

## 2021 年春季学期/数理统计/第七周/课后作业解答

龚梓阳

更新：2021 年 5 月 14 日

3 证明. (a). 由于,

$$E(X) = \frac{1}{N} \sum_{k=0}^{n-1} k = \frac{N-1}{2}$$

即  $N = 2E(X) + 1$ , 也就是  $\hat{N} = 2\bar{X} + 1$ 。同时由于  $N$  是正整数, 故  $\theta$  的矩估计为  $\hat{N} = [2\bar{X}] + 1$ , 其中  $[*]$  为向上取整。

(b). 由于,

$$\begin{aligned} E(X) &= \sum_{k=2}^{+\infty} k \cdot (k-1)\theta^2(1-\theta)^{k-2} \\ &= \theta^2 \sum_{k=2}^{+\infty} \frac{\partial^2}{\partial \theta^2} (1-\theta)^k \\ &= \theta^2 \frac{\partial^2}{\partial \theta^2} \left[ \sum_{k=2}^{+\infty} (1-\theta)^k \right] \\ &= \theta^2 \frac{\partial^2}{\partial \theta^2} \left\{ \lim_{t \rightarrow +\infty} \frac{(1-\theta)^2 [1 - (1-\theta)^t]}{1 - (1-\theta)} \right\} \\ &= \theta^2 \frac{\partial^2}{\partial \theta^2} \left[ \frac{(1-\theta)^2}{1 - (1-\theta)} \right] \\ &= \theta^2 \frac{\partial^2}{\partial \theta^2} \left( \frac{1}{\theta} - 2 + \theta \right) \\ &= \theta^2 \cdot \frac{2}{\theta^3} = \frac{2}{\theta} \end{aligned}$$

即  $\theta = \frac{2}{E(X)}$ , 故  $\theta$  的矩估计为  $\hat{\theta} = \frac{2}{\bar{X}}$ 。

4 证明. (a). 由于,

$$E(X) = \int_0^\theta x \cdot \frac{2}{\theta^2}(\theta - x)dx = \frac{2}{\theta^2} \left( \theta \cdot \frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^\theta = \frac{\theta}{3}$$

即  $\theta = 3E(X)$ , 故  $\theta$  的矩估计为  $\hat{\theta} = 3\bar{X}$ 。

(b). 由于,

$$E(X) = \int_0^1 x \cdot (\theta + 1)x^\theta dx = (\theta + 1) \cdot \frac{x^{\theta+2}}{\theta + 2} \Big|_0^1 = \frac{\theta + 1}{\theta + 2}$$

即  $\theta = \frac{2E(X)-1}{1-E(X)}$ , 故  $\theta$  的矩估计为  $\hat{\theta} = \frac{2\bar{X}-1}{1-\bar{X}}$ 。

(c). 由于,

$$E(X) = \int_0^1 x \cdot \sqrt{\theta} x^{\sqrt{\theta}-1} dx = \sqrt{\theta} \cdot \frac{x^{\sqrt{\theta}+1}}{\sqrt{\theta}+1} \Big|_0^1 = \frac{\sqrt{\theta}}{\sqrt{\theta}+1}$$

即  $\theta = \left[ \frac{E(X)}{1-E(X)} \right]^2$ , 故  $\theta$  的矩估计为  $\hat{\theta} = \left( \frac{\bar{X}}{1-\bar{X}} \right)^2$ 。

(d). 由于,

$$\begin{aligned} E(X) &= \int_{\mu}^{+\infty} x \cdot \frac{1}{\theta} e^{\frac{x-\mu}{\theta}} dx \\ &= \int_{\mu}^{+\infty} x \cdot (-1) de^{\frac{x-\mu}{\theta}} \\ &= -x e^{\frac{x-\mu}{\theta}} \Big|_{\mu}^{+\infty} + \int_{\mu}^{+\infty} e^{\frac{x-\mu}{\theta}} dx \\ &= \mu - \theta e^{\frac{x-\mu}{\theta}} \Big|_{\mu}^{+\infty} \\ &= \mu + \theta \\ E(X^2) &= \int_{\mu}^{+\infty} x^2 \cdot \frac{1}{\theta} e^{\frac{x-\mu}{\theta}} dx \\ &= \int_{\mu}^{+\infty} x^2 \cdot (-1) de^{\frac{x-\mu}{\theta}} \\ &= -x^2 e^{\frac{x-\mu}{\theta}} \Big|_{\mu}^{+\infty} + \int_{\mu}^{+\infty} 2x e^{\frac{x-\mu}{\theta}} dx \\ &= \mu^2 + 2\theta E(X) \\ &= \mu^2 + 2\mu\theta + 2\theta^2 \end{aligned}$$

因此,

$$E(X) = \mu + \theta, \quad \text{Var}(X) = E(X^2) - [E(X)]^2 = \theta^2$$

即

$$\theta = \sqrt{\text{Var}(X)}, \quad \mu = E(X) - \sqrt{\text{Var}(X)}$$

故  $(\theta, \mu)$  的矩估计为

$$\hat{\theta} = \sqrt{S^2}, \quad \hat{\mu} = \bar{X} - \sqrt{S^2}$$

5 证明. 由于,

$$p = P\{X > 0\} = P\{X - \mu > -\mu\} = 1 - \Phi(-\mu) = \Phi(\mu)$$

即  $\mu = \Phi^{-1}(p)$ , 故  $\mu$  的矩估计为  $\hat{\mu} = \Phi^{-1}(\hat{p}) = \Phi^{-1}\left(\frac{k}{n}\right)$ 。

7 证明. 由于,

$$E(X) = mp, \quad \text{Var}(X) = mp(1-p)$$

即

$$p = 1 - \frac{\text{Var}(X)}{E(X)}, \quad m = \frac{E(X)}{p} = \frac{[E(X)]^2}{E(X) - \text{Var}(X)}$$

故  $(m, p)$  的矩估计为

$$\hat{m} = \frac{\bar{X}^2}{\bar{X} - S^2}, \quad \hat{p} = 1 - \frac{S^2}{\bar{X}}$$

同时，由于  $m$  为正整数，故  $m$  的矩估计应修改为

$$\hat{m} = \lceil \frac{\bar{X}^2}{\bar{X} - S^2} \rceil$$