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# EXISTENCE, UNIQUENESS, AND COMPUTATION OF ROBUST NASH EQUILIBRIA IN A CLASS OF MULTI-LEADER-FOLLOWER GAMES*

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**Abstract.** The multi-leader-follower game can be looked on as a generalization of the Nash equilibrium problem, which contains several leaders and followers. Recently, the multi-leader-follower game has been drawing more and more attention, for example, in power markets. On the other hand, in such real-world problems, uncertainty normally exists and sometimes cannot simply be ignored. To handle mathematical programming problems with uncertainty, the robust optimization technique assumes that the uncertain data belong to some sets, and the objective function is minimized with respect to the worst-case scenario. In this paper, we focus on a class of multi-leader single-follower games under uncertainty with some special structure. We particularly assume that the follower's problem contains only equality constraints. By means of the robust optimization technique, we first formulate the game as the robust Nash equilibrium problem and then as the generalized variational inequality (GVI) problem. We then establish some results on the existence and uniqueness of a robust leader-follower (L/F) Nash equilibrium. We also apply the forward-backward splitting method to solve the GVI formulation of the problem and present some numerical examples, including the one with multiple followers, to illustrate the behavior of robust L/F Nash equilibria.

**Key words.** robust optimization, Nash equilibrium problem, multi-leader-follower game, generalized variational inequality problem

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## 1. Introduction.

As a solid mathematical methodology to deal with many social problems, such as economics, management, and political science, game theory studies the strategic solutions where an individual makes a choice by taking into account the others' choices. Game theory was developed in 1950 when John Nash introduced the well-known concept of Nash equilibrium in noncooperative games [28, 29], which means no player can obtain any more benefit by changing his/her current strategy unilaterally (other players keep their current strategies). Since then, the Nash equilibrium problem (NEP), or the Nash game, has received a lot of academic attention from researchers. It has also been playing an important role in many application areas of economics, engineering, and so on [4, 12, 36].

The multi-leader-follower game can be looked on as a generalization of the NEP, which arises in various real-world conflict situations such as oligopolistic competition in a deregulated electricity market. It may further be divided into that which contains only one follower, called the multi-leader single-follower game, and that which contains multiple followers, called the multi-leader multi-follower game. In the multi-leader-follower game, several distinctive players called the leaders solve their own optimization problems in the upper level, where the leaders compete in a Nash game. At the same time, given the leaders' strategies, the remaining players, called the

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followers, also solve their own optimization problems in the lower level, where the followers also compete in a Nash game which is parameterized by the strategy tuple of the leaders. In particular, the leaders can anticipate the responses of the followers and then use this ability to select their own optimal strategies. On the other hand, each follower selects his/her optimal strategy responding to the strategies of the leaders and the other followers. When no player can improve his/her status by changing strategy unilaterally, we call the current set of leaders' and followers' strategies a leader-follower (L/F) Nash equilibrium.

The multi-leader-follower game has been studied by some researchers and used to model several problems in applications. A particular type of multi-leader multi-follower game was first studied by Sherali [35], who established an existence result about the equilibrium by assuming that each leader can exactly anticipate the aggregate follower reaction curve. Sherali [35] also ensured the uniqueness of equilibrium for a special case where all leaders share an identical cost function. Su [38] considered a forward market equilibrium model, extending the existence result of Sherali [35] under some weaker assumptions. Pang and Fukushima [31] introduced a class of remedial models for the multi-leader-follower game that can be formulated as a generalized Nash equilibrium problem (GNEP) with convexified strategy sets. Moreover, they also proposed some oligopolistic competition models in electricity power markets that lead to multi-leader-follower games. Based on the strong stationarity conditions of each leader in a multi-leader-follower game, Leyffer and Munson [26] derived a family of nonlinear complementarity problem (NCP), nonlinear program (NLP), and mathematical program with equilibrium constraints (MPEC) formulations of the multi-leader-follower games. They also reformulated the game as a square nonlinear complementarity problem by imposing an additional restriction. By considering the equivalent implicit program formulation, Hu and Ralph [23] established an existence result about the equilibrium of a multi-leader multi-follower game which arose from a restructured electricity market model.

In the above mentioned two equilibrium concepts, Nash equilibrium and L/F Nash equilibrium, each player is assumed to have complete information about the game. This means that in a NEP, each player can observe his/her opponents' strategies and choose his/her own strategy exactly, while in a multi-leader-follower game, each leader can anticipate each follower's response to the leaders' strategies exactly. However, in many real-world problems, such strong assumptions are not always satisfied. Another kind of game with uncertain data and the corresponding concept of equilibria needs to be considered.

There has been important work about games with uncertain data. Under the assumption on probability distributions called the Bayesian hypothesis, Harsanyi [18, 19, 20] considered a game with incomplete information, where the players have no complete information about some important parameters of the game. Further assuming all players share some common knowledge about those probability distributions, the game was finally reformulated as a game with complete information, called the Bayes equivalent of the original game. DeMiguel and Xu [10] considered a stochastic multi-leader multi-follower game applied in the telecommunications industry and established the existence and uniqueness of the equilibrium. Shanbhag, Infanger, and Glynn [34] considered a class of stochastic multi-leader multi-follower games and established the existence of a local equilibrium by a related simultaneous stochastic Nash game.

Besides the probability distribution models, the distribution-free models based on the worst-case scenario have received attention in recent years [1, 21, 30]. In the latter

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models, each player makes a decision according to the concept of robust optimization [5, 6, 7, 11]. Basically, in robust optimization, uncertain data are assumed to belong to some set called an uncertainty set, and then a solution is sought by taking into account the worst case in terms of the objective function value and/or the constraint violation. In a NEP containing some uncertain parameters, we may also define an equilibrium called robust Nash equilibrium. Namely, if each player has chosen a strategy pessimistically and no player can obtain more benefit by changing his/her own current strategy unilaterally (i.e., the other players hold their current strategies), then the tuple of the current strategies of all players is defined as a robust Nash equilibrium, and the problem of finding a robust Nash equilibrium is called a robust NEP. Such an equilibrium problem was studied by Hayashi, Yamashita, and Fukushima [21], who considered the bimatrix game with uncertain data and proposed a new concept of equilibrium called robust Nash equilibrium. Under some assumptions on the uncertainty sets, they presented some existence results about robust Nash equilibria. Furthermore, the authors showed that such a robust NEP can be reformulated as a second-order cone complementarity problem (SOCCP) by converting each player's problem into a second-order cone program. Aghassi and Bertsimas [1] considered a robust Nash equilibrium in an $N$-person NEP with bounded polyhedral uncertainty sets, where each player solves a linear programming problem. They also proposed a method of computing robust Nash equilibria. Note that both of these models [1, 21] particularly deal with linear objective functions in players' optimization problems.

More recently, Nishimura, Hayashi, and Fukushima [30] considered a more general NEP with uncertain data, where each player solves an optimization problem with a nonlinear objective function. Under some mild assumptions on the uncertainty sets, the authors presented some results about the existence and uniqueness of the robust Nash equilibrium. They also proposed to compute a robust Nash equilibrium by reformulating the problem as an SOCCP.

In this paper, inspired by previous work on the robust NEP, we extend the idea of robust optimization for the NEP to the multi-leader single-follower game.[1] We propose a new concept of equilibrium for the multi-leader single-follower game with uncertain data, called robust L/F Nash equilibrium. In particular, we show some results about the existence and uniqueness of the robust L/F Nash equilibrium. We also consider computation of the equilibrium by reformulating the problem as a generalized variational inequality (GVI) problem. The idea for this paper also comes from Hu and Fukushima [22], who considered a class of multi-leader single-follower games with complete information and showed some existence and uniqueness results for the L/F Nash equilibrium by way of the VI formulation. A remarkable feature of the multi-leader single-follower game studied in this paper is that the leaders anticipate the follower's response under their respective uncertain circumstances, and hence the follower's responses estimated by the leaders are generally different from each other.

The organization of this paper is as follows. In the next section, we describe the robust multi-leader single-follower game and define the corresponding robust L/F Nash equilibrium. In section 3, we show sufficient conditions to guarantee the existence of a robust L/F Nash equilibrium by reformulating it as a robust NEP. In section 4, we consider a particular class of robust multi-leader single-follower games

[1] We will focus on the multi-leader single-follower game. This is, however, for simplicity of presentation. In fact, the obtained results can naturally be extended to some multi-leader multi-follower games with considerable notational complication. See Remark 3.1.

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with uncertain data and discuss the uniqueness of the robust Nash equilibrium by way of the GVI formulation. In section 5, we show results of numerical experiments where the GVI formulation is solved by the forward-backward splitting method. Finally, we conclude the paper in section 6.

Throughout this paper, we use the following notation. The gradient $\nabla f(x)$ of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is regarded as a column vector. For any set $X$, $\mathcal{P}(X)$ denotes the set comprising of all the subsets of $X$. $\mathbb{R}^n_+$ denotes the $n$-dimensional nonnegative orthant in $\mathbb{R}^n$, that is, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n | x_i \geq 0, i = 1, \ldots, n\}$. For any vector $x \in \mathbb{R}^n$, its Euclidean norm is denoted by $\|x\| := \sqrt{x^\top x}$. If a vector $x$ consists of several subvectors, $x^1, \ldots, x^N$, it is denoted for simplicity of notation as $(x^1, \ldots, x^N)$ instead of $((x^1)^\top, \ldots, (x^N)^\top)^\top$.

## 2. Preliminaries.

### 2.1. NEPs with uncertainty.

In this subsection, we describe the NEP with uncertainty and its solution concept, robust Nash equilibrium. First, we introduce the NEP and Nash equilibrium.

In a NEP, there are $N$ players labeled by integers $\nu = 1, \ldots, N$. Player $\nu$'s strategy is denoted by vector $x^\nu \in \mathbb{R}^{n_\nu}$ and his/her cost function $\theta_\nu(x)$ depends on all players' strategies, which are collectively denoted by the vector $x \in \mathbb{R}^n$ consisting of subvectors $x^\nu \in \mathbb{R}^{n_\nu}$, $\nu = 1, \ldots, N$, and $n := n_1 + \cdots + n_N$. Player $\nu$'s strategy set $X^\nu \subseteq \mathbb{R}^{n_\nu}$ is independent of the other players' strategies, which are denoted collectively as $x^{-\nu} := (x^1, \ldots, x^{\nu-1}, x^{\nu+1}, \ldots, x^N) \in \mathbb{R}^{n_{-\nu}}$, where $n_{-\nu} := n - n_\nu$. For every fixed but arbitrary vector $x^{-\nu} \in X^{-\nu} := \prod_{\nu'=1, \nu' \neq \nu}^N X^{\nu'}$, which consists of all the other players' strategies, player $\nu$ solves the following optimization problem for his own variable $x^\nu$:

$$
\begin{aligned}
&\underset{x^\nu}{\text{minimize}} \quad && \theta_\nu(x^\nu, x^{-\nu}) \\
&\text{subject to} \quad && x^\nu \in X^\nu,
\end{aligned}
\tag{2.1}
$$

where we denote $\theta_\nu(x) = \theta_\nu(x^\nu, x^{-\nu})$ to emphasize the particular role of $x^\nu$ in this problem. A tuple of strategies $x^* := (x^{*,\nu})_{\nu=1}^N \in X := \prod_{\nu=1}^N X^\nu$ is called a Nash equilibrium if for all $\nu = 1, \ldots, N$,

$$
\theta_\nu(x^{*,\nu}, x^{*,-\nu}) \leq \theta_\nu(x^\nu, x^{*,-\nu}) \quad \text{for all } x^\nu \in X^\nu.
$$

For the $N$-person noncooperative NEP, we have the following well-known result about the existence of a Nash equilibrium.

