Contents

Part I

Calculus

Chapter 1 Limit Theory

Definition 1.1. Mapping

Let $X : \Omega_1 \to \Omega_2$ be a mapping.

1. *For every subset $B \in \Omega_2$, the inverse image of B is*

$$X^{-1}(B) = \{\omega : \omega \in \Omega_1, X(\omega) \in B\} := \{X \in B\}.$$

2. *For every class*

Chapter 2 Differential Calculus

Chapter 3 Integral Calculus

Part II

Real Analysis

Chapter 4 Measure Theory

4.1 Semi-algebras, Algebras and Sigma-algebras

> **Definition 4.1. Semi-algebra**
>
> *A nonempty class of $\mathcal{S}$ of subsets of Ω is an **semi-algebra** on Ω that satisfy*
>
> *1. if $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$.*
>
> *2. if $A \in \mathcal{S}$, then A^C is a finite disjoint union of sets in $\mathcal{S}$, i.e.,*
> $$A^C = \sum_{i=1}^{n} A_i, \text{ where } A_i \in \mathcal{S}, A_i \cap A_j = \emptyset, i \neq j.$$

> **Definition 4.2. Algebra**
>
> *A nonempty class $\mathcal{A}$ of subsets of Ω is an **algebra** on Ω that satisfy*
>
> *1. if $A \in \mathcal{A}$, then $A^C \in \mathcal{A}$.*
>
> *2. if $A_1, A_2 \in \mathcal{A}$, then $A_1 \cup A_2 \in \mathcal{A}$.*

> **Definition 4.3. σ-algebra**
>
> *A nonempty class $\mathcal{F}$ of subsets of Ω is a σ-**algebra** on Ω that satisfy*
>
> *1. if $A \in \mathcal{F}$, then $A^C \in \mathcal{F}$.*
>
> *2. if $A_i \in \mathcal{F}$ is a countable sequence of sets, then $\cup_i A_i \in \mathcal{F}$.*

Example 4.1 Special σ-algebra

1. **Trival σ-algebra** := $\{\emptyset, \Omega\}$. This is smallest σ-algebra.

2. **Power Set** := all subsets of σ, denoted by $\mathcal{P}(\Omega)$. This is the largest σ-algebra.

3. **The smallest σ-algebra containing** $A \in \Omega := \{\emptyset, A, A^C, \Omega\}$.

It is easy to define (Lesbegue) measure on the semi-algebra $\mathcal{S}$, and then easily to extend it to the algebra $\overline{\mathcal{S}}$, finally, we can extend it further t o some σ-algebra (mostly consider the smallest one containing $\mathcal{S}$).

> **Lemma 4.1**
>
> *If $\mathcal{S}$ is a semi-algebra, then*
> $$\overline{\mathcal{S}} = \{\text{finite disjoint unions of sets in } \mathcal{S}\}$$

is an algebra, denoted by $\mathcal{A}(\mathcal{S})$, called **the algebra generated by** $\mathcal{S}$.

Proof Let $A, B \in \overline{\mathcal{S}}$, then $A = \sum_{i=1}^{n} A_i$, $B = \sum_{j=1}^{m} B_j$ with $A_i, B_i \in \mathcal{S}$.

Intersection: For $A_i \cap B_j \in \mathcal{S}$ by the definition of semi-algebra $\mathcal{S}$, thus

$$A \cap B = \sum_{i=1}^{n} \sum_{j=1}^{m} A_i \cap B_j \in \overline{\mathcal{S}}.$$

So $\overline{\mathcal{S}}$ is closed under (finite) intersection.

Complement: For DeMorgan's Law, $A_i^C \in \mathcal{S}$ by the definition of semi-algebra $\mathcal{S}$ and $\overline{\mathcal{S}}$ closed under (finite) intersection that we just shown, thus

$$A^C = (\sum_{i=1}^{n} A_i)^C = \cap_{i=1}^{n} A_i^C \in \overline{\mathcal{S}}.$$

So $\overline{\mathcal{S}}$ is closed under complement.

Union: For DeMorgan's Law and $\overline{\mathcal{S}}$ closed under (finite) intersection and complement that we just shown, thus

$$A \cup B = (A^C \cap B^C)^C \in \overline{\mathcal{S}}.$$

So $\overline{\mathcal{S}}$ is closed under (finite) union.

Hence, $\overline{\mathcal{S}}$ is an algebra.

Theorem 4.1

*For any class $\mathcal{A}$, there exists a unique minimal σ-algebra containing $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$, called **the σ-algebra generated by** $\mathcal{A}$. In other words,*

1. *$\mathcal{A} \subset \sigma(\mathcal{A})$.*
2. *For any σ-algebra $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{B}$, $\sigma(\mathcal{A}) \subset \mathcal{B}$.*

and $\sigma(\mathcal{A})$ is unique.

Proof **Existence**:

Uniqueness:

Example 4.2 Borel σ-algebras generated from semi-algebras

1.

4.2 Measure

> **Definition 4.4. Measure**
>
> *_Measure_ is a nonnegative countably additive set function, that is, a function $\mu : \mathcal{A} \to \mathbf{R}$ with*
>
> 1. *$\mu(A) \geq \mu(\emptyset) = 0$ for all $A \in \mathcal{A}$.*
> 2. *if $A_i \in \mathcal{A}$ is a countable sequence of disjoint sets, then*
> $$\mu(\cup_i A_i) = \sum_i \mu(A_i).$$
> ♣

> **Definition 4.5. Measure Space**
>
> *If μ is a measure on a σ-algebra $\mathcal{A}$ of subsets of Ω, the triplet $(\Omega, \mathcal{A}, \mu)$ is a _measure space_.*
> ♣

Note *A measure space $(\Omega, \mathcal{A}, \mu)$ is a **probability space**, if $P(\Omega) = 1$.*

Property *Let μ be a measure on a σ-algebra $\mathcal{A}$*

1. *__monotonieity__ if $A \subset B$, then $\mu(A) \leq \mu(B)$.*
2. *__subadditivity__ if $A \subset \cup_{m=1}^{\infty} A_m$, then $\mu(A) \leq \sum_{m=1}^{\infty} u(A_m)$.*
3. *__continuity from below__ if $A_i \uparrow A$ (i.e. $A_1 \subset A_2 \subset \ldots$ and $\cup_i A_i = A$), then $\mu(A_i) \uparrow \mu(A)$.*
4. *__continuity from above__ if $A_i \downarrow A$ (i.e. $A_1 \supset A_2 \supset \ldots$ and $\cap_i A_i = A$), then $\mu(A_i) \downarrow \mu(A)$.*

Proof

Chapter 5 Lebesgue Integration

5.1 Properties of the Integral

Theorem 5.1. Jensen's Inequality

Let (Ω, A, μ) be a probability space. If f is a real-valued function that is μ-integrable, and if φ is a convex function on the real line, then:

$$\varphi\left(\int_\Omega f\,\mathrm{d}\mu\right) \leq \int_\Omega \varphi(f)\mathrm{d}\mu. \tag{5.1}$$

Proof Let $x_0 = \int_\Omega f\,\mathrm{d}\mu$. Since the existence of subderivatives for convex functions, $\exists a, b \in R$, such that,

$$\forall x \in R, \varphi(x) \geq ax + b \text{ and } ax_0 + b = \varphi(x_0).$$

Then, we got

$$\int_\Omega \varphi(f)\mathrm{d}\mu \geq \int_\Omega af + b\,\mathrm{d}\mu = a\int_\Omega f\,\mathrm{d}\mu + b = ax_0 + b = \varphi\left(\int_\Omega f\,\mathrm{d}\mu\right).$$

Theorem 5.2. Hölder's Inequality

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. Then, for all measurable functions f and g on Ω,

$$\int_\Omega |f \cdot g|\,\mathrm{d}\mu \leq \|f\|_p \|g\|_q. \tag{5.2}$$

Proof

Theorem 5.3. Minkowski's Inequality

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $p \in [1, \infty]$. Then, for all measurable functions f and g on Ω,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \tag{5.3}$$

Proof Since $\varphi(x) = x^p$ is a convex function for $p \in [1, \infty)$. By it's definition,

$$|f + g|^p = \left|2 \cdot \frac{f}{2} + 2 \cdot \frac{g}{2}\right|^p \leq \frac{1}{2}|2f|^p + \frac{1}{2}|2g|^p = 2^{p-1}\left(|f|^p + |g|^p\right).$$

Therefore,

$$|f + g|^p < 2^{p-1} (|f|^p + |g|^p) < \infty.$$

By Hölder's Inequality (5.2),

$$
\begin{aligned}
\|f + g\|_p^p &= \int |f + g|^p \mathrm{d}\mu \\
&= \int |f + g| \cdot |f + g|^{p-1} \mathrm{d}\mu \\
&\leq \int (|f| + |g|) |f + g|^{p-1} \mathrm{d}\mu \\
&= \int |f| |f + g|^{p-1} \mathrm{d}\mu + \int |g| |f + g|^{p-1} \mathrm{d}\mu \\
&\leq \left(\left(\int |f|^p \mathrm{d}\mu \right)^{\frac{1}{p}} + \left(\int |g|^p \mathrm{d}\mu \right)^{\frac{1}{p}} \right) \left(\int |f + g|^{(p-1)\left(\frac{p}{p-1}\right)} \mathrm{d}\mu \right)^{1 - \frac{1}{p}} \\
&= \left(\|f\|_p + \|g\|_p \right) \frac{\|f + g\|_p^p}{\|f + g\|_p}
\end{aligned}
$$

which means, as $p \in [1, \infty)$,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

When $p = \infty$,

$$a$$

Theorem 5.4. Bounded Convergence Theorem

$\heartsuit$

Theorem 5.5. Fatou's Lemma

$\heartsuit$

Theorem 5.6. Monotone Convergence Theorem

$\heartsuit$

5.2 Product Measures

Theorem 5.7. Fubini's Theorem

$\heartsuit$

Part III

Functional Analysis

Part IV

Probability Theory

Chapter 6 Random Variables

6.1 Probability Space

> **Definition 6.1. Probability Space**
>
> *A probability space is a triple $(\Omega, \mathcal{F}, P)$ consisting of:*
>
> 1. *the sample space Ω: an arbitrary non-empty set.*
> 2. *the σ-algebra $\mathcal{F} \subseteq 2^{\Omega}$: a set of subsets of Ω, called events.*
> 3. *the probability measure $P : \mathcal{F} \to [0, 1]$: a function on $\mathcal{F}$ which is a measure function.*
> ♣

6.2 Random Variables

> **Definition 6.2. Random Variable**
>
> *A random variable is a measurable function $X : \Omega \to S$ from a set of possible outcomes $(\Omega, \mathcal{F})$ to a measurable space $(S, \mathcal{S})$, that is,*
>
> $$X^{-1}(B) \equiv \{\omega : X(\omega) \in B\} \in \mathcal{F} \quad \forall B \in \mathcal{S}. \tag{6.1}$$
>
> *Typically, $(S, \mathcal{S}) = (R^d, \mathcal{R}^d) \quad (d > 1)$.*
> ♣

How to prove that functions are measurable?

