Fluids – Lecture 17 Notes

1. Prandtl-Meyer Waves
2. Shock-Expansion Theory (Supersonic Airfoils)

Reading: Anderson 9.6, 9.7

Prandtl-Meyer Waves

Wave flow relations

An expansion fan, sometimes also called a *Prandtl-Meyer expansion wave*, can be considered as a continuous sequence of infinitesimal Mach expansion waves. To analyze this continuous change, we will now consider the flow angle θ to be a flowfield variable, like M or V.

Across each Mach wave of the fan, the flow direction changes by $d\theta$, while the speed changes by dV. Oblique-shock analysis dictates that only the normal velocity component u can change across any wave, so that dV must be entirely due to the normal-velocity change du.
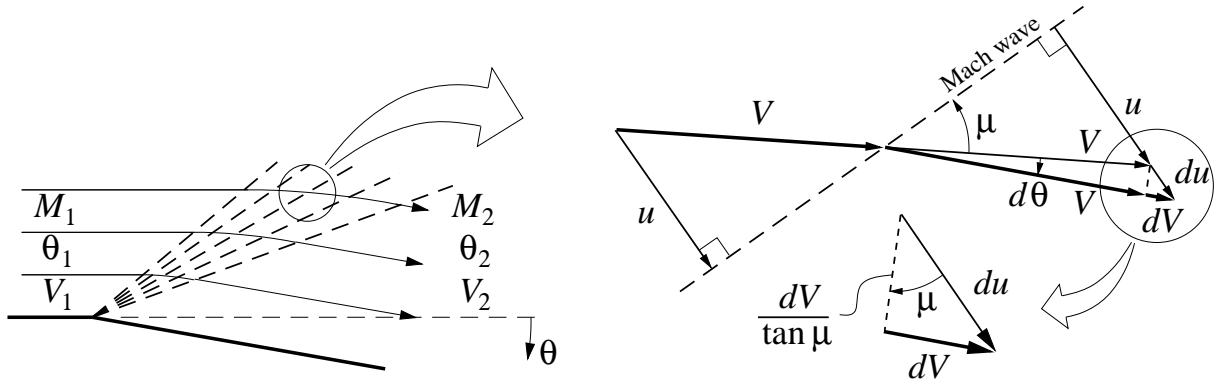
From the u-V and du-dV velocity triangles, it is evident that $d\theta$ and dV are related by

$$d\theta \;=\; \frac{dV}{\tan\mu}\,\frac{1}{V}$$

assuming $d\theta$ is a small angle. With $\;\sin\mu = 1/M\;$, we have

$$\frac{1}{\tan\mu} \;=\; \frac{\cos\mu}{\sin\mu} \;=\; \frac{\sqrt{1-\sin^2\mu}}{\sin\mu} \;=\; \frac{\sqrt{1-1/M^2}}{1/M} \;=\; \sqrt{M^2-1}$$

and so the flow relation above becomes

$$d\theta \;=\; \sqrt{M^2-1}\,\frac{dV}{V} \tag{1}$$

This is a differential equation which relates a change $d\theta$ in the flow angle to a change dV in the flow speed throughout the expansion fan.

Prandtl-Meyer Function

The differential equation (1) can be integrated if we first express V in terms of M.

$$V \;=\; M\,a \;=\; M\,a_o\left(1 + \frac{\gamma-1}{2}M^2\right)^{-1/2}$$

$$\ln V \;=\; \ln M \;+\; \ln a_o \;-\; \frac{1}{2}\,\ln\!\left(1 \;+\; \frac{\gamma-1}{2}M^2\right)$$

$$\frac{dV}{V} \;=\; \frac{dM}{M} \;-\; \frac{1}{2}\left(1 \;+\; \frac{\gamma-1}{2}M^2\right)^{-1}\frac{\gamma-1}{2}\,2M\;dM$$