Lemma 2.1 (see [2, Theorem 9.1.1]). *Suppose that for each player $\nu$,*

(a) *the strategy set $X^\nu$ is nonempty, convex, and compact;*

(b) *the objective function $\theta_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \to \mathbb{R}$ is continuous;*

(c) *the function $\theta_\nu$ is convex with respect to $x^\nu$.*

*Then, the NEP comprising of the players' problems* (2.1) *has at least one Nash equilibrium.*

In the NEP with complete information, all players are in an equal position. Nash equilibrium is well-defined when all players seek their own optimal strategies simultaneously by observing and estimating the opponents' strategies, as well as the values of their own objective functions, exactly. However, in many real-world models, such information may contain some uncertain parameters because of observation errors or estimation errors.

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To deal with some uncertainty in the NEP, Nishimura, Hayashi, and Fukushima [30] considered a NEP with uncertainty and defined the corresponding equilibrium called robust Nash equilibrium, which we briefly explain under the following assumption: A parameter $u^\nu \in \mathbb{R}^{l_\nu}$ is involved in player $\nu$'s objective function, which is now expressed as $\theta_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \times \mathbb{R}^{l_\nu} \to \mathbb{R}$. Although the player $\nu$ does not know the exact value of parameter $u^\nu$, he/she can confirm that it must belong to a given nonempty set $U^\nu \subseteq \mathbb{R}^{l_\nu}$.

Then, player $\nu$ solves the following optimization problem with parameter $u^\nu$ for his/her own variable $x^\nu$:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} \quad \theta_\nu(x^\nu, x^{-\nu}, u^\nu) \\ &\text{subject to} \quad x^\nu \in X^\nu, \end{aligned} \tag{2.2}$$

where $u^\nu \in U^\nu$. According to the robust optimization paradigm, we assume that each player $\nu$ tries to minimize the worst value of his/her objective function. Under this assumption, each player $\nu$ considers the worst cost function $\tilde{\theta}_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \to (-\infty, +\infty]$ defined by

$$\tilde{\theta}_\nu(x^\nu, x^{-\nu}) := \sup\{\theta_\nu(x^\nu, x^{-\nu}, u^\nu) \mid u^\nu \in U^\nu\}$$

and solves the following optimization problem:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} \quad \tilde{\theta}_\nu(x^\nu, x^{-\nu}) \\ &\text{subject to} \quad x^\nu \in X^\nu. \end{aligned} \tag{2.3}$$

Since this is regarded as a NEP with complete information, we can define the equilibrium of the NEP with uncertain parameters as follows.

DEFINITION 2.2. *A strategy tuple $x = (x^\nu)_{\nu=1}^N$ is called a robust Nash equilibrium of the noncooperative game comprising problems* (2.2) *if $x$ is a Nash equilibrium of the NEP comprising problems* (2.3).

**2.2. Multi-leader single-follower games with uncertainty.** In this subsection, we describe a multi-leader single-follower game with uncertainty and then define the corresponding robust L/F Nash equilibrium based on the above discussions about the robust Nash equilibrium.

First, we introduce the multi-leader single-follower game. Let $X^\nu \subseteq \mathbb{R}^{n_\nu}$ denote the strategy set of leader $\nu$, $\nu = 1, \dots, N$. We assume that the strategy set of each leader is independent of the other rival leaders. We also denote each leader's objective function by $\theta_\nu(x^\nu, x^{-\nu}, y)$, $\nu = 1, \dots, N$, which is dependent of his/her own strategy $x^\nu$ and all the other rival leaders' strategies $x^{-\nu} \in X^{-\nu} := \prod_{\nu'=1,\nu'\neq\nu}^N X^{\nu'}$, as well as the follower's strategy, denoted by $y$.

Let $\gamma(x, y)$ and $K(x)$ denote, respectively, the follower's objective function and strategy set that depend on the leaders' strategies $x = (x^\nu)_{\nu=1}^N$. For given strategies $x$ of the leaders, the follower chooses his/her strategy by solving the following optimization problem for variable $y$:

$$\begin{aligned} &\underset{y}{\text{minimize}} \quad \gamma(x, y) \\ &\text{subject to} \quad y \in K(x). \end{aligned} \tag{2.4}$$

For the multi-leader single-follower game described above, we can define L/F Nash equilibrium [22] under the assumption that all the leaders can anticipate the follower's

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responses, observe and estimate their opponents' strategies, and evaluate their own objective functions exactly. However, in many real-world models, the information may contain uncertainty, due to some observation errors or estimation errors. In this paper, we particularly consider a multi-leader single-follower game with uncertainty, where each leader $\nu = 1, \dots, N$ tries to solve the following uncertain optimization problem for his/her own variable $x^\nu$:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} \quad \theta_\nu(x^\nu, x^{-\nu}, y, u^\nu) \\ &\text{subject to} \quad x^\nu \in X^\nu, \end{aligned} \tag{2.5}$$

where $y$ is an optimal solution of the follower's optimization problem (2.4) parameterized by $x = (x^\nu)_{\nu=1}^N$. In this problem, an uncertain parameter $u^\nu \in \mathbb{R}^{l_\nu}$ appears in the objective function $\theta_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \times \mathbb{R}^m \times \mathbb{R}^{l_\nu} \to \mathbb{R}$. We assume that although leader $\nu$ does not know the exact value of parameter $u^\nu$, he/she can confirm that it must belong to a given nonempty set $U^\nu \subseteq \mathbb{R}^{l_\nu}$.

Here we assume that although the follower responds to the leaders' strategies with his/her optimal strategy, each leader cannot anticipate the response of the follower exactly because of some observation errors and/or estimation errors. Consequently, each leader $\nu$ estimates that the follower solves the following uncertain optimization problem for variable $y$:

$$\begin{aligned} &\underset{y}{\text{minimize}} \quad \gamma_\nu(x, y, v^\nu) \\ &\text{subject to} \quad y \in K(x), \end{aligned} \tag{2.6}$$

where an uncertain parameter $v^\nu \in \mathbb{R}^{k_\nu}$ appears in the objective function $\gamma_\nu : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{k_\nu} \to \mathbb{R}$ conceived by leader $\nu$. We assume that although leader $\nu$ cannot know the exact value of $v^\nu$, he/she can estimate that it belongs to a given nonempty set $V^\nu \subseteq \mathbb{R}^{k_\nu}$. It should be emphasized that the uncertain parameter $v^\nu$ is associated with leader $\nu$, which means the leaders may estimate the follower's problem differently. Hence, the follower's response anticipated by a leader may be different from the one anticipated by another leader.

In the follower's problem (2.6) anticipated by leader $\nu$, we assume that for any fixed $x \in X$ and $v^\nu \in V^\nu$, $\gamma_\nu(x, \cdot, v^\nu)$ is a strictly convex function and $K(x)$ is a nonempty, closed, convex set. That is, problem (2.6) is a strictly convex optimization problem parameterized by $x$ and $v^\nu$. We denote its unique optimal solution by $y^\nu(x, v^\nu)$, which we assume to exist.

Therefore, the above multi-leader single-follower game with uncertainty can be reformulated as a robust NEP where each player $\nu$ solves the following uncertain optimization problem for his/her own variable $x^\nu$:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} \quad \theta_\nu(x^\nu, x^{-\nu}, y^\nu(x^\nu, x^{-\nu}, v^\nu), u^\nu) \\ &\text{subject to} \quad x^\nu \in X^\nu, \end{aligned} \tag{2.7}$$

where uncertain parameters $u^\nu \in U^\nu$ and $v^\nu \in V^\nu$.

By means of the robust optimization paradigm, we define the worst cost function $\tilde{\Theta}_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \to (-\infty, +\infty]$ for each player $\nu$ as follows:

$$\tilde{\Theta}_\nu(x^\nu, x^{-\nu}) := \sup\{\Theta_\nu(x^\nu, x^{-\nu}, v^\nu, u^\nu) |\ u^\nu \in U^\nu, v^\nu \in V^\nu\}, \tag{2.8}$$

where $\Theta_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \times \mathbb{R}^{k_\nu} \times \mathbb{R}^{l_\nu} \to \mathbb{R}$ is defined by $\Theta_\nu(x^\nu, x^{-\nu}, v^\nu, u^\nu) := \theta_\nu(x^\nu, x^{-\nu}, y^\nu(x^\nu,\ x^{-\nu}, v^\nu), u^\nu)$.

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Thus, we obtain a NEP with complete information, where each player $\nu$ solves the following optimization problem:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} \quad \tilde{\Theta}_\nu(x^\nu, x^{-\nu}) \\ &\text{subject to} \quad x^\nu \in X^\nu. \end{aligned} \tag{2.9}$$

Moreover, we can define an equilibrium for the multi-leader single-follower game with uncertainty comprising problems (2.5) and (2.6) as follows.

DEFINITION 2.3. *A strategy tuple $x = (x^\nu)_{\nu=1}^N \in X$ is called a robust L/F Nash equilibrium of the multi-leader single-follower game with uncertainty comprising problems* (2.5) *and* (2.6) *if $x$ is a robust Nash equilibrium of the NEP with uncertainty comprising problems* (2.7)*, i.e., a Nash equilibrium of the NEP comprising problems* (2.9)*.*

**2.3. GVI problem.** The GVI problem GVI$(S, \mathcal{F})$ is to find a vector $x^* \in S$ such that

$$\exists\, \xi \in \mathcal{F}(x^*), \quad \xi^\top (x - x^*) \geq 0 \quad \text{for all } x \in S, \tag{2.10}$$

where $S \subseteq \mathbb{R}^n$ is a nonempty closed convex set and $\mathcal{F} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a given set-valued mapping. If the set-valued mapping $\mathcal{F}$ happens to be a vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^n$, i.e., $\mathcal{F}(x) = \{F(x)\}$, then GVI (2.10) reduces to the following VI problem VI$(S, F)$:

$$F(x^*)^\top (x - x^*) \geq 0 \quad \text{for all } x \in S. \tag{2.11}$$

The VI and GVI problems have wide applications in various areas, such as transportation systems, mechanics, and economics [16, 27].

