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ORIGINAL RESEARCH

Robust Nash equilibria in vector-valued games with uncertainty

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Abstract
We study a vector-valued game with uncertainty in the pay-off functions. We reduce the notion of Nash equilibrium to a robust set optimization problem and we define accordingly the notions of robust Nash equilibria and weak robust Nash equilibria. Existence results for the latter are proved and a comparison between the former and the analogous notion in Yu and Liu (J Optim Theory Appl 159:272–280, 2013) is shown with an example. The proposed definition of weak robust Nash equilibrium is weaker than that already introduced in Yu and Liu (2013). On the contrary, the robust Nash equilibrium we introduce is not comparable with the notion of robust equilibrium in Yu and Liu (2013), that is defined componentwise. Nevertheless, by means of an example, we show that our notion has some advantages, avoiding some pitfalls that occurs with the other.

Keywords Vector-valued game · Uncertainty · Robust solution · Nash equilibria · Set optimization

Mathematics Subject Classification 91A10 · 93D09 · 90C29

1 Introduction

Since the seminal paper by Ben-Tal and Nemirovski (1998), robust optimization has gained an increasing interest as a tool to deal with scalar optimization problems where the objective function (and possibly also the constraints) are affected by uncertainty. More recently, the robust optimization approach has been extended to the vector case, see e.g. Kuroiwa and

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Lee (2012) and the references therein. One of the issues, when dealing with vector-valued problems with uncertainty, is the "worst case" may not be defined in a unique way. This has lead to different approaches to robust vector optimization. In the first one the "worst case" is considered componentwise, while in the second one set optimization theory is used to define a robust solution.

Recently, vector-valued games have been investigated, see e.g. Ding (2012), Ding (2006), Kim and Ding (2003), Levaggi and Pusillo (2017), Patriche (2014) and Pusillo (2017). Moreover robust optimization techniques have been applied to Game Theory. In Aghassi and Bertsimas (2006), matrix games with scalar uncertain payoffs have been considered. Further investigations can be found in Crespi et al. (2017b). Matrix games with multiobjective uncertain payoffs are investigated in Yu and Liu (2013). Here, the authors present the notions of robust and weak robust equilibria, using the componentwise approach to the "worst case". In this paper, we define new notions of equilibria in the sense of set optimization. We prove the existence of the weak robust Nash equilibria for matrix games with uncertain payoffs and we compare these notions with those proposed in Yu and Liu (2013).

The outline of the paper is the following. In Sect. 1 we recall the basic concepts of set optimization, in Sect. 2 we define the concept of Nash equilibrium in the sense of set optimization, while Sect. 3 is devoted to the existence result. Finally, in Sect. 4 the comparison with the equilibrium concept by Yu and Liu (2013) is considered.

2 Preliminaries

Let Y be a topological vector space, K a closed convex cone with nonempty interior, Y^* the continuous dual space of Y, $\langle y^*, y\rangle$ the value of $y^* \in Y^*$ at $y \in Y$ and K^+ the non-negative polar cone of K, that is, $K^+ = \{y^* \in Y^* \mid \langle y^*, y\rangle \geq 0, \forall y \in K\}$. We assume that $K^+ \neq \{0\}$. For any two given subsets $A,\ B \subseteq Y$, the set $A - B$ is defined as follows:

$$A - B = \{a - b \mid a \in A, b \in B\}.$$

Let $\mathcal{C}_0 := \mathcal{C}_0(Y)$ be the family of all nonempty compact subsets of Y. On the family $\mathcal{C}_0$, we define two binary relations as follows: For any $A, B \in \mathcal{C}_0$,

1. $A \preceq_K B$ if and only if $A \subset B - K$, and
2. $A \prec_K B$ if and only if $A \subset B - \mathrm{int}K$,

where $\mathrm{int}K$ is the interior of K. These are called u-types set relations, as part of the more general framework of set optimization given in Kuroiwa (2001).

These u-type set relations enjoy the following properties.

