Brouwer Fixed-Point Theorem

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1 Introduction to Fixed Points

Fixed points have many applications. One of their prime applications is in the mathematical field of game theory; here, they are involved in finding equilibria. The existence and location of the fixed point(s) is important in determining the location of any equilibria. They are then applied to some economics, and used to justify the existence of economic equilibriums in the market, as well as equilibria in dynamical systems

Definition 1.0.1. Fixed Point: For a function $f : \mathcal{X} \to \mathcal{X}$, a fixed point $c \in \mathcal{X}$ is a point where $f(c) = c$.

When a function has a fixed point, c, the point (c, c) is on its graph. The function $f(x) = x$ is composed entirely of fixed points, but it is largely unique in this respect. Many other functions may not even have one fixed point.

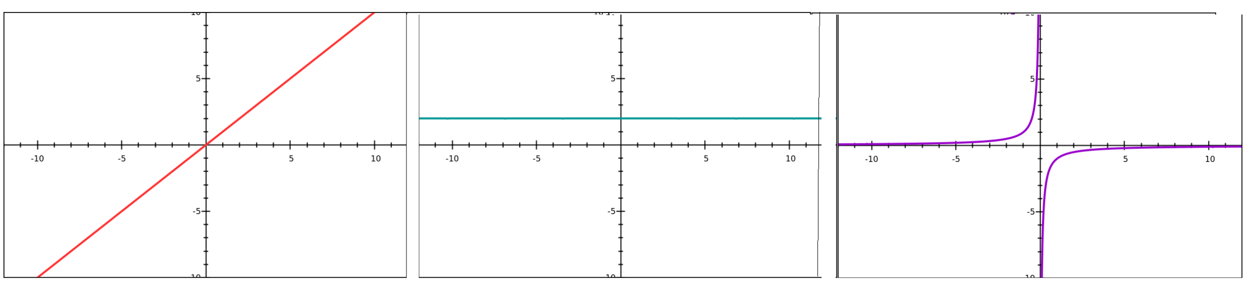

Figure 1: $f(x) = x$, $f(x) = 2$, and $f(x) = -{}^{1}/_{x}$, respectively. The first is entirely fixed points, the second has one fixed point at 2, and the last has none.

Fixed points came into mathematical focus in the late 19$^{\text{th}}$ century. The mathematician Henri Poincaré began using them in topological analysis of nonlinear problems, moving fixed-point theory towards the front of topology. Luitzen Egbertus Jan Brouwer, of the University of Amsterdam, worked with algebraic topology. He formulated his fixed-point theorem, which was first published relating only to the three-dimensional case in 1909, though other proofs for this specific case already existed.

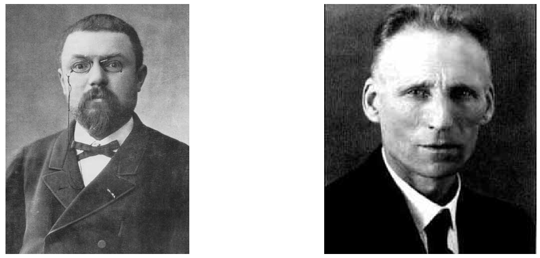

Figure 2: Henri Poincaré, left, and Luitzen Egbertus Jan Brouwer, right.

Brouwer, in 1910, presented his fixed point theorem:

1. Brouwer Fixed-Point Theorem in $\mathbb{R}$: Given that set $K \subset \mathbb{R}^n$ is compact and convex, and that function $f : K \to K$ is continuous, then there exists some $c \in K$ such that $f(c) = c$; that is, c is a fixed point.

The original wording of theorem gave this result for n-simplexes—a specific class of compact and convex sets, an n-simplex is the 'simplest' polygon in n dimensions, that has $n+1$ vertices. However, here we will be focusing on unit intervals and discs instead.

2 General definitions

Definition 2.0.1. Topological Space A topological space is a set, $\mathcal{X}$, equipped with an collection of its subsets, $\mathcal{T}$. The collection of subsets must include $\mathcal{X}$ and the empty set $\emptyset$. It must also be so that, for any arbitrary collection $U_\alpha \in \mathcal{T}$ for $\alpha \in \Lambda$, the union $\bigcup_{\alpha \in \Lambda} U_\alpha$ must also be part of the collection $\mathcal{T}$. Finally, the intersection of any two $U_1, U_2 \in \mathcal{T}$ must, again, be part of $\mathcal{T}$. Here, $\mathcal{T}$ is called the topology of $\mathcal{X}$, and all members of $\mathcal{T}$ are considered open in $\mathcal{X}$.