Recall that a vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone (strictly monotone) on a nonempty convex set $S \subseteq \mathbb{R}^n$ if $(F(x) - F(y))^\top (x - y) \geq (>)\ 0$ for all $x, y \in S$ (for all $x, y \in S$ such that $x \neq y$). It is well known that if $F$ is a strictly monotone function, VI (2.11) has at most one solution [13]. The GVI problem has a similar property. To see this, we first introduce the monotonicity of a set-valued mapping.

DEFINITION 2.4 (see [40]). *Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set. A set-valued mapping $\mathcal{F} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be monotone (strictly monotone) on $S$ if the inequality*

$$(\xi - \eta)^\top (x - y) \geq (>)\ 0$$

*holds for all $x, y \in S$ (for all $x, y \in S$ such that $x \neq y$) and any $\xi \in \mathcal{F}(x)$, $\eta \in \mathcal{F}(y)$. Moreover, $\mathcal{F}$ is called maximal monotone if its graph*

$$\text{gph}\mathcal{F} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n |\ \xi \in \mathcal{F}(x)\}$$

*is not properly contained in the graph of any other monotone mapping on $\mathbb{R}^n$.*

PROPOSITION 2.5 (see [15]). *Suppose that the set-valued mapping $\mathcal{F} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is strictly monotone on $S$. Then the GVI* (2.10) *has at most one solution.*

Maximal monotone mappings have been studied extensively; e.g., see [32]. A well-known example of the monotone set-valued mapping is $T = \partial f$, where $\partial f$ is the subdifferential of a proper closed convex function. Another important example is $T = F + N_S$, where $F$ is a vector-valued, continuous maximal monotone mapping, $S$

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is a nonempty closed convex set in $\mathbb{R}^n$, and $N_S$ is the normal cone mapping defined by $N_S(x) := \{d \in \mathbb{R}^n | \ d^\top(y-x) \leq 0 \text{ for all } y \in S\}$. Then, from inequality (2.11), we can easily see that a vector $x^* \in S$ solves VI$(S, F)$ if and only if $0 \in F(x^*) + N_S(x^*)$. For the GVI problem, a similar property holds. A vector $x^* \in S$ solves GVI$(S, \mathcal{F})$ if and only if $0 \in \mathcal{F}(x^*) + N_S(x^*)$. In section 5, we will solve the GVI formulation of our game by applying a splitting method to this generalized equation.

**3. Existence of robust L/F Nash equilibrium.** In this section, we discuss the existence of a robust L/F Nash equilibrium for a multi-leader single-follower game with uncertainty.

*Assumption* 3.1. For each leader $\nu$, the following conditions hold:

(a) The functions $\theta_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \times \mathbb{R}^m \times \mathbb{R}^{l_\nu} \to \mathbb{R}$ and $y^\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \times \mathbb{R}^{k_\nu} \to \mathbb{R}^m$ are both continuous.

(b) The uncertainty sets $U^\nu \subseteq \mathbb{R}^{l_\nu}$ and $V^\nu \subseteq \mathbb{R}^{k_\nu}$ are both nonempty and compact.

(c) The strategy set $X^\nu$ is nonempty, compact, and convex.

(d) The function $\Theta_\nu(\cdot, x^{-\nu}, v^\nu, u^\nu) : \mathbb{R}^{n_\nu} \to \mathbb{R}$ is convex for any fixed $x^{-\nu}$, $v^\nu$, and $u^\nu$.

Under Assumption 3.1, we have the following property for function $\tilde{\Theta}_\nu$ defined by (2.8).

PROPOSITION 3.2. *For each leader $\nu$, under Assumption* 3.1 *we have that*

(a) *$\tilde{\Theta}_\nu(x)$ is finite for any $x \in X$, and the function $\tilde{\Theta}_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \to \mathbb{R}$ is continuous;*

(b) *the function $\tilde{\Theta}_\nu(\cdot, x^{-\nu}) : \mathbb{R}^{n_\nu} \to \mathbb{R}$ is convex on $X^\nu$ for any fixed $x^{-\nu} \in X^{-\nu}$.*

*Proof.* The results follow directly from Theorem 1.4.16 in [3] and Proposition 1.2.4(c) in [9]. ∎

Now we establish the existence of a robust L/F Nash equilibrium.

THEOREM 3.3. *If Assumption* 3.1 *holds, then the multi-leader single-follower game with uncertainty comprising problems* (2.5) *and* (2.6) *has at least one robust L/F Nash equilibrium.*

*Proof.* For each leader $\nu$, since Assumption 3.1 holds, the function $\tilde{\Theta}_\nu$ is continuous and finite at any $x \in X$ and it is also convex with respect to $x^\nu$ on $X^\nu$ from Proposition 3.2. Therefore, from Lemma 2.1, the NEP comprising problems (2.9) has at least one Nash equilibrium. That is, the NEP with uncertainty comprising problems (2.7) has at least one robust Nash equilibrium. This also means, by Definition 2.2, that the multi-leader single-follower game with uncertainty comprising problems (2.5) and (2.6) has at least one robust L/F Nash equilibrium. ∎

*Remark* 3.1. As the referees suggested, we may also consider a more general case with multiple followers, where each leader $\nu = 1, \ldots, N$ solves the following optimization problem in his/her own variable $x^\nu$:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} && \theta_\nu(x^\nu, x^{-\nu}, y, u^\nu) \\ &\text{subject to} && x^\nu \in X^\nu. \end{aligned}$$

Here $y := (y^\omega)_{\omega=1}^M$ with $y^\omega := y^\omega(x)$ is a solution of the NEP parameterized by $x$, which is made up of $M$ followers, where each follower $\omega = 1, \ldots, M$ solves the following optimization problem in his/her variable $y^\omega$:

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$$\begin{aligned} &\underset{y^\omega}{\text{minimize}} \quad \gamma_\omega(x, y^\omega, y^{-\omega}) \\ &\text{subject to} \quad y^\omega \in K^\omega(x). \end{aligned}$$

When leaders lose complete anticipation of the strategies of followers, we assume leader $\nu$ anticipates that follower $\omega$ solves the following optimization problem with uncertainty in his/her variable $y^\omega$:

$$\begin{aligned} &\underset{y^{\nu,\omega}}{\text{minimize}} \quad \gamma_{\nu,\omega}(x, y^{\nu,\omega}, y^{-\nu,\omega}, v^\nu) \\ &\text{subject to} \quad y^{\nu,\omega} \in K^\omega(x). \end{aligned}$$

Under the convexity assumption, for a given strategy tuple $x = (x^\nu)_{\nu=1}^N$ and uncertain parameter $v^\nu$, the set of Nash equilibria $y^\nu := (y^{\nu,\omega})_{\omega=1}^M$ with $y^{\nu,\omega} := y^{\nu,\omega}(x, v^\nu)$ anticipated by leader $\nu$ can be described as the solution set of the VI problem $\mathrm{VI}_\nu(K(x), F^\nu(x,\, y^\nu, v^\nu))$, where

$$K(x) := \prod_{\omega=1}^M K^\omega(x) \quad \text{and} \quad F^\nu(x, y^\nu, v^\nu) := \begin{pmatrix} \nabla_{y^{\nu,1}}\gamma_{\nu,1}(x, y^\nu, v^\nu) \\ \vdots \\ \nabla_{y^{\nu,M}}\gamma_{\nu,M}(x, y^\nu, v^\nu) \end{pmatrix}.$$

By further assuming the strict monotonicity of mappings $F^\nu$ for $\nu = 1, \dots, N$, we can determine the anticipated responses of the followers for each leader uniquely. And then we can reformulate this more general multi-leader multi-follower game as a NEP with uncertainty, which is similar to the NEP with uncertainty comprising problems (2.7). Under an assumption similar to Assumption 3.1, we can also establish the existence of a robust L/F Nash equilibrium of this game. A uniqueness result about the robust L/F Nash equilibrium can also be established that is similar to that in section 4. Yet we can imagine the presentation will become excessively complicated, which is the main reason we focus on the single-follower case. Nevertheless, we will show some numerical results for a multi-leader multi-follower game in section 5.

**4. A uniqueness result for a robust L/F Nash equilibrium model.** In this section, we discuss the uniqueness of a robust L/F Nash equilibrium for a special class of multi-leader single-follower games with uncertainty. In this game, each leader $\nu = 1, \dots, N$ solves the following optimization problem:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} \quad \theta_\nu(x^\nu, x^{-\nu}, y, u^\nu) := \omega_\nu(x^\nu, x^{-\nu}, u^\nu) + \varphi_\nu(x^\nu, y) \\ &\text{subject to} \quad x^\nu \in X^\nu, \end{aligned}$$

where $y$ is an optimal solution of the following follower's problem parameterized by the leaders' strategy tuple $x = (x^\nu)_{\nu=1}^N$:

$$\begin{aligned} &\underset{y}{\text{minimize}} \quad \gamma(x, y) := \psi(y) - \sum_{\nu=1}^N \varphi_\nu(x^\nu, y) \\ &\text{subject to} \quad y \in \mathcal{Y}. \end{aligned}$$

In this game, the objective functions of $N$ leaders and the follower contain some related terms. In particular, the last term of each leader's objective function appears in the follower's objective function in the negated form. Therefore, the game partly contains a kind of zero-sum structure between each leader and the follower. An application of such special multi-leader single-follower games with complete information

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has been presented with some illustrative numerical examples in [22]. Here, in each leader $\nu$'s problem, we assume that the strategy set $X^\nu$ is nonempty, compact, and convex. Due to some estimation errors, leader $\nu$ cannot evaluate his/her objective function exactly but knows only that it contains some uncertain parameter $u^\nu$ belonging to a fixed uncertainty set $U^\nu \subseteq \mathbb{R}^{l_\nu}$. We further assume that functions $\omega_\nu$, $\varphi_\nu$, $\psi$ and the set $\mathcal{Y}$ have the following explicit representations:

$$\begin{aligned}
\omega_\nu(x^\nu, x^{-\nu}, u^\nu) &:= \frac{1}{2}(x^\nu)^\top H_\nu x^\nu + \sum_{\nu'=1,\nu'\neq\nu}^{N} (x^\nu)^\top E_{\nu\nu'} x^{\nu'} + (x^\nu)^\top R_\nu u^\nu,\\
\varphi_\nu(x^\nu, y) &:= (x^\nu)^\top D_\nu y,\\
\psi(y) &:= \frac{1}{2} y^\top B y + c^\top y,\\
\mathcal{Y} &:= \{y \in \mathbb{R}^m | Ay + a = 0\},
\end{aligned}$$

where $H_\nu \in \mathbb{R}^{n_\nu \times n_\nu}$ is symmetric, $D_\nu \in \mathbb{R}^{n_\nu \times m}$, $R_\nu \in \mathbb{R}^{n_\nu \times l_\nu}$, $E_{\nu\nu'} \in \mathbb{R}^{n_\nu \times n_{\nu'}}$, $\nu, \nu' = 1, \ldots, N$, and $c \in \mathbb{R}^m$. In the case that $N = 2$, since there is no ambiguity, for convenience we write $E_\nu$ instead of $E_{\nu\nu'}$. Matrix $B \in \mathbb{R}^{m\times m}$ is assumed to be symmetric and positive definite. Moreover, $A \in \mathbb{R}^{p_0 \times m}$, $a \in \mathbb{R}^{p_0}$, and $A$ has full row rank.

