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GENERALIZATION OF A THEOREM BY v. NEUMANN CONCERNING ZERO SUM TWO PERSON GAMES

BY ABRAHAM WALD

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1. Introduction

In the theory of games developed by John v. Neumann [1], [2] the normalized form of a zero sum two person game is defined as follows (see section 14.1 in [2]): There are two players and there is a function $K(\tau_1, \tau_2)$ of two variables τ_1 and τ_2 given where τ_1 and τ_2 can take only a finite number of values. Player 1 chooses a value τ_1 and player 2 chooses a value τ_2, each choice being made in complete ignorance of the other, and then players 1 and 2 get the amounts $K(\tau_1, \tau_2)$ and $-K(\tau_1, \tau_2)$, respectively. Obviously, player 1 wishes to maximize $K(\tau_1, \tau_2)$ and player 2 wishes to minimize $K(\tau_1, \tau_2)$.

As v. Neumann has shown (see section 14.5 in [2]), the choice of τ_1 by player 1 and the choice of τ_2 by player 2 can be rationalized if the game is strictly determined, i.e., if

$$\text{Max}_{\tau_1} \text{Min}_{\tau_2} K(\tau_1, \tau_2) = \text{Min}_{\tau_2} \text{Max}_{\tau_1} K(\tau_1, \tau_2). \tag{1.1}$$

If (1.1) is fulfilled, a good way for 1 to play the game is to choose a value τ_1 for which $\text{Min}_{\tau_2} K(\tau_1, \tau_2)$ assumes its maximum value, and a good way for 2 to play the game is to choose a value τ_2 for which $\text{Max}_{\tau_1} K(\tau_1, \tau_2)$ assumes its minimum value.

There are games for which (1.1) is not fulfilled To overcome this difficulty, the problem is reformulated as follows (see section 17 in [2]): Instead of choosing a particular value of τ_i, player i considers all possible values of τ_i and chooses only the probabilities with which he is going to use them, respectively. In other words, if the possible values of τ_i are $1, 2, \cdots, \beta_i$, player i does not choose any particular number in this set, but chooses a set of probabilities $\rho_1, \cdots, \rho_{\beta_i}$ and the value of τ_i is then determined by a chance mechanism constructed in such a way that the probability that $\tau_i = j$ is equal to ρ_j. Thus, the choice of player 1 is now characterized by a vector $\overrightarrow{\xi} = (\xi_1, \cdots, \xi_{\beta_1})$ and the choice of 2 is characterized by a vector $\overrightarrow{\eta} = (\eta_1, \cdots, \eta_{\beta_2})$. Of course, the vectors $\overrightarrow{\xi}$ and $\overrightarrow{\eta}$ are subject to the restrictions: $\xi_i \geqq 0$, $\sum_{i=1}^{\beta_1} \xi_i = 1$, $\eta_j \geqq 0$ and $\sum_{j=1}^{\beta_2} \eta_j = 1$. The mathematical expectation of the outcome $K(\tau_1, \tau_2)$ is given by

$$K^*(\overrightarrow{\xi}, \overrightarrow{\eta}) = \sum_{j=1}^{\beta_2} \sum_{i=1}^{\beta_1} K(i, j)\xi_i \eta_j. \tag{1.2}$$

The main theorem proved by v. Neumann (see section 17.6 in [2]) states that for any arbitrary function $K(\tau_1, \tau_2)$ the game corresponding to $K^*(\vec{\xi}, \vec{\eta})$ is always strictly determined, i.e.,

$$\text{(1.3)} \qquad \operatorname{Max}_{\vec{\xi}} \operatorname{Min}_{\vec{\eta}} K^*(\vec{\xi}, \vec{\eta}) = \operatorname{Min}_{\vec{\eta}} \operatorname{Max}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}).$$

The above relation was proved by v. Neumann under the restriction that τ_1 and τ_2 can take only a finite number of values. The purpose of the present paper is to investigate the validity of (1.3) when τ_1 or τ_2 or both can take infinitely many values. In what follows in this paper we shall assume that $K(\tau_1, \tau_2)$ is a bounded function of τ_1 and τ_2 and that the number of different values the variable τ_i $(i = 1, 2)$ can take is denumerable. Since the domain of τ_i is not assumed to be finite, the existence of a maximum or minimum of $K^*(\vec{\xi}, \vec{\eta})$ with respect to $\vec{\xi}$ or $\vec{\eta}$ is not guaranteed. However, supremum (least upper bound) and infimum (greatest lower bound) of $K^*(\vec{\xi}, \vec{\eta})$ with respect to $\vec{\xi}$ or $\vec{\eta}$ will always exist. Therefore, instead of (1.3) we shall consider the relation

$$\text{(1.4)} \qquad \operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}} K^*(\vec{\xi}, \vec{\eta}) = \operatorname{Inf}_{\vec{\eta}} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta})$$

and we shall investigate the conditions under which (1.4) holds.