> **Theorem 6.1. I**
>
> $\{\omega : X(\omega) \in A\} \in \mathcal{F}$ *for all $A \in \mathcal{A}$ and $\mathcal{A}$ generates $\mathcal{S}$, then X is measurable.*
> ♡

1.

6.3 Distributions

6.3.1 Definition of Distributions

> **Definition 6.3. Distribution**
>
> *A distribution of random variable X is a probability function $P : \mathcal{R} \to \mathbb{R}$ by setting*
>
> $$\mu(A) = P(X \in A) = P\left(X^{-1}(A)\right), \quad for A \in \mathcal{R}. \tag{6.2}$$

> **Definition 6.4. Distribution Function**
>
> *The distribution of a random variable X is usually described by giving its **distribution function**,*
>
> $$F(x) = P(X \leq x). \tag{6.3}$$

> **Definition 6.5. Density Function**
>
> *If the distribution function $F(x) = P(X \leq x)$ has the form*
>
> $$F(x) = \int_{-\infty}^{x} f(y)\mathrm{d}y,$$
>
> *that X has density function f.*

6.3.2 Properties of Distributions

> **Theorem 6.2. Properties of Distribution Function**
>
> *Any distribution function F has the following properties,*
>
> 1. *F is nondecreasing.*
> 2. *$\lim_{x \to \infty} F(x) = 1, \lim_{x \to -\infty} F(x) = 0$.*
> 3. *F is right continuous, i.e., $\lim_{y \downarrow x} F(y) = F(x)$.*
> 4. *If $F(x-) = \lim_{y \uparrow x} F(y)$, then $F(x-) = P(X < x)$.*
> 5. *$P(X = x) = F(x) - F(x-)$.*

Proof

> **Theorem 6.3**
>
> *If F satisfies (1), (2), and (3) in Theorem 6.2, then it is the distribution function of some random variable.*

Proof

> **Theorem 6.4**
> *A distribution function has at most countably many discontinuities*

Proof

6.3.3 Families of Distributions

6.4 Expected Value

> **Definition 6.6. Expectation**
> ♣

> **Theorem 6.5. Bounded Convergence Theorem**
> ♡

> **Theorem 6.6. Fatou's Lemma**
> *If $X_n \geq 0$, then*
> $$\liminf_{n\to\infty} E X_n \geq E\left(\liminf_{n\to\infty} X_n\right). \qquad (6.4)$$

> **Theorem 6.7. Monotone Convergence Theorem**
> *If $0 \leq X_n \uparrow X$, then*
> $$E X_n \uparrow E X. \qquad (6.5)$$

> **Theorem 6.8. Dominated Convergence Theorem**
> *If $X_n \to X$ a.s., $|X_n| \leq Y$ for all n, and $E Y < \infty$, then*
> $$E X_n \to E X. \qquad (6.6)$$

6.5 Independence

6.5.1 Definition of Independence

> **Definition 6.7. Independence**
>
> 1. *Two events A and B are independent if $P(A \cap B) = P(A)P(B)$.*
> 2. *Two random variables X and Y are independent if for all $C, D \in \mathcal{R}$*
>
> $$P(X \in C, Y \in D) = P(X \in C)P(Y \in D). \qquad (6.7)$$
>
> 3. *Two σ-fields $\mathcal{F}$ and $\mathcal{G}$ are independent if for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$ the events A and B are independent.* ♣

The second definition is a special case of the third.

> **Theorem 6.9**
>
> 1. *If X and Y are independent then $\sigma(X)$ and $\sigma(Y)$ are independent.*
> 2. *Conversely, if $\mathcal{F}$ and $\mathcal{G}$ are independent, $X \in \mathcal{F}$ and $Y \in \mathcal{G}$, then X and Y are independent.* ♡

The first definition is, in turn, a special case of the second.

> **Theorem 6.10**
>
> 1. *If A and B are independent, then so are A^c and B, A and B^c, and A^c and B^c.*
> 2. *Conversely, events A and B are independent if and only if their indicator random variables 1_A and 1_B are independent.* ♡

The definition of independence can be extended to the infinite collection.

> **Definition 6.8**
>
> *An infinite collection of objects (σ-fields, random variables, or sets) is said to be independent if every finite subcollection is,*
>
> 1. *σ-fields $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ are independent if whenever $A_i \in \mathcal{F}_i$ for $i = 1, \ldots, n$, we have*
>
> $$P\left(\cap_{i=1}^n A_i\right) = \prod_{i=1}^n P\left(A_i\right). \qquad (6.8)$$
>
> 2. *Random variables $X_1, \ldots, X_n$ are independent if whenever $B_i \in \mathcal{R}$ for $i =$*

$1, \ldots, n$ we have

$$P\left(\cap_{i=1}^n \{X_i \in B_i\}\right) = \prod_{i=1}^n P\left(X_i \in B_i\right). \tag{6.9}$$

3. *Sets $A_1, \ldots, A_n$ are independent if whenever $I \subset \{1, \ldots, n\}$ we have*

$$P\left(\cap_{i \in I} A_i\right) = \prod_{i \in I} P\left(A_i\right). \tag{6.10}$$

6.5.2 Sufficient Conditions for Independence

6.5.3 Independence, Distribution, and Expectation

Theorem 6.11

Suppose $X_1, \ldots, X_n$ are independent random variables and X_i has distribution μ_i, then $(X_1, \ldots, X_n)$ has distribution $\mu_1 \times \cdots \times \mu_n$.

Theorem 6.12

If $X_1, \ldots, X_n$ are independent and have

1. *$X_i \geq 0$ for all i, or*
2. *$E|X_i| < \infty$ for all i.*

then

$$E\left(\prod_{i=1}^n X_i\right) = \prod_{i=1}^n EX_i \tag{6.11}$$

6.5.4 Sums of Independent Random Variables

Theorem 6.13. Convolution for Random Variables

1. *If X and Y are independent, $F(x) = P(X \leq x)$, and $G(y) = P(Y \leq y)$, then*

$$P(X + Y \leq z) = \int F(z - y) \mathrm{d}G(y). \tag{6.12}$$

2. *If X and Y are independent, X with density f and Y with distribution function G, then $X + Y$ has density*

$$h(x) = \int f(x - y) \mathrm{d}G(y). \tag{6.13}$$

Suppose Y has density g, the last formula can be written as

$$h(x) = \int f(x - y)g(y)\mathrm{d}y. \tag{6.14}$$

3. If X and Y are independent, integral-valued random variables, then

$$P(X + Y = n) = \sum_m P(X = m)P(Y = n - m). \tag{6.15}$$

6.6 Moments

Lemma 6.1

If $Y > 0$ and $p > 0$, then

$$E(Y^p) = \int_0^\infty py^{p-1}P(Y > y)\mathrm{d}y. \tag{6.16}$$

6.7 Characteristic Functions

6.7.1 Definition of Characteristic Functions

Definition 6.9. Characteristic Function

If X is a random variable, we define its characteristic function (ch.f) by

$$\varphi(t) = E\left(e^{itX}\right) = E\left(\cos tX\right) + iE\left(\sin tX\right). \tag{6.17}$$

Note *Eular Equation.*

6.7.2 Properties of Characteristic Functions

Theorem 6.14. Properties of Characteristic Function

Any characteristic function has the following properties:

1. $\varphi(0) = 1$,
2. $\varphi(-t) = \overline{\varphi(t)}$,
3. $|\varphi(t)| = |Ee^{itX}| \le E|e^{itX}| = 1$,
4. $\varphi(t)$ is uniformly continuous on $(-\infty, \infty)$,

> 5. $Ee^{it(aX+b)} = e^{itb}\varphi(at)$,
>
> 6. If X_1 and X_2 are independent and have ch.f.'s φ_1 and φ_2, then $X_1 + X_2$ has ch.f. $\varphi_1(t)\varphi_2(t)$.

Proof

6.7.3 The Inversion Formula

The characteristic function uniquely determines the distribution. This and more is provided by:

Theorem 6.15. The Inversion Formula

Let $\varphi(t) = \int e^{itx}\mu(\mathrm{d}x)$ where μ is a probability measure. If $a < b$, then

$$\lim_{T\to\infty}(2\pi)^{-1}\int_{-T}^{T}\frac{e^{-ita} - e^{-itb}}{it}\varphi(t)dt = \mu(a,b) + \frac{1}{2}\mu(\{a,b\}) \qquad (6.18)$$

Proof

Theorem 6.16

If $\int |\varphi(t)|\mathrm{d}t < \infty$, then μ has bounded continuous density

$$f(y) = \frac{1}{2\pi}\int e^{-ity}\varphi(t)\mathrm{d}t. \qquad (6.19)$$

Proof

6.7.4 Convergence in Distribution

Theorem 6.17. Lèvy' s Continuity Theorem

Let $\mu_n, 1 \leq n \leq \infty$ be probability measures with ch.f. φ_n.

1. If $\mu_n \xrightarrow{d} \mu_\infty$, then $\varphi_n(t) \to \varphi_\infty(t)$ for all t.

2. If $\varphi_n(t)$ converges pointwise to a limit $\varphi(t)$ that is continuous at 0, then the associated sequence of distributions μ_n is tight and converges weakly to the measure μ with characteristic function φ.