We assume that although the follower can respond to all leaders' strategies exactly, each leader $\nu$ cannot exactly know the follower's problem but only can anticipate it as follows:

$$\begin{aligned}
&\underset{y}{\text{minimize}} \quad \gamma^\nu(x, y, v^\nu) := \frac{1}{2} y^\top B y + (c + v^\nu)^\top y - \sum_{\nu=1}^{N} \varphi_\nu(x^\nu, y)\\
&\text{subject to} \quad y \in \mathcal{Y}.
\end{aligned}$$

Here, the uncertain parameter $v^\nu$ belongs to some fixed uncertainty set $V^\nu \subseteq \mathbb{R}^m$.

In the remainder of the paper, for simplicity we will mainly consider the following game with two leaders, labeled I and II. The results presented below can be extended to the case of more than two leaders in a straightforward manner.[2] In this game, leader $\nu$ solves the following problem:

$$\begin{aligned}
&\underset{x^\nu}{\text{minimize}} \quad \frac{1}{2}(x^\nu)^\top H_\nu x^\nu + (x^\nu)^\top E_\nu x^{-\nu} + (x^\nu)^\top R_\nu u^\nu + (x^\nu)^\top D_\nu y\\
&\text{subject to} \quad x^\nu \in X^\nu,
\end{aligned} \tag{4.1}$$

where $y$ is an optimal solution of the following follower's problem anticipated by leader $\nu$:

$$\begin{aligned}
&\underset{y}{\text{minimize}} \quad \frac{1}{2} y^\top B y + (c + v^\nu)^\top y - (x^{\mathrm{I}})^\top D_{\mathrm{I}} y - (x^{\mathrm{II}})^\top D_{\mathrm{II}} y\\
&\text{subject to} \quad Ay + a = 0,
\end{aligned} \tag{4.2}$$

where $u^\nu \in U^\nu$ and $v^\nu \in V^\nu$, $\nu = \mathrm{I}, \mathrm{II}$.

Since the follower's problems estimated by two leaders are both strictly convex quadratic programming problems with equality constraints, each of them is equivalent to finding a pair $(y, \lambda) \in \mathbb{R}^m \times \mathbb{R}^{p_0}$ satisfying the following KKT system of linear equations for $\nu = \mathrm{I}, \mathrm{II}$:

[2] We will give a numerical example with three leaders in section 5.

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$$
\begin{aligned}
By + c + v^\nu - (D_{\mathrm{I}})^\top x^{\mathrm{I}} - (D_{\mathrm{II}})^\top x^{\mathrm{II}} + A^\top \lambda &= 0,\\
Ay + a &= 0.
\end{aligned}
$$

Note that under the given assumptions, a KKT pair $(y, \lambda)$ exists uniquely for each $(x^{\mathrm{I}}, x^{\mathrm{II}}, v^\nu)$ and is denoted by $(y^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, v^\nu), \lambda^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, v^\nu))$. For each $\nu = \mathrm{I}, \mathrm{II}$, by direct calculations we have

$$
\begin{aligned}
y^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, v^\nu) = &- B^{-1}(c + v^\nu) - B^{-1}A^\top(AB^{-1}A^\top)^{-1}(a - AB^{-1}(c + v^\nu))\\
&+ [B^{-1}(D_{\mathrm{I}})^\top - B^{-1}A^\top(AB^{-1}A^\top)^{-1}AB^{-1}(D_{\mathrm{I}})^\top]x^{\mathrm{I}}\\
&+ [B^{-1}(D_{\mathrm{II}})^\top - B^{-1}A^\top(AB^{-1}A^\top)^{-1}AB^{-1}(D_{\mathrm{II}})^\top]x^{\mathrm{II}},\\
\lambda^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, v^\nu) = &\ (AB^{-1}A^\top)^{-1}(a - AB^{-1}(c + v^\nu)) + (AB^{-1}A^\top)^{-1}AB^{-1}(D_{\mathrm{I}})^\top x^{\mathrm{I}}\\
&+ (AB^{-1}A^\top)^{-1}AB^{-1}(D_{\mathrm{II}})^\top x^{\mathrm{II}}.
\end{aligned}
$$

Let $P = I - B^{-\frac{1}{2}}A^\top(AB^{-1}A^\top)^{-1}AB^{-\frac{1}{2}}$. Then, by substituting each $y^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, v^\nu)$ for $y$ in the respective leader's problem, leader $\nu$'s objective function can be rewritten as

$$
\begin{aligned}
&\Theta_\nu(x^\nu, x^{-\nu}, v^\nu, u^\nu)\\
&\quad := \ \theta_\nu(x^\nu, x^{-\nu}, y^\nu(x^\nu, x^{-\nu}, v^\nu), u^\nu)\\
&\quad = \ \frac{1}{2}(x^\nu)^\top H_\nu x^\nu + (x^\nu)^\top D_\nu G_\nu x^\nu + (x^\nu)^\top R_\nu u^\nu + (x^\nu)^\top D_\nu r\\
&\qquad + (x^\nu)^\top(D_\nu G_{-\nu} + E_\nu)x^{-\nu} - (x^\nu)^\top D_\nu B^{-\frac{1}{2}}PB^{-\frac{1}{2}}v^\nu.
\end{aligned}
\tag{4.3}
$$

Here, $G_{\mathrm{I}} \in \mathbb{R}^{m\times n_{\mathrm{I}}}$, $G_{\mathrm{II}} \in \mathbb{R}^{m\times n_{\mathrm{II}}}$, and $r \in \mathbb{R}^m$ are given by

$$
\begin{aligned}
G_{\mathrm{I}} &= B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^T,\\
G_{\mathrm{II}} &= B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^T,\\
r &= -B^{-\frac{1}{2}}PB^{-\frac{1}{2}}c - B^{-1}A^T(AB^{-1}A^T)^{-1}a.
\end{aligned}
$$

With the functions $\Theta_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \times \mathbb{R}^m \times \mathbb{R}^{l_\nu} \to \mathbb{R}$ defined by (4.3), we can formulate the above multi-leader single-follower game with uncertainty as a NEP with uncertainty where as the $\nu$th player, leader $\nu$ solves the following optimization problem:

$$
\begin{aligned}
&\underset{x^\nu}{\text{minimize}} \quad \Theta_\nu(x^\nu, x^{-\nu}, v^\nu, u^\nu)\\
&\text{subject to} \quad x^\nu \in X^\nu.
\end{aligned}
$$

Here, $u^\nu \in U^\nu$ and $v^\nu \in V^\nu$, $\nu = \mathrm{I}, \mathrm{II}$.

By means of the robust optimization technique, we construct the robust counterpart of the above NEP with uncertainty, which is a NEP with complete information, where leader $\nu$ solves the following optimization problem:

$$
\begin{aligned}
&\underset{x^\nu}{\text{minimize}} \quad \tilde{\Theta}_\nu(x^\nu, x^{-\nu})\\
&\text{subject to} \quad x^\nu \in X^\nu.
\end{aligned}
\tag{4.4}
$$

Here, functions $\tilde{\Theta}_\nu : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \to \mathbb{R}$ and $\tilde{\Theta}_{-\nu} : \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \to \mathbb{R}$ are defined by

$$
\begin{aligned}
\tilde{\Theta}_\nu(x^\nu, x^{-\nu}) &:= \sup\{\Theta_\nu(x^\nu, x^{-\nu}, v^\nu, u^\nu) | u^\nu \in U^\nu, v^\nu \in V^\nu\}\\
&= \frac{1}{2}(x^\nu)^\top H_\nu x^\nu + (x^\nu)^\top D_\nu G_\nu x^\nu + (x^\nu)^\top D_\nu r\\
&\quad + (x^\nu)^\top(D_\nu G_{-\nu} + E_\nu)x^{-\nu} + \phi_\nu(x^\nu),
\end{aligned}
$$

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where $\phi_\nu : \mathbb{R}^{n_\nu} \to \mathbb{R}$ are defined by

$$\begin{aligned}\phi_\nu(x^\nu) := &\sup\{(x^\nu)^\top R_\nu u^\nu | u^\nu \in U^\nu\} \\ &+ \sup\{-(x^\nu)^\top D_\nu B^{-\frac{1}{2}} P B^{-\frac{1}{2}} v^\nu | v^\nu \in V^\nu\}.\end{aligned} \tag{4.5}$$

In what follows, based on the analysis of the previous section, we first show the existence of a robust L/F Nash equilibrium.

THEOREM 4.1. *Suppose that for each* $\nu = \mathrm{I}, \mathrm{II}$, *the strategy set* $X^\nu$ *is nonempty, compact, and convex, the matrix* $H_\nu \in \mathbb{R}^{n_\nu \times n_\nu}$ *is symmetric and positive semidefinite, and the uncertainty sets* $U^\nu$ *and* $V^\nu$ *are nonempty and compact. Then, the multi-leader single-follower game with uncertainty comprising problems* (4.1) *and* (4.2) *has at least one robust L/F Nash equilibrium.*

*Proof.* We will show that the conditions in Assumption 3.1 hold. Since conditions (a)–(c) clearly hold, we only confirm that condition (d) holds. In fact, recalling that $P$ is a projection matrix, it is easy to see that $D_{\mathrm{I}} G_{\mathrm{I}}$ and $D_{\mathrm{II}} G_{\mathrm{II}}$ are both positive semidefinite. Since $H_{\mathrm{I}}$ and $H_{\mathrm{II}}$ are also positive semidefinite, the functions $\Theta_{\mathrm{I}}$ and $\Theta_{\mathrm{II}}$ are convex with respect to $x^{\mathrm{I}}$ and $x^{\mathrm{II}}$, respectively. Therefore, Assumption 3.1 holds. Hence, by Theorem 3.3, the proof is complete. ∎

In order to investigate the uniqueness of a robust L/F Nash equilibrium, we reformulate the robust Nash equilibrium counterpart comprising problems (4.4) as a GVI problem.

