1-parameter families.

straight-line homotopy.

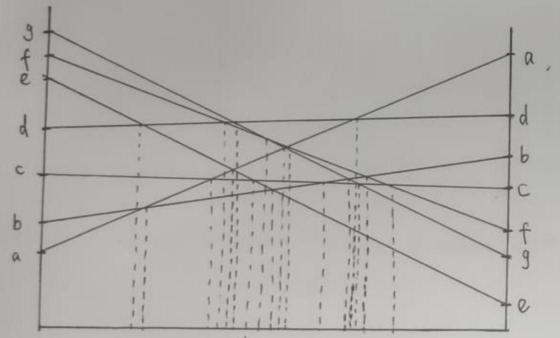
$f, g: K \to \mathbb{R}$ monotonic.

$F: K \times [0,1] \longrightarrow \mathbb{R}$ straight-line homotopy.

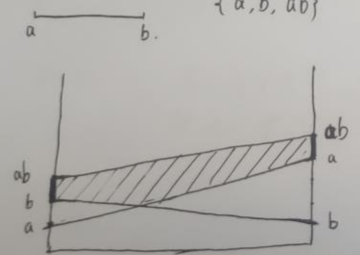
$F(\sigma, t) = (1-t) f(\sigma) + t\, g(\sigma)$

obviously, for $\forall t \in [0,1]$ $F(\ ,t) = f_t$ is monotonic.

①



②



$\{a, b, ab\}$

f: a, b, ab : 1, 2, 3

g: a, b, ab : 4, 1, 5

$f \Rightarrow [f(b), f(ab)]$

$g \Rightarrow [g(a), g(ab)]$

We have the total order of the simplex in K, that is define by f_t. And we know the order is same, when $\forall t \in [t_i, t_{i+1}]$.

$\Rightarrow \mathrm{Dgm}(f_t) = \mathrm{Dgm}(f_{t'})$, $\forall t \in [t_i, t_{i+1}]$, for all i.

So, we only consider the location of $[t_i - \varepsilon, t_i + \varepsilon]$.

Simplify, we assume that there are at most two simplexes that satisfy the condition of $f_t(\sigma) = f_t(\tau)$. $t \in (0,1)$.

$f_t \Rightarrow$ boundary matrix $\partial_t \Rightarrow$ Reduced R_t

Reduced: $\mathrm{low}(j_1) \neq \mathrm{low}(j_0)$, $j_0 \neq j_1$)

$f_{t_i-\varepsilon}(\sigma) < f_{t_i-\varepsilon}(\tau) \Rightarrow f_{t_i+\varepsilon}(\tau) < f_{t_i+\varepsilon}(\sigma)$

$\partial_{t_i-\varepsilon} \Rightarrow P \partial_{t_i-\varepsilon} P = \partial_{t_i+\varepsilon}$

$$P = \begin{bmatrix} I_{i-1} & & \\ & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \\ & & I_{n-i-1} \end{bmatrix}$$

Given, $R = \partial V \Rightarrow \partial = RU$ — R: reduced, upper triangular, invertible. U: upper triangular, invertible.

Goal: decomposition $\partial_{t_i+\varepsilon} = P \partial_{t_i-\varepsilon} P = R_{t_i+\varepsilon} \cdot U_{t_i+\varepsilon}$

$R = \partial V \Rightarrow \partial = RU$.

Note $P^2 = I$. $\Rightarrow P\partial P = PRUP = \underbrace{PRP}_{\text{non-reduced}}\underbrace{PUP}_{\text{non-triangular}}$

~~non-triangular~~

PR may change the ~~non-triangular~~ reduced property of R, so we consider how one fix the deficiency.