Proof

6.7.5 Moments and Derivatives

Theorem 6.18

If $\int |x|^n \mu(dx) < \infty$, then its characteristic function φ has a continuous derivative of order n given by

$$\varphi^{(n)}(t) = \int (ix)^n e^{itx} \mu(dx). \qquad (6.20)$$

Theorem 6.19

If $E|X|^2 < \infty$ then

$$\varphi(t) = 1 + itEX - t^2 E\left(X^2\right)/2 + o\left(t^2\right). \qquad (6.21)$$

Theorem 6.20

If $\limsup_{h\downarrow 0}\{\varphi(h) - 2\varphi(0) + \varphi(-h)\}/h^2 > -\infty$, then

$$E|X|^2 < \infty. \qquad (6.22)$$

Chapter 7 Convergence of Random Variables

Introduction

❏ *Convergence in Mean*

❏ *Convergence in Probability*

❏ *Convergence in Distribution*

❏ *Almost Sure Convergence*

7.1 Convergence in Mean

Definition 7.1. Convergence in Mean

*A sequence $\{X_n\}$ of real-valued random variables **converges in the r-th mean** ($r \geq 1$) towards the random variable X, if*

1. *The r-th absolute moments $E(|X_n|^r)$ and $E(|X|^r)$ of $\{X_n\}$ and X exist,*
2. *$\lim_{n\to\infty} E(|X_n - X|^r) = 0$.*

Convergence in the r-th mean is denoted by

$$X_n \xrightarrow{L^r} X. \tag{7.1}$$ ♣

7.2 Convergence in Probability

7.2.1 Definition of Convergence in Probability

Definition 7.2. Convergence in Probability

*A sequence $\{X_n\}$ of real-valued random variables **converges in probability** towards the random variable X, if*

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0, \quad \forall \varepsilon > 0. \tag{7.2}$$

Convergence in probability is denoted by

$$X_n \xrightarrow{p} X. \tag{7.3}$$ ♣

Definition 7.3. Convergence in Uninform

♣

7.2.2 Properties of Convergence in Probability

7.3 Convergence in Distribution

7.3.1 Definition of Convergence in Distribution

> **Definition 7.4. Convergence in Distribution**
>
> *A sequence $\{X_n\}$ of real-valued random variables is said to **converge in distribution**, or **converge weakly**, or **converge in law** to a random variable X, if*
>
> $$\lim_{n\to\infty} F_n(x) = F(x), \tag{7.4}$$
>
> *for every number at $x \in \mathbb{R}$ which F is continuous. Here F_n and F are the cumulative distribution functions of random variables X_n and X, respectively. Convergence in distribution can be denoted as*
>
> $$X_n \xrightarrow{d} X, \text{ or } X_n \Rightarrow X. \tag{7.5}$$ ♣

⚖ Note

- *Convergence in Distribution is the weakest form of convergence typically discussed, since it is implied by all other types of convergence mentioned in this chapter.*
- *Convergence in Distribution does not imply that the sequence of corresponding probability density functions will also converge. However, according to Scheffé's theorem, convergence of the probability density functions implies convergence in distribution.*

7.3.2 Properties of Convergence in Distribution

> **Lemma 7.1**
>
> *If $F_n \xrightarrow{d} F_\infty$, then there are random variables $Y_n, 1 \le n \le \infty$, with distribution F_n so that*
>
> $$Y_n \xrightarrow{a.s.} Y_\infty. \tag{7.6}$$ ♡

> **Theorem 7.1. Portmanteau Lemma**
>
> *$\{X_n\}$ converges in distribution to X, if and only if any of the following statements are true,*

- $P(X_n \leq x) \to P(X \leq x)$, *for all continuity points of the distribution of* X.
- $Ef(X_n) \to Ef(X)$, *for all bounded, continuous (Lipschitz) functions* f.
- $\liminf_{n \to \infty} P(X_n \in G) \geq P(X_\infty \in G)$, *for all open sets* G.
- $\limsup_{n \to \infty} P(X_n \in K) \leq P(X_\infty \in K)$, *for all closed sets* K.
- $\lim_{n \to \infty} P(X_n \in A) = P(X_\infty \in A)$, *for all Borel sets* A *with* $P(X_\infty \in \partial A) = 0$.

Proof

Theorem 7.2. Continuous Mapping Theorem

Let g *be a measurable function and* $D_g = \{x : g \text{ is discontinuous at } x\}$ *with* $P(X \in D_g) = 0$, *then,*

$$
\begin{aligned}
X_n \xrightarrow{d} X &\Rightarrow g(X_n) \xrightarrow{d} g(X), \\
X_n \xrightarrow{p} X &\Rightarrow g(X_n) \xrightarrow{p} g(X), \\
X_n \xrightarrow{a.s.} X &\Rightarrow g(X_n) \xrightarrow{a.s.} g(X).
\end{aligned}
\tag{7.7}
$$

If in addition g *is bounded, then*

$$
Eg(X_n) \to Eg(X). \tag{7.8}
$$

Proof

Theorem 7.3

If $X_n \xrightarrow{p} X$, *then*

$$
X_n \xrightarrow{d} X, \tag{7.9}
$$

and that, conversely, if $X_n \xrightarrow{d} c$, *where* c *is a constant, then*

$$
X_n \xrightarrow{p} c. \tag{7.10}
$$

Proof

1. $\forall \varepsilon > 0$, at fixed point x, since if $X_n \leq x$ and $|X_n - X| \leq \varepsilon$, then $X \leq x + \varepsilon$, then

$$
\{X \leq x + \varepsilon\} \subset \{X_n \leq x\} \cup \{|X_n - X| > \varepsilon\},
$$

similarily, if $X \leq x - \varepsilon$ and $|X_n - X| \leq \varepsilon$, then $X_n \leq x$, then

$$
\{X_n \leq x\} \subset \{X \leq x - \varepsilon\} \cup \{|X_n - X| > \varepsilon\},
$$

then, by the union bound,

$$P\left(X \leq x + \varepsilon\right) \leq P\left(X_n \leq x\right) + P\left(|X_n - X| > \varepsilon\right),$$

$$P\left(X_n \leq x\right) \leq P\left(X \leq x - \varepsilon\right) + P\left(|X_n - X| > \varepsilon\right).$$

So, we got

$$P\left(X \leq x + \varepsilon\right) - P\left(|X_n - X| > \varepsilon\right) \leq P\left(X_n \leq x\right)$$

$$\leq P\left(X \leq x - \varepsilon\right) + P\left(|X_n - X| > \varepsilon\right)$$

As $n \to \infty$, $P\left(|X_n - X| > \varepsilon\right) \to 0$, then

$$P\left(X \leq x - \varepsilon\right) \leq \lim_{n \to \infty} P\left(X_n \leq x\right) \leq P\left(X \leq x + \varepsilon\right)$$

$$\Rightarrow F(x - \varepsilon) \leq \lim_{n \to \infty} F_n(x) \leq F(x + \varepsilon)$$

By the property of distribution (Theorem 6.2), as $\varepsilon \to 0$, then

$$\lim_{n \to \infty} F_n(x) = F(x),$$

which means,

$$X_n \xrightarrow{d} X.$$

2. Since $X_n \xrightarrow{d} c$, where c is a constant, then $\forall \varepsilon > 0$,

$$\lim_{n \to \infty} P(X_n \leq c + \varepsilon) = 1 \Rightarrow \lim_{n \to \infty} P(X_n > c + \varepsilon) = 0$$

$$\lim_{n \to \infty} P(X_n \leq c - \varepsilon) = 0.$$

Therefore,

$$P\left(|X_n - c| < \varepsilon\right) = 0,$$

which means

$$X_n \xrightarrow{p} c.$$

Theorem 7.4. Slutsky's Theorem

Let X_n, Y_n be sequences of random variables. If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$, then

1. $X_n + Y_n \xrightarrow{d} X + c$.
2. $X_n Y_n \xrightarrow{d} cX$.
3. $X_n / Y_n \xrightarrow{d} X/c$, provided that c is invertible.

♡

Proof

1. Since

2.

3.

Note *However that convergence in distribution of $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ does in general not imply convergence in distribution of $X_n + Y_n \xrightarrow{d} X + Y$ or of $X_n Y_n \xrightarrow{d} XY$.*

> **Theorem 7.5. Cramér-Wold Theorem**

> **Theorem 7.6. Delta Method**
>
> *Let $\{X_n\}$ be a sequence of random variables with*
>
> $$\sqrt{n}\,[X_n - \theta] \xrightarrow{d} \sigma\chi,$$
>
> *where θ and σ are finite, then for any function g with the property that $g'(\theta)$ exists and is non-zero valued,*
>
> $$\sqrt{n}\,[g(X_n) - g(\theta)] \xrightarrow{d} \sigma g'(\theta)\chi.$$

Proof Under the assumption that $g'(\theta)$ is continuous.

Since, $g'(\theta)$ exists, with the first-order Taylor Approximation:

$$g(X_n) = g(\theta) + g'(\tilde{\theta})(X_n - \theta),$$

where $\tilde{\theta}$ lies between X_n and θ.

Since $X_n \xrightarrow{p} \theta$, and $|\tilde{\theta} - \theta| < |X_n - \theta|$, then

$$\tilde{\theta} \xrightarrow{p} \theta,$$

Since $g'(\theta)$ is continuous, by Continuous Mapping Theorem (7.2),

$$g'(\tilde{\theta}) \xrightarrow{p} g'(\theta).$$

and,

$$\sqrt{n}\,(g(X_n) - g(\theta)) = \sqrt{n}g'(\tilde{\theta})(X_n - \theta),$$

$$\sqrt{n}\,[X_n - \theta] \xrightarrow{d} \sigma\chi,$$

by Slutsky's Theorem (7.4),

$$\sqrt{n}\,[g(X_n) - g(\theta)] \xrightarrow{d} \sigma g'(\theta)\chi.$$

7.3.3 Limits of Sequences of Distributions $\{F_n\}$

> **Theorem 7.7. Helly's Selection Theorem**
>
> *For every sequence F_n of distribution functions, there is a subsequence $F_{n(k)}$ and a right continuous nondecreasing function F so that $\lim_{k \to \infty} F_{n(k)}(y) = F(y)$ at all continuity points y of F.*

> **Theorem 7.8**
>
> *Every subsequential limit is the distribution function of a probability measure if and only if the sequence F_n is tight, i.e., for all $\epsilon > 0$ there is an M_ϵ so that*
>
> $$\limsup_{n \to \infty} 1 - F_n(M_\epsilon) + F_n(-M_\epsilon) \le \epsilon. \tag{7.11}$$

7.4 Almost Sure Convergence

7.4.1 Definition of Almost Sure Convergence

> **Definition 7.5. Almost Sure Convergence**
>
> *A sequence $\{X_n\}$ of real-valued random variables converges **almost sure** or **almost everywhere** or **with probability 1** or **strongly** towards the random variable X, if*
>
> $$P\left(\lim_{n \to \infty} X_n = X\right) = 1. \tag{7.12}$$
>
> *Almost sure convergence is denoted by*
>
> $$X_n \overset{a.s.}{\to} X. \tag{7.13} \clubsuit$$

Note

7.4.2 Properties of Almost Sure Convergence

> **Theorem 7.9**
>
> *If $X_n \overset{a.s.}{\to} X$, then*
>
> $$X_n \overset{p}{\to} X. \tag{7.14}$$

