2022年春季学期/数理统计/第七周/课后作业解答

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更新：2022 年 4 月 21 日

3.(2) **证明.** 由于，

$$\begin{aligned}E(X)=&\sum_{k=2}^{+\infty}k\cdot(k-1)\theta^2(1-\theta)^{k-2}\\=&\theta^2\sum_{k=2}^{+\infty}\frac{\partial^2}{\partial\theta^2}(1-\theta)^k\\=&\theta^2\frac{\partial^2}{\partial\theta^2}\left[\sum_{k=2}^{+\infty}(1-\theta)^k\right]\\=&\theta^2\frac{\partial^2}{\partial\theta^2}\left\{\lim_{t\to+\infty}\frac{(1-\theta)^2\left[1-(1-\theta)^t\right]}{1-(1-\theta)}\right\}\\=&\theta^2\frac{\partial^2}{\partial\theta^2}\left[\frac{(1-\theta)^2}{1-(1-\theta)}\right]\\=&\theta^2\frac{\partial^2}{\partial\theta^2}\left(\frac{1}{\theta}-2+\theta\right)\\=&\theta^2\cdot\frac{2}{\theta^3}=\frac{2}{\theta}\end{aligned}$$

即 $\theta=\frac{2}{E(X)}$，故 θ 的矩估计为 $\hat{\theta}=\frac{2}{\bar{X}}$。

4.(2) **证明.** 由于，

$$E(X)=\int_0^1 x\cdot(\theta+1)x^\theta\mathrm{d}x=(\theta+1)\cdot\left.\frac{x^{\theta+2}}{\theta+2}\right|_0^1=\frac{\theta+1}{\theta+2}$$

即 $\theta=\frac{2E(X)-1}{1-E(X)}$，故 θ 的矩估计为 $\hat{\theta}=\frac{2\bar{X}-1}{1-\bar{X}}$。

4.(3) **证明.** 由于，

$$E(X)=\int_0^1 x\cdot\sqrt{\theta}x^{\sqrt{\theta}-1}\mathrm{d}x=\sqrt{\theta}\cdot\left.\frac{x^{\sqrt{\theta}+1}}{\sqrt{\theta}+1}\right|_0^1=\frac{\sqrt{\theta}}{\sqrt{\theta}+1}$$

即 $\theta=\left[\frac{E(X)}{1-E(X)}\right]^2$，故 θ 的矩估计为 $\hat{\theta}=\left(\frac{\bar{X}}{1-\bar{X}}\right)^2$。

4.(4) **证明.** 由于,

$$
\begin{aligned}
E(X) &= \int_{\mu}^{+\infty} x \cdot \frac{1}{\theta} e^{\frac{x-\mu}{\theta}} \mathrm{d}x \\
&= \int_{\mu}^{+\infty} x \cdot (-1) \mathrm{d} e^{\frac{x-\mu}{\theta}} \\
&= - x e^{\frac{x-\mu}{\theta}} \Big|_{\mu}^{+\infty} + \int_{\mu}^{+\infty} e^{\frac{x-\mu}{\theta}} \mathrm{d}x \\
&= \mu - \theta e^{\frac{x-\mu}{\theta}} \Big|_{\mu}^{+\infty} \\
&= \mu + \theta
\end{aligned}
$$

$$
\begin{aligned}
E\left(X^2\right) &= \int_{\mu}^{+\infty} x^2 \cdot \frac{1}{\theta} e^{\frac{x-\mu}{\theta}} \mathrm{d}x \\
&= \int_{\mu}^{+\infty} x^2 \cdot (-1) \mathrm{d} e^{\frac{x-\mu}{\theta}} \\
&= - x^2 e^{\frac{x-\mu}{\theta}} \Big|_{\mu}^{+\infty} + \int_{\mu}^{+\infty} 2x e^{\frac{x-\mu}{\theta}} \mathrm{d}x \\
&= \mu^2 + 2\theta E(X) \\
&= \mu^2 + 2\mu\theta + 2\theta^2
\end{aligned}
$$

因此,

$$
E(X) = \mu + \theta, \quad \operatorname{Var}(X) = E\left(X^2\right) - [E(X)]^2 = \theta^2
$$

即

$$
\theta = \sqrt{\operatorname{Var}(X)}, \quad \mu = E(X) - \sqrt{\operatorname{Var}(X)}
$$

故 (θ, μ) 的矩估计为

$$
\hat{\theta} = \sqrt{S^2}, \quad \hat{\mu} = \bar{X} - \sqrt{S^2}
$$

5 **证明.** 由于,

$$
p = P\{X > 0\} = P\{X - \mu > -\mu\} = 1 - \Phi(-\mu) = \Phi(\mu)
$$

即 $\mu = \Phi^{-1}(p)$, 故 μ 的矩估计为 $\hat{\mu} = \Phi^{-1}(\hat{p}) = \Phi^{-1}\left(\frac{k}{n}\right)$。