2022 年春季学期/数理统计/第三周/课后作业解答

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4 **证明.** 样本容量为 $n+1$ 时的样本均值 $\bar{x}_{n+1}$：

$$
\begin{aligned}
\bar{x}_{n+1} =& \frac{1}{n+1} \sum_{i=1}^{n+1} x_i = \frac{1}{n+1} \left(\sum_{i=1}^{n} x_i + x_{n+1} \right) = \frac{1}{n+1} \left(n\bar{x}_n + x_{n+1} \right) \\
=& \frac{1}{n+1} \left[(n+1)\bar{x}_n - \bar{x}_n + x_{n+1} \right] = \bar{x}_n + \frac{1}{n+1} \left(x_{n+1} - \bar{x}_n \right)
\end{aligned}
$$

样本容量为 $n+1$ 时的样本方差 s_{n+1}^2：

$$
\begin{aligned}
s_{n+1}^2 =& \frac{1}{n} \sum_{i=1}^{n+1} \left(x_i - \bar{x}_{n+1} \right)^2 \\
=& \frac{1}{n} \sum_{i=1}^{n} \left\{ x_i - \left[\bar{x}_n + \frac{1}{n+1} \left(x_{n+1} - \bar{x}_n \right) \right] \right\}^2 \\
=& \frac{1}{n} \sum_{i=1}^{n+1} \left[\left(x_i - \bar{x}_n \right) - \frac{1}{n+1} \left(x_{n+1} - \bar{x}_n \right) \right]^2 \\
=& \frac{1}{n} \sum_{i=1}^{n+1} \left\{ \left(x_i - \bar{x}_n \right)^2 - \frac{2}{n+1} \left(x_i - \bar{x}_n \right) \left(x_{n+1} - \bar{x}_n \right) + \frac{1}{(n+1)^2} \left(x_{n+1} - \bar{x}_n \right)^2 \right\} \\
=& \frac{1}{n} \left[\sum_{i=1}^{n} \left(x_i - \bar{x}_n \right)^2 \right] + \frac{1}{n} \left(x_{n+1} - \bar{x}_n \right)^2 - \frac{2}{n(n+1)} \left(x_{n+1} - \bar{x}_n \right) \left[\sum_{i=1}^{n} \left(x_i - \bar{x}_n \right) \right] \\
& - \frac{2}{n(n+1)} \left(x_{n+1} - \bar{x}_n \right)^2 + \frac{1}{n(n+1)} \left(x_{n+1} - \bar{x}_n \right)^2 \\
=& \frac{n-1}{n} s_n^2 + \frac{1}{n+1} \left(x_{n+1} - \bar{x}_n \right)^2
\end{aligned}
$$

（另一种思路）

$$s_{n+1}^2 = \frac{1}{n} \sum_{i=1}^{n+1} (x_i - \bar{x}_{n+1})^2 = \frac{1}{n} \sum_{i=1}^{n+1} \left[(x_i - \bar{x}_n) + (\bar{x}_n - \bar{x}_{n+1}) \right]^2$$

$$= \frac{1}{n} \left[\sum_{i=1}^{n+1} (x_i - \bar{x}_n)^2 \right] + \frac{2}{n} (\bar{x}_n - \bar{x}_{n+1}) \sum_{i=1}^{n+1} (x_i - \bar{x}_n) + \frac{1}{n} \sum_{i=1}^{n+1} (\bar{x}_n - \bar{x}_{n+1})^2$$

$$= \frac{1}{n} \left[\sum_{i=1}^{n+1} (x_i - \bar{x}_n)^2 \right] - \frac{n+1}{n} (\bar{x}_n - \bar{x}_{n+1})^2$$

$$= \frac{1}{n} \left[\sum_{i=1}^{n} (x_i - \bar{x}_n)^2 + (x_{n+1} - \bar{x}_n)^2 \right] - \frac{n+1}{n} \frac{1}{(n+1)^2} (x_{n+1} - \bar{x}_n)^2$$

$$= \frac{1}{n} \left[(n-1) \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x}_n)^2 + \frac{n}{n+1} (x_{n+1} - \bar{x}_n)^2 \right]$$

$$= \frac{n-1}{n} s_n^2 + \frac{1}{n+1} (x_{n+1} - \bar{x}_n)^2$$

6 **证明.** 样本 B 的均值 $\bar{y}_B$：

$$\bar{y}_B = \frac{1}{n} \sum_{i=1}^{n} y_i = \frac{1}{n} \sum_{i=1}^{n} (ax_i + b) = \frac{1}{n} \left(a \sum_{i=1}^{n} x_i + nb \right)$$

$$= a \cdot \frac{1}{n} \sum_{i=1}^{n} x_i + b = a\bar{x}_A + b$$

样本 B 的标准差 s_B：

$$s_B = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y}_B)^2} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (ax_i + b - a\bar{x}_A - b)^2}$$

$$= |a| \cdot \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x}_A)^2} = |a| s_A$$

样本 B 的极差 R_B：

$$R_B = y_{(n)} - y_{(1)} = ax_{(n)} + b - ax_{(1)} - b = a \left[x_{(n)} - x_{(1)} \right] = aR_A$$

样本 B 的中位数 $m_{0.5B}$：
(a). 当 n 为偶数时，
$$m_{0.5B} = y_{\left(\frac{n+1}{2} \right)} = ax_{\left(\frac{n+1}{2} \right)} + b = am_{0.5A} + b$$