Proof

Theorem 7.10

$X_n \xrightarrow{p} X$ *if and only if for all subsequence $X_{n(m)}$ exists a further subsequence $X_{n(m_k)}$,*
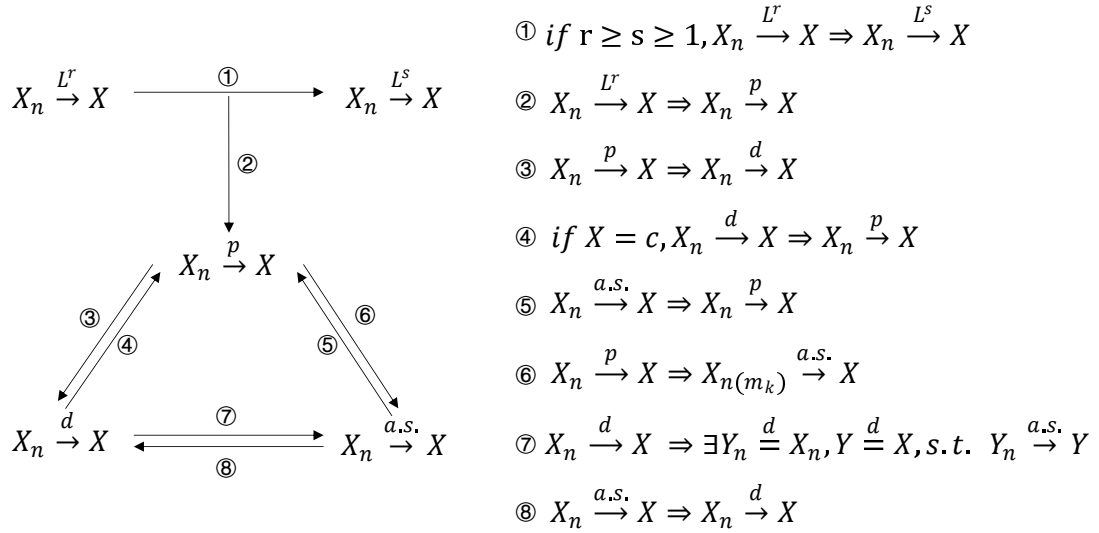*such that*

$$X_{n(m_k)} \xrightarrow{a.s.} X. \tag{7.15}$$

$$X_n \xrightarrow{L^r} X \xrightarrow{\text{①}} X_n \xrightarrow{L^s} X$$

$$\downarrow \text{②}$$

$$X_n \xrightarrow{p} X$$

① *if* $r \geq s \geq 1, X_n \xrightarrow{L^r} X \Rightarrow X_n \xrightarrow{L^s} X$

② $X_n \xrightarrow{L^r} X \Rightarrow X_n \xrightarrow{p} X$

③ $X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X$

④ *if* $X = c, X_n \xrightarrow{d} X \Rightarrow X_n \xrightarrow{p} X$

⑤ $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p} X$

⑥ $X_n \xrightarrow{p} X \Rightarrow X_{n(m_k)} \xrightarrow{a.s.} X$

⑦ $X_n \xrightarrow{d} X \Rightarrow \exists Y_n \overset{d}{=} X_n, Y \overset{d}{=} X, s.t. \ Y_n \xrightarrow{a.s.} Y$

⑧ $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{d} X$

$$X_n \xrightarrow{d} X \quad \underset{\text{⑧}}{\overset{\text{⑦}}{\rightleftarrows}} \quad X_n \xrightarrow{a.s.} X$$

Figure 7.1: Relations of Convergence of Random Variables

Chapter 8 Law of Large Numbers

8.1 Weak Law of Large Numbers

Lemma 8.1

If $p > 0$ and $E\,|Z_n|^p \to 0$, then

$$Z_n \xrightarrow{d} 0. \tag{8.1}$$

Proof

Theorem 8.1. Weak Law of Large Numbers with Finite Variances

Let $X_1, X_2, \ldots$ be i.i.d. random variables with $EX_i = \mu$ and $Var(X_i) \leq C < \infty$. Suppose $S_n = X_1 + X_2 + \ldots + X_n$, then

$$S_n/n \xrightarrow{L^2} \mu, \quad S_n/n \xrightarrow{p} \mu. \tag{8.2}$$

Proof

Theorem 8.2. Weak Law of Large Numbers without i.i.d.

Let $X_1, X_2, \ldots$ be random variables, Suppose $S_n = X_1 + X_2 + \ldots + X_n$, $\mu_n = ES_n$, $\sigma_n^2 = Var(S_n)$, if $\sigma_n^2/b_n^2 \to 0$, then

$$\frac{S_n - \mu_n}{b_n} \xrightarrow{p} 0. \tag{8.3}$$

Proof

Theorem 8.3. Weak Law of Large Numbers for Triangular Arrays

For each n, let $X_{n,m}, 1 \leq m \leq n$, be independent random variables. Suppose $b_n > 0$ with $b_n \to \infty$, $\bar{X}_{n,m} = X_{n,m} I_{(X_{n,m} \leq b_n)}$, if

  1. $\sum_{m=1}^{n} P\,(|X_{n,m}| > b_n) \to 0$, and

  2. $b_n^{-2} \sum_{m=1}^{n} E\bar{X}_{n,m}^2 \to 0$.

Suppose $S_n = X_{n,1} + \cdots + X_{n,n}$ and $a_n = \sum_{m=1}^{n} E \bar{X}_{n,m}$, then

$$\frac{S_n - a_n}{b_n} \xrightarrow{p} 0. \tag{8.4}$$

Proof

> **Theorem 8.4. Weak Law of Large Numbers by Feller**
>
> Let $X_1, X_2, \ldots$ be i.i.d. random variables with
>
> $$\lim_{x \to 0} x P(|X_i| > x) = 0. \tag{8.5}$$
>
> Suppose $S_n = X_1 + X_2 + \ldots + X_n$, $\mu_n = E\left(X_1 I_{(|X_1| < n)}\right)$, then
>
> $$S_n/n - \mu_n \xrightarrow{p} 0. \tag{8.6}$$

Proof

> **Theorem 8.5. Weak Law of Large Numbers**
>
> Let $X_1, X_2, \ldots$ be i.i.d. random variables with $E|X_i| < \infty$. Suppose $S_n = X_1 + X_2 + \ldots + X_n$, $\mu = E X_i$, then
>
> $$S_n/n \xrightarrow{p} \mu. \tag{8.7}$$

Proof

Note $E|X_i| = \infty$

8.2 Strong Law of Large Numbers

8.2.1 Borel-Cantelli Lemmas

> **Lemma 8.2. Borel-Cantelli Lemma**
>
> If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then
>
> $$P(A_n \text{ i.o. }) = 0. \tag{8.8}$$

> **Lemma 8.3. The Second Borel-Cantelli Lemma**
>
> If $\{A_n\}$ are independent with $\sum_{n=1}^{\infty} P(A_n) = \infty$, then,
>
> $$P(A_n \text{ i.o. }) = 1. \tag{8.9}$$

> ### Corollary 8.1
>
> Suppose $\{A_n\}$ are independent with $P(A_n) < 1, \forall n$. If $P\left(\cup_{n=1}^{\infty} A_n\right) = 1$ then
>
> $$\sum_{n=1}^{\infty} P(A_n) = \infty, \tag{8.10}$$
>
> and hence $P(A_n \text{ i.o. }) = 1$

Proof

8.2.2 Strong Law of Large Numbers

> ### Theorem 8.6. Strong Law of Large Numbers
>
> Let $X_1, X_2, \ldots$ be i.i.d. random variables with $E|X_i| < \infty$. Suppose $S_n = X_1 + X_2 + \ldots + X_n$, $\mu = EX_i$, then
>
> $$S_n/n \overset{a.s.}{\to} \mu. \tag{8.11}$$

8.3 Uniform Law of Large Numbers

> ### Theorem 8.7. Uniform Law of Large Numbers
>
> *Suppose*
>
> 1. Θ is compact.
> 2. $g(X_i, \theta)$ is continuous at each $\theta \in \Theta$ almost sure.
> 3. $g(X_i, \theta)$ is dominated by a function $G(X_i)$, i.e. $|g(X_i, \theta)| \leq G(X_i)$.
> 4. $EG(X_i) < \infty$.
>
> *Then*
>
> $$\sup_{\theta \in \Theta}\left| n^{-1}\sum_{i=1}^{n} g(X_i, \theta) - Eg(X_i, \theta)\right| \overset{p}{\to} 0. \tag{8.12}$$

Proof Suppose

$$\Delta_\delta(X_i, \theta_0) = \sup_{\theta \in B(\theta_0, \delta)} g(X_i, \theta) - \inf_{\theta \in B(\theta_0, \delta)} g(X_i, \theta).$$

Since (i) $\Delta_\delta(X_i, \theta_0) \overset{a.s.}{\to} 0$ by condition (2), (ii) $\Delta_\delta(X_i, \theta_0) \leq 2\sup_{\theta \in \Theta}|g(X_i, \theta)| \leq 2G(X_i)$ by condition (3) and (4). Then

$$E\Delta_\delta(X_i, \theta_0) \to 0, \text{ as } \delta \to 0.$$

So, for all $\theta \in \Theta$ and $\varepsilon > 0$, there exists $\delta_\varepsilon(\theta)$ such that

$$E\left[\Delta_{\delta_\varepsilon(\theta)}(X_i, \theta)\right] < \varepsilon.$$

Since Θ is compact, we can find a finite subcover, such that Θ is covered by

$$\cup_{k=1}^{K} B\left(\theta_k, \delta_\varepsilon(\theta_k)\right).$$

$$
\sup_{\theta \in \Theta} \left[n^{-1} \sum_{i=1}^{n} g(X_i, \theta) - E g(X_i, \theta) \right]
$$

$$
= \max_{k} \sup_{\theta \in B(\theta_k, \delta_\varepsilon(\theta_k))} \left[n^{-1} \sum_{i=1}^{n} g(X_i, \theta) - E g(X_i, \theta) \right] .
$$

$$
\leq \max_{k} \left[n^{-1} \sum_{i=1}^{n} \sup_{\theta \in B(\theta_k, \delta_\varepsilon(\theta_k))} g(X_i, \theta) - E \inf_{\theta \in B(\theta_k, \delta_\varepsilon(\theta_k))} g(X_i, \theta) \right]
$$

Since

$$
E \left| \sup_{\theta \in B(\theta_k, \delta_c(\theta_k))} g(X_i, \theta) \right| \leq E G(X_i) < \infty,
$$

by the Weak Law of Large Numbers (Theorem 8.5),

$$
= o_p(1) + \max_{k} \left[E \sup_{\theta \in B(\theta_k, \delta_\varepsilon(\theta_k))} g(X_i, \theta) - E \inf_{\theta \in B(\theta_k, \delta_\varepsilon(\theta_k))} g(X_i, \theta) \right]
$$

$$
= o_p(1) + \max_{k} E \Delta_{\delta_\varepsilon(\theta_k)}(X_i, \theta_k)
$$

$$
\leq o_p(1) + \varepsilon
$$

By analogous argument,

$$
\inf_{\theta \in \Theta} \left[n^{-1} \sum_{i=1}^{n} g(X_i, \theta) - E g(X_i, \theta) \right] \geq o_p(1) - \varepsilon.
$$

The desired result follows from the above equation by the fact that ε is chosen arbitrarily.