Definition 2.0.2. Open Set: In Euclidian metric space $\mathbb{R}^n$, the basic open sets are open intervals, discs, or balls for $n = 1, 2, 3, \ldots$ respectively. The full collection of open sets—the topology—consists of the basic sets and their infinite unions and finite intersections. This also includes $\mathbb{R}^n$ itself and $\emptyset$, the empty set.

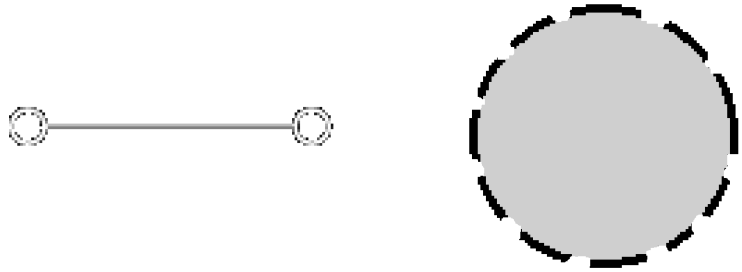

Figure 3: Basic open sets in $\mathbb{R}$ and $\mathbb{R}^2$. Note that the interval is open, as the endpoints are not filled in, and that the disc has a dotted boundary, indicating that the points on the boundary are not actually in the set.

Definition 2.0.3. Closed Set: A set $F \subseteq \mathbb{R}^n$ is closed if its complement, F^C, is open. A set is also closed if it is the arbitrary intersection or finite union of closed sets. In $\mathbb{R}$, closed intervals and singletons are closed.

Definition 2.0.4. Convex: A set $G \subseteq \mathbb{R}^n$ is said to be convex if, for any two points $g_1, g_2 \in G$, all points on the straight line segment connecting g_1 and g_2 are also in G.

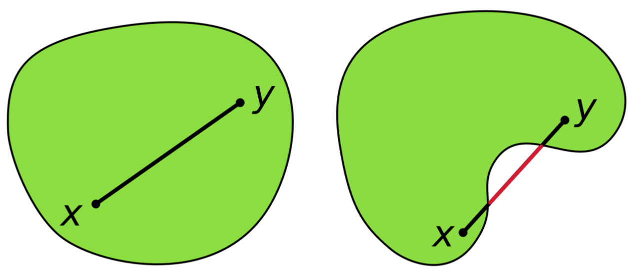

Figure 4: The left figure is convex, whereas the right is not.

Definition 2.0.5. Open Cover: A collection $\mathcal{A}$ of open sets in $\mathbb{R}^n$ is an open cover for set A if the union of all sets in $\mathcal{A}$ has A as a subset.

Definition 2.0.6. Compact: Let $(\mathcal{X}, \mathcal{T})$ be a topological space; if every open covering $\mathcal{A}$ of A contains a finite subcovering—a finite subcollection of $\mathcal{A}$ that is still an open cover for A—then A is compact.

In the most familiar of cases, the real numbers with the usual topology, a set must simply be closed and bounded in order to be compact, as shown be the Heine-Borel Theorem. In particular, this is true for $\mathbb{R}^n$ with the usual topology.

Definition 2.0.7. Continuous: Let $(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{T}_{\mathcal{Y}})$ be topological spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is continuous if, for V an open subset of $\mathcal{Y}$, $f^{-1}(V)$ is open in $\mathcal{X}$. The open sets in $\mathcal{X}$ and $\mathcal{Y}$ are the member sets of $\mathcal{T}_{\mathcal{X}}$ and $\mathcal{T}_{\mathcal{Y}}$, respectively.

Definition 2.0.8. Open Map: Let $(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{T}_{\mathcal{Y}})$ be topological spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is an open map if, for U an open subset of $\mathcal{X}$, $f(U)$ is open in $\mathcal{Y}$.

Definition 2.0.9. Bijection: A function $f : \mathcal{X} \to \mathcal{Y}$ is a bijection if it is injective and surjective; that is to say, f is a bijection if for all $y \in \mathcal{Y}$ there exists $x \in \mathcal{X}$ such that $f(x) = y$, and if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. A function that is bijective will have a well-defined inverse; that is, its inverse will be a function.