Deficiency: 1.

$$R \quad \begin{array}{c|cc} & k & l \\ \hline i & 1 & 1 \\ i+1 & 0 & 1 \end{array}$$

$low(k) = i$, $low(l) = i+1$

2

$$\begin{array}{c|cc} & k & l \\ \hline i & 1 & 1 \\ i+1 & 1 & 0 \end{array}$$

$low(k) = i+1$, $low(l) = \cancel{i+1}\ i$

Fixing: Add the k-column to the l-column. before the transposition.

$$\tilde{R} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

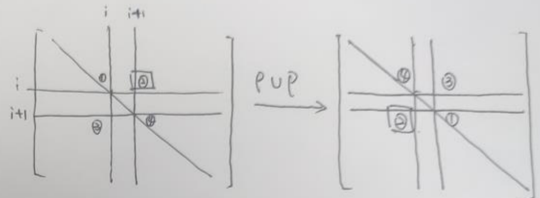
$R = \partial V$ $\quad \tilde{R} = \partial \cdot V \cdot (I + E_{kl}) = \partial \tilde{V}$

$\Rightarrow \partial = \tilde{R} \cdot (\tilde{V})^{-1} \triangleq \tilde{R} \cdot \hat{U}$

$\hat{U}$ is ~~tri~~ upper-triangular

$\tilde{R}$ isn't reduced.

Deficiency: PUP non-triangular, if $U[i, i+1] = 1$



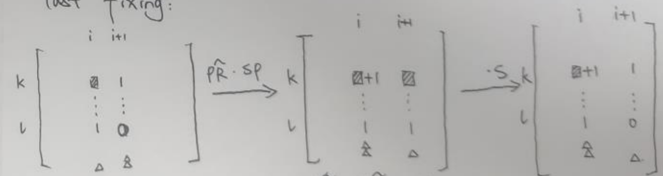
Fixing: Add the i+1-row to i-row $\quad S = I + E_{i,i+1}$ $(S^2 = I)$

$\partial' = P\tilde{R}\hat{U}P = \boxed{P\tilde{R}SP}\boxed{PS\hat{U}P}$

$PS\hat{U}P$ is triangular.

but $P\tilde{R}SP$ may not reduced, because of S.

last fixing:



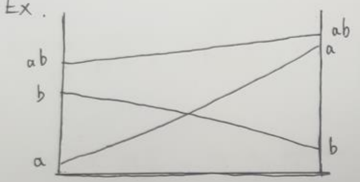
$P\tilde{R}$ $\qquad$ $P\tilde{R}\cdot SP$ $\qquad$ $P\tilde{R}SPS$

$P\partial P = \underbrace{P\tilde{R}SPS}\ \underbrace{SPS\hat{U}P}$ (Encounter deficiency)

$\partial_{t,t\varepsilon} = R_{t,t\varepsilon} \cdot U_{t,t\varepsilon}$.

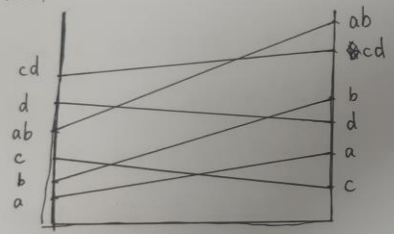
3.

Ex.

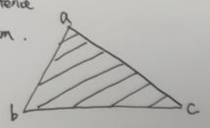


$(b, ab) \to (a, ab)$

Ex.



persistence diagram.



$K = \{a, b, c, ab, ac, bc, abc\}$

$f \quad \Downarrow$

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
(ex:	1	3	5	7	9	11	13)
(ex:	1	2	3	4	5	6	7)

satisfy : $f(\sigma) \le f(\tau)$, if σ is a face of τ.

let $K_0 = \emptyset$, $K_i = f^{-1}(-\infty, a_i]$ $i = 1, \cdots, 7$

ex. $f^{-1}(-\infty, a_5] = K_5 = \{a, b, c, ab, ac\}$

filtration : $\emptyset = K_0 \subset K_1 \subset \cdots \subset K_7 = K$

$\Downarrow$

$\tilde{H}_p(K_0) \to \tilde{H}_p(K_1) \to \cdots \to \tilde{H}_p(K_7)$

let. $f_p^{i,j} : H_p(K_i) \to H_p(K_j)$ (by inclusion)