Chapter 9 Central Limit Theorems

9.1 The De Moivre-Laplace Theorem

> **Lemma 9.1. Stirling's Formula**
>
> $$n! \sim \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n} \text{ as } n \to \infty. \tag{9.1}$$

Proof

> **Lemma 9.2**
>
> If $c_j \to 0$, $a_j \to \infty$ and $a_j c_j \to \lambda$, then
>
> $$\left(1 + c_j\right)^{a_j} \to e^{\lambda}. \tag{9.2}$$

Proof

> **Theorem 9.1. The De Moivre-Laplace Theorem**
>
> Let $X_1, X_2, \ldots$ be i.i.d. with $P\left(X_1 = 1\right) = P\left(X_1 = -1\right) = 1/2$ and let $S_n = X_1 + \cdots + X_n$. If $a < b$, then as $m \to \infty$
>
> $$P\left(a \le S_m/\sqrt{m} \le b\right) \to \int_a^b (2\pi)^{-1/2}e^{-x^2/2}\mathrm{d}x. \tag{9.3}$$

Proof If n and k and integers

$$P\left(S_{2n} = 2k\right) = \binom{2n}{n+k} 2^{-2n}$$

By lemma 9.1, we have

$$\binom{2n}{n+k} = \frac{(2n)!}{(n+k)!(n-k)!}$$

$$\sim \frac{(2n)^{2n}}{(n+k)^{n+k}(n-k)^{n-k}} \cdot \frac{(2\pi(2n))^{1/2}}{(2\pi(n+k))^{1/2}(2\pi(n-k))^{1/2}}$$

Hence,

$$P(S_{2n} = 2k) = \binom{2n}{n+k} 2^{-2n}$$

$$\sim \left(1 + \frac{k}{n}\right)^{-n-k} \cdot \left(1 - \frac{k}{n}\right)^{-n+k}$$

$$\cdot (\pi n)^{-1/2} \cdot \left(1 + \frac{k}{n}\right)^{-1/2} \cdot \left(1 - \frac{k}{n}\right)^{-1/2}$$

$$= \left(1 - \frac{k^2}{n^2}\right)^{-n} \cdot \left(1 + \frac{k}{n}\right)^{-k} \cdot \left(1 - \frac{k}{n}\right)^{k}$$

$$\cdot (\pi n)^{-1/2} \cdot \left(1 + \frac{k}{n}\right)^{-1/2} \cdot \left(1 - \frac{k}{n}\right)^{-1/2}$$

Let $2k = x\sqrt{2n}$, i.e., $k = x\sqrt{\frac{n}{2}}$. By lemma 9.2, we have

$$\left(1 - \frac{k^2}{n^2}\right)^{-n} = \left(1 - x^2/2n\right)^{-n} \to e^{x^2/2}$$

$$\left(1 + \frac{k}{n}\right)^{-k} = (1 + x/\sqrt{2n})^{-x\sqrt{n/2}} \to e^{-x^2/2}$$

$$\left(1 - \frac{k}{n}\right)^{k} = (1 - x/\sqrt{2n})^{x\sqrt{n/2}} \to e^{-x^2/2}$$

For this choice of k, $k/n \to 0$, so

$$\left(1 + \frac{k}{n}\right)^{-1/2} \cdot \left(1 - \frac{k}{n}\right)^{-1/2} \to 1.$$

Putting things together, we have

$$P(S_{2n} = 2k) \sim (\pi n)^{-1/2} e^{-x^2/2}, \text{ as } \frac{2k}{\sqrt{2n}} \to x.$$

Therefore,

$$P\left(a\sqrt{2n} \le S_{2n} \le b\sqrt{2n}\right) = \sum_{m \in [a\sqrt{2n}, b\sqrt{2n}] \cap 2\mathbb{Z}} P(S_{2n} = m)$$

Let $m = x\sqrt{2n}$, we have that this is

$$\approx \sum_{x \in [a,b] \cap (2\mathbb{Z}/\sqrt{2n})} (2\pi)^{-1/2} e^{-x^2/2} \cdot (2/n)^{1/2}$$

where $2\mathbb{Z}/\sqrt{2n} = \left\{ 2z/\sqrt{2n} : z \in \mathbb{Z} \right\}$. As $n \to \infty$, the sum just shown is

$$\approx \int_a^b (2\pi)^{-1/2} e^{-x^2/2} \mathrm{d}x.$$

To remove the restriction to even integers, observe $S_{2n+1} = S_{2n} \pm 1$.

Let $m = 2n$, as $m \to \infty$,

$$P\left(a \le S_m/\sqrt{m} \le b\right) \to \int_a^b (2\pi)^{-1/2} e^{-x^2/2} \mathrm{d}x.$$

9.2 Central Limit Theorems

> **Theorem 9.2. Central Limit Theorem for i.i.d. Sequences**
>
> Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = \mu$, $Var(X_i) = \sigma^2 \in (0, \infty)$. Let $S_n = X_1 + X_2 + \ldots + X_n$, then
>
> $$\frac{S_n - n\mu}{\sigma n^{\frac{1}{2}}} \xrightarrow{d} \chi, \tag{9.4}$$
>
> where χ has the standard normal distribution.

Proof

> **Theorem 9.3. The Linderberg-Feller Central Limit Theorem**
>
> For each n, let $X_{n,m}, 1 \le m \le n$, be independent random variables with $EX_{n,m} = 0$. If
>
> 1. $\sum_{m=1}^n EX_{n,m}^2 \to \sigma^2 > 0$.
> 2. $\forall \epsilon > 0, \lim_{n\to\infty} \sum_{m=1}^n E\left(|X_{n,m}|^2 ; |X_{n,m}| > \epsilon\right) = 0$
>
> Then $S_n = X_{n,1} + \cdots + X_{n,n} \xrightarrow{d} \sigma\chi$ as $n \to \infty$.

Proof

Chapter 10 Exercises for Probability Theory and Examples

10.1 Measure Theory

✐ **Exercise 10.1**
1. Show that if $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ are σ-algebras, then $\cup_i \mathcal{F}_i$ is an algebra.
2. Give an example to show that $\cup_i \mathcal{F}_i$ need not be a σ-algebra.

Solution

1. ***Complement**: Suppose $A \in \cup_i \mathcal{F}_i$, since $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$, assume $A \in \mathcal{F}_i$. And each $\mathcal{F}_i$ is σ-algebra,*

$$A^c \in \mathcal{F}_i \subset \cup_i \mathcal{F}_i.$$

   ***Finite Union**: Suppose $A_1, A_2 \in \cup_i \mathcal{F}_i$, since $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$, assume $A_1 \in \mathcal{F}_i, A_2 \in \mathcal{F}_j$, such that,*

$$A_1, A_2 \in \mathcal{F}_{\max(i,j)}.$$

   Since each $\mathcal{F}_i$ is σ-algebra,

$$A_1 \cup A_2 \in \mathcal{F}_i \subset \cup_i \mathcal{F}_i.$$

2. *Let $\mathcal{F}_i$ be a Borel Set of $[1, 2 - \frac{1}{i}]$. Suppose $A_i = [1, 2 - \frac{1}{i}] \in \mathcal{F}_i$,*

$$\cup_i A_i = [1, 2) \notin \cup_i \mathcal{F}_i.$$

10.2 Laws of Large Numbers

10.3 Central Limit Theorems

✐ **Exercise 10.2** Let $g \geq 0$ be continuous. If $X_n \overset{d}{\to} X_\infty$, then

$$\liminf_{n \to \infty} E g(X_n) \geq E g(X_\infty).$$

Solution *Let $Y_n \overset{d}{=} X_n, 1 \leq n \leq \infty$ with $Y_n \overset{a.s.}{\to} Y_\infty$ (Lemma 7.1). Since $g \geq 0$ be continuous,*

$g(Y_n) \overset{a.s.}{\to} g(Y_\infty)$ and $g(Y_n) \geq 0$ *(Theorem 7.2), and the Fatou's Lemma (6.6) implies,*

$$\liminf_{n\to\infty} Eg(X_n) = \liminf_{n\to\infty} Eg(Y_n) \geq E\left(\liminf_{n\to\infty} g(Y_n)\right)$$

$$= Eg(Y_\infty) = Eg(X_\infty).$$

✎ **Exercise 10.3** Suppose g, h are continuous with $g(x) > 0$, and $|h(x)|/g(x) \to 0$ as $|x| \to \infty$. If $F_n \overset{d}{\to} F$ and $\int g(x)\mathrm{d}F_n(x) \leq C < \infty$, then

$$\int h(x)\mathrm{d}F_n(x) \to \int h(x)\mathrm{d}F(x).$$

Solution

$$\left|\int h(x)\mathrm{d}F_n(x) - \int h(x)\mathrm{d}F(x)\right| = \left|\int_{x\in[-M,M]} h(x)\mathrm{d}F_n(x) + \int_{x\notin[-M,M]} h(x)\mathrm{d}F_n(x)\right.$$

$$\left. - \int_{x\in[-M,M]} h(x)\mathrm{d}F(x) - \int_{x\notin[-M,M]} h(x)\mathrm{d}F(x)\right|$$

$$\leq \left|\int_{x\in[-M,M]} h(x)\mathrm{d}F_n(x) - \int_{x\in[-M,M]} h(x)\mathrm{d}F(x)\right|$$

$$+ \left|\int_{x\notin[-M,M]} h(x)\mathrm{d}F_n(x) - \int_{x\notin[-M,M]} h(x)\mathrm{d}F(x)\right|$$

Let $X_n, 1 \leq n < \infty$, with distribution F_n, so that $X_n \overset{a.s.}{\to} X$ (Lemma 7.1).