Definition 2.0.10. Homeomorphism: A homeomorphism is a function that is continuous, an open map, and bijective. It is clear in this context, then, how being an open map relates to it having a continuous inverse, and how all of this relates to structures defined through open sets. The existence of a homeomorphism between two sets is sufficient to show that the two sets are homeomorphic. If two sets are homeomorphic, then they are topologically equivalent. Thus, topological properties that hold for one set will hold for any set homeomorphic to it—in fact, it is this quality that makes a property topological.

3 Case of 1 dimension

The most simple case to consider the fixed point theorem is when the set $K \subset \mathbb{R}$ has $\mathbb{R}$ only having 1 dimension, and is in fact the unit 'square' $I = [0, 1]$. For a continuous function $f : [0,1] \to [0,1]$ to have fixed points, it must be so that there is a point $c \in X$ where $f(c) = c$. While K is one-dimensional, however, the actual work will be done in $[0,1]^2$, which has two dimensions.

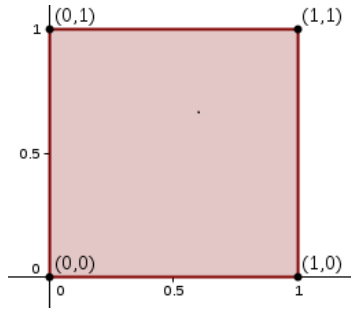

Figure 5: The unit square; the unit interval crossed with itself

This proof relies on the Intermediate Value Theorem:

Theorem 3.1. Intermediate Value Theorem: Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}$. Given a function $f : \mathcal{X} \to \mathcal{Y}$ that is continuous on $[a,b] \subseteq \mathcal{X}$, there exists for every $d \in (f(a), f(b)) \subseteq \mathcal{Y}$ (assuming, without loss of generality, that $f(a) \leq f(b)$) some $c \in (a,b)$ such that $f(c) = d$.

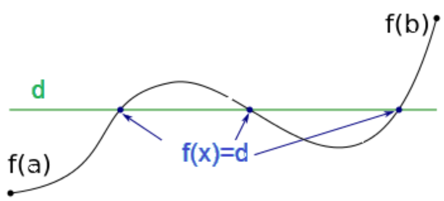

Figure 6: A pictoral representation of the Intermediate Value Theorem.

When dealing with one dimension, any closed and convex subset of $\mathbb{R}$ is homeomorphic to $[0,1]$. We can then show that any one-dimensional case for the Brouwer Fixed Point Theorem is equivalent to the case in $[0,1]$, and thus, the Theorem applies there.

3.1 Basic Proof of the Brouwer Fixed-Point Theorem on Set $[0, 1]$

Given that set K is compact and convex, and that function $f : K \to K$ is continuous, then there exists some $c \in K$ such that $f(c) = c$; that is, c is a fixed point.

Proof: Let $f : [0, 1] \to [0, 1]$ be continuous function on the unit square. The function $g(x) = f(x) - x$ is also continuous on the unit square as well, as it is the difference of continuous f and the identity function $i(x) = x$ which is also continuous. When $x = 0$, $f(0) \geq 0$, and $g(0) = f(0) - 0$, so $g(0)$ is either positive or 0. Now when $x = 1$, $f(1) \leq 1$, and $g(1) = f(1) - 1$. Similarly, $g(1)$ is either negative or 0.

Since g is a continuous function on a closed set, the Intermediate Value Theorem applies. Then we have $g(0) \geq 0$ and $g(1) \leq 0$, and it must be so that there is a $c \in [0, 1]$ such that $g(c) = d$, for any d, but in particular when $d = 0$. Thus, we have a point c where $g(c) = f(c) - c = 0$; thus, $f(c) = c$, and therefore c is our fixed point.

$\square$

To consider it through explanation, note a fixed point requires passing through the line $f(x) = x$. Thus, a function without a fixed point cannot intersect this line. That, however, leaves something such as the figure below, which isn't continuous.

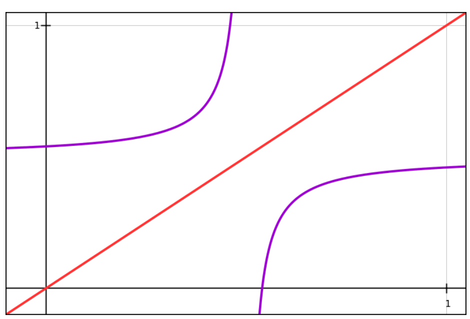

Figure 7: The darker function has no fixed point as it does not intersect $f(x) = x$ (lighter), but it is absolutely not continuous.

