度函数为

$$p(x) = \frac{\Gamma\left(\frac{k+m}{2}\right)\left(\frac{k}{m}\right)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{m}{2}\right)} x^{\frac{k}{2}-1} \left(1 + \frac{k}{m}x\right)^{-\frac{k+m}{2}}, \quad x > 0.$$

因此,

$$\begin{aligned} E(F^r) &= \frac{\Gamma\left(\frac{k+m}{2}\right)\left(\frac{k}{m}\right)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{m}{2}\right)} \int_0^{+\infty} x^r \cdot x^{\frac{k}{2}-1} \left(1 + \frac{k}{m}x\right)^{-\frac{k+m}{2}} dx \\ &= \frac{\Gamma\left(\frac{k+m}{2}\right)\left(\frac{k}{m}\right)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{m}{2}\right)} \int_0^{+\infty} x^{\frac{k}{2}+r-1} \left(1 + \frac{k}{m}x\right)^{-\frac{k+m}{2}} dx \end{aligned}$$

令 $t = (1 + \frac{k}{m}x)^{-1}$, 则 $x = \frac{m}{k}(\frac{1}{t} - 1)$, $dx = \frac{m}{k} \cdot (-\frac{1}{t^2}) dt$, 因此,

$$\begin{aligned}
&= \frac{\Gamma\left(\frac{k+m}{2}\right) \left(\frac{k}{m}\right)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{m}{2}\right)} \int_1^0 \left(\frac{m}{k}\right)^{\frac{k}{2}+r-1} \left(\frac{1-t}{t}\right)^{\frac{k}{2}+r-1} \cdot t^{\frac{k+m}{2}} \cdot \frac{m}{k} \left(-\frac{1}{t^2}\right) dt \\
&= \frac{\Gamma\left(\frac{k+m}{2}\right) \left(\frac{k}{m}\right)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{m}{2}\right)} \left(\frac{m}{k}\right)^{\frac{k}{2}+r} \int_0^1 t^{\frac{m}{2}-r-1} (1-t)^{\frac{k}{2}+r-1} dt \\
&= \frac{\Gamma\left(\frac{k+m}{2}\right) \left(\frac{k}{m}\right)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{m}{2}\right)} \left(\frac{m}{k}\right)^{\frac{k}{2}+r} B\left(\frac{m}{2}-r, \frac{k}{2}+r\right) \\
&= \frac{\Gamma\left(\frac{k+m}{2}\right) \left(\frac{k}{m}\right)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{m}{2}\right)} \left(\frac{m}{k}\right)^{\frac{k}{2}+r} \frac{\Gamma\left(\frac{m}{2}-r\right) \Gamma\left(\frac{k}{2}+r\right)}{\Gamma\left(\frac{m+k}{2}\right)} \\
&= \frac{m^r \Gamma\left(\frac{k}{2}+r\right) \Gamma\left(\frac{m}{2}-r\right)}{k^r \Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{m}{2}\right)}
\end{aligned}$$

当 $r = 1$ 时, 由于 $k > 0$, 只要 $m > 2$, 就有

$$E(F) = \frac{m \Gamma\left(\frac{k}{2}+1\right) \Gamma\left(\frac{m}{2}-1\right)}{k \Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{m}{2}\right)} = \frac{m \cdot \frac{k}{2}}{k \left(\frac{m}{2}-1\right)} = \frac{m}{m-2}$$

当 $r = 2$ 时, 由于 $k > 0$, 只要 $m > 4$, 就有

$$E(F^2) = \frac{m^2 \Gamma\left(\frac{k}{2}+2\right) \Gamma\left(\frac{m}{2}-2\right)}{k^2 \Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{m}{2}\right)} = \frac{m^2(k+2)}{k(m-2)(m-4)}$$

因此,

$$\text{Var}(F) = E(F^2) - [E(F)]^2 = \frac{m^2(k+2)}{k(m-2)(m-4)} - \left(\frac{m}{m-2}\right)^2 = \frac{2m^2(m+k-2)}{k(m-2)^2(m-4)}$$

19 **证明.** $Y_i = -2 \ln F(X_i)$ 的分布函数为

$$\begin{aligned}
F_Y(y) &= P(-2 \ln F(X_i) \leq y) \\
&= P\left(X_i \geq F^{-1}\left(e^{-\frac{y}{2}}\right)\right) \\
&= 1 - F\left[F^{-1}\left(e^{-\frac{y}{2}}\right)\right] \\
&= 1 - e^{-\frac{y}{2}}, \quad y > 0
\end{aligned}$$

因此,

$$Y_i \sim \text{Exp}\left(\frac{1}{2}\right) = \chi^2(2).$$

因 X_1, X_2, \dots, X_n 相互独立, 有 Y_1, Y_2, \dots, Y_n 相互独立, 由 χ^2 分布的可加性, 可知

$$T = -2 \sum_{i=1}^n \ln F(X_i) \sim \chi^2(2n).$$