(b). 当 n 为奇数时，
$$m_{0.5B} = \frac{1}{2} \left[y_{\left(\frac{n}{2} \right)} + y_{\left(\frac{n}{2}+1 \right)} \right] = \frac{1}{2} \left[ax_{\left(\frac{n}{2} \right)} + b + ax_{\left(\frac{n}{2}+1 \right)} + b \right]$$
$$= \frac{a}{2} \left[x_{\left(\frac{n}{2} \right)} + x_{\left(\frac{n}{2}+1 \right)} \right] + b = am_{0.5A} + b$$

因此，样本 B 的中位数 $m_{0.5B}$ 为

$$m_{0.5B} = am_{0.5A} + b.$$

23 **证明.**（可参考例 3.3.4 与例 3.3.5）

$$P\left(X \leq k\right) = \sum_{i=1}^{k} pq^{i-1} = \frac{p\left(1 - q^k\right)}{1 - q} = 1 - q^k, \quad k = 1, 2, \ldots,$$

对于 $X_{(n)}$，有

$$\begin{aligned}
P\left(X_{(n)} = k\right) &= P\left(X_{(n)} \leq k\right) - P\left(X_{(n)} \leq k - 1\right) \\
&= \prod_{i=1}^{n} P\left(X_i \leq k\right) - \prod_{i=1}^{n} P\left(X_i \leq k - 1\right) \\
&= \left(1 - q^k\right)^n - \left(1 - q^{k-1}\right)^n
\end{aligned}$$

对于 $X_{(1)}$，有

$$\begin{aligned}
P\left(X_{(1)} = k\right) &= P\left(X_{(1)} \leq k\right) - P\left(X_{(1)} \leq k - 1\right) \\
&= 1 - P\left(X_{(1)} > k\right) - \left[1 - P\left(X_{(1)} > k - 1\right)\right] \\
&= \prod_{i=1}^{n} P\left(X_i > k - 1\right) - \prod_{i=1}^{n} P\left(X_i > k\right) \\
&= q^{n(k-1)} - q^{nk} = q^{n(k-1)}\left(1 - q^n\right)
\end{aligned}$$

32 **证明.** 总体 X 的密度函数和分布函数分别为

$$p(x) = \begin{cases} 3x^2, & 0 < x < 1; \\ 0, & \text{其他.} \end{cases} \qquad F(x) = \begin{cases} 0, & x < 0; \\ x^3, & 0 \leq x < 1; \\ 1, & x \geq 1. \end{cases}$$

所以，根据 P242 (5.3.16)，$\left(X_{(2)}, X_{(4)}\right)$ 的联合密度函数为

$$\begin{aligned}
p(x, y) &= \frac{5!}{1! \cdot 1! \cdot 1!} \left[F\left(x\right)\right]^{2-1} \left[F\left(y\right) - F\left(x\right)\right]^{4-2-1} \left[1 - F\left(y\right)\right]^{5-4} p\left(x\right) p\left(y\right) \\
&= 120 x^3 \left(y^3 - x^3\right) \left(1 - y^3\right) \cdot 3x^2 \cdot 3y^2 \\
&= 1080 x^5 y^2 \left(y^3 - x^3\right) \left(1 - y^3\right), \quad 0 < x < y < 1
\end{aligned}$$

为求 $\left(\frac{X_{(2)}}{X_{(4)}}, X_{(4)}\right)$ 的联合密度，令

$$\begin{cases} \mu = \frac{x}{y} \\ \nu = y \end{cases} \implies \begin{cases} x = \mu\nu \\ y = \nu \end{cases}$$

则，其雅可比行列式为

$$J = \frac{\partial\left(x, y\right)}{\partial\left(\mu, \nu\right)} = \begin{vmatrix} \nu & \mu \\ 0 & 1 \end{vmatrix} = \nu$$

由 $0 < x < y < 1$ 可得 $0 < \mu < 1, 0 < \nu < 1$。于是，$\left(\frac{X_{(2)}}{X_{(4)}}, X_{(4)}\right)$ 的联合密度函数为

$$\begin{aligned}
p\left(\mu, \nu\right) &= p\left(\mu\nu, \nu\right) \cdot |J| = 1080\left(\mu\nu\right)^5 \nu^2 \left[\nu^3 - \left(\mu\nu\right)^3\right]\left(1 - \nu^3\right) \\
&= 1080 \mu^5 \left(1 - \mu^3\right) \cdot \nu^{11}\left(1 - \nu^3\right)
\end{aligned}$$

可求得其边际密度函数为，

$$U = \frac{X_{(2)}}{X_{(4)}} \sim p(\mu) = \int_0^1 p(\mu, \nu)\, \mathrm{d}\nu = 1080\mu^5 \left(1 - \mu^3\right) \int_0^1 \nu^{11} \left(1 - \nu^3\right)\, \mathrm{d}\nu$$
$$= 18\mu^5 \left(1 - \mu^3\right), \quad 0 < \mu < 1$$

类似可得

$$V = X_{(4)} \sim p(\nu) = 60\nu^{11} \left(1 - \nu^3\right), \quad 0 < \nu < 1$$

因此，可以证明

$$p(\mu, \nu) = p(\mu) \cdot p(\nu)$$

即，$\frac{X_{(2)}}{X_{(4)}}$ 与 $X_{(4)}$ 独立。

评论 也可利用变量独立的一个推论，判断 $p(\mu, \nu)$ 可分离变量，省去求解边际密度函数的步骤。

推论 令 $(X, Y) \sim p(x, y)$，则 X 与 Y 相互独立的充要条件是，$p(x, y)$ 可分离变量，即存在非负函数 $g(x)$，$h(y)$ 使得 $p(x, y) = g(x) \cdot h(y)$。