$\beta_p^{a_i, a_j} = \text{rank Im} f_p^{i,j}$ be called persistent homology.

consider $\mu_p^{a_i, a_j} = (\beta_p^{a_i, a_{j-1}} - \beta_p^{a_i, a_j}) - (\beta_p^{a_{i-1}, a_{j-1}} - \beta_p^{a_{i-1}, a_j})$

is multiplicity of (a_i, a_j)

boundary matrix.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ & & & 1 & 0 & 1 & 0 \\ & & & & 1 & 1 & 0 \\ & & & & & & 1 \\ & & & & & & 1 \\ & & & & & & 1 \\ & & & & & & \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 \\ & & & & & & 1 \\ & & & & & & 1 \\ & & & & & & 1 \\ & & & & & & \end{bmatrix}$$

$low(i) = j$ (row: j ; column: i)

$low(1) = 0$ or don't define it.

$low(2) = 0$

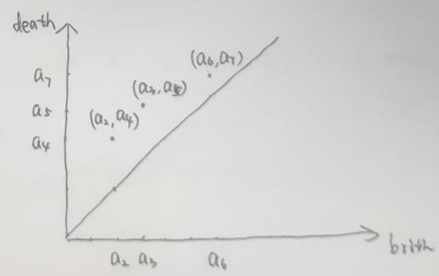
$low(3) = 0$

$low(4) = 2$, represent a homology class which birth: 2, death: 4 $\Rightarrow (a_2, a_4)$

$low(5) = 3$ (a_3, a_5)

$low(6) = 0$ [illegible] produce a new homology class (S')

$low(7) = 6$ (a_6, a_7)



(0,+∞)

考试科目

Stability.

Bottleneck distance.

$\mathrm{Dgm}(f) = X \quad \mathrm{Dgm}(g) = Y$ (contain diagonal line 多 / each point with infinite multiplicity)

$\gamma : X \to Y$ is a bijection.

$W_\infty(X, Y) = \inf\limits_{\gamma: X \to Y} \sup\limits_{x \in X} \| x - \gamma(x) \|_\infty$ (Remark: $\exists \gamma_0 : X \to Y$ s.t. $W_\infty(X,Y) = \sup\limits_{x\in X} \| x - \gamma_0(x) \|_\infty$)

refer to P18 | Figure Computational Topology.

Fact. 1. $W_\infty(X,Y) = 0$ iff $X = Y$

2. $W_\infty(X,Y) = W_\infty(Y,X)$

3: $W_\infty(X,Z) \leq W_\infty(X,Y) + W(Y,Z)$

Result: W_∞ is a distance.

Thm 1. Let K be a simplicial complex, and $f, g : K \to \mathbb{R}$ are two monotonic function. For every p, we have inequality

$$W_\infty(X,Y) \leq \| f - g \|.$$

Proof.

Let: $F(\sigma, t) = (1-t) f(\sigma) + t\, g(\sigma)$

$F : K \times I \to \mathbb{R}$.

let $f_t(\sigma) = F(\sigma, t)$, $X_t = \mathrm{Dgm}(f_t)$

Simplify, we assume that there are at most two simplexes that satisfy the condition of $f_t(\sigma) = f_t(\tau)$, $\forall t \in (0,1)$

let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$, $\exists\, \sigma, \tau \in K$, s.t. $f_{t_i}(\sigma) = f_{t_i}(\tau)$ $i = 1, \cdots, n-1$

We have some facts:

1: $W_\infty(X_{t_i}, X_{t_{i+1}}) \leq \| f_{t_i} - f_{t_{i+1}} \|_\infty$

2: $W_\infty(X_0, X_1) \leq \sum_i W_\infty(X_{t_i}, X_{t_{i+1}})$

3: $\| f_{t_i} - f_{t_{i+1}} \|_\infty = |f(\sigma) - g(\sigma)| (t_{i+1} - t_i)$ for some σ.