It is impossible for a continuous function to not intersect the line $i(x) = x$; however, to intersect that line is to have a fixed point, as all points on $i(x) = x$ are in fact fixed points.

For $g(x) = f(x) - x$, instead of trying to not intersect $i(x) = x$, we are trying to not intersect the zero line $h(x) = 0$. It is easier to show, using the intermediate value theorem, that g intersects the constant function h that it is to show that f intersects i.

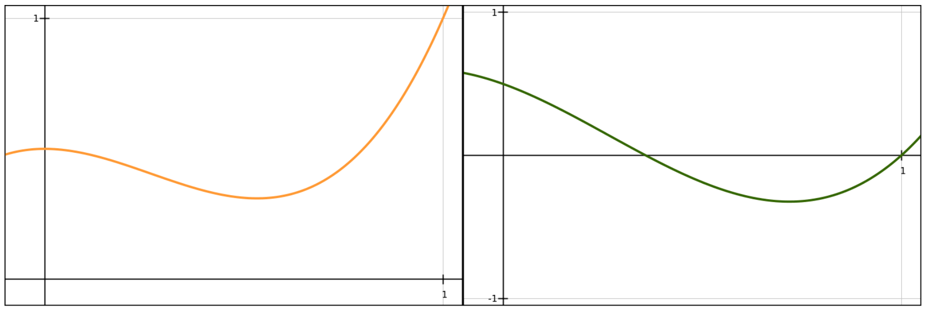

Figure 8: $f(x) = 2.5x^3 - 2x^2 + 0.5$, left, and $g(x) = f(x) - x$, right.

3.2 Extension to Homeomorphic Sets

In order to determine whether a fixed point is guaranteed for some other compact convex interval K, then one must determine whether or not a homeomorphism can be found between K and $[0, 1]$. If it in fact is, then K also has a fixed point for any continuous functions from K into itself.

In higher dimensions, we can show that $f : K \to K$ has a fixed point under the same conditions: compactness and convexness of the set K, and continuity of the function f.

4 Proof of the Brouwer Fixed-Point Theorem for Disc in 2D

Definition 4.0.1. Closure: Let $(\mathcal{X}, \mathcal{T})$be a topological space, and let $G \subseteq \mathcal{X}$. The closure of G, written $\overline{G}$, is the intersection of all closed sets that fully contain G. The closure of a set will always be closed.

Definition 4.0.2. Retraction: Let set $S \subseteq \mathbb{R}^2$ with $B \subseteq S$. We call $r : S \to B$ a retraction if it is continuous and $r(b) = b$ for all $b \in B$.

We will consider this with S being a disc, and B being the 'surface' or boundary of that disk. Or, rather, we will consider the lack of existence of such a retraction.

In $\mathbb{R}^2$, the unit disc can be defined by $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| \leq 1\}$ and the unit circle as $C = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| = 1\}$.

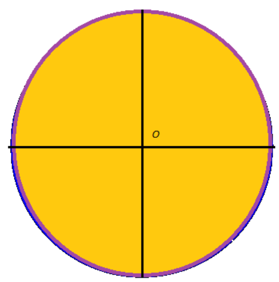

Figure 9: The unit disc $\mathbb{D}$. Its boundary C is the purple line.

Theorem 4.1. No Retraction Theorem: There does not exist any retraction from a closed unit disc $\mathbb{D}$ to its boundary, C.

We will need the No-Retraction Theorem in order to classify a function without a fixed point as a retraction that violates the above theorem. Because so much of the proof of the Brouwer Fixed-Point Theorem rests on the No-Retraction theorem, we also present its proof here for $\mathbb{D} \subset \mathbb{R}^2$. [3]

Proof: Let $r : \mathbb{D} \to C$ be a retraction from the unit disk $\mathbb{D}$ to its boundary, C. Consider $a, b \in C$; by removing these from C, we create two disjoint open arcs that compose $C \setminus \{a, b\}$. Now let $A = r^{-1}(a)$, and $B = r^{-1}(b)$. Since r is a retraction, $a \in A$ and $b \in B$, and so A and B intersect C. Since r is continuous, and $\{a\}$ and $\{b\}$ are closed, A and B must also be closed. Furthermore, a and b can be the only points where A and B, respectively, can intersect C, as they are the only elements of A and B that are in C. Note that $\overline{(C \setminus \{a, b\})} = C$. We can, then, find a subset of $\mathbb{D} \setminus (A \cup B)$ whose closure will contain C. Let us call this set P. We can choose it so that it is open and path-connected P.