Lemma 2.1 *For any $A, B, C \in \mathcal{C}_0$,*

1. *$A \prec_K B$ implies $A \preceq_K C$,*
2. *$A \prec_K B$ and $B \preceq_K C$ implies $A \prec_K C$,*
3. *$A \prec_K B$ implies $\sup_{a\in A} \langle w, a\rangle < \sup_{b\in B} \langle w, b\rangle$ for any $w \in K^+ \setminus \{0\}$,*
4. *$A \not\prec_K A$.*

Proof Both statements 1 and 2 follow from the definitions, $\mathrm{int}K \subset K$ and $\mathrm{int}K + K = \mathrm{int}K$. Assume $A \prec_K B$ to show statement 3. Since A is compact and $B - \mathrm{int}K$ is open, there exists an open neighborhood V of 0 in Y such that $A + V \subseteq B - \mathrm{int}K$. For any $w \in K^+ \setminus \{0\}$,

$$\sup_{a\in A} \langle w, A\rangle < \sup_{a\in A, v\in V} \langle w, a + v\rangle \leq \sup_{b\in B, y\in \mathrm{int}K} \langle w, b - y\rangle \leq \sup_{b\in B} \langle w, b\rangle .$$

The first inequality follows from $\sup_{v \in V} \langle w, v \rangle > 0$ because $w \neq 0$. Statement 4 is clear from statement 3. $\square$

By using these set-relations, we consider the following vector-valued minimization problem with uncertainty:

$$\begin{array}{ll} \mathrm{P}_u & \text{Minimize } f(x, u), \quad x \in X \\ & \text{subject to } x \in X \end{array}$$

where $f : X \times \mathcal{U} \to Y$, X is a set, $\mathcal{U}$ is a set of all uncertainty, and $u \in \mathcal{U}$ is a certain realization. Except the trivial cases, 'the worst case' of possible values of $f(x, \cdot)$ may not be determined with a vector optimization approach. Kuroiwa and Lee (2012) studied the problem when $Y = \mathbb{R}^m$ and $K = \mathbb{R}^m_+$ and defined the robust solutions by considering the robust counterpart problem, i.e. an optimization problem where the objective function is given by componentwise maximization of $f(x, \cdot)$ with respect to $u \in \mathcal{U}$. Ide et al. (2014) stated robust solutions in the sense of set optimization, based on set-relations, as follows: let

$$f(x, \mathcal{U}) = \{f(x, u) \mid u \in \mathcal{U}\}, \quad x \in X,$$

represent all possible realizations of the loss function when decision x is assumed, and let $f(x, \mathcal{U}) \in \mathcal{C}_0$ for every $x \in X$. An element $x^* \in X$ is said to be a robust minimal solution if

$$f(x, \mathcal{U}) \preceq_K f(x^*, \mathcal{U}) \text{ implies } f(x^*, \mathcal{U}) \preceq_K f(x, \mathcal{U}),$$

and $x^* \in X$ is said to be a weak robust minimal solution if

$$\nexists x \in X \text{ such that } f(x, \mathcal{U}) \prec_K f(x^*, \mathcal{U}).$$

Properties of these solution concepts have been investigated in Crespi et al. (2017a).

The notions in Ide et al. (2014) are different from those in Kuroiwa and Lee (2012), see Suzuki and Kuroiwa (2018). The following relatioship between the two notions holds:

Lemma 2.2 *If x^* is a robust minimal solution, then x^* is also a robust weak minimal solution.*

Proof Assume that x^* is a robust minimal solution, but not a robust weak minimal solution. Then there exists $x \in X$ such that $f(x, \mathcal{U}) \prec_K f(x^*, \mathcal{U})$. Since $f(x, \mathcal{U}) \preceq_K f(x^*, \mathcal{U})$ also holds from 1 of Lemma 2.1 and x^* is a robust minimal solution, $f(x^*, \mathcal{U}) \preceq_K f(x, \mathcal{U})$ holds. This is a contradiction from 2 and 4 in Lemma 2.1. $\square$

We study robust Nash equilibria of a vector-valued game with uncertainty based on the above robust solutions in Sect. 4.

We conclude this preliminary section with the fixed point theorem we need to prove the existence of equilibria of games.