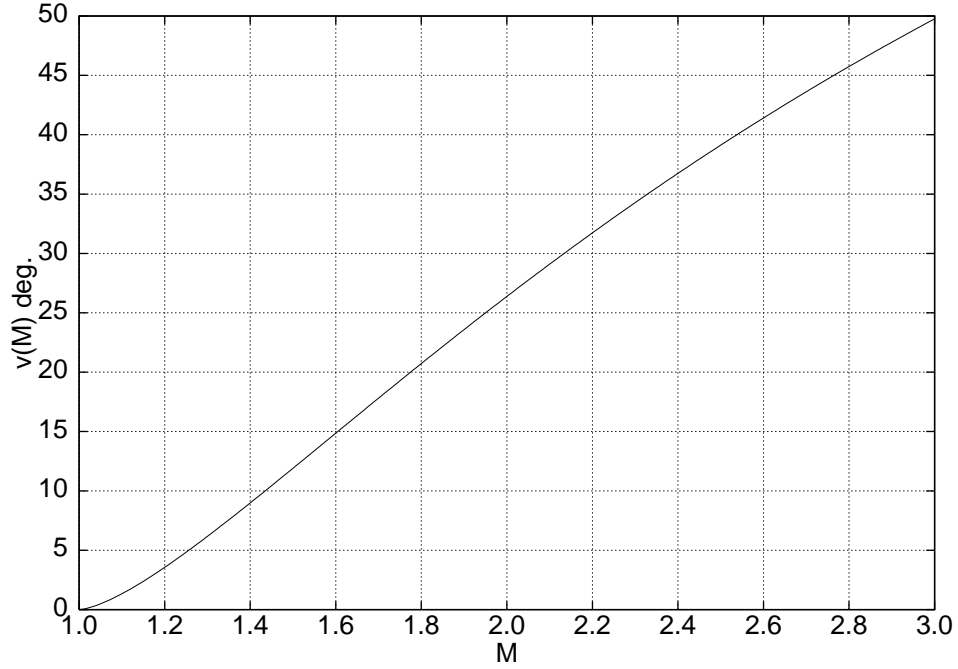$$\frac{dV}{V} \;=\; \frac{1}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

Equation (1) then becomes

$$d\theta \;=\; \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M} \tag{2}$$

which can now be integrated from any point 1 to any point 2 in the Prandtl-Meyer wave.

$$\int_{\theta_1}^{\theta_2} d\theta \;=\; \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

$$\theta_2 - \theta_1 \;=\; \nu(M_2) \;-\; \nu(M_1) \tag{3}$$

where
$$\nu(M) \;\equiv\; \sqrt{\frac{\gamma+1}{\gamma-1}}\;\arctan\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} \;-\; \arctan\sqrt{M^2-1} \tag{4}$$

Here, $\nu(M)$ is called the *Prandtl-Meyer function*, and is shown plotted for $\gamma = 1.4$.

Equation (3) can be applied to any two points within an expansion fan, but the most common use is to relate the two flow conditions before and after the fan. Reverting back to our previous notation where θ is the total turning of the corner, the relation between θ and the upstream and downstream Mach number is

$$\theta \;=\; \nu(M_2) \;-\; \nu(M_1) \tag{5}$$
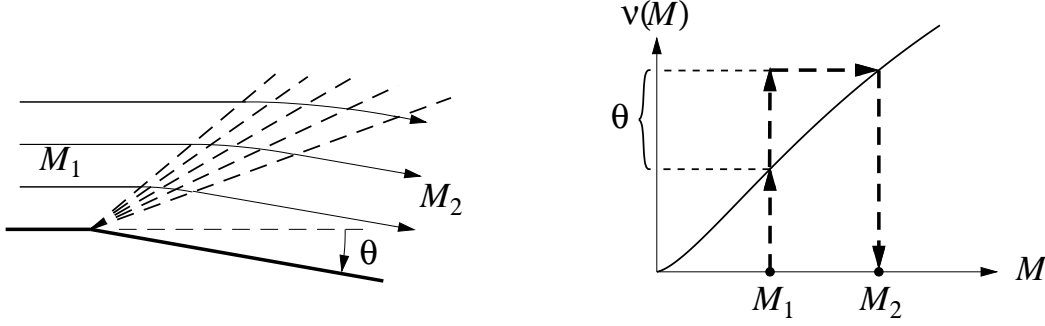
This can be considered an implicit definition of $M_2(M_1, \theta)$, which can be evaluated graphically using the $\nu(M)$ function plot, as shown in the figure.