Consider a closed arc of C, called C_a, that contains a. Let C_a have endpoints x_a, y_a. Both x_a and y_a will be in $\overline{P}$; thus, there exists a path that connects them. Furthermore, since we have defined P as a subset of $\mathbb{D} \setminus (A \cup B)$, this path cannot intersect A or B. However, unioning this path with $C \setminus \{a, b\}$ results in another a connected set. This implies that the retraction image of that union of the path and $C \setminus \{a, b\}$ is $C \setminus \{a, b\}$, because the path avoided A and B. But the image of a connected set under a continuous function cannot be disconnected; a contradiction. Therefore, it must be that r, the retraction, cannot exist.

$\square$

The No-Retraction Theorem proved above will be the cornerstone for the following proof for the Brouwer fixed-point theorem on $\mathbb{D}$.

2. Brouwer Fixed-Point Theorem on $\mathbb{D} \subset \mathbb{R}^2$: Given that function $f : \mathbb{D} \to \mathbb{D}$ is continuous, then there exists some $c \in \mathbb{D}$ such that $f(c) = c$; that is, c is a fixed point.

Proof: Let $\mathbb{D}$ be the unit disk in $\mathbb{R}^2$. Let $f : \mathbb{D} \to \mathbb{D}$ be continuous, but suppose that it does not have a fixed point. Now let $r : \mathbb{D} \to \mathbb{D}$ be another function that, for each $x \in \mathbb{D}$, assigns it to the tip of the ray that extends from the boundary of $\mathbb{D}$—the unit disk C—and passes through $f(x)$, then x This will be well-defined since $f(x) \neq x$ for all $x \in \mathbb{D}$. As r is defined in terms of f, and f is continuous, r will also be continuous.

However, consider x_0, a point which itself lies on C. In this situation, $r(x)$ must equal x, and thus, r is a retraction. But no such retraction can exist, due to the No-Retraction Theorem. This contradicts that f can exist as it is, with no fixed points.

Therefore, it must be so that any $f : \mathbb{D} \to \mathbb{D}$ must in fact have a fixed point.

$\square$

Again, this will also be true for any sets in $\mathbb{R}^2$ that are homeomorphic to $\mathbb{D}$—that is to say, compact convex sets. Thus, this actually satisfies any possible case of a compact convex set in $\mathbb{R}^2$.

5 General Proof

Now, we will move on to proving the Brouwer Fixed-Point Theorem in any-dimensional $\mathbb{R}^n$. First, however, a few things must be defined.

Definition 5.0.1. C^1**:** A C_1 function is continuous, and has a continuous derivative.

Theorem 5.1. Stone-Weierstrauss Theorem: Given a continuous function, it can be approximated to any degree with a subalgebra which separates points. That is, one may get as close as one likes to the original function. A polynomial—which is C^1—is a point-separating subalgebra; we will only be using Stone-Weirestrauss to give us polynomials; thus, while Stone-Weirestrauss does allow for other functions to be used as approximations, we will not mind those.

Theorem 5.2. Inverse Function Theorem: Let $\mathcal{X} \subseteq \mathbb{R}^n$ be open, and let function $f : \mathcal{X} \to \mathbb{R}^n$ be continuously differentiable and have that its derivative—expressed as a matrix of partial derivatives—is invertible at point $c \in \mathcal{X}$, then it is also invertible in a neighborhood about c.

First, as it is again such an integral part, we will prove the general-dimensional case of the no-retraction theorem, before using it on the Brouwer Fixed-Point Theorem. The addition of more dimensions requires changes to the theorem, as derivatives are now used to prove it and non-C^1 functions do not have usable derivatives. We proceed by assuming that such a retraction can exist, and then disproving its existence through contradiction.

Theorem 5.3. No Retraction Theorem: There can be no C^1 retraction from the unit n-dimensional ball $\mathbb{D}^n$ to its boundary, the unit 'sphere' B^{n-1}.

Proof: Let $r : \mathbb{D}^n \to B^{n-1}$ be a C^1 retraction from the unit n-dimensional disk $\mathbb{D}^n$ to its boundary, B^{n-1}. Let $g(x) = r(x) - x$, let $t \in [0,1]$ be fixed, and let $f_l(x) = x + lg(x) = x(1-l)+lr(x)$. For $x \in \mathbb{D}^n$, note be the Triangle inequality that $\|f_l(x)\| \leq \|x\|(1-l)+l\|r(x)\|$, because l and $1-l$ both have magnitude less than 1. Furthermore, because x and $r(x)$ must also have magnitude less than 1, $\|f_l(x)\| \leq (1-l)+l = 1$; this makes f_l a function from $\mathbb{D}^n$ to $\mathbb{D}^n$. Furthermore, $f_l(x) = x(1-l)+lr(x) = x(1-l)+lx = x$ if $x \in B^{n-1}$, due to r being a retraction. This makes all point of in B^{n-1} fixed point of f_l.