Notice that the functions $\tilde{\Theta}_\nu$ are convex with respect to $x^\nu$. Let us define the mappings $T_{\mathrm{I}} : \mathbb{R}^{n_{\mathrm{I}}} \times \mathbb{R}^{n_{\mathrm{II}}} \to \mathbb{R}^{n_{\mathrm{I}}}$ and $T_{\mathrm{II}} : \mathbb{R}^{n_{\mathrm{I}}} \times \mathbb{R}^{n_{\mathrm{II}}} \to \mathbb{R}^{n_{\mathrm{II}}}$ as

$$\begin{aligned} T_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) &:= H_{\mathrm{I}} x^{\mathrm{I}} + D_{\mathrm{I}} r + 2 D_{\mathrm{I}} G_{\mathrm{I}} x^{\mathrm{I}} + (D_{\mathrm{I}} G_{\mathrm{II}} + E_{\mathrm{I}}) x^{\mathrm{II}}, \\ T_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) &:= H_{\mathrm{II}} x^{\mathrm{II}} + D_{\mathrm{II}} r + (D_{\mathrm{II}} G_{\mathrm{I}} + E_{\mathrm{II}}) x^{\mathrm{I}} + 2 D_{\mathrm{II}} G_{\mathrm{II}} x^{\mathrm{II}}. \end{aligned} \tag{4.6}$$

Then, the subdifferentials of $\tilde{\Theta}_\nu$ with respect to $x^\nu$ can be written as

$$\begin{aligned} \partial_{x^{\mathrm{I}}} \tilde{\Theta}_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) &= T_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + \partial \phi_{\mathrm{I}}(x^{\mathrm{I}}), \\ \partial_{x^{\mathrm{II}}} \tilde{\Theta}_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) &= T_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + \partial \phi_{\mathrm{II}}(x^{\mathrm{II}}), \end{aligned}$$

where $\partial \phi_\nu$ denotes the subdifferentials of $\phi_\nu$, $\nu = \mathrm{I}, \mathrm{II}$. By [8, Proposition B.24(f)], for each $\nu = \mathrm{I}, \mathrm{II}$, $x^{*,\nu}$ solves the problem (4.4) if and only if there exists a subgradient $\xi^\nu \in \partial_{x^\nu} \tilde{\Theta}_\nu(x^{*,\nu}, x^{-\nu})$ such that

$$(\xi^\nu)^\top (x^\nu - x^{*,\nu}) \geq 0 \quad \text{for all } x^\nu \in X^\nu. \tag{4.7}$$

Therefore, we can investigate the uniqueness of a robust L/F Nash equilibrium by considering the following GVI problem, which is formulated by concatenating the above first-order optimality conditions (4.7) of all leaders' problems: Find a vector $x^* = (x^{*,\mathrm{I}}, x^{*,\mathrm{II}}) \in X := X^{\mathrm{I}} \times X^{\mathrm{II}}$ such that

$$\exists\, \xi \in \tilde{\mathcal{F}}(x^*), \quad \xi^\top (x - x^*) \geq 0 \quad \text{for all } x \in X,$$

where $\xi = (\xi^{\mathrm{I}}, \xi^{\mathrm{II}}) \in \mathbb{R}^n, x = (x^{\mathrm{I}}, x^{\mathrm{II}}) \in \mathbb{R}^n$, and the set-valued mapping $\tilde{\mathcal{F}} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is defined by $\tilde{\mathcal{F}}(x) := \partial_{x^{\mathrm{I}}} \tilde{\Theta}_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \times \partial_{x^{\mathrm{II}}} \tilde{\Theta}_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}})$.

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In what follows, we show that $\tilde{\mathcal{F}}$ is strictly monotone under suitable conditions. Then, by Proposition 2.5, we can ensure the uniqueness of a robust L/F Nash equilibrium. Since the subdifferentials $\partial\phi_{\rm I}$ and $\partial\phi_{\rm II}$ are monotone, we only need to establish the strict monotonicity of mapping $T : \mathbb{R}^{n_{\rm I}+n_{\rm II}} \to \mathbb{R}^{n_{\rm I}+n_{\rm II}}$ defined by

$$T(x) := \begin{pmatrix} T_{\rm I}(x^{\rm I}, x^{\rm II}) \\ T_{\rm II}(x^{\rm I}, x^{\rm II}) \end{pmatrix}. \tag{4.8}$$

For this purpose, we assume that the matrix

$$\mathcal{J} := \begin{pmatrix} H_{\rm I} & E_{\rm I} \\ E_{\rm II} & H_{\rm II} \end{pmatrix} \tag{4.9}$$

is positive definite. Note that the transpose of matrix $\mathcal{J}$ is the Jacobian of the so-called pseudogradient of the first two terms $\frac{1}{2}(x^\nu)^\top H_\nu x^\nu + (x^\nu)^\top E_\nu x^{\nu'}$ in the objective functions of problems (4.1) and (4.2). The positive definiteness of such a matrix is often assumed in the study on NEP and GNEP [24, 25, 33].

LEMMA 4.2. *Suppose that matrix $\mathcal{J}$ defined by* (4.9) *is positive definite. Then, the mapping $T$ defined by* (4.8) *is strictly monotone.*

*Proof.* For any $x = (x^{\rm I}, x^{\rm II}), \tilde{x} = (\tilde{x}^{\rm I}, \tilde{x}^{\rm II}) \in X$ such that $x \neq \tilde{x}$, we have

$$\begin{aligned} &(x-\tilde{x})^\top (T(x) - T(\tilde{x})) \\ &\quad = (x-\tilde{x})^\top \begin{pmatrix} H_{\rm I} & E_{\rm I} \\ E_{\rm II} & H_{\rm II} \end{pmatrix} (x-\tilde{x}) + (x-\tilde{x})^\top \begin{pmatrix} 2D_{\rm I}G_{\rm I} & D_{\rm I}G_{\rm II} \\ D_{\rm II}G_{\rm I} & 2D_{\rm II}G_{\rm II} \end{pmatrix} (x-\tilde{x}). \end{aligned}$$

It can be shown [22, Lemma 4.1] that the matrix $\left(\begin{smallmatrix} 2D_{\rm I}G_{\rm I} & D_{\rm I}G_{\rm II} \\ D_{\rm II}G_{\rm I} & 2D_{\rm II}G_{\rm II} \end{smallmatrix}\right)$ is positive semidefinite. Hence, the mapping $T$ is strictly monotone since matrix $\mathcal{J}$ is positive definite by assumption. The proof is complete. ∎

Now, we are ready to establish the uniqueness of a robust L/F Nash equilibrium.

THEOREM 4.3. *Suppose that matrix $\mathcal{J}$ defined by* (4.9) *is positive definite and the uncertainty sets $U^\nu$ and $V^\nu$ are nonempty and compact. Then the multi-leader single-follower game with uncertainty comprising problems* (4.1) *and* (4.2) *has a unique robust L/F Nash equilibrium.*

*Proof.* The proof follows directly from Theorem 4.1, Proposition 2.5, and Lemma 4.2. We omit the details. ∎

*Remark* 4.1. In our current framework, it is impossible to deal with the case where the follower's problem contains inequality constraints since in this case the leaders' problems will become nonconvex from the complementarity conditions in the KKT system of the follower's problem.

**5. Numerical experiments.** In this section, we present some numerical results for the robust L/F Nash equilibrium model described in section 4. For this purpose, we use a splitting method for finding a zero of the sum of two maximal monotone mappings $\mathcal{A}$ and $\mathcal{B}$. The splitting method solves a sequence of subproblems, each of which involves only one of the two mappings $\mathcal{A}$ and $\mathcal{B}$. In particular, the forward-backward splitting method [17] may be regarded as a generalization of the gradient projection method for constrained convex optimization problems and monotone VI problems. In the case where $\mathcal{B}$ is vector-valued, the forward-backward splitting method for finding a zero of the mapping $\mathcal{A} + \mathcal{B}$ uses the recursion

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$$x^{k+1} = (I+\mu\mathcal{A})^{-1}(I-\mu\mathcal{B})(x^k) \\ := J_{\mu\mathcal{A}}((I-\mu\mathcal{B})(x^k)), \quad k=0,1,\ldots, \tag{5.1}$$

where the mapping $J_{\mu\mathcal{A}} := (I+\mu\mathcal{A})^{-1}$ is called the resolvent of $\mathcal{A}$ (with constant $\mu > 0$), which is a vector-valued mapping from $\mathbb{R}^n$ to dom$\mathcal{A}$.

In what follows, we assume that in the robust multi-leader-follower game comprising problems (4.1) and (4.2), for each leader $\nu = \mathrm{I}, \mathrm{II}$, the uncertainty sets $U^\nu \in \mathbb{R}^{l_\nu}$ and $V^\nu \in \mathbb{R}^m$ are given by

$$U^\nu := \{u^\nu \in \mathbb{R}^{l_\nu} \,|\, \|u^\nu\| \le \rho^\nu\}$$

and

$$V^\nu := \{v^\nu \in \mathbb{R}^{m} \,|\, \|v^\nu\| \le \sigma^\nu\}$$

with given uncertainty bounds $\rho^\nu > 0$ and $\sigma^\nu > 0$. Here we assume that the uncertainty sets are specified in terms of the Euclidean norm, but we may also use different norms such as the $l_\infty$ norm; see Example 5.3. We further assume that the constraints $x^\nu \in X^\nu$ are explicitly written as $g^\nu(x^\nu) := A_\nu^\top x^\nu + b_\nu \le 0$, where $A_\nu \in \mathbb{R}^{n_\nu \times l_\nu}$ and $b_\nu \in \mathbb{R}^{l_\nu}$, $\nu = \mathrm{I}, \mathrm{II}$.