$$\left|\int_{x\in[-M,M]} h(x)\mathrm{d}F_n(x) - \int_{x\in[-M,M]} h(x)\mathrm{d}F(x)\right| = \left|E\left(h(X_n) - h(X)\right) I_{x\in[-M,M]}\right|.$$

By Continuity Mapping Theorem (7.2), $\lim_{n\to\infty} \left|E\left(h(X_n) - h(X)\right) I_{x\in[-M,M]}\right| = 0.$

Since

$$h(x)I_{x\notin[-M,M]} \leq g(x) \sup_{x\notin[-M,M]} \frac{h(x)}{g(x)},$$

and by Exercise 10.2

$$Eg(X) \leq \liminf_{n\to\infty} Eg(X_n) = \liminf_{n\to\infty} \int g(x)\mathrm{d}F_n(x) \leq C < \infty,$$

$$\left|\int_{x\notin[-M,M]} h(x)\mathrm{d}F_n(x) - \int_{x\notin[-M,M]} h(x)\mathrm{d}F(x)\right| = \left|E\left(h(X_n) - h(X)\right) I_{x\notin[-M,M]}\right|$$

$$\leq 2E\max(h(Xn), h(X))I_{x\notin[-M,M]} \leq 2C \sup_{x\notin[-M,M]} \frac{h(x)}{g(x)}.$$

Hence, let $M \to \infty$,

$$\lim_{n\to\infty} \left|\int h(x)\mathrm{d}F_n(x) - \int h(x)\mathrm{d}F(x)\right| \leq 2C \sup_{x\notin[-M,M]} \frac{h(x)}{g(x)} \to 0,$$

which means,

$$\int h(x)\mathrm{d}F_n(x) \to \int h(x)\mathrm{d}F(x).$$

✎ **Exercise 10.4** Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0$ and $EX_i^2 = \sigma^2 \in (0, \infty)$. Then

$$\sum_{m=1}^{n} X_m \Big/ \left(\sum_{m=1}^{n} X_m^2 \right)^{1/2} \xrightarrow{d} \chi.$$

✎ **Exercise 10.5** Show that if $|X_i| \leq M$ and $\sum_n \mathrm{Var}(X_n) = \infty$, then

$$(S_n - E S_n) \big/ \sqrt{\mathrm{Var}(S_n)} \xrightarrow{d} \chi.$$

✎ **Exercise 10.6** Suppose $EX_i = 0$, $EX_i^2 = 1$ and $E|X_i|^{2+\delta} \leq C$ for some $0 < \delta, C < \infty$. Show that

$$S_n \big/ \sqrt{n} \xrightarrow{d} \chi.$$

Part V

Stochastic Process

Chapter 11 Exercises for Probability Theory and Examples

Part VI

Statistics Inference

Chapter 12 Introduction

12.1 Populations and Samples

12.2 Statistics

12.2.1 Sufficient Statistics

> **Definition 12.1. Sufficient Statistics**
>
> *A statistic T is said to be sufficient for X, or for the family $\mathcal{P} = \{P_\theta, \theta \in \Omega\}$ of possible distributions of X, or for θ, if the conditional distribution of X given $T = t$ is independent of θ for all t.* ♣

> **Theorem 12.1. Fisher–Neyman Factorization Theorem**
>
> *If the probability density function is $p_\theta(x)$, then T is sufficient for θ if and only if nonnegative functions g and h can be found such that*
>
> $$p_\theta(x) = h(x)g_\theta[T(x)].$$
> ♡

Proof

12.2.2 Complete Statistics

> **Definition 12.2. Complete Statistics**
>
> *A statistic T is said to be complete, if $Eg(T) = 0$ for all θ and some function g implies that $P(g(T) = 0 \mid \theta) = 1$ for all θ.* ♣

12.3 Estimators

12.3.1 Definition of Estimators

> **Definition 12.3. Estimator**
>
> *An estimator is a real-valued function defined over the sample space, that is*
>
> $$\delta : X \to \mathbb{R}. \tag{12.1}$$
>
> *It is used to estimate an estimand, θ, a real-valued function of the parameter.* ♣

12.3.2 Properties of Estimators

Unbiasedness

> **Definition 12.4. Unbiasedness**
>
> *An estimator $\hat{\theta}$ of θ is unbiased if*
>
> $$E\hat{\theta} = \theta, \quad \forall \theta \in \Theta. \tag{12.2}$$ ♣

⚗ **Note**

- *Unbiased estimators of θ may not exist.*

-

Example 12.1 Nonexistence of Unbiased Estimator

Consistency

> **Definition 12.5. Consistency**
>
> *An estimator $\hat{\theta}_n$ of θ is consistent if*
>
> $$\lim_{n \to \infty} P\left(\left|\hat{\theta}_n - \theta\right| > \varepsilon\right) = 0, \quad \forall \varepsilon > 0. \tag{12.3}$$ ♣

Example 12.2 Unbiased But Consistent

Example 12.3 Biased But Not Consistent

Asymptotic Normality

> **Definition 12.6. Asymptotic Normality**
>
> An estimator $\hat{\theta}_n$ of θ is asymptotic normality if
> $$\sqrt{n}\left(\hat{\theta} - \theta\right) \overset{d}{\to} N\left(0, \sigma_\theta^2\right). \tag{12.4}$$
> ♣

Efficiency

> **Definition 12.7. Efficiency**
>
> ♣

Robustness

> **Definition 12.8. Robustness**
>
> ♣

Chapter 13 Maximum Likelihood Estimator

Suppose that $\mathbf{X}_n = (X_1, \ldots, X_n)$, where the X_i are i.i.d. with common density $p(x; \theta_0) \in \mathcal{P} = \{p(x; \theta) : \theta \in \Theta\}$.

We assume that

θ_0 is identified in the sense that if $\theta \neq \theta_0$ and $\theta \in \Theta$, then $p(x; \theta) \neq p(x; \theta_0)$ with respect to the dominating measure μ.

For fixed $\theta \in \Theta$, the joint density of $\mathbf{X}_n$ is equal to the product of the individual densities, i.e.,

$$p(\mathbf{X}_n; \theta) = \prod_{i=1}^{n} p(x_i; \theta). \tag{13.1}$$

The maximum likelihood estimate for observed $\mathbf{X}_n$ is the value $\theta \in \Theta$ which maximizes $L(\theta; X_n) := p(\mathbf{X}_n; \theta)$, i.e.,

$$\hat{\theta}(\mathbf{X}_n) = \max_{\theta \in \Theta} L(\theta; X_n). \tag{13.2}$$

Equivalently, the MLE can be taken to be the maximum of the standardized log-likelihood,

$$\frac{l(\theta; \mathbf{X}_n)}{n} = \frac{\log L(\theta; \mathbf{X}_n)}{n} = \frac{1}{n} \sum_{i=1}^{n} \log p(X_i; \theta) = \frac{1}{n} \sum_{i=1}^{n} l(\theta; X_i). \tag{13.3}$$

Define

$$Q(\theta; \mathbf{X}_n) := \frac{1}{n} \sum_{i=1}^{n} l(\theta; X_i),$$
$$\hat{\theta}(\mathbf{X}_n) := \max_{\theta \in \Theta} Q(\theta; \mathbf{X}_n). \tag{13.4}$$

13.1 Consistency of MLE

By the Weak Law of Large Numbers (Theorem 8.5), we can get,

$$\frac{1}{n} \sum_{i=1}^{n} l(\theta; X_i) \overset{p}{\to} E[l(\theta; X)]. \tag{13.5}$$

Suppose $Q_0(\theta) = E\left[l(\theta; X)\right]$, then we will show that $Q_0(\theta)$ is maximized at θ_0 (i.e., the truth).

> **Lemma 13.1**
>
> *If θ_0 is identified and $E_{\theta_0}\left[|\log p(X;\theta)|\right] < \infty, \forall \theta \in \Theta$, then $Q_0(\theta)$ is uniquely maximized at $\theta = \theta_0$.*

Proof

> **Theorem 13.1. Consistency of MLE**
>
> *Suppose that $Q(\theta; X_n)$ is continuous in θ and there exists a function $Q_0(\theta)$ such that*
>
> 1. *$Q_0(\theta)$ is uniquely maximized at θ_0.*
> 2. *Θ is compact.*
> 3. *$Q_0(\theta)$ is continuous in θ.*
> 4. *$Q(\theta; X_n)$ converges uniformly in probability to $Q_0(\theta)$.*
>
> *then*
> $$\hat{\theta}(X_n) \xrightarrow{p} \theta_0. \tag{13.6}$$

Proof $\forall \varepsilon > 0$, let

$$\Theta(\epsilon) = \{\theta : \|\theta - \theta_0\| < \epsilon\}.$$

Since $\Theta(\epsilon)$ is an open set, then $\Theta \cap \Theta(\epsilon)^C$ is a compact set (Assumption 2).

Since $Q_0(\theta)$ is a continuous function (Assumption 3), then

$$\theta^* := \sup_{\theta \in \Theta \cap \Theta(\epsilon)^C} \{Q_0(\theta)\}$$

is a achieved for a θ in the compact set.

Since θ_0 is the unique maximized, let

$$Q_0(\theta_0) - Q_0(\theta^*) = \delta > 0.$$

1. For $\theta \in \Theta \cap \Theta(\epsilon)^C$. Let $A_n = \left\{\sup_{\theta \in \Theta \cap \Theta(\epsilon)^C} |Q(\theta; X_n) - Q_0(\theta)| < \frac{\delta}{2}\right\}$, then

$$\begin{aligned}
A_n \Rightarrow Q(\theta; X_n) &< Q_0(\theta) + \frac{\delta}{2} \\
&\leq Q_0(\theta^*) + \frac{\delta}{2} \\
&= Q_0(\theta_0) - \frac{\delta}{2}
\end{aligned}$$

2. For $\theta \in \Theta(\epsilon)$. Let $B_n = \left\{ \sup_{\theta \in \Theta(\epsilon)} |Q(\theta; \boldsymbol{X}_n) - Q_0(\theta)| < \frac{\delta}{2} \right\}$, then

$$B_n \Rightarrow Q(\theta; \boldsymbol{X}_n) > Q_0(\theta) - \frac{\delta}{2}, \forall \theta \in \Theta(\epsilon)$$

By Assumption 1,

$$Q(\theta_0; \boldsymbol{X}_n) > Q_0(\theta_0) - \frac{\delta}{2}$$

If both A_n and B_n hold, then

$$\hat{\theta} \in \Theta(\epsilon).$$

By Assumption 4, we can concluded that $P(A_n \cap B_n) \to 1$, so

$$P(\hat{\theta} \in \Theta(\epsilon)) \to 1,$$

which means,

$$\hat{\theta}(\mathbf{X}_n) \xrightarrow{p} \theta_0.$$

13.2 Asymptotic Normality of MLE

13.3 Efficiency of MLE

Chapter 14 Minimum-Variance Unbiased Estimator

> **Definition 14.1. UMVU Estimators**
>
> *An unbiased estimator $\delta(X)$ of $g(\theta)$ is the uniform minimum variance unbiased (UMVU) estimator of $g(\theta)$ if*
>
> $$Var_\theta \delta(X) \le Var_\theta \delta'(X), \quad \forall \theta \in \Theta, \tag{14.1}$$
>
> *where $\delta'(X)$ is any other unbiased estimator of $g(\theta)$.* ♣

Note *If there exists an unbiased estimator of g, the estimand g will be called U-estimable.*

1. If $T(\mathbf{X})$ is a complete sufficient statistic, estimator $\delta(\mathbf{X})$ that only depends on $T(\mathbf{X})$, then for any U-estimable function $g(\theta)$ with

   $$E_\theta \delta(T(\mathbf{X})) = g(\theta), \quad \forall \theta \in \Theta, \tag{14.2}$$

   hence, $\delta(T(\mathbf{X}))$ is the unique UMVU estimator of $g(\theta)$.