Since r is C^1, h must also be C^1, and there must then exist some C, a constant, where $\|g(x_2) - g(x_1)\| \leq C\|x_2 - x_1\|$.

Suppose that there are $x_1, x_2 \in \mathbb{D}^n$ with $x_1 \neq x_2$ but also with $f_l(x_1) = f_l(x_2)$. Using the definition of f_l, $f_l(x_1) = x_1 + lg(x_1) = x_2 + lg(x_2) = f_l(x_2)$, and from those we can derive $x_1 - x_2 = lg(x_2) - lg(x_1)$. Then, though, we have that $\|x_1 - x_2\| = l\|g(x_2) - g(x_1)\| \leq lC\|x_1 - x_2\|$, which means that $lC \geq 1$.

When $l < C^{-1}$, r_l must be injective, because in that case $lC < 1$ and $\|x_1 - x_2\| \leq lC|x_1 - x_2\|$ only if $x_1 - x_2 = 0$. Let $U_l = f_l[\mathbb{D}^n]$, and note that $f_l'(x) = (1,1,1,\ldots,1) + lg'(x)$. We also know, due to g being C^1, that there exists some l_0 for which f_l' has a positive determinant when expressed as a matrix of partials for all $l \leq l_0$. This allows for the use of the inverse function theorem, so f_l is also invertible near that point. This allows for U_l to be open for sufficiently small l, as the continuity of f makes its inverse an open map. Let $l \in [0, l_0]$ be fixed yet arbitrary from here on. We now have a bijection

However, suppose that $U_l = f_l(\mathbb{D}^n) \neq \mathbb{D}^n$. Clearly, $\mathbb{D}^n \subset U_l$, as f_l does not map outside of $\mathbb{D}^n$. It must then be so that the boundary of U_l will intersect the interior of $\mathbb{D}^n$—that is, the boundary of U_l must intersect a point that is not on the boundary of $\mathbb{D}^n$. Let us call that point x_0. We have compactness, and moreover sequential compactness. Since y_0 is in the boundary of U_l, it is in the closure of U_l, and it is then a limit point. We can then find a sequence in U_l that converges to y; let us define this sequence in $\mathbb{D}^n$ as $(x_n) \subset \mathbb{D}^n$ for which $f(x_n) \to y_0$. But, as we have compactness, we can find a convergent subsequence of (x_n) as well. Suppose that $x_{n_m} \to x_0$; since f is continuous, this means that $f(x_{n_m}) \to f(x_0)$. However, $f(x_n) \to y_0$, and so $f(x_0) = y_0$. Yet, y_0 cannot be in U_l, as U_l is open and thus cannot contain its boundary. It must then be so that x_0 is in B^{n-1} the boundary of $\mathbb{D}^n$; otherwise it could not map to the boundary of U_l. But, as we have a retraction, $f(x_0) = x_0$; therefore, $x_0 = y_0$. This, however, would imply that $y_0 \in B^{n-1}$, despite our initial condition that y_0 not be in the boundary of $\mathbb{D}^n$. Therefore, we have a contradiction, and so $f(\mathbb{D}^n) = U_l = \mathbb{D}^n$ for $l \in [0, l_0]$; that is, f_l is surjective. Thus, when $l \in [0, l_0]$ and $l < C^{-1}$, we have the f_l is both injective and surjective; it is a bijection. From here on, we will only consider f_l where it is a bijection.

Because we have f_l continuous, we can have $F : [0, l] \to \mathbb{R}$ defined by $F(l) = \displaystyle\int_{\mathbb{D}_n} \det f_l'(x)dx$. This is with $f_l' = (1, 1, 1, \ldots, 1) + lg'(x)$ being constructed as a (square) matrix. This will actually be n integrals, however, we will let dx serve as $dx_1 dx_2 dx_3 \ldots dx_n$ for these n dimensions. The determinant of a matrix can be written in the form of a polynomial. Note that F is a function of l (x being completely removed during the integration process), and so we can consider its determinant as a polynomial of l. But F is an integral of f_l, and it will grant the volume of $f_l(\mathbb{D}^n)$ (if $l < C^{-1}$). As $\mathbb{D}^n$ is a bijection, $f_l(\mathbb{D}^n) = \mathbb{D}^n$, and so this provides us a range for which the polynomial is constant. However, a polynomial that is constant on some interval is constant everywhere. We can now conlcude that $F(l)$ gives the volume of $\mathbb{D}^n$ for all $l \in [0, 1]$.