If we accept three fact, then

$$\begin{aligned} W_\infty(X_0, X_1) &\leq \sum_i W_\infty(X_{t_i}, X_{t_{i+1}}) \\ &\leq \sum_i \| f_{t_i} - f_{t_{i+1}} \|_\infty \\ &= \sum_i | f(\sigma(i)) - g(\sigma(i)) | \cdot (t_{i+1} - t_i) \\ &\leq \sum_i (t_{i+1} - t_i) \cdot \| f - g \|_\infty = \| f - g \|_\infty. \end{aligned}$$

Proof of 2: obviously.

Proof of 3: $f_{t_i} = (1 - t_i) f + t_i g$

$$\begin{aligned} \| f_{t_i} - f_{t_{i+1}} \|_\infty &= \| [(1-t_i) \cdot f + t_i g] - [(1 - t_{i+1}) \cdot f + t_{i+1} g] \|_\infty \\ &= \| (t_{i+1} - t_i)(f - g) \|_\infty \\ &= (t_{i+1} - t_i) \cdot \| f - g \|_\infty = (t_{i+1} - t_i) \cdot | f(\sigma) - g(\sigma) | \end{aligned}$$

for some σ.

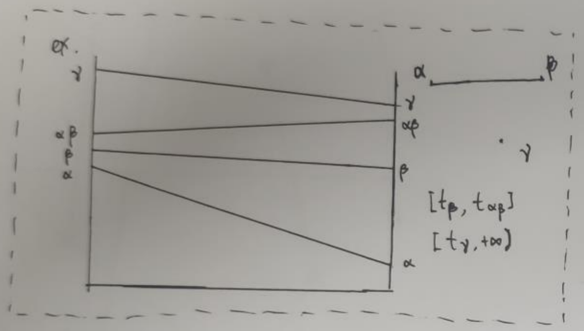
$(0,+\infty)$

Proof 1:

Let $\varphi, \psi : K \to \mathbb{R}$ be monotonic and

$\cancel{\varphi(\sigma) \le \psi(\sigma) \Leftrightarrow}$

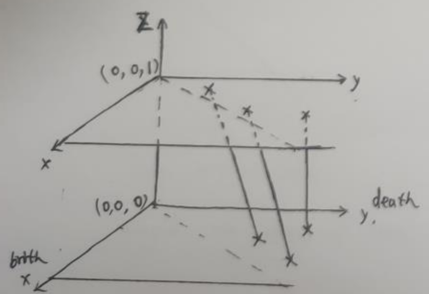
$\varphi(\sigma) \le \varphi(\tau) \Leftrightarrow \psi(\sigma) \le \psi(\tau)$



then $(\varphi(\sigma), \varphi(\tau)) \in Dgm(\varphi)$

$\Leftrightarrow (\psi(\sigma), \psi(\tau)) \in Dgm(\psi)$

persistence diagram $Dgm(\varphi)$ and $Dgm(\psi)$ are similar in some extent.



$$W_\infty(Dgm\varphi, Dgm\psi) = \inf_{\gamma: Dgm\varphi \to Dgm\psi} \left\{ \sup_{x \in Dgm\varphi} \|x - \gamma(x)\|_\infty \right\}$$

$$\le \sup_{x \in Dgm\varphi} \|x - \gamma(x)\|_\infty \quad \text{for } \gamma: Dgm\varphi \to Dgm\psi$$