2. If $T(\mathbf{X})$ is a complete sufficient statistic and $\delta(\mathbf{X})$ is any unbiased estimator of $g(\theta)$, then the UMVU estimator of $g(\theta)$ can be obtained by

   $$E\left[\delta(\mathbf{X}) \mid T(\mathbf{X})\right]. \tag{14.3}$$

Example 14.1 Estimating Polynomials of a Normal Variance Let $X_1, \ldots, X_n$ be distributed with joint density

$$\frac{1}{(\sqrt{2\pi}\sigma)^n} \exp\left[-\frac{1}{2\sigma^2} \sum (x_i - \xi)^2\right]. \tag{14.4}$$

Discussing the UMVU estimators of $\xi^r, \sigma^r, \xi/\sigma$.

Solution

1. *σ is known:*

   Since $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ is the complete sufficient statistic of X_i, and

   $$E(\bar{X}) = \xi,$$

   then the UMVU estimator of ξ is $\bar{X}$.

   Therefore, the UMVU estimator of ξ^r is $\bar{X}^r$ and the UMVU estimator of ξ/σ is $\bar{X}/\sigma$.

2. *ξ is known:*

Since $s^r = \sum (x_i - \xi)^r$ is the complete sufficient statistic of X_i.

Assume

$$E\left[\frac{s^r}{\sigma^r}\right] = \frac{1}{K_{n,r}},$$

where $K_{n,r}$ is a constant depends on n, r.

Since $s^2/\sigma^2 \sim Ga(n/2, 1/2) = \chi^2(n)$, then

$$E\left[\frac{s^r}{\sigma^r}\right] = E\left[\left(\frac{s^2}{\sigma^2}\right)^{\frac{r}{2}}\right] = \int_0^\infty x^{\frac{r}{2}} \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} dx = \frac{\Gamma\left(\frac{n+r}{2}\right)}{\Gamma(\frac{n}{2})} \cdot 2^{\frac{r}{2}}.$$

therefore,

$$K_{n,r} = \frac{\Gamma(\frac{n}{2})}{2^{\frac{r}{2}} \cdot \Gamma\left(\frac{n+r}{2}\right)}.$$

Hence,

$$E\left[s^r K_{n,r}\right] = \sigma^r \text{ and } E[\xi s^{-1} K_{n,-1}] = \xi/\sigma,$$

which means the UMVU estimator of σ^r is $s^r K_{n,r}$ and the UMVU estimator of ξ/σ is $\xi s^{-1} K_{n,-1}$.

3. **Both ξ and σ is unknown:**

   Since $(\bar{X}, s_x^r)$ are the complete sufficient statistic of X_i, where $s_x^2 = \sum (x_i - \bar{X})^r$.

   Since $s_x^2/\sigma^2 \sim \chi^2(n-1)$, then

   $$E\left[\frac{s_x^r}{\sigma^r}\right] = \frac{1}{K_{n-1,r}}.$$

   Hence,

   $$E\left[s_x^r K_{n-1,r}\right] = \sigma^r,$$

   which means the UMVU estimator of σ^r is $s_x^r K_{n-1,r}$, and

   $$E(\bar{X}^r) = \xi^r,$$

   which means the UMVU estimator of ξ^r is $\bar{X}^r$.

   Since $\bar{X}$ and s_x^r are independent, then

   $$E[\bar{X} s_x^{-1} K_{n-1,-1}] = \xi/\sigma$$

   which means the UMVU estimator of ξ/σ is $\bar{X} s_x^{-1} K_{n-1,-1}$.

Example 14.2 Let $X_1, \ldots, X_n$ be i.i.d sample from $U(\theta_1 - \theta_2, \theta_1 + \theta_2)$, where $\theta_1 \in \mathbb{R}, \theta_2 \in \mathbb{R}^+$. Discussing the UMVU estimators of θ_1, θ_2.

Solution *Let $X_{(i)}$ be the i-th order statistic of X_i, then $(X_{(1)}, X_{(n)})$ is the complete and sufficient statistic for (θ_1, θ_2). Thus it suffices to find a function $(X_{(1)}, X_{(n)})$, which is unbiased of (θ_1, θ_2).*

Let

$$Y_i = \frac{X_i - (\theta_1 - \theta_2)}{2\theta_2} \sim U(0, 1),$$

and

$$Y_{(i)} = \frac{X_{(i)} - (\theta_1 - \theta_2)}{2\theta_2},$$

be the i-th order statistic of Y_i, then we got

$$
\begin{aligned}
E[X_{(1)}] &= 2\theta_2 E[Y_{(1)}] + (\theta_1 - \theta_2) \\
&= 2\theta_2 \int_0^1 ny(1-y)^{n-1}\mathrm{d}y + (\theta_1 - \theta_2) \\
&= \theta_1 - \frac{3n+1}{n+1}\theta_2 \\
E[X_{(n)}] &= 2\theta_2 E[Y_{(n)}] + (\theta_1 - \theta_2) \\
&= 2\theta_2 \int_0^1 ny^n\mathrm{d}y + (\theta_1 - \theta_2) \\
&= \theta_1 + \frac{n-1}{n+1}\theta_2
\end{aligned}
$$

.

Thus,

$$\theta_1 = E\left[\frac{n-1}{4n}X_{(1)} + \frac{3n+1}{4n}X_{(n)}\right],$$

$$\theta_2 = E\left[-\frac{n+1}{4n}X_{(1)} + \frac{n+1}{4n}X_{(n)}\right],$$

which means the UMVU estimator is

$$\hat{\theta}_1 = \frac{n-1}{4n}X_{(1)} + \frac{3n+1}{4n}X_{(n)}, \quad \hat{\theta}_2 = -\frac{n+1}{4n}X_{(1)} + \frac{n+1}{4n}X_{(n)}.$$

Chapter 15 Bayes Estimator

We shall look for some estimators that make the risk function $R(\theta, \delta)$ small in some overall sense. There are two way to solve it: minimize the average risk, minimize the maximum risk.

This chapter will discuss the first method, also known as, Bayes Estimator.

> **Definition 15.1. Bayes Estimator**
>
> *The Bayes Estimator δ with respect to Λ is minimizing the Bayes Risk of δ*
>
> $$r(\Lambda, \delta) = \int R(\theta, \delta)\, d\Lambda(\theta) \tag{15.1}$$
>
> *where Λ is the probability distribution.* ♣

In Bayesian arguments, it is important to keep track of which variables are being conditioned on. Hence, the notations are as followed:

- The density of X will be denoted by $X \sim f(x \mid \theta)$.
- The prior distribution will be denoted by $\Pi \sim \pi(\theta \mid \lambda)$ or $\Lambda \sim \gamma(\lambda)$, where λ is another parameter (sometimes called a hyperparameter).
- The posterior distribution, which calculate the conditional distributions as that of θ given x and λ, or λ given x, which is denoted by $\Pi \sim \pi(\theta \mid x, \lambda)$ or $\Lambda \sim \gamma(\lambda \mid x)$, that is
  $$\pi(\theta \mid x, \lambda) = \frac{f(x \mid \theta)\, \pi(\theta \mid \lambda)}{m(x \mid \lambda)}, \tag{15.2}$$
  where marginal distributions $m(x \mid \lambda) = \int f(x \mid \theta)\, \pi(\theta \mid \lambda)\, d\theta$.

> **Theorem 15.1**
>
> *Let Θ have distribution Λ, and given $\Theta = \theta$, let X have distribution P_θ. Suppose, the following assumptions hold for the problem of estimating $g(\Theta)$ with non-negative loss function $L(\theta, d)$,*
>
> - *There exists an estimator δ_0 with finite risk.*
> - *For almost all x, there exists a value $\delta_\Lambda(x)$ minimizing*
>
> $$E\{L[\Theta, \delta(x)] \mid X = x\}. \tag{15.3}$$
>
> *Then, $\delta_\Lambda(x)$ is a Bayes Estimator.* ♡

⚗ **Note** *Improper prior*

Corollary 15.1

Suppose the assumptions of Theorem 15.1 hold.

1. *If $L(\theta, d) = [d - g(\theta)]^2$, then*

$$\delta_\Lambda(x) = E[g(\Theta) \mid x]. \tag{15.4}$$

2. *If $L(\theta, d) = w(\theta)[d - g(\theta)]^2$, then*

$$\delta_\Lambda(x) = \frac{E[w(\theta)g(\Theta) \mid x]}{E[w(\theta) \mid x]}. \tag{15.5}$$

3. *If $L(\theta, d) = |d - g(\theta)|$, then $\delta_\Lambda(x)$ is any median of the conditional distribution of Θ given x.*

4. *If*

$$L(\theta, d) = \begin{cases} 0 \text{ when } |d - \theta| \leq c \\ 1 \text{ when } |d - \theta| > c \end{cases},$$

then $\delta_\Lambda(x)$ is the midpoint of the interval I of length $2c$ which maxmizes $P(\Theta \in I \mid x)$.

Proof

Theorem 15.2

Necessiary condition for Bayes Estimator

Methodologies have been developed to deal with the difficulty which sometimes incorporate frequentist measures to assess the choic of Λ.

- Empirical Bayes.
- Hierarchical Bayes.
- Robust Bayes.
- Objective Bayes.