Of particular note is that $F(1)$ gives us this volume, and that this volume will be greater than 0. However, consider the inner product (sometimes called the dot product) of f_l with itself, notated $\langle f_l, f_l \rangle$. Note that $f_l(x) = f(x)$ when in B^{n-1} for any x; hence $\langle f_l, f_l \rangle$ for

$l = 1$ is simply $\|f_1(x)\| = 1$. Consider any arbitrary vector $v \in \mathbb{R}^n$; the inner product of $vf_1'(x)$ and $f(x)$, is equal to the derivative, with respect to t, of the inner product of ${}^1\!/_2\langle f_1(xt + vt), f_1(x + tv)\rangle$. However, this results in the derivative of ${}^1\!/_2(1)$, and the derivitive of a constant is always 0. From this, we can see that the determinant of f_1' itself will be 0, implying that $F(1) = 0$ However, that is in contradiction to the earlier claim that $F(1) > 0$. Therefore, it must be so that r, through which f is defined, cannot exist; there can be no C^1 retraction from the unit 'ball' $\mathbb{D}^n$ to its boundary B^{n-1}.

□

This is the proof outlined—as a lemma for the Milnor-Rogers proof of the Brouwer Fixed-Point Theorem—for the general-dimensional No-Retraction Theorem. It is the Milnor-Rogers method that I will follow. [7] [5] It is a topological method for proving the theorem; there are many others that are combinatorial. Let us state the theorem one more time:

3. Brouwer Fixed-Point Theorem on $\mathbb{D}^n \subset \mathbb{R}^n$: Given that function $f : \mathbb{D}^n \to DDD^n$ is continuous, then there exists some $c \in \mathbb{D}^n$ such that $f(c) = c$; that is, c is a fixed point.

Proof: Let ${}^\varepsilon\!/_2 > 0$ be fixed yet arbitrary, and let $f : \mathbb{D}^n \to \mathbb{D}^n$ be continuous; $\mathbb{D}^n$ is the n-dimensional unit 'ball', as before. The Stone-Weierstrauss Theorem gives a sequence of C^1 functions $p_l : \mathbb{D}^n \to \mathbb{R}^n$ where $\|p_l(x) - f(x)\| \leq {}^1\!/_l$ for all $x \in \mathbb{D}^n$, with $l \in \mathbb{N}$. Let $q_l = \left(1 + {}^1\!/_l\right)^{-1} p_l$ for each $l \in \mathbb{N}$. Then we have that $\|q_l(x) - f(x)\| = \left\|\left(1 + {}^1\!/_l\right)^{-1} p_l(x) - f(x)\right\| \leq 1 + {}^1\!/_l$ for any $x \in \mathbb{D}^n$, through substitution. We can choose, for ${}^\varepsilon\!/_2$, an L_1 so that $\|p_l(x) - f(x)\| \leq {}^\varepsilon\!/_3$ for all $l \geq L_1$. Let $L_2 = L_1 + 1$, then $\|p_l(x) - f(x)\| < {}^\varepsilon\!/_2$ for all $l \geq L_2$, for any $x \in \mathbb{D}^n$. Thus, $q_l \to f$ uniformly. It is, then, also so that subsequence $q_{l_k} \to f$ uniformly; let $L_2 = K_1$.

Let us define $h_l : \mathbb{D}^n \to B^{n-1}$ be the function that draws a straight line that touches $q_l(x)$, then x, and then returns the point where the line intersects B^{n-1}. For those h_l that have no fixed points, each is a C^1 map; it is derived from C^1 function q_l. However, it is also a retraction. It thus cannot exist, which means that q_l must have a fixed point, for all l, as otherwise h_l would be an impossible retraction.

Let $\{x_l\}_{l=1} \subset \mathbb{D}^n$ be the sequence of fixed points for q_l. Now, we are in a sequentially compact space, and thus $\{x_l\}$ must have a convergent subsequence. Let $x_{l_k k=m}$ converge to $x_0 \in \mathbb{D}^n$; then for all $^{\varepsilon}/_2$ there exists an $K_2 \in \mathbb{N}$ for which $\|x_{l_k} - x\| < {}^{\varepsilon}/_2$ for all $k \geq K_2$.