Lemma 2.3 (Kakutani–Glicksberg–Fan, Kakutani 1941; Glicksberg 1952; Fan 1952) *Let X be a compact convex subset of a Hausdorff locally convex topological vector space E. Assume that a set-valued map $F : X \rightrightarrows X$ is upper semicontinuous with nonempty compact convex values. Then there exists $x^* \in X$ such that $x^* \in F(x^*)$.*

3 A vector-valued game with uncertainty

Let $N = \{1, 2, \ldots, n\}$ be the set of players. For each $i \in N$, let X_i a nonempty compact convex subset of a Hausdorff locally convex topological vector space E_i, Y_i a topological vector space, and K_i a closed convex cone with nonempty interior in Y_i, and assume that $K_i^+ \neq \{0\}$. Let $\mathcal{U}$ be a nonempty set, which explains uncertainty, and $X = \prod_{i \in N} X_i$ the set of strategies of the players. For each $i \in N$, let f_i be a function from $X \times \mathcal{U}$ to Y_i, and assume that $f_i(x, \mathcal{U}) \in \mathcal{C}_0(Y_i)$ for all $x \in X$. A vector-valued game with uncertainty is given by the following quadruple form:

$$G = (N, \{X_i\}_{i \in N}, \{f_i\}_{i \in N}, \mathcal{U}).$$

For this game, Yu and Liu (2013) defined notions of robust Nash equilibria which are similar to the notions of robust solution in multiobjective optimization by Kuroiwa and Lee (2012). In this paper, we define other notions of robust Nash equilibria in the sense of set optimization. For $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in X$, $i \in N$, and $x_i \in X_i$, we set

$$(x_i, x_{-i}^*) = \left(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*\right).$$

Definition 3.1 An element $x^* = (x_1^*, \ldots, x_n^*) \in X$ is called

1. A robust Nash equilibrium in the sense of set optimization if for all $i \in N$, $f_i(x_i, x_{-i}^*, \mathcal{U}) \preceq_{K_i} f_i(x^*, \mathcal{U})$ implies $f_i(x^*, \mathcal{U}) \preceq_{K_i} f_i(x_i, x_{-i}^*, \mathcal{U})$,
2. A robust weak Nash equilibrium in the sense of set optimization if for all $i \in N$, there does not exist $x_i \in X_i$ such that $f_i(x_i, x_{-i}^*, \mathcal{U}) \prec_{K_i} f_i(x^*, \mathcal{U})$.

Remark 3.1 1. In the sequel, we will omit "in the sense of set optimization" if no confusion occurs.
2. The notions in Definition 3.1 are different from those in Yu and Liu (2013), see Sect. 5.
3. An element $x^* = (x_1^*, \ldots, x_n^*) \in X$ is a robust Nash or robust weak Nash equilibrium if and only if $x_i^* \in X_i$ is a minimal or a weak minimal solution, respectively, of

$$\begin{array}{ll} \mathrm{P}_i & \text{Minimize } f_i(x_i, x_{-i}^*, \mathcal{U}), \quad x_i \in X_i \\ & \text{subject to } x_i \in X_i \end{array}$$

for each $i \in N$.

From statement 3 of Remark 3.1 and Lemma 2.2, we have the following lemma immediately:

Lemma 3.1 *A robust Nash equilibrium is also a robust weak Nash equilibrium.*

4 The existence of Nash equilibria of vector-valued game with uncertainty

In this section we give existence theorems for Nash equilibria of the vector-valued game with uncertainty.

For a given $w_i \in K_i^+ \backslash \{0\}$, we define a set-valued map $C_i^{w_i} : X_{-i} \rightrightarrows 2^{X_i}$, $i \in N$, by

$$C_i^{w_i}(x_{-i}) = \left\{ x_i \in X_i \;\middle|\; \sup_{u \in \mathcal{U}} \langle w_i, f_i(x, u) \rangle \leq \inf_{z_i \in X_i} \sup_{u \in \mathcal{U}} \langle w_i, f_i(z_i, x_{-i}, u) \rangle \right\}.$$

Lemma 4.1 *Assume that $x_i^* \in C_i^{w_i}(x_{-i}^*)$ for all $i \in N$. Then $x^* = (x_1^*, \ldots, x_n^*)$ is a robust weak Nash equilibrium.*

Proof If $x^* = (x_1^*, \ldots, x_n^*)$ is not a robust weak Nash equilibrium, then there exist $i \in N$ and $x_i \in X_i$ such that $f_i(x_i, x_{-i}^*, \mathcal{U}) \prec_{K_i} f_i(x^*, \mathcal{U})$. By using 3 of Lemma 2.1, we have

$$\sup_{u\in\mathcal{U}} \left\langle w_i, f_i(x_i, x_{-i}^*, u)\right\rangle < \sup_{u\in\mathcal{U}} \left\langle w_i, f_i(x^*, u)\right\rangle.$$