$$(\varphi(\sigma), \varphi(\tau)) \mapsto (\psi(\sigma), \psi(\tau))$$

$$\le \|\varphi - \psi\|_\infty$$

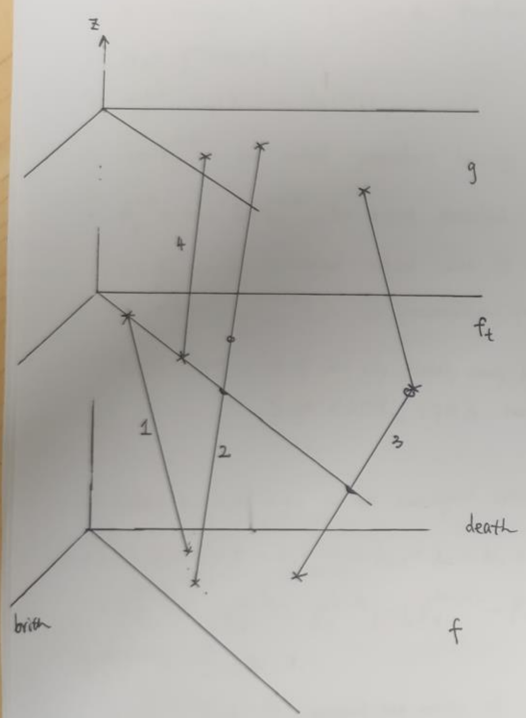
For f_{t_i}, $f_{t_{i+1}}$, we consider $f_{t_i} = \lim_{\varepsilon \to 0^+} f_{t_i + \varepsilon}$

$f_{t_{i+1}} = \lim_{\varepsilon \to 0^+} f_{t_{i+1} - \varepsilon}$

Then: $W_\infty(X_{t_i}, X_{t_{i+1}}) \le \| f_{t_i} - f_{t_{i+1}} \|_\infty$

$\blacksquare$

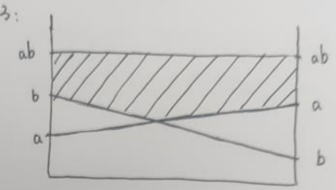
Considering the evolution of the persistence diagram.



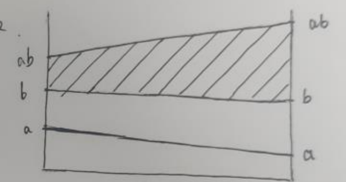
vine: polygonal path.

vinegard: the multiset of vines.

3:



2.



1.4. I think these cases don't exist.

Because, if it is true, "4" is true,

$\forall k$, $0<k<t$, assume $(f_k(\sigma), f_k(\tau))$ in $dgm(f_k)$

$f_t(\sigma) = f_t(\tau)$ and $\dim\sigma + 1 = \dim\tau$

σ is the proper face of τ.

对 $\forall l$, $t<l<1$, we have $f_l(\sigma) > f_l(\tau)$, a contradiction.

Tame function.

- X : triangulable $f: X \to \mathbb{R}$ function.
- $X_a = f^{-1}(-\infty, a]$
- $f_p^{a,b} : H_p(X_a) \to H_p(X_b)$
- $\operatorname{Im} f_p^{a,b}$ be called persistent homology group.
- $\beta_p^{a,b} = \operatorname{rank} \operatorname{Im} f_p^{a,b}$ be called persistent Betti number.
- $a \in \mathbb{R}$ is a homological critical value, if there is no $\varepsilon > 0$, for which $f_p^{a-\varepsilon, a+\varepsilon}$ is an isomorphism. for any dimension p.
- f is tame : ① f has only finitely many homological critical values
  ② for $\forall a \in \mathbb{R}$, $\forall p \in \mathbb{Z}$, $\operatorname{rank} H_p(X_a) < \infty$.

- Let~~ting~~ $a_1 < a_2 < \cdots < a_n$ be ~~the~~ homological critical values of f.

$-\infty = b_{-1} < b_0 < a_1 < b_1 < a_2 < \cdots < b_{n-1} < a_n < b_n < b_{n+1} = +\infty$

- $\mu_p^{a_i, a_j} = (\beta_p^{b_i, b_{j-1}} - \beta_p^{b_i, b_j}) - (\beta_p^{b_{i-1}, b_{j-1}} - \beta_p^{b_{i-1}, b_j})$ is the multiplicity of (a_i, a_j)

  $\beta_p^{b_i, b_{j-1}} - \beta_p^{b_i, b_j}$: represent the number of homological class which is in $H_p(X_{b_i})$ and died in $H_p(X_{b_j})$.