15.1 Single-Prior Bayes

The Single-Prior Bayes model in a general form as

$$X \mid \theta \sim f(x \mid \theta),$$
$$\Theta \mid \gamma \sim \pi(\theta \mid \lambda), \tag{15.6}$$

where we assume that the functional form of the prior and the value of λ is known (we will write it as $\gamma = \gamma_0$).

Given a loss function $L(\theta, d)$, we would then determine the estimator that minimizes

$$\int L(\theta, d(x)) \pi(\theta \mid x) \, d\theta, \qquad (15.7)$$

where $\pi(\theta \mid x)$ is posterior distribution given by

$$\pi(\theta \mid x) = \frac{f(x \mid \theta) \pi(\theta \mid \gamma_0)}{\int f(x \mid \theta) \pi(\theta \mid \gamma_0) \, d\theta}.$$

In general, this Bayes estimator under squared error loss is given by

$$E(\Theta \mid x) = \frac{\int \theta f(x \mid \theta) \pi(\theta \mid \gamma_0) \, d\theta}{\int f(x \mid \theta) \pi(\theta \mid \gamma_0) \, d\theta}. \qquad (15.8)$$

Example 15.1 Consider

$$X_i \overset{\text{i.i.d}}{\sim} N(\mu, \Gamma^{-1}), \quad i = 1, 2, \ldots, n$$

$$\mu \sim N(0, 1),$$

$$\Gamma \sim \text{Gamma}(2, 1),$$

calculate the Single-Prior Bayes estimator under squared error loss.

Solution

$$p(X \mid \mu, \Gamma) = \Gamma^n (2\pi)^{-\frac{n}{2}} \exp\left[-2\Gamma^2 \sum_{i=1}^{n} (x_i - \mu)^2\right],$$

$$p(\mu) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\mu^2}{2}\right),$$

$$p(\Gamma) = \frac{1}{\Gamma(2)} \Gamma \exp(-\Gamma).$$

Therefore,

$$h(X, \mu, \Gamma) = C \Gamma^n \exp\left[-2\Gamma^2 \sum_{i=1}^{n} (x_i - \mu)^2\right] \exp\left(-\frac{\mu^2}{2}\right) \Gamma \exp(-\Gamma),$$

where $C = \frac{(2\pi)^{-\frac{n+1}{2}}}{\Gamma(2)}$.

For μ, we have

$$\pi(\mu \mid X, \Gamma) = \frac{h(X, \mu, \Gamma)}{p(\mu \mid X)}$$

For exponential families

Theorem 15.3

15.2 Hierarchical Bayes

In a Hierarchical Bayes model, rather than specifying the prior distribution as a single function, we specify it in a **hierarchy**. Thus, the Hierarchical Bayes model in a general form as

$$X \mid \theta \sim f(x \mid \theta),$$
$$\Theta \mid \gamma \sim \pi(\theta \mid \lambda), \tag{15.9}$$
$$\Gamma \sim \psi(\gamma),$$

where we assume that $\psi(\cdot)$ is known and not dependent on any other unknown hyperparameters.

Note *We can continue this hierarchical modeling and add more stages to the model, but this is not ofthen done in practice.*

Given a loss function $L(\theta, d)$, we would then determine the estimator that minimizes

$$\int L(\theta, d(x)) \pi(\theta \mid x) \, d\theta, \tag{15.10}$$

where $\pi(\theta \mid x)$ is posterior distribution given by

$$\pi(\theta \mid x) = \frac{\int f(x \mid \theta) \pi(\theta \mid \gamma) \psi(\gamma) \, d\gamma}{\int \int f(x \mid \theta) \pi(\theta \mid \gamma) \psi(\gamma) \, d\theta d\gamma}.$$

Note *The posterior distribution can also be writed as*

$$\pi(\theta \mid x) = \int \pi(\theta \mid x, \gamma) \pi(\gamma \mid x) \, d\gamma,$$

where $\pi(\gamma \mid x)$ is the posterior distribution of Γ, unconditional on θ. The equation 15.10 can be writed as

$$\int L(\theta, d(x)) \pi(\theta \mid x) \, d\theta = \int \left[\int L(\theta, d(x)) \pi(\theta \mid x, \gamma) \, d\theta \right] \pi(\gamma \mid x) \, d\gamma.$$

*which shows that **the Hierarchical Bayes estimator can be thought of as a mixture of Single-Prior estimators**.*

Example 15.2 Poisson Hierarchy Consider

$$X_i \mid \lambda \overset{i.i.d}{\sim} \text{Poisson}(\lambda), \quad i = 1, 2 \dots, n$$
$$\lambda \mid b \sim \text{Gamma}(a, b), \; a \text{ known}, \tag{15.11}$$
$$\frac{1}{b} \sim \text{Gamma}(k, \tau),$$

calculate the Hierarchical Bayes estimator under squared error loss.

> **Theorem 15.4**
>
> *For the Hierarchical Bayes model (15.9),*
>
> $$K\left[\pi\left(\lambda \mid x\right), \psi\left(\lambda\right)\right] < K\left[\pi\left(\theta \mid x\right), \pi\left(\theta\right)\right], \qquad (15.12)$$
>
> *where K is the Kullback-Leibler information for discrimination between two densities.*

Proof

Note

15.3 Empirical Bayes

15.4 Bayes Prediction

Chapter 16 Hypothesis Testing

Part VII

Convex Optimization

Chapter 17 Convex Sets

17.1 Affine and Convex Sets

17.1.1 Affine Sets

> **Definition 17.1. Affine Set**
>
> *A nonempty set C is a **affine set** that satisfy*
>
> $$\forall x_1, x_2 \in C, \theta \in \mathbf{R}, \theta x_1 + (1 - \theta)x_2 \in C.$$

17.1.2 Convex Sets

> **Definition 17.2. Convex Set**
>
> *A nonempty set C is a **convex set** that satisfy*
>
> $$\forall x_1, x_2 \in C, \theta \in [0, 1], \theta x_1 + (1 - \theta)x_2 \in C.$$

> **Definition 17.3. Convex Hull**
>
> *The **convex hull** of a set C, denoted by conv C is a set of all convex combinations of points in C,*
>
> $$conv\ C = \{\theta_1 x_1 + \ldots + \theta_k x_k | x_i \in C; \theta_i \geq 0, i = 1, \ldots, k; \theta_1 + \ldots + \theta_k = 1\}.$$

Note *The convex hull conv C is always convex, which is the minimal convex set that contains C.*

17.1.3 Cones

> **Definition 17.4. Cone**
>
> *A nonempty set C is a **cone** that satisfy*
>
> $$\forall x \in C, \theta \geq 0, \theta x \in C.$$

> **Definition 17.5. Convex Cone**
>
> *A nonempty set C is a **convex cone** that satisfy*
>
> $$\forall x_1, x_2 \in C, \theta_1, \theta_2 \geq 0, \theta_1 x_1 + \theta_2 x_2 \in C.$$
>
> ♣

17.2 Some Important Examples

> **Definition 17.6. Hyperplane**
>
> *A hyperplane is a set of the form*
>
> $$\{x | a^T x = b\},$$
>
> *where $a \in \mathbf{R}^n, a \neq 0, b \in \mathbf{R}$.*
>
> ♣

> **Definition 17.7. Halfspace**
>
> *A hyperplane is a set of the form*
>
> $$\{x | a^T x \leq b\},$$
>
> *where $a \in \mathbf{R}^n, a \neq 0, b \in \mathbf{R}$.*
>
> ♣

> **Definition 17.8. (Euclidean) Ball**
>
> *A (Euclidean) ball in $\mathbf{R}^n$ with center x_c and radius r has the form*
>
> $$B(x_c, r) = \{x | \|x - x_c\|_2 \leq r\} = \{x_c + ru | \|u\|_2 \leq 1\},$$
>
> *where $r > 0$.*
>
> ♣

> **Definition 17.9. Ellipsoid**
>
> *A Ellipsoid in $\mathbf{R}^n$ with center x_c has the form*
>
> $$\mathcal{E} = \{x | (x - x_c)^T P^{-1} (x - x_c) \leq 1\} = \{x_c + Au | \|u_2\| \leq 1\},$$
>
> *where $P \in \mathbf{S}^n_{++}$ (symmetric positive definite).*
>
> ♣

> **Definition 17.10. Norm Ball**
>
> ♣

> **Definition 17.11. Norm Cone**
>
> ♣

> **Definition 17.12. Polyhedra**
>
> ♣

Definition 17.13. Positive Semidefinite Cone

♣

17.3 Operations that Preserve Convexity

17.3.1 Intersection

17.3.2 Affine Functions

17.3.3 Perspective Function

17.3.4 Linear-fractional Function

17.4 Generalized Inequalities

17.5 Separating and Supporting Hyperplanes

17.6 Dual Cones and Generalized Inequalities

Chapter 18 Exercises for Convex Optimization

18.1 Convex Sets

✐ **Exercise 18.1** Solution set of a quadratic inequality Let $C \subseteq \mathbf{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbf{R}^n | x^T A x + b^T x + c \leq 0\}$$

with $A \in \mathbf{S}^n$, $b \in \mathbf{R}^n$, and $c \in \mathbf{R}$.

1. Show that C is convex if $A \succeq 0$.

Solution

1. *We have to show that $\theta x + (1 - \theta) y \in C$ for all $\theta \in [0, 1]$ and $x, y \in C$.*

$$(\theta x + (1 - \theta) y)^T A (\theta x + (1 - \theta) y) + b^T (\theta x + (1 - \theta) y) + c$$
$$= \theta^2 x^T A x + \theta(1 - \theta)(y^T A x + x^T A y) + (1 - \theta)^2 y^T A y + \theta b^T x + (1 - \theta) b^T y + c$$
$$= \theta^2 (x^T A x + b^T x + c) + (1 - \theta)^2 (y^T A y + b^T y + c) - \theta^2 (b^T x + c)$$
$$\quad - (1 - \theta)^2 (b^T y + c) + \theta(1 - \theta)(y^T A x + x^T A y) + \theta b^T x + (1 - \theta) b^T y + c$$
$$\leq - \theta^2 (b^T x + c) - (1 - \theta)^2 (b^T y + c) + \theta(1 - \theta)(y^T A x + x^T A y)$$
$$\quad + \theta b^T x + (1 - \theta) b^T y + c$$
$$= \theta(1 - \theta)[(b^T x + c) + (b^T y + c) + x^T A x + y^T A y]$$
$$\leq \theta(1 - \theta)(-x^T A x - y^T A y + x^T A x + y^T A y) \leq 0$$

Therefore, $\theta x + (1 - \theta) y \in C$, which shows that C is convex if $A \succeq 0$.