Under these assumptions, the functions $\phi_\nu, \nu = \mathrm{I}, \mathrm{II}$, defined by (4.5) can be written explicitly as

$$\phi_\nu(x^\nu) := \rho^\nu \|R_\nu^\top x^\nu\| + \sigma^\nu \|B^{-\frac{1}{2}} P B^{-\frac{1}{2}} D_\nu^\top x^\nu\|, \quad \nu = \mathrm{I}, \mathrm{II}.$$

Hence, for player $\nu = \mathrm{I}, \mathrm{II}$, we can rewrite the problem (4.4) as follows:

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \frac{1}{2}(x^\nu)^\top (H_\nu + 2D_\nu G_\nu)x^\nu + (x^\nu)^\top(D_\nu G_{-\nu} + E_\nu)x^{-\nu} \\ & + (x^\nu)^\top D_\nu r + \rho^\nu\|R_\nu^\top x^\nu\| + \sigma^\nu \|B^{-\frac{1}{2}} P B^{-\frac{1}{2}} D_\nu^\top x^\nu\| \\ \text{subject to} \quad & A_\nu^\top x^\nu + b_\nu \le 0. \end{aligned} \tag{5.2}$$

To apply the forward-backward splitting method to the NEP with the leaders' problems (5.2), we let the mappings $\mathcal{A}$ and $\mathcal{B}$ be specified by

$$\mathcal{A}(x) := \begin{pmatrix} \partial\phi_{\mathrm{I}}(x^{\mathrm{I}}) \\ \partial\phi_{\mathrm{II}}(x^{\mathrm{II}}) \end{pmatrix} + N_X(x),$$
$$\mathcal{B}(x) := T(x),$$

where $T(x)$ is given by (4.8). Note that $\mathcal{A}$ is set-valued, while $\mathcal{B}$ is vector-valued.

Under the assumption of Lemma 4.2, the mapping $\mathcal{B} := T$ is strictly monotone. Since $T$ is affine under the current setting (see (4.6) and (4.8)), it is strongly monotone as well, which along with the Lipschitz continuity implies $T$ is co-coercive [14, p. 164]. Moreover the mapping $\mathcal{A}$ is maximal monotone, since it is the sum of two maximal monotone mappings. Then the convergence of the forward-backward splitting method given by (5.1) is guaranteed provided the constant $\mu$ is chosen sufficiently small [14, Theorem 12.4.6]. Moreover, it may be worth mentioning that the convergence is not affected by the specific structure of the uncertainty sets $U^\nu$ and $V^\nu$, since it solely relies on the fact that the functions $\phi_\nu$ defined by (4.5) are convex functions.

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In order to evaluate the iterative point $x^{k+1} := (x^{\mathrm{I},k+1}, x^{\mathrm{II},k+1})$ in (5.1), we first compute $z^{\nu,k} := x^{\nu,k} - \mu T_\nu(x^k)$. Then $x^{\nu,k+1}$ can be evaluated by solving the following problem:

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \frac{1}{2\mu}\|x^\nu - z^{\nu,k}\|^2 + \rho^\nu \|R_\nu^\top x^\nu\| + \sigma^\nu \|B^{-\frac{1}{2}} P B^{-\frac{1}{2}} D_\nu^\top x^\nu\| \\ \text{subject to} \quad & A_\nu^\top x^\nu + b_\nu \le 0. \end{aligned}$$

Note that these problems can be rewritten as linear second-order cone programming problems, for which efficient solvers are available [37, 39]. In what follows, we show some numerical results to observe the behavior of robust L/F Nash equilibria with different uncertainty bounds. To compute those equilibria, we use the forward-backward splitting method with $\mu = 0.2$.

*Example* 5.1. The problem data are given as follows:

$$H_{\mathrm{I}} = \begin{pmatrix} 1.7 & 1.6 \\ 1.6 & 2.8 \end{pmatrix}, \quad H_{\mathrm{II}} = \begin{pmatrix} 2.7 & 1.3 \\ 1.3 & 3.6 \end{pmatrix}, \quad D_{\mathrm{I}} = \begin{pmatrix} 2.3 & 1.4 & 2.6 \\ 1.3 & 2.1 & 1.7 \end{pmatrix},$$

$$D_{\mathrm{II}} = \begin{pmatrix} 2.5 & 1.9 & 1.4 \\ 1.3 & 2.4 & 1.6 \end{pmatrix}, \quad E_{\mathrm{I}} = \begin{pmatrix} 1.8 & 1.4 \\ 1.5 & 2.7 \end{pmatrix}, \quad E_{\mathrm{II}} = \begin{pmatrix} 1.3 & 1.7 \\ 2.4 & 0.3 \end{pmatrix},$$

$$R_{\mathrm{I}} = \begin{pmatrix} 1.2 & 1.8 \\ 1.6 & 1.7 \end{pmatrix}, \quad R_{\mathrm{II}} = \begin{pmatrix} 1.8 & 2.3 \\ 1.4 & 1.7 \end{pmatrix}, \quad B = \begin{pmatrix} 2.5 & 1.8 & 0.2 \\ 1.8 & 3.6 & 2.1 \\ 0.2 & 2.1 & 4.6 \end{pmatrix},$$

$$A_{\mathrm{I}} = \begin{pmatrix} 1.6 & 0.8 & 1.3 \\ 2.6 & 2.2 & 1.7 \end{pmatrix}, \quad A_{\mathrm{II}} = \begin{pmatrix} 1.8 & 1.6 & 1.4 \\ 1.3 & 1.2 & 2.7 \end{pmatrix}, \quad c = \begin{pmatrix} 1.4 \\ 2.6 \\ 2.1 \end{pmatrix},$$

$$A = \begin{pmatrix} 1.3 & 2.4 & 1.8 \end{pmatrix}, \quad a = 1.3, \quad b_{\mathrm{I}} = \begin{pmatrix} 1.6 \\ 1.2 \\ 0.4 \end{pmatrix}, \quad b_{\mathrm{II}} = \begin{pmatrix} 1.6 \\ 1.5 \\ 2.6 \end{pmatrix}.$$

Table 5.1 shows the computational results. In the table, $(x^{*,\mathrm{I}}, x^{*,\mathrm{II}})$ denotes the leaders' optimal strategies and $(y^{*,\mathrm{I}}, y^{*,\mathrm{II}})$ denotes the follower's responses estimated respectively by the two leaders, at the computed equilibria for various values of the uncertainty bounds $\rho = (\rho^{\mathrm{I}}, \rho^{\mathrm{II}})$ and $\sigma = (\sigma^{\mathrm{I}}, \sigma^{\mathrm{II}})$. In particular, when there is no uncertainty ($\rho = 0, \sigma = 0$), the follower's response anticipated by the two leaders' naturally coincide, i.e., $y^{*,\mathrm{I}} = y^{*,\mathrm{II}}$, which is denoted $\bar{y}^*$ in the table. ValL1 and ValL2 denote the optimal objective values of the two leaders' respective optimization problems. Iter denotes the number of iterations required by the forward-backward splitting method to compute each equilibrium.

Both ValL1 and ValL2 increase as uncertainty increases, indicating that the leaders have to pay additional costs that compensate for the loss of information.

Moreover, the two leaders' estimates of the follower's response tend not only to deviate from the estimate under complete information but also have a larger gap between them.

*Example* 5.2. We further show some numerical experiments that uniqueness of L/F Nash equilibrium does not hold. For this propose, we consider the following example, which is generated from Example 5.1 by introducing the extra variable $x^{\mathrm{III}}$ and replacing $x^{\mathrm{II}}$ by $x^{\mathrm{II}} + x^{\mathrm{III}}$ in the game. Then the problem must have infinitely many equilibria. The problem data are the same as those in Example 5.1 except that $H_\nu$, $D_\nu$, $E_\nu$, $R_\nu$, and $A_\nu$, $\nu = \mathrm{I}, \mathrm{II}, \mathrm{III}$, are given by

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TABLE 5.1
*Computational results for Example* 5.1.

| $(\rho;\sigma)$ | $(0.0, 0.0; 0.0, 0.0)$ | $(0.6, 0.6; 0.6, 0.6)$ | $(1.2, 1.2; 1.2, 1.2)$ | $(1.8, 1.8; 1.8, 1.8)$ |
|---|---|---|---|---|
| $x^{*,\mathrm{I}}$ | $\begin{pmatrix} 0.216787064 \\ -0.748792040 \end{pmatrix}$ | $\begin{pmatrix} 0.105530863 \\ -0.680326685 \end{pmatrix}$ | $\begin{pmatrix} 0.065920927 \\ -0.655951340 \end{pmatrix}$ | $\begin{pmatrix} 0.064979902 \\ -0.655372249 \end{pmatrix}$ |
| $x^{*,\mathrm{II}}$ | $\begin{pmatrix} -0.352272721 \\ -0.780303041 \end{pmatrix}$ | $\begin{pmatrix} -0.352272287 \\ -0.780303621 \end{pmatrix}$ | $\begin{pmatrix} -0.269585166 \\ -0.890553113 \end{pmatrix}$ | $\begin{pmatrix} -0.193410901 \\ -0.992118800 \end{pmatrix}$ |
| ValL1 | 2.771204467 | 3.678656718 | 4.637566703 | 5.580803697 |
| ValL2 | 3.247031626 | 5.397518304 | 7.347156678 | 9.138817545 |
| $y^{*,\mathrm{I}}$ | $\begin{pmatrix} -0.279775659 \\ -0.325691328 \\ -0.085906920 \end{pmatrix}$ | $\begin{pmatrix} 0.087483058 \\ -0.673770231 \\ 0.112955878 \end{pmatrix}$ | $\begin{pmatrix} 0.129947739 \\ -0.900612585 \\ 0.384743413 \end{pmatrix}$ | $\begin{pmatrix} 0.144223848 \\ -1.062906561 \\ 0.590824858 \end{pmatrix}$ |
| $\Vert y^{*,\mathrm{I}} - \bar{y}^*\Vert$ | 0 | 0.543676649 | 0.848480672 | 1.086843035 |
| $y^{*,\mathrm{II}}$ | $\begin{pmatrix} -0.279775659 \\ -0.325691328 \\ -0.085906920 \end{pmatrix}$ | $\begin{pmatrix} -0.950427761 \\ 0.253806425 \\ -0.374210739 \end{pmatrix}$ | $\begin{pmatrix} -1.427476127 \\ 0.644464496 \\ -0.550553237 \end{pmatrix}$ | $\begin{pmatrix} -1.80466852 \\ 0.893151590 \\ -0.609719301 \end{pmatrix}$ |
| $\Vert y^{*,\mathrm{II}} - \bar{y}^*\Vert$ | 0 | 0.932046661 | 1.572995513 | 2.021201548 |
| $\Vert y^{*,\mathrm{I}} - y^{*,\mathrm{II}}\Vert$ | 0 | 1.474784265 | 2.384871507 | 3.010921989 |
| Iter | 6 | 6 | 12 | 11 |

$$H_{\mathrm{I}} = \begin{pmatrix} 1.7 & 1.6 & 1.6 \\ 1.6 & 2.8 & 2.8 \\ 1.6 & 2.8 & 2.8 \end{pmatrix}, \quad H_{\mathrm{II}} = \begin{pmatrix} 2.7 & 1.3 & 1.3 \\ 1.3 & 3.6 & 3.6 \\ 1.3 & 3.6 & 3.6 \end{pmatrix}, \quad D_{\mathrm{I}} = \begin{pmatrix} 2.3 & 1.4 & 2.6 \\ 1.3 & 2.1 & 1.7 \\ 1.3 & 2.1 & 1.7 \end{pmatrix},$$

