MAT-MEK4270 mandatory assignment 1

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Problem 1.2.3

We are given

$$u(t,x,y) = e^{\mathbf{i}(k_x x + k_y y - \omega t)}. \tag{1}$$

Where $\mathbf{i} = \sqrt{-1}$, $k_x = \pi m_x$, $k_y = \pi m_y$) and ω is the dispersion coefficient. And we want to show that equation (1), satisfies the wave equation which is defined as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u. \tag{2}$$

To do this, we first compute

$$\begin{aligned}
\nabla^2 u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
&= -k_x^2 e^{\mathbf{i}(k_x x + k_y y - \omega t)} - k_y^2 e^{\mathbf{i}(k_x x + k_y y - \omega t)} \\
&= -e^{\mathbf{i}(k_x x + k_y y - \omega t)}(k_x^2 + k_y^2).
\end{aligned} \tag{3}$$

Next we compute

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 e^{\mathbf{i}(k_x x + k_y y - \omega t)}. \tag{4}$$

Now in order to show that equation (3) is equal to equation (4), we need to determine ω. To this we insert the Dirichlet problem, from equation (1.4) in the assignment, $u(t,x,y) = \sin(k_x x)\sin(k_y y)\cos(\omega t)$ into equation (2). We then get that

$$-\omega^2 \sin(k_x x)\sin(k_y y)\cos(\omega t) = c^2(-k_x^2 \sin(k_x x)\sin(k_y y)\cos(\omega t) - k_y^2 \sin(k_x x)\sin(k_y y)\cos(\omega t)).$$

Solving for ω in the above, we get that

$$\omega = c \cdot \sqrt{k_x^2 + k_y^2}. \tag{5}$$

Using this, and inserting equations (3, 4) into the wave equation we get that

$$-c^2(k_x^2 + k_y^2)e^{\mathbf{i}(k_x x + k_y y - \omega t)} = -c^2(k_x^2 + k_y^2)e^{\mathbf{i}(k_x x + k_y y - \omega t)}. \tag{6}$$

From this we conclude that equation (1) satisfies the wave equation in equation (2).

Problem 1.2.4

We now assume $m_x = m_y$ and $k_x = k_y = k$, we can then discretize equation (1) as

$$u_{ij}^n = e^{\mathbf{i}(kh(i+j) - \tilde{\omega} n \Delta t)}. \tag{7}$$

Where $\tilde{\omega}$ is a numerical dispersion coefficient, i.e., the numerical approximation of the exact ω. Equation (1.3) in the mandatory assignment is given as

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = c^2 \left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2} \right). \tag{8}$$

We now want to insert equation (7) in equation (1.3), and start by computing

$$\begin{aligned}
\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} &= \frac{1}{\Delta t^2}(e^{\mathbf{i}(kh(i+j) - \tilde{\omega}(n+1)\Delta t)} - 2e^{\mathbf{i}(kh(i+j) - \tilde{\omega} n \Delta t)} + e^{\mathbf{i}(kh(i+j) - \tilde{\omega}(n-1)\Delta t)}) \\
&= \frac{e^{\mathbf{i}kh(i+j)}}{\Delta t^2}(e^{-\mathbf{i}\tilde{\omega}(n+1)\Delta t} - 2e^{-\mathbf{i}\tilde{\omega} n \Delta t} + e^{-\mathbf{i}\tilde{\omega}(n-1)\Delta t}) \\
&= \frac{e^{\mathbf{i}(kh(i+j) - \mathbf{i}\tilde{\omega} n \Delta t)}}{\Delta t^2}(e^{-\mathbf{i}\Delta t \tilde{\omega}} - 2 + e^{\mathbf{i}\Delta t \tilde{\omega}}).
\end{aligned} \tag{9}$$

Next we compute

$$\begin{aligned}\frac{u^n_{i+1,j}-2u^n_{i,j}+u^n_{i-1,j}}{h^2} &= \frac{1}{h^2}(e^{\mathbf{i}(kh(i+j+1)-\tilde{\omega}n\Delta t)}-2e^{\mathbf{i}(kh(i+j)-\tilde{\omega}n\Delta t)}+e^{\mathbf{i}(kh(i+j-1)-\tilde{\omega}n\Delta t)})\\ &= \frac{e^{\mathbf{i}(kh(i+j)-\tilde{\omega}n\Delta t)}}{h^2}(e^{\mathbf{i}kh}-2+e^{-\mathbf{i}kh}). \end{aligned} \tag{10}$$

And similarly

$$\frac{u^n_{i,j+1}-2u^n_{i,j}+u^n_{i,j-1}}{h^2} = \frac{e^{\mathbf{i}(kh(i+j)-\tilde{\omega}n\Delta t)}}{h^2}(e^{\mathbf{i}kh}-2+e^{-\mathbf{i}kh}). \tag{11}$$

Inserting equations (9, 10, 11) into equation (1.3) from the mandatory assignment we get that

$$\frac{e^{\mathbf{i}(kh(i+j)-\mathbf{i}\tilde{\omega}n\Delta t)}}{\Delta t^2}(e^{-\mathbf{i}\Delta t\tilde{\omega}}-2+e^{\mathbf{i}\Delta t\tilde{\omega}}) = 2c^2\frac{e^{\mathbf{i}(kh(i+j)-\tilde{\omega}n\Delta t)}}{h^2}(e^{\mathbf{i}kh}-2+e^{-\mathbf{i}kh}).$$

Which leads to the following

$$\frac{e^{-\mathbf{i}\Delta t\tilde{\omega}}-2+e^{\mathbf{i}\Delta t\tilde{\omega}}}{\Delta t^2} = \frac{2c^2(e^{\mathbf{i}kh}-2+e^{-\mathbf{i}kh})}{h^2}. \tag{12}$$

We are now given that the cfl number $C=\frac{c\Delta t}{h}=\frac{1}{\sqrt{2}}$. If we multiply both side of equation (12) with Δt^2, and use this we have that

$$e^{-\mathbf{i}\Delta t\tilde{\omega}}+e^{\mathbf{i}\Delta t\tilde{\omega}} = e^{\mathbf{i}kh}+e^{-\mathbf{i}kh}.$$

From the above, it is clear that $\Delta t\tilde{\omega}=kh$. If we now make the observation $\omega = c\cdot\sqrt{k^2+k^2}=ck\sqrt{2}$, we can make the following observations

$$\Delta t\tilde{\omega}=kh \to \frac{c\Delta t\tilde{\omega}}{h}=ck \to \frac{\tilde{\omega}}{\sqrt{2}}=ck \to \tilde{\omega}=ck\sqrt{2}=\omega.$$

Thus we conclude that for $C=\frac{1}{\sqrt{2}}$, we have that $\tilde{\omega}=\omega$.