From the assumption of this lemma, we have

$$\sup_{u\in\mathcal{U}} \left\langle w_i, f_i(x^*, u)\right\rangle \leq \inf_{z_i\in X_i} \sup_{u\in\mathcal{U}} \left\langle w_i, f_i(z_i, x_{-i}^*, u)\right\rangle \leq \sup_{u\in\mathcal{U}} \left\langle w_i, f_i(x_i, x_{-i}^*, u)\right\rangle.$$

This is a contradiction. □

Lemma 4.2 *Assume that X_i and $\mathcal{U}$ are compact. If $(x, u) \mapsto \langle w_i, f_i(x, u)\rangle$ is continuous, then $C_i^{w_i}$ is a nonempty compact-valued upper semicontinuous set-valued map.*

Proof At first, we can see that the function $x \mapsto \sup_{u\in\mathcal{U}} \langle w_i, f_i(x, u)\rangle$ is continuous, see (Berge 1963). Since X_i is compact, it is easy to see that $C_i^{w_i}(x_{-i})$ is a nonempty compact subset for each $x_{-i} \in X_{-i}$. Next we show that $C_i^{w_i}$ is upper semicontinuous. Choose an arbitrary $x_{-i} \in X_{-i}$ and an open neighborhood U_i of the origin in E_i. To show upper semicontinuity of $C_i^{w_i}$ at x_{-i}, we must show that there exists a neighborhood $V(x_{-i})$ of x_{-i} such that for any $z_{-i} \in V(x_{-i})$, $C_i^{w_i}(z_{-i}) \subset C_i^{w_i}(x_{-i}) + U_i$. If not, there is a net $\{x_{-i}^\alpha\} \subset X_{-i}$ converging to x_{-i} such that $C_i^{w_i}(x_{-i}^\alpha) \not\subset C_i^{w_i}(x_{-i}) + U_i$. Also we can find a net $\{x_i^\alpha\}$ such that $x_i^\alpha \in C_i^{w_i}(x_{-i}^\alpha)$ and $x_i^\alpha \notin C_i^{w_i}(x_{-i}) + U_i$. Since X is compact and $\{x^\alpha\} \subset X$, we may assume that $\{x^\alpha\}$ converges to some $x \in X$ without loss of generality. From $x_i^\alpha \in C_i^{w_i}(x_{-i}^\alpha)$,

$$\sup_{u\in\mathcal{U}} \left\langle w_i, f_i(x^\alpha, u)\right\rangle \leq \sup_{u\in\mathcal{U}} \left\langle w_i, f_i(z_i, x_{-i}^\alpha, u)\right\rangle$$

for any $z_i \in X_i$. Since $\{x^\alpha\}$ converges to x and $\sup_{u\in\mathcal{U}} \langle w_i, f_i(\cdot, u)\rangle$ is continuous, we have

$$\sup_{u\in\mathcal{U}} \langle w_i, f_i(x, u)\rangle \leq \sup_{u\in\mathcal{U}} \langle w_i, f_i(z_i, x_{-i}, u)\rangle$$

for any $z_i \in X_i$, that is $x_i \in C_i^{w_i}(x_{-i})$, which contradicts to $x_i^\alpha \notin C_i^{w_i}(x_{-i}) + U_i$ and $x_i^\alpha \to x_i$. Hence C_i is upper semicontinuous on X_{-i}. □

Lemma 4.3 *If X_i is convex and $x_i \mapsto \langle w_i, f_i(x_i, x_{-i}, u)\rangle$ is quasiconvex for each $(x_{-i}, u) \in X_{-i} \times \mathcal{U}$, then $C_i^{w_i}(x_{-i})$ is convex for each $x_{-i} \in X_{-i}$.*

Proof From the assumption, the map $x_i \mapsto \sup_{u\in\mathcal{U}} \langle w_i, f_i(x_i, x_{-i}, u)\rangle$ is quasiconvex for each $x_{-i} \in X_{-i}$. Since $C_i^{w_i}(x_{-i})$ is a sublevel set of this map, it is convex. □