It will be shown that if one of the variables τ_1 and τ_2 has a finite domain, equation (1.4) always holds. If both variables τ_1 and τ_2 can take infinitely many values, (1.4) does not always hold. A simple example where (1.4) does not hold is the following: Assume that τ_1 and τ_2 can take all positive integral values. Let $K(\tau_1, \tau_2) = 1$ if $\tau_1 > \tau_2$, $K(\tau_1, \tau_2) = 0$ if $\tau_1 = \tau_2$ and $K(\tau_1, \tau_2) = -1$ if $\tau_1 < \tau_2$. Then, as can readily be seen, $\operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}} K^*(\vec{\xi}, \vec{\eta}) = -1$ and $\operatorname{Inf}_{\vec{\eta}} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}) = 1$. Thus, (1.4) does not hold. In section 4 we shall give a necessary and sufficient condition for the validity of (1.4) when each of the variables τ_1 and τ_2 can take infinitely many values.

2. Some Lemmas

In this section we shall prove several lemmas which will then be used in sections 3 and 4.

LEMMA 1. $\operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}} K^*(\vec{\xi}, \vec{\eta}) \leqq \operatorname{Inf}_{\vec{\eta}} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta})$.

PROOF: Assume that Lemma 1 is not true. Then

$$\text{(2.1)} \qquad \operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}} K^*(\vec{\xi}, \vec{\eta}) = \operatorname{Inf}_{\vec{\eta}} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}) + \delta \qquad (\delta > 0).$$

From (2.1) it follows that there exists a vector $\vec{\xi_0}$ such that

$$\text{(2.2)} \qquad K^*(\vec{\xi_0}, \vec{\eta}) \geqq \operatorname{Inf}_{\vec{\eta}} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}) + \tfrac{1}{2}\delta \quad \text{for all } \eta.$$

From (2.2) we obtain

$$\text{(2.3)} \qquad \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}) \geqq \operatorname{Inf}_{\vec{\eta}} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}) + \tfrac{1}{2}\delta \quad \text{for all } \eta$$

which is impossible. Hence we arrived at a contradiction and Lemma 1 is proved.

LEMMA 2. *If* $\lim_{u=\infty} \overrightarrow{\xi_u} = \overrightarrow{\xi}$ *then* $\lim_{u=\infty} K^*(\overrightarrow{\xi_u}, \overrightarrow{\eta}) = K^*(\overrightarrow{\xi}, \overrightarrow{\eta})$ *uniformly in* $\overrightarrow{\eta}$. *Similarly, if* $\lim_{u=\infty} \overrightarrow{\eta_u} = \overrightarrow{\eta}$, $\lim_{u=\infty} K^*(\overrightarrow{\xi}, \overrightarrow{\eta_u}) = K^*(\overrightarrow{\xi}, \overrightarrow{\eta})$ *uniformly in* $\overrightarrow{\xi}$.

PROOF: It is sufficient to prove the first half of the lemma. We can assume without loss of generality that τ_i can take only positive integral values, i.e., $\tau_i = 1, 2, 3, \cdots$, ad inf. ($i = 1, 2$). Let A be an upper bound of $| K(\tau_1, \tau_2) |$. Then for any $\overrightarrow{\eta} = (\eta_1, \eta_2, \cdots, \text{ad inf.})$

$$\left| \sum_{j=1}^{\infty} K(i, j)\eta_j \right| \leq A. \tag{2.4}$$

Let $\overrightarrow{\xi_u} = (\xi_{u1}, \xi_{u2}, \cdots)$ and $\overrightarrow{\xi} = (\xi_1, \xi_2, \cdots)$. From (2.4) it follows readily that for any positive integer r

$$\lim_{u=\infty} \sum_{j=1}^{\infty} \sum_{i=1}^{r} K(i, j)\xi_{ui}\eta_j = \sum_{j=1}^{\infty} \sum_{i=1}^{r} K(i, j)\xi_i \eta_j \tag{2.5}$$

uniformly in $\overrightarrow{\eta}$. For any $\epsilon > 0$ there exists a positive integer r_ϵ such that $\sum_{i=1}^{r_\epsilon} \xi_i \geqq 1 - \epsilon$ and $\sum_{i=1}^{r_\epsilon} \xi_{ui} \geqq 1 - \epsilon$ for all u. Then it follows from (2.4) that

$$\left| \sum_{j=1}^{\infty} \sum_{i=1}^{r_\epsilon} K(i, j)\xi_{ui}\eta_j - \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} K(i, j)\xi_{ui}\eta_j \right| \leqq \epsilon A \quad \text{all for } \eta \text{ and all } u, \tag{2.6}$$

and

$$\left| \sum_{j=1}^{\infty} \sum_{i=1}^{r_\epsilon} K(i, j)\xi_i\eta_j - \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} K(i, j)\xi_i\eta_j \right| \leqq \epsilon A \quad \text{for all } \eta. \tag{2.7}$$

