Oblig 4: Burgers and Chaos: Due 24 Nov. 2020

Problem 1: Burgers equation

A landslide in a fjord generates a large wave (tsunami) which propagates down the fjord, threatening houses along the coast. We model this as a Burgers equation:

$$\frac{\partial}{\partial t}h + h\frac{\partial}{\partial x}h = \nu_t \frac{\partial^2}{\partial x^2}h$$

Here ν_t is a *turbulent viscosity*, which acts like a molecular viscosity but has a larger value. An analytical solution to the equations is:

$$h = \frac{a}{2} - \frac{a}{2}tanh(a(\frac{x - at/2}{4\nu_t}))$$

if the initial amplitude is a. Assume the fjord extends from x=-10 to x=30.

a) Plot the analytical solution with $a = 2$ and $\nu_t = 1$ at $t = 1, 10, 20$. Now plot the solution with $\nu_t = 0.1$. How are these different? Plot the solution with $a = 4$ and $\nu_t = 0.1$, and discuss how this differs. How is the larger amplitude wave more of a challenge for evacuating the coast?

b) Write the equation in finite difference and then matrix form, using the FTCS scheme and five interior grid points (with two boundary points). Note that the left boundary condition (at x=-10) is u=2, not u=0. So you need to have an additional term to account for this. You can find this term by writing the equation for j=2.

c) Write a code for this using the FTCS scheme. You can use the energy form of the equation, where the advection is in terms of $E = h^2/2$.

d) Run the code for the two cases above with $a = 2$. Use the analytical solution at t=1 as the initial condition, and calculate the responses at t=10 and 20. Plot the results for $s = 0.4$, with $\nu_t = 1$ and $\nu_t = 0.1$. How do these compare to the analytical solution? Plot the result too with $\nu_t = 0.1$ and $s = 0.6$.

Problem 2: Chaos

An alternate equation which exhibits irregular behavior is the following:

$$\frac{d}{dt}u + ru^2 = ru - u$$

So now the "forcing" is proportional to the velocity, u, itself.

a) Write this as a map, using $dt = 1$.

b) What are the steady state value(s) for this mapping, as function of r?

c) Write a code for the map.

d) For what range of r value do you get a steady state solution (no oscillations)? You can use just one decimal place (e.g. $0 \leq r < 10.1$). Which root is the steady solution? Use $u(0) = 0.2$ and plot an example.

e) For what range of r is there a single oscillation (decaying or not)? Plot an example.

f) Plot an example of a double oscillation (two frequencies).

g) For what range of r do you get irregular (chaotic) oscillations. Plot an example. What are the maximum and minimum values of u?

h) For what range of r is the mapping numerically unstable? Plot an example.