Theorem 4.1 *Let X_i be a nonempty compact convex subset of a Hausdorff locally convex topological vector space E_i, Y_i a topological vector space, and K_i a closed convex cone with nonempty interior in Y_i, for each $i \in N$, and $\mathcal{U}$ a compact set. Assume that there exist $w_1 \in K_1^+ \setminus \{0\}, \ldots, w_n \in K_n^+ \setminus \{0\}$ such that $(x, u) \mapsto \langle w_i, f_i(x, u)\rangle$ is continuous and $x_i \mapsto \langle w_i, f_i(x_i, x_{-i}, u)\rangle$ is quasiconvex for each $(x_{-i}, u) \in X_{-i} \times \mathcal{U}$, for every $i \in N$. Then there exists a robust weak Nash equilibrium.*

Proof Let $w = (w_1, \ldots, w_n)$ and define a set-valued map $C^w : X \rightrightarrows X$ by

$$C^w(x) = C_1^{w_i}(x_{-1}) \times C_2^{w_i}(x_{-2}) \times \cdots \times C_n^{w_i}(x_{-n}).$$

From Lemmas 4.2 and 4.3, C^w is upper semicontinuous and nonempty compact convex valued. By using Lemma 2.3, there exists $x^* = (x_1^*, \ldots, x_n^*) \in X$ such that $x^* \in C^w(x^*)$. This shows $x_i^* \in C_i^{w_i}(x_{-i}^*)$ for any $i \in N$, that is, x^* is a robust weak Nash equilibrium from Lemma 4.1. □

By using the previous result, we can show an existence theorem for a vector-valued matrix game with uncertainty. Let $N = \{1, 2, \dots, n\}$, the number of players, $m_i \in \mathbb{N}$, the number of the strategies for player $i \in N$, and $\mathcal{U}$ a nonempty set, the set of possible uncertainties. Let $a_i(s^1, s^2, \dots, s^n, u) \in Y_i$ be the vector-valued loss of player $i \in N$ when player i chooses strategy s^i and a realization $u \in \mathcal{U}$ occurs, where Y_i is a topological vector space, for each $i \in N$. Also let K_i be a closed convex cone with nonempty interior in Y_i representing the preference of player i, and assume that $K_i^+ \neq \{0\}$. for each $i \in N$. Since in this game we consider mixed strategies, put $X_i = \operatorname{conv}\{e_1, \dots, e_{m_i}\}$, where $\{e_1, \dots, e_{m_i}\}$ is the standard basis for $\mathbb{R}^{m_i}$. Now we define the vector-valued loss function for player $i \in N$ as follows:

$$f_i(x, u) = \sum_{j_1=1}^{m_1} \cdots \sum_{j_n=1}^{m_n} x_{j_1}^1 \cdots x_{j_n}^n a_i\left(s_{j_1}^1, \dots, s_{j_n}^n, u\right) \tag{1}$$

where $x_i = \sum_{j_i=1}^{m_i} x_{j_i}^i e_{j_i} \in X_i$, $\sum_{j_i=1}^{m_i} x_{j_i}^i = 1$, $x_{j_i}^i \geq 0$, and $u \in \mathcal{U}$. Then we have the following existence theorem:

Theorem 4.2 *Assume that $\mathcal{U}$ is a compact set and $u \mapsto a_i(s^1, s^2, \dots, s^n, u)$ is continuous for every $i \in N$ and $s^1, s^2, \dots, s^n$. Then the game $(N, \{f_i\}, \{X_i\}, \mathcal{U})$ has a robust weak Nash equilibrium.*

Proof For each $i \in N$ and $w_i \in K_i^+ \setminus \{0\}$, the function

$$(x, u) \mapsto \langle w_i, f_i(x, u)\rangle$$

is continuous, and the function

$$x_i \mapsto \langle w_i, f_i(x_i, x_{-i}, u)\rangle$$

is affine, that is quasiconvex, for every $(x_{-i}, u) \in X_{-i} \times \mathcal{U}$. By using Theorem 4.1, we conclude that there exists a robust weak Nash equilibrium. □

Remark 4.1 If $\mathcal{U}$ is finite, since the assumption of Theorem 4.2 holds, then the game $(N, \{f_i\}, \{X_i\}, \mathcal{U})$ has a robust weak Nash equilibrium.