Shock-Expansion Theory

The combination of oblique-shock relations and Prandtl-Meyer wave relations constitutes *Shock-Expansion Theory*, which can be used to determine the flow properties and forces for simple 2-D shapes in supersonic flow.

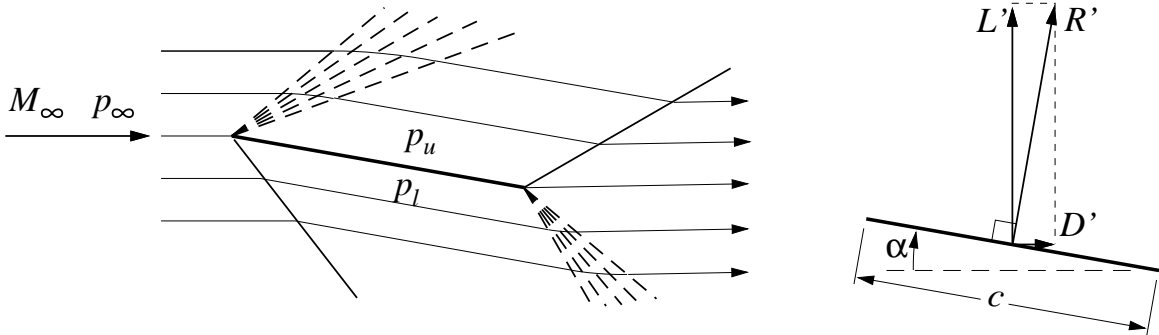### Flat-plate supersonic airfoil

A flat plate is the simplest supersonic airfoil. When set at an angle of attack α, the leading edge point effectively is a convex corner to the upper surface flow, with turning angle $\theta = \alpha$. The upper flow then passes through the resulting Prandtl-Meyer expansion, which increases its Mach number from $M_1 = M_\infty$, to a larger value $M_2 = M_u$. The latter is implicitly determined via the Prandtl-Meyer relation (5).

$$\alpha \;=\; \nu(M_u) - \nu(M_\infty) \qquad \rightarrow \qquad M_u(M_\infty, \alpha)$$

The corresponding upper-surface pressure is then given by the isentropic relation.

$$p_u \;=\; p_\infty \left(\frac{1 + \frac{\gamma-1}{2} M_\infty^2}{1 + \frac{\gamma-1}{2} M_u^2} \right)^{\gamma/(\gamma-1)}$$

Conversely, the leading edge is a concave corner to the bottom surface flow, which then sees a pressure rise through the resulting oblique shock. The lower-surface Mach number M_ℓ and pressure p_ℓ are obtained from oblique-shock relations, with $M_1 = M_\infty$, $\theta = \alpha$ as the inputs.

The pressure difference produces a resultant force/span R' acting normal to the plate, which can be resolved into lift and drag components.

$$
\begin{aligned}
R' &= (p_\ell - p_u)\, c \\
L' &= (p_\ell - p_u)\, c \, \cos\alpha \\
D' &= (p_\ell - p_u)\, c \, \sin\alpha
\end{aligned}
$$

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It's worthwhile to note that this supersonic airfoil has a nonzero drag even though the flow is being assumed inviscid. We shall see that when we consider inviscid airfoils in subsonic flow the drag is zero. This is known as D'Alambert's paradox. Hence, d'Alembert's Paradox definitely does not apply to supersonic 2-D flow. The drag D' computed here is associated with the viscous dissipation occurring in the oblique shock waves, and hence is called *wave drag*. This wave drag is an additional drag component in supersonic flow, and must be added to the usual viscous friction drag, and also the induced drag in 3-D cases.