  $\beta_p^{b_{i-1}, b_{j-1}} - \beta_p^{b_{i-1}, b_j}$: represent the number of homological class which is in $H_p(X_{b_{i-1}})$ and died in $H_p(X_{b_j})$

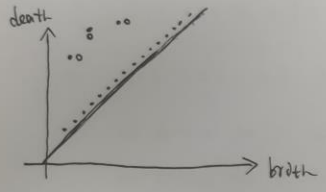
Thm 2. X: triangulable, $f, g: X \to \mathbb{R}$ be tame functions.

$X = \operatorname{Dgm}_p(f)$, $Y = \operatorname{Dgm}_p(g)$ for $\forall p \in \mathbb{Z}$

$W_\infty(X, Y) \leq \|f - g\|_\infty$

Wasserstein distance. $W_q(X,Y)=\left[\inf_{\gamma: X\to Y} \sum_{x\in X} \|x-\gamma(x)\|_\infty^q\right]^{\frac{1}{q}}$

Compare Wasserstein distance with bottleneck distance



bottleneck distance : for $\forall x \in X$, $\gamma(x)$ is close to x.

Wasserstein distance : most of the points, $\gamma(x)$ and x, are close enough.

f is Lipschitz : for $\forall x,y\in \mathbb{X}$, $|f(x)-f(y)| < C\,\|x-y\|$, constant C.
$\|\cdot\|$ is distance function in $\mathbb{X}$. ($\mathbb{X}$ is a metric space)

$\mathbb{X}$: triangulable, K, homeomorphism $\phi: |K| \to \mathbb{X}$

mesh : maximum distance between the images of two points of the same simplex in K.

$N(r)$: minimum number of simplices in triangulation with mesh at most r.

grow polynomially : if there are constants c and j, s.t. $N(r) \le \frac{c}{r^j}$

degree-k ~~totally~~ total persistence : $\Phi^k(X) = \sum_{x\in X} \mathrm{pers}(x)^k$, X is a persistence diagram, $\mathrm{pers}(x) = (x_2-x_1)$, $x=(x_1,x_2)$, (require $(x_2-x_1)<\infty$).

Lemma. $f: \mathbb{X}\to\mathbb{R}$: Lipschitz, $\mathbb{X}$: a metric space, it's triangulation grow polynomially with constant exponent j, then $\Phi^k(\mathrm{dgm}_p(f))$ is bounded from above by a constant for every $k>j$. ▨

Thm 3. $f,g: \mathbb{X}\to\mathbb{R}$: tame, Lipschitz

$\mathbb{X}$: a metric space, its triangulation grow polynomially with constant exponent j.

Then there are constants C and $k>j$

$W_q(\mathrm{Dgm}_p(f), \mathrm{Dgm}_p(g)) \le C\cdot \|f-g\|_\infty^{1-\frac{k}{q}}$ $\quad \forall q\ge k$.

Proof. Let $\eta: \mathrm{Dgm}_p(f)\to \mathrm{Dgm}_p(g)$ be bijection, ~~s.t.~~

s.t. $W_\infty(\mathrm{Dgm}_p(f), \mathrm{Dgm}_p(g)) = \sup_{x\in X}\|x-\eta(x)\|_\infty$

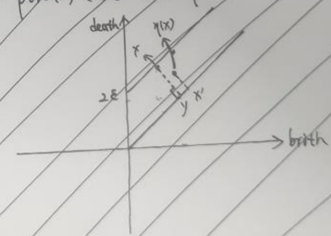
$\Rightarrow \|x-\eta(x)\|_\infty \le \|f-g\|_\infty = \varepsilon \quad \forall x\in \mathrm{Dgm}_p(f)$.