5 Observations the Nash equilibria in vector-valued game with uncertainty

At first we introduce the robust Nash equilibria in vector-valued games with uncertainty defined in Yu and Liu (2013) to compare these notions with our notions. In this section, we consider the vector-valued matrix game with uncertainty defined in Sect. 4, and assume the same situation in (1). Additionally, we need the assumptions $Y_i = \mathbb{R}^{l_i}$ and $K_i = \mathbb{R}_+^{l_i}$, and define

$$\varphi_i(x) = (\sup_{u \in \mathcal{U}} f_{i,1}(x, u), \dots, \sup_{u \in \mathcal{U}} f_{i,l_i}(x, u)),$$

where $f_i(x, u) = (f_{i,1}(x, u), \dots, f_{i,l_i}(x, u))$, for all $i \in N$. Now we give the definitions in Yu and Liu (2013). An element $x^* = (x_1^*, \dots, x_n^*) \in X$ is called

1. A robust Nash equilibrium in the sense of Yu and Liu (2013) if for all $i \in N$, $\varphi_i(x_i, x_{-i}^*) \in \varphi_i(x^*) - \mathbb{R}_+^{l_i}$ implies $\varphi_i(x^*) \in \varphi_i(x_i, x_{-i}^*) - \mathbb{R}_+^{l_i}$,

2. A robust weak Nash equilibrium in the sense of Yu and Liu (2013) if for all $i \in N$, there does not exist $x_i \in X_i$ such that $\varphi_i(x_i, x^*_{-i}) \in \varphi_i(x^*) - \mathrm{int}\mathbb{R}^{l_i}_+$.

Remark 5.1 A robust weak Nash equilibrium in the sense of Yu and Liu (2013) is also a robust weak Nash equilibrium in the sense of set optimization. We give a proof by contraposition. If $x^* = (x^*_1, \ldots, x^*_n) \in X$ is not a robust weak Nash equilibrium in the sense of set optimization, then there exist $i \in N$ and $x \in X$ such that $f_i(x_i, x^*_{-i}, \mathcal{U}) \prec_{K_i} f_i(x^*, \mathcal{U})$, that is, $f_i(x_i, x^*_{-i}, \mathcal{U}) \subset f_i(x^*, \mathcal{U}) - \mathrm{int}\mathbb{R}^{l_i}_+$. This implies that $\varphi_i(x_i, x^*_{-i}) \in \varphi_i(x^*, \mathcal{U}) - \mathrm{int}\mathbb{R}^{l_i}_+$, therefore, $x^* = (x^*_1, \ldots, x^*_n) \in X$ is not a robust weak Nash equilibrium in the sense of Yu and Liu (2013). But the reverse implication does not hold in general. See the following example:

Example 5.1 Let us consider a game with two players which have two strategies, two loss objectives, and two uncertainties as follows:

f_1 :

	s^2_0	s^2_1
s^1_0	$(1-u)(1,2) + u(2,1)$	$(1-u)(2,3) + u(3,2)$
s^1_1	$(1-u)(3,0) + u(0,3)$	$(1-u)(4,1) + u(1,4)$

f_2 :

	s^2_0	s^2_1
s^1_0	$(1-u)(3,2) + u(2,3)$	$(1-u)(2,1) + u(1,2)$
s^1_1	$(1-u)(1,4) + u(4,1)$	$(1-u)(0,3) + u(3,0)$

where $u \in \mathcal{U} = \{0, 1\}$. Then we have

$f_1(\cdot, \cdot, \mathcal{U})$:

	s^2_0	s^2_1
s^1_0	$\{(1,2), (2,1)\}$	$\{(2,3), (3,2)\}$
s^1_1	$\{(3,0), (0,3)\}$	$\{(4,1), (1,4)\}$

$f_2(\cdot, \cdot, \mathcal{U})$:

	s^2_0	s^2_1
s^1_0	$\{(3,2), (2,3)\}$	$\{(2,1), (1,2)\}$
s^1_1	$\{(1,4), (4,1)\}$	$\{(0,3), (3,0)\}$