Since ϵ can be chosen arbitrarily small, Lemma 2 follows from (2.5), (2.6) and (2.7).

LEMMA 3. *If* $\lim_{u=\infty} \overrightarrow{\xi_u} = \overrightarrow{\xi}$ *and* $\lim_{u=\infty} \overrightarrow{\eta_u} = \overrightarrow{\eta}$ *then* $\lim_{u=\infty} K^*(\overrightarrow{\xi_u}, \overrightarrow{\eta_u}) = K^*(\overrightarrow{\xi}, \overrightarrow{\eta})$.

PROOF: From Lemma 2 it follows that

$$\lim_{u=\infty} \{K^*(\overrightarrow{\xi_u}, \overrightarrow{\eta_u}) - K^*(\overrightarrow{\xi_u}, \overrightarrow{\eta})\} = 0 \tag{2.8}$$

and

$$\lim_{u=\infty} \{K^*(\overrightarrow{\xi_u}, \overrightarrow{\eta}) - K^*(\overrightarrow{\xi}, \overrightarrow{\eta})\} = 0. \tag{2.9}$$

Lemma 3 is a consequence of (2.8) and (2.9).

In what follows a superscript k attached to a vector $\overrightarrow{\zeta}$, i.e., $\overrightarrow{\zeta^k}$ will mean that the j^{th} component of $\overrightarrow{\zeta}$ is zero for all $j > k$.

LEMMA 4. $\lim_{k=\infty} \operatorname{Inf}_{\vec{\eta}^k} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}^k) = \operatorname{Inf}_{\vec{\eta}} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta})$.

PROOF: Clearly, $\operatorname{Inf}_{\vec{\eta}^k} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}^k)$ is monotonically decreasing with increasing k. Hence $\lim_{k=\infty} \operatorname{Inf}_{\vec{\eta}^k} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}^k)$ exists. We shall denote this limit by ρ. Then

$$\operatorname{Inf}_{\vec{\eta}^k} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}^k) \geqq \rho \quad \text{for all } k \tag{2.10}$$

and

$$\operatorname{Inf}_{\vec{\eta}} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}) = \rho - \delta \qquad (\delta \geqq 0). \tag{2.11}$$

Assume that $\delta > 0$ and we shall derive a contradiction. Then it follows from (2.11) that there exists a vector $\vec{\eta}_0$ such that

$$K^*(\xi, \vec{\eta}_0) \leqq \rho - \tfrac{1}{2}\delta \quad \text{for all } \xi. \tag{2.12}$$

Let $\{\vec{\eta}_k^k\}$ $(k = 1, 2, \cdots, \text{ad inf.})$ be a sequence of vectors such that

$$\lim_{k=\infty} \vec{\eta}_k^k = \vec{\eta}_0 . \tag{2.13}$$

Then, according to Lemma 2,

$$\lim_{k=\infty} K^*(\xi, \vec{\eta}_k^k) = K^*(\vec{\xi}, \vec{\eta}_0) \tag{2.14}$$

uniformly in $\vec{\xi}$. Hence there exists a finite k such that

$$K^*(\vec{\xi}, \vec{\eta}_k^k) \leqq \rho - \tfrac{1}{3}\delta \quad \text{for all } \xi. \tag{2.15}$$

But this is in contradiction to (2.10). Hence Lemma 4 is proved.

3. The case where one of the variables τ_1 and τ_2 can take only a finite number of values.

We shall prove the following theorem.

THEOREM 3.1. *If one of the variables τ_1 and τ_2 takes inly a finite number of values, the relation*

$$\operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}} K^*(\vec{\xi}, \vec{\eta}) = \operatorname{Inf}_{\vec{\eta}} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta})$$

always holds.