We can require $\|x-\eta(x)\|_\infty \le \frac{1}{2}[\mathrm{pers}(x)+\mathrm{pers}(\eta(x))]$.

if this inequality doesn't hold, i.e. $\exists x\in \mathrm{Dgm}_p(f)$ s.t. $\|x-\eta(x)\|_\infty > \frac{1}{2}[\mathrm{pers}(x)+\mathrm{pers}(\eta(x))]$

$\Rightarrow \varepsilon \ge \|x-\eta(x)\|_\infty > \frac{1}{2}[\mathrm{per}(x)+\mathrm{pers}(\eta(x))] \Rightarrow \mathrm{pers}(x)\le 2\varepsilon$, $\mathrm{pers}(\eta(x))\le 2\varepsilon$

$\Rightarrow$ $\mathrm{pers}(x)\le 2\varepsilon$ and $\mathrm{pers}(\eta(x))\le 2\varepsilon$



modify η, $\begin{cases}\hat{\eta}(x)=y \\ \hat{\eta}(x')=\eta(x)\end{cases}$, the rest be not modified.

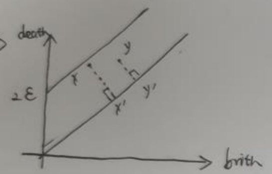
let $y = \eta(x)$. then $x = (x_1, x_2)$ $y = (y_1, y_2)$

$$\begin{cases} \|(x_1 - y_1, x_2 - y_2)\|_\infty \leq \varepsilon \\ \frac{1}{2}\left[(x_2 - x_1) + (y_2 - y_1)\right] < \|(x_1 - y_1, x_2 - y_2)\|_\infty \end{cases}$$

$\Rightarrow$ $x_2 - x_1 < 2\varepsilon$ and $y_2 - y_1 < 2\varepsilon$.

$\Rightarrow$ the distance for x to $l: x - y = 0$ is $\sqrt{2}\varepsilon$

the distance for y to $l: x - y = 0$ is $\sqrt{2}\varepsilon$

$\Rightarrow$ 

modify η. $\begin{cases} \eta(x) = x' \\ \eta(y') = y \end{cases}$, the rest remain

$\Rightarrow$ η satisfy $\begin{cases} \|x - \eta(x)\|_\infty \leq \|f - g\|_\infty = \varepsilon \\ \|x - \eta(x)\|_\infty \leq \frac{1}{2}\left[\mathrm{pers}(x) + \mathrm{pers}(\eta(x))\right] \end{cases}$

$\cancel{W_q(X, Y)^q = \inf_{\gamma: X \to Y}}$

$$W_q(\mathrm{dgm}_p(f), \mathrm{dgm}_p(g))^q = \inf_{\gamma: \mathrm{dgm}_p(f) \to \mathrm{dgm}_p(g)} \sum_{x \in \mathrm{dgm}_p(f)} \|x - \gamma(x)\|_\infty^q$$

$$\leq \sum_{x \in \mathrm{dgm}_p(f)} \|x - \eta(x)\|_\infty^q$$

$$\leq \varepsilon^{q-k} \sum_{x \in \mathrm{dgm}_p(f)} \|x - \eta(x)\|_\infty^k$$

$$\leq \frac{\varepsilon^{q-k}}{2^k} \sum_{x \in \mathrm{dgm}_p(f)} \left[\mathrm{pers}(x) + \mathrm{pers}(\eta(x))\right]^k$$

$$\leq \frac{\varepsilon^{q-k}}{2^k} \sum_{x \in \mathrm{dgm}_p(f)} \left[\left[2\,\mathrm{pers}(x)\right]^k + \left[2\,\mathrm{pers}(\eta(x))\right]^k\right]$$

$\Rightarrow$ $W_q(X, Y)^q \leq \varepsilon^{q-k}\left[\Phi^k(\mathrm{dgm}_p(f)) + \Phi^k(\mathrm{dgm}_p(g))\right]$

$\Rightarrow$ Thm 3.