$$D_{\mathrm{II}} = \begin{pmatrix} 2.5 & 1.9 & 1.4 \\ 1.3 & 2.4 & 1.6 \\ 1.3 & 2.4 & 1.6 \end{pmatrix}, \quad E_{\mathrm{I}} = \begin{pmatrix} 1.8 & 1.4 & 1.4 \\ 1.5 & 2.7 & 2.7 \\ 1.5 & 2.7 & 2.7 \end{pmatrix}, \quad E_{\mathrm{II}} = \begin{pmatrix} 1.3 & 1.7 & 1.7 \\ 2.4 & 0.3 & 0.3 \\ 2.4 & 0.3 & 0.3 \end{pmatrix},$$

$$R_{\mathrm{I}} = \begin{pmatrix} 1.2 & 1.8 \\ 1.6 & 1.7 \\ 1.6 & 1.7 \end{pmatrix}, \quad R_{\mathrm{II}} = \begin{pmatrix} 1.8 & 2.3 \\ 1.4 & 1.7 \\ 1.4 & 1.7 \end{pmatrix}, \quad A_{\mathrm{I}} = \begin{pmatrix} 1.6 & 0.8 & 1.3 \\ 2.6 & 2.2 & 1.7 \\ 2.6 & 2.2 & 1.7 \end{pmatrix},$$

$$A_{\mathrm{II}} = \begin{pmatrix} 1.8 & 1.6 & 1.4 \\ 1.3 & 1.2 & 2.7 \\ 1.3 & 1.2 & 2.7 \end{pmatrix}.$$

From the construction of the game, it has infinitely many L/F Nash equilibria. The computational results are shown in Tables 5.2 and 5.3 with two different starting points $(0.2, 0.2, 0.2, 0.2, 0.2, 0.2)^\top$ and $(1.2, 0.2, 1.2, 0.2, 1.2, 0.2)^\top$, respectively. In addition to observations similar to those in Example 5.1, we can confirm that $x^{\mathrm{II}} + x^{\mathrm{III}}$ in Example 5.2 equals $x^{\mathrm{II}}$ in Example 5.1.

*Example* 5.3. In this example, the uncertainty sets are specified by the $l_\infty$ norm as

$$U^\nu := \{u^\nu \in \mathbb{R}^{l_\nu} \mid \|u^\nu\|_\infty \le \rho^\nu\}$$

and

$$V^\nu := \{v^\nu \in \mathbb{R}^m \mid \|v^\nu\|_\infty \le \sigma^\nu\}$$

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TABLE 5.2
*Computational results for Example* 5.2 *with starting point* $(0.2, 0.2, 0.2, 0.2, 0.2, 0.2)^\top$.

| $(\rho;\sigma)$ | $(0.0, 0.0; 0.0, 0.0)$ | $(0.6, 0.6; 0.6, 0.6)$ | $(1.2, 1.2; 1.2, 1.2)$ | $(1.8, 1.8; 1.8, 1.8)$ |
|---|---|---|---|---|
| $x^{*,\mathrm{I}}$ | $\begin{pmatrix} 0.216786827 \\ -0.374395947 \\ -0.374395947 \end{pmatrix}$ | $\begin{pmatrix} 0.105517660 \\ -0.340159280 \\ -0.340159280 \end{pmatrix}$ | $\begin{pmatrix} 0.065921166 \\ -0.327975744 \\ -0.327975744 \end{pmatrix}$ | $\begin{pmatrix} 0.064980047 \\ -0.327686169 \\ -0.327686169 \end{pmatrix}$ |
| $x^{*,\mathrm{II}}$ | $\begin{pmatrix} -0.352272722 \\ -0.390151520 \\ -0.390151520 \end{pmatrix}$ | $\begin{pmatrix} -0.352272329 \\ -0.390151782 \\ -0.390151782 \end{pmatrix}$ | $\begin{pmatrix} -0.269592550 \\ -0.445271634 \\ -0.445271634 \end{pmatrix}$ | $\begin{pmatrix} -0.193429058 \\ -0.496047295 \\ -0.496047295 \end{pmatrix}$ |
| ValL1 | 2.771204468 | 3.678656657 | 4.637556559 | 5.580778795 |
| ValL2 | 3.247032184 | 5.397549415 | 7.347156108 | 9.138817185 |
| $y^{*,\mathrm{I}}$ | $\begin{pmatrix} -0.279775961 \\ -0.325691035 \\ -0.085907093 \end{pmatrix}$ | $\begin{pmatrix} 0.087462920 \\ -0.673753783 \\ 0.112948491 \end{pmatrix}$ | $\begin{pmatrix} 0.129983950 \\ -0.900638618 \\ 0.384751972 \end{pmatrix}$ | $\begin{pmatrix} 0.144264549 \\ -1.062939416 \\ 0.590839269 \end{pmatrix}$ |
| $\|y^{*,\mathrm{I}} - \bar{y}^*\|$ | 0 | 0.543650270 | 0.848520986 | 1.086890596 |
| $y^{*,\mathrm{II}}$ | $\begin{pmatrix} -0.279775961 \\ -0.325691035 \\ -0.085907093 \end{pmatrix}$ | $\begin{pmatrix} -0.950444578 \\ 0.253822732 \\ -0.374220337 \end{pmatrix}$ | $\begin{pmatrix} -1.427485800 \\ 0.644474594 \\ -0.550559714 \end{pmatrix}$ | $\begin{pmatrix} -1.804717734 \\ 0.893213826 \\ -0.609766737 \end{pmatrix}$ |
| $\|y^{*,\mathrm{II}} - \bar{y}^*\|$ | 0 | 0.932071417 | 1.573010260 | 2.021288054 |
| $\|y^{*,\mathrm{I}} - y^{*,\mathrm{II}}\|$ | 0 | 1.474782572 | 2.384930777 | 3.011066627 |
| Iter | 7 | 6 | 4 | 4 |

TABLE 5.3
*Computational results for Example* 5.2 *with starting point* $(1.2, 0.2, 1.2, 0.2, 1.2, 0.2)^\top$.

| $(\rho;\sigma)$ | $(0.0, 0.0; 0.0, 0.0)$ | $(0.6, 0.6; 0.6, 0.6)$ | $(1.2, 1.2; 1.2, 1.2)$ | $(1.8, 1.8; 1.8, 1.8)$ |
|---|---|---|---|---|
| $x^{*,\mathrm{I}}$ | $\begin{pmatrix} 0.216787993 \\ -0.874394922 \\ 0.125602311 \end{pmatrix}$ | $\begin{pmatrix} 0.105517725 \\ -0.840159755 \\ 0.159841154 \end{pmatrix}$ | $\begin{pmatrix} 0.065921292 \\ -0.827975698 \\ 0.172024132 \end{pmatrix}$ | $\begin{pmatrix} 0.064980580 \\ -0.827686360 \\ 0.172313695 \end{pmatrix}$ |
| $x^{*,\mathrm{II}}$ | $\begin{pmatrix} -0.352272720 \\ 0.109844629 \\ -0.890147672 \end{pmatrix}$ | $\begin{pmatrix} -0.352272515 \\ 0.109848063 \\ -0.890151378 \end{pmatrix}$ | $\begin{pmatrix} -0.269604279 \\ 0.054736278 \\ -0.945263909 \end{pmatrix}$ | $\begin{pmatrix} -0.193427075 \\ 0.003951386 \\ -0.996048621 \end{pmatrix}$ |
| ValL1 | 2.771204471 | 3.678656376 | 4.637540452 | 5.580781510 |
| ValL2 | 3.247029440 | 5.397549253 | 7.347155806 | 9.138815871 |
| $y^{*,\mathrm{I}}$ | $\begin{pmatrix} -0.279774478 \\ -0.325692473 \\ -0.085906246 \end{pmatrix}$ | $\begin{pmatrix} 0.087462817 \\ -0.673753709 \\ 0.112948466 \end{pmatrix}$ | $\begin{pmatrix} 0.129994416 \\ -0.900645726 \\ 0.384753890 \end{pmatrix}$ | $\begin{pmatrix} 0.144489556 \\ -1.063118095 \\ 0.590915003 \end{pmatrix}$ |
| $\|y^{*,\mathrm{I}} - \bar{y}^*\|$ | 0 | 0.543647910 | 0.848529759 | 1.087144664 |
| $y^{*,\mathrm{II}}$ | $\begin{pmatrix} -0.279774479 \\ -0.325692473 \\ -0.085906246 \end{pmatrix}$ | $\begin{pmatrix} 0.087462817 \\ -0.673753709 \\ 0.112948466 \end{pmatrix}$ | $\begin{pmatrix} 0.129994416 \\ -0.900645726 \\ 0.384753890 \end{pmatrix}$ | $\begin{pmatrix} 0.144489556 \\ -1.063118095 \\ 0.590915003 \end{pmatrix}$ |
| $\|y^{*,\mathrm{II}} - \bar{y}^*\|$ | 0 | 0.932073807 | 1.573037101 | 2.021279729 |
| $\|y^{*,\mathrm{I}} - y^{*,\mathrm{II}}\|$ | 0 | 1.474782609 | 2.384967911 | 3.011347537 |
| Iter | 7 | 8 | 6 | 4 |

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Table 5.4
*Computational results for Example* 5.3.