In this game, just pairs (s^1_0, s^2_1) and (s^1_1, s^2_1) are robust Nash equilibria and robust weak Nash equilibria in the sense of set optimization. Now we observe the result. Since s^2_0 is dominated by s^2_1 in $f_2(\cdot, \cdot, \mathcal{U})$, Player 2 should choose s^2_1. Then Player 1's losses will be $(2,3)$, $(3,2)$, $(4,1)$, or $(1,4)$. But we could not compare these vectors because these are in the trade-off relationship. So it is very natural that the robust Nash equilibria are (s^1_0, s^2_1) and (s^1_1, s^2_1). On the other hand, since

φ_1 :

	s^2_0	s^2_1
s^1_0	$(2,2)$	$(3,3)$
s^1_1	$(3,3)$	$(4,4)$

φ_2 :

	s^2_0	s^2_1
s^1_0	$(3,3)$	$(2,2)$
s^1_1	$(4,4)$	$(3,3)$

just (s^1_0, s^2_1) is the robust Nash equilibrium and robust weak Nash equilibrium in the sense of Yu and Liu (2013), but (s^1_1, s^2_1) is not. This shows that the reverse implication of Remark 5.1 does not hold in general.

Remark 5.2 Robust Nash equilibria in the sense of set optimization and robust Nash equilibria in the sense of Yu and Liu (2013) have not any relations, that is, both implications do not hold in general. It is clear that the notion in the sense of set optimization does not imply the one of Yu and Liu (2013) from Example 5.1. The reverse implication also fails. See the following example:

Example 5.2 Let us consider a game with two players which have two strategies, two loss objectives, and two uncertainties as follows:

$f_1 = f_2$:

	s_0^2	s_1^2
s_0^1	$(1-u)(4,0)+u(0,4)$	$(1-u)(4,1)+u(1,4)$
s_1^1	$(1-u)(4,2)+u(2,4)$	$(1-u)(4,3)+u(3,4)$

where $u \in \mathcal{U} = \{0, 1\}$. Then we have

$f_1(\cdot,\cdot,\mathcal{U}) = f_2(\cdot,\cdot,\mathcal{U})$:

	s_0^2	s_1^2
s_0^1	$\{(4,0),(0,4)\}$	$\{(4,1),(1,4)\}$
s_1^1	$\{(4,2),(2,4)\}$	$\{(4,3),(3,4)\}$

and then, only (s_0^1, s_0^2) is the robust Nash equilibrium in the sense of set optimization. The result should be natural. On the other hand, since

$\varphi_1 = \varphi_2$:

	s_0^2	s_1^2
s_0^1	$(4,4)$	$(4,4)$
s_1^1	$(4,4)$	$(4,4)$

that is, φ_1 and φ_2 are constant, all pairs (s_0^1, s_0^2), (s_0^1, s_1^2), (s_1^1, s_0^2), and (s_1^1, s_1^2) are the robust Nash equilibrium in the sense of Yu and Liu (2013). The essence of this problem was hidden by componentwise maximization in $u \in \mathcal{U}$ in this case.

6 Conclusions

In vector-valued games the notion of Nash equilibria has been studied in several papers, see e.g. Ding (2012, 2006), Kim and Ding (2003), Levaggi and Pusillo (2017), Patriche (2014) and Pusillo (2017). However, fewer results have been published when uncertainty on the pay-off functions is involved, see e.g. Yu and Liu (2013). The paper proposes a robust optimization approach to the solution of these families of games, introducing the notion of robust weak Nash equilibrium. We prove the existence of such a solution for general vector-valued games with n players and a compact uncertainty set. To obtain our main result, we apply linear scalarization to the vector valued pay-off functions, requesting some mild regularity of the scalarized function.

Moreover, the notion of equilibrium can be strengthened in the form of robust Nash equilibrium providing two new concepts of Nash equilibria to be compared with those proposed in Yu and Liu (2013). The different approach we follow in the definitions makes the robust weak Nash equilibrium in this paper more general than the weak robust equilibrium in Definition 2.1 of Yu and Liu (2013).

On the other side, the componentwise approach used in Definition 2.1 of Yu and Liu (2013) to introduce their notion of robust equilibrium, allows for pathological games where every strategy becomes a Nash equilibrium. We manage to overcome this pitfall with our definition of robust Nash equilibrium, as shown in Example 5.2. However this makes the two stronger notions incomparable, while both sounds reasonable and have their advantages.

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