PROOF: It is sufficient to prove Theorem 3.1 for the case when τ_1 can take only a finite number of values. The case where τ_2 can take only a finite number of values can be reduced to the previous case by interchanging the players 1 and 2 and by substituting $-K(\tau_1, \tau_2)$ for $K(\tau_1, \tau_2)$. We can assume without loss of generality that τ_1 can take the values 1, 2, $\cdots$, r and τ_2 can take any positive integral value. Then

$$K^*(\vec{\xi}, \vec{\eta}) = \sum_{j=1}^{\infty} \sum_{i=1}^{r} K(i, j)\xi_i \eta_j .$$

To prove Theorem 3.1 it is sufficient to show that

$$\lim_{k=\infty} \operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}^k} K^*(\vec{\xi}, \vec{\eta}^k) = \operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}} K^*(\vec{\xi}, \vec{\eta}). \tag{3.1}$$

In fact, according to v. Neumann's theorem for the finite case we have

$$\operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}^k} K^*(\vec{\xi}, \vec{\eta}^k) = \operatorname{Inf}_{\vec{\eta}^k} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}^k). \tag{3.2}$$

Theorem 3.1 is an immediate consequence of equations (3.1), (3.2) and Lemma 4. To prove (3.1) we shall assume that (3.1) does not hold and we shall derive a contradiction. Then

$$\operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}} K^*(\vec{\xi}, \vec{\eta}) = \rho - \delta \quad (\delta > 0) \tag{3.3}$$

where

$$\rho = \lim_{k=\infty} \operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}^k} K^*(\vec{\xi}, \vec{\eta}^k). \tag{3.4}$$

Since the domain of τ_1 is finite, for each positive integer k there exists a pair $(\vec{\xi}_k, \vec{\eta}^k_k)$ such that

$$\operatorname{Inf}_{\vec{\eta}^k} K^*(\vec{\xi}_k, \vec{\eta}^k) = K^*(\vec{\xi}_k, \vec{\eta}^k_k) = \operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}^k} K^*(\vec{\xi}, \vec{\eta}^k) \geqq \rho. \tag{3.5}$$

Since the domain of τ_1 is finite, there exists a subsequence $\{k'\}$ of the sequence $\{k\}$ such that the sequence $\{\vec{\xi}_{k'}\}$ converges. Let

$$\lim_{k=\infty} \vec{\xi}_{k'} = \vec{\xi}_0 . \tag{3.6}$$

From (3.3) it follows that there exists a vector $\vec{\eta}_0$ such that

$$K^*(\vec{\xi}_0, \vec{\eta}_0) \leqq \rho - \tfrac{1}{2}\delta. \tag{3.7}$$

Let $\{\vec{\eta}^k_0\}$ ($k = 1, 2, \cdots$, ad inf.) a sequence of vectors such that

$$\lim_{k=\infty} \vec{\eta}^k_0 = \eta_0 . \tag{3.8}$$

From equations (3.6), (3.8) and Lemma 3 we obtain

$$K^*(\vec{\xi}_{k'}, \vec{\eta}^{k'}_0) \leqq \rho - \tfrac{1}{3}\delta \tag{3.9}$$

for almost all values of k. But (3.9) is in contradiction to (3.5). Hence Theorem 3.1 is proved.

4. The case where both τ_1 and τ_2 can take infinitely many values

As we have seen in section 1, the relation (1.4) does not always hold when both τ_1 and τ_2 can take infinitely many values. In this section we shall give a necessary and sufficient condition for the validity of (1.4). We shall prove the following theorem.

THEOREM 4.1. *A necessary and sufficient condition for the validity of*

$$\operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}} K^*(\vec{\xi}, \vec{\eta}) = \operatorname{Inf}_{\vec{\eta}} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}) \tag{4.1}$$

is that

$$\lim_{k=\infty} \operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}^k} K^*(\vec{\xi}, \vec{\eta}^k) = \operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}} K^*(\vec{\xi}, \vec{\eta}). \tag{4.2}$$

PROOF: According to Theorem 3.1 we have

$$\operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}^k} K^*(\vec{\xi}, \vec{\eta}^k) = \operatorname{Inf}_{\vec{\eta}^k} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}^k). \tag{4.3}$$

From (4.3) and Lemma 4 it follows that

$$\lim_{k=\infty} \operatorname{Sup}_{\vec{\xi}} \operatorname{Inf}_{\vec{\eta}^k} K^*(\vec{\xi}, \vec{\eta}^k) = \operatorname{Inf}_{\vec{\eta}} \operatorname{Sup}_{\vec{\xi}} K^*(\vec{\xi}, \vec{\eta}). \tag{4.4}$$

Hence (4.1) implies (4.2) and (4.2) implies (4.1). This proves the theorem.

COLUMBIA UNIVERSITY

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