We can, then, combine these, and see that for any ε, there exists a $K = \max\{K_1, K_2\}$ for which $\|q_{l_k}(x_{l_k}) - f(x_0)\| < \varepsilon$ for all $k \geq K$. However, $q_{l_k}(x_{l_k})$ is fixed, so $q_{l_k}(x_{l_k}) = x_{l_k}$ for each k. Thus, we have $\|x_{l_k} - f(x_0)\| < \varepsilon$. However, $x_{l_k} \to x_0$. Therefore, $x_0 = f(x_0)$, and so f has a fixed point.

$\square$

Now, we have only shown this result for one particular set. However, it is homeomorphic to any other compact and convex set; thus, on all compact and convex sets K, the Brouwer Fixed-Point Theorem applies.

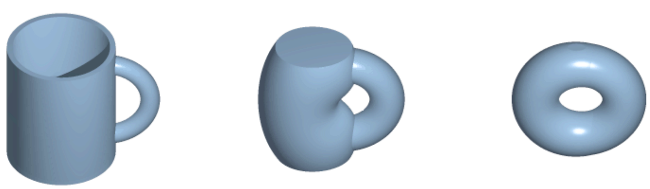

Figure 10: One of the more classic examples of two homeomorphic spaces, the coffee cup and the doughnut. Both are three-dimensional spaces with a hole in the middle. Neither is convex.

6 Applications

The Brouwer Fixed-Point Theorem is often used in the proving of the existance of Nash equilibriums. A Nash equilibirum occurs, in Game Theory, when the players know what strategies their opponents will use, know their strategies will not change, and also know that the current startegy they themselves are using is the best one to use. That is, both know what the other is planning to do, and both know that their own current plans are the best strategy considering what their opponent is planning. They are incredibly important in Game Theory, being used to analyze problems or games where the different players act near-simultaneously. A specific example of this is modeling the market; Nash Equilibriums are used to predict and model actions taken during market crises.

Another application is in Dynamical Systems. Equilibriums, stable or unstable can be considered to be fixed points. Thus, in certaint spaces, one is guaranteed to have an equilibria.

A particular application of this is to economics, this time more directly than through Nash Equilibriums. Fixed points are used to prove the existence of equilibria in the free market (for example, the meeting of supply and demand).

Other applications include coincidence theory and the Bass conjecture, and game theory in convex-valued multi-maps.

It is important to note, however, that not all of these rely on the Brouwer Fixed-Point Theorem. Indeed, his was not even the first fixed-point theorem. The Brouwer Theorem applies to any compact and convex space; not just the one that I have used, the standard Euclidian space. There are even more general theorems, however, such as the Banach and the Kakatani equivalent theorems. There has also been work done by Fan, and Browder. Furthermore, even many who work specifically with the Brouwer Fixed-Point Theorem my prefer to to use a Combinatorial lens instead

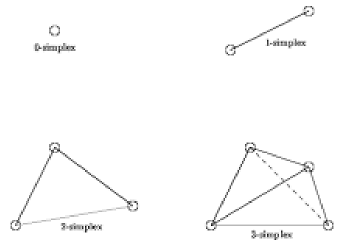

Figure 11: 1,2,3,and 4-simplexes, respectively.

References

[1] Kim C. Border. *Fixed Point Theorems with applications to Economics and Game Theory*. Cambridge University Press, 1985.

[2] J. L. Casti. *Five Golden Rules*. Wiley & Sons, 1996.

[3] Jack Coughlin. The no retraction theorem and a generalization. `https://www.math.washington.edu/~morrow/336_11/papers/jack.pdf`, May 20, 2011.

[4] Andrzej Granas. *Fixed Point Theory*. Springer, 2003.

[5] Ralph Howard. The milnor-rogers proof of the brouwer fixed point theorem. `http://people.math.sc.edu/howard/Notes/brouwer.pdf`, 2004.

[6] Jong Bum Lee. Topological fixed point theory. July 2013.

[7] John Milnor. Analytic proofs of the "hairy ball theorem" and the brouwer fixed point theorem. 1978.

[8] Sehie Park. Ninety years of the brouwer fixed point theorem. May 15, 1999.

[9] Matt Young. The stone-weierstrass theorem. `http://www.mast.queensu.ca/~speicher/Section14.pdf`.