| $(\rho;\sigma)$ | $(0.0, 0.0; 0.0, 0.0)$ | $(0.6, 0.6; 0.6, 0.6)$ | $(1.2, 1.2; 1.2, 1.2)$ | $(1.8, 1.8; 1.8, 1.8)$ |
|---|---|---|---|---|
| $x^{*,\mathrm{I}}$ | $\begin{pmatrix} 0.216787400 \\ -0.748792246 \end{pmatrix}$ | $\begin{pmatrix} 0.374532694 \\ -0.845866273 \end{pmatrix}$ | $\begin{pmatrix} 0.527398068 \\ -0.939937273 \end{pmatrix}$ | $\begin{pmatrix} 0.662190056 \\ -1.022886188 \end{pmatrix}$ |
| $x^{*,\mathrm{II}}$ | $\begin{pmatrix} -0.352272727 \\ -0.780303030 \end{pmatrix}$ | $\begin{pmatrix} -0.352272727 \\ -0.780303030 \end{pmatrix}$ | $\begin{pmatrix} -0.342434266 \\ -0.793420979 \end{pmatrix}$ | $\begin{pmatrix} -0.296159749 \\ -0.855120335 \end{pmatrix}$ |
| ValL1 | 2.771204449 | 4.457474463 | 6.740324310 | 9.593635545 |
| ValL2 | 3.247030819 | 5.542234844 | 7.828858707 | 10.016028089 |
| $y^{*,\mathrm{I}}$ | $\begin{pmatrix} -0.279775240 \\ -0.325691736 \\ -0.085906679 \end{pmatrix}$ | $\begin{pmatrix} 0.437517352 \\ -1.004824855 \\ 0.301559496 \end{pmatrix}$ | $\begin{pmatrix} 1.160047269 \\ -1.686007511 \\ 0.687975875 \end{pmatrix}$ | $\begin{pmatrix} 1.899242021 \\ -2.375209993 \\ 1.073049643 \end{pmatrix}$ |
| $\Vert y^{*,\mathrm{I}} - \bar{y}^*\Vert$ | 0 | 1.061065735 | 2.126603412 | 3.208086823 |
| $y^{*,\mathrm{II}}$ | $\begin{pmatrix} -0.279775240 \\ -0.325691736 \\ -0.085906679 \end{pmatrix}$ | $\begin{pmatrix} -0.608758003 \\ -0.077625335 \\ -0.179063218 \end{pmatrix}$ | $\begin{pmatrix} -0.932362313 \\ 0.165657967 \\ -0.269726729 \end{pmatrix}$ | $\begin{pmatrix} -1.230555251 \\ 0.378270341 \\ -0.337848329 \end{pmatrix}$ |
| $\Vert y^{*,\mathrm{II}} - \bar{y}^*\Vert$ | 0 | 0.422427199 | 0.837307269 | 1.209553567 |
| $\Vert y^{*,\mathrm{I}} - y^{*,\mathrm{II}}\Vert$ | 0 | 1.478306214 | 2.953648114 | 4.400899681 |
| Iter | 6 | 6 | 10 | 11 |

with given uncertainty bounds $\rho^\nu > 0$ and $\sigma^\nu > 0$, $\nu = \mathrm{I}, \mathrm{II}$. In the forward-backward splitting method, $x^{\nu,k+1}$ can be obtained by solving the following optimization problems:

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \dfrac{1}{2\mu}\|x^\nu - z^{\nu,k}\|^2 + \rho^\nu \|R_\nu^\top x^\nu\|_1 + \sigma^\nu \|B^{-\frac{1}{2}} P B^{-\frac{1}{2}} (D_\nu)^\top x^\nu\|_1 \\ \text{subject to} & A_\nu^\top x^\nu + b_\nu \le 0, \end{array}$$

where $\|\cdot\|_1$ denotes the $l_1$ norm. These problems can further be rewritten as convex quadratic programming problems. We use the same problem data as that in Example 5.1.

The computational results are shown in Table 5.4. In addition to observations similar to those in Example 5.1, it may be interesting to notice that the optimal values of leaders in Example 5.3 are always larger than those in Example 5.1 under the same value of uncertainty data $(\rho;\sigma)$ except $(\rho,\sigma) = (0;0)$. This is probably because the worst case in Example 5.3 tends to be more pessimistic than that in Example 5.1, as the former usually occurs at a vertex of the box uncertainty set, while the latter occurs on the boundary of the inscribed sphere.

*Example* 5.4. We show some numerical results for a multi-leader multi-follower game with three leaders and two followers.[3] More specifically, each leader $\nu$ solves the following problem:

[3]This example is somewhat simpler than the one mentioned in Remark 3.1. We may consider a Nash game in the lower level where multiple followers compete with each other in a noncooperative manner. In this case, each follower $\omega$'s objective function also contains his/her rivals' strategies $y^{-\omega}$. Then, we may compute the explicit response of the followers for any given strategy tuple $x$ of leaders from a system of linear equations formed by concatenating the KKT systems of all followers' optimization problems. Here, we would not go into detail since the presentation will become excessively complicated.

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$$\begin{aligned}(5.3)\qquad \underset{x^\nu}{\text{minimize}}\quad & \frac{1}{2}(x^\nu)^\top H_\nu x^\nu + \sum_{\nu'=\mathrm{I},\nu'\neq\nu}^{\mathrm{III}} (x^\nu)^\top E_{\nu,\nu'} x^{\nu'} + (x^\nu)^\top R_\nu u^\nu \\ & + \sum_{\omega=1}^{2} (x^\nu)^\top D_\nu^\omega y^\omega \\ \text{subject to}\quad & A_\nu^\top x^\nu + b_\nu \le 0,\end{aligned}$$

where $H_\nu \in \mathbb{R}^{n_\nu \times n_\nu}$ is symmetric, and $D_\nu^\omega \in \mathbb{R}^{n_\nu \times m}, \omega = 1, 2$. The variable $y^\omega \in \mathbb{R}^m$ is an optimal solution of the following follower $\omega$'s problem anticipated by leader $\nu$:

$$\begin{aligned}(5.4)\qquad \underset{y^\omega}{\text{minimize}}\quad & \frac{1}{2}(y^\omega)^\top B_\omega y^\omega + (c^\omega + v^{\omega,\nu})^\top y^\omega - \sum_{\nu=\mathrm{I}}^{\mathrm{III}} (x^\nu)^\top D_\nu^\omega y^\omega \\ \text{subject to}\quad & A_\omega y^\omega + a^\omega = 0,\end{aligned}$$

where $B_\omega \in \mathbb{R}^{m \times m}$ is symmetric and positive definite and $c^\omega \in \mathbb{R}^m$. The uncertainty parameters are $u^\nu \in U^\nu = \{u^\nu \in \mathbb{R}^{l_\nu} | \|u^\nu\| \le \rho^\nu\}$ and $v^{\omega,\nu} \in V^\nu = \{v^{\omega,\nu} \in \mathbb{R}^m | \|v^{\omega,\nu}\| \le \sigma^\nu\}$, $\omega = 1, 2$, $\nu = \mathrm{I}, \mathrm{II}, \mathrm{III}$.

Under these settings, we can finally reformulate the game comprising the problems (5.3) and (5.4) as a NEP, where each player $\nu = \mathrm{I}, \mathrm{II}, \mathrm{III}$ solves the following problem:

$$\begin{aligned}(5.5)\qquad \underset{x^\nu}{\text{minimize}}\quad & \frac{1}{2}(x^\nu)^\top \left(H_\nu + 2\sum_{\omega=1}^{2} D_\nu^\omega G_\nu^\omega\right) x^\nu + \sum_{\omega=1}^{2} (x^\nu)^\top D_\nu^\omega r^\omega \\ & + \sum_{\nu'=\mathrm{I},\nu'\neq\nu}^{\mathrm{III}} (x^\nu)^\top \left(\sum_{\omega=1}^{2} D_\nu^\omega G_{\nu'}^\omega + E_{\nu,\nu'}\right) x^{\nu'} + \rho^\nu \|R_\nu^\top x^\nu\| \\ & + \sum_{\omega=1}^{2} \sigma^{\omega,\nu} \|(B_\omega)^{-\frac{1}{2}}(P_\omega)(B_\omega)^{-\frac{1}{2}}(D_\nu^\omega)^\top x^\nu\| \\ \text{subject to}\quad & A_\nu^\top x^\nu + b_\nu \le 0,\end{aligned}$$

where $G_\nu^\omega \in \mathbb{R}^{m \times n_\nu}$ and $r^\omega \in \mathbb{R}^m$ are given by

$$\begin{aligned} G_\nu^\omega &= B_\omega^{-\frac{1}{2}} P_\omega B_\omega^{-\frac{1}{2}} (D_\nu^\omega)^\top, \\ r^\omega &= -(B_\omega)^{-\frac{1}{2}} P_\omega (B_\omega)^{-\frac{1}{2}} c^\omega - (B_\omega)^{-1}(A_\omega)^\top (A_\omega (B_\omega)^{-1}(A_\omega)^\top)^{-1} a^\omega, \\ P_\omega &= I - B_\omega^{-\frac{1}{2}} (A_\omega)^\top (A_\omega (B_\omega)^{-1} (A_\omega)^\top)^{-1} A_\omega (B_\omega)^{-\frac{1}{2}}. \end{aligned}$$

In the forward-backward splitting method, the iterative point $x^{k+1} := (x^{\mathrm{I},k+1}, x^{\mathrm{II},k+1}, x^{\mathrm{III},k+1})$ in (5.1) is obtained by letting $z^{\nu,k} := x^{\nu,k} - \mu T_\nu(x^k)$, $\nu = \mathrm{I}, \mathrm{II}, \mathrm{III}$, and then by solving the following optimization problems for $\nu = \mathrm{I}, \mathrm{II}, \mathrm{III}$:

$$\begin{aligned}\underset{x^\nu}{\text{minimize}}\quad & \frac{1}{2\mu}\|x^\nu - z^{\nu,k}\|^2 + \rho^\nu \|R_\nu^\top x^\nu\| + \sum_{\omega=1}^{2} \sigma^\nu \|(B_\omega)^{-\frac{1}{2}}(P_\omega)(B_\omega)^{-\frac{1}{2}}(D_\nu^\omega)^\top x^\nu\| \\ \text{subject to}\quad & A_\nu^\top x^\nu + b_\nu \le 0.\end{aligned}$$

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Table 5.5
*Computational results for Example* 5.4.

| $\begin{pmatrix}\rho\\ \sigma^1\\ \sigma^2\end{pmatrix}$ | $\begin{pmatrix}0.0, 0.0, 0.0;\\ 0.0, 0.0, 0.0;\\ 0.0, 0.0, 0.0\end{pmatrix}$ | $\begin{pmatrix}0.5, 0.5, 0.5;\\ 0.5, 0.5, 0.5;\\ 0.5, 0.5, 0.5\end{pmatrix}$ | $\begin{pmatrix}1.5, 1.5, 1.5;\\ 1.5, 1.5, 1.5;\\ 1.5, 1.5, 1.5\end{pmatrix}$ | $\begin{pmatrix}2.5, 2.5, 2.5;\\ 2.5, 2.5, 2.5;\\ 2.5, 2.5, 2.5\end{pmatrix}$ |
|---|---|---|---|---|
| $x^{*,\mathrm{I}}$ | $\begin{pmatrix}-1.597902084\\ 0.223776190\end{pmatrix}$ | $\begin{pmatrix}-1.597902084\\ 0.223776189\end{pmatrix}$ | $\begin{pmatrix}-1.597902091\\ 0.223776205\end{pmatrix}$ | $\begin{pmatrix}-1.597902092\\ 0.223776207\end{pmatrix}$ |
| $x^{*,\mathrm{II}}$ | $\begin{pmatrix}-1.645161294\\ 0.688172047\end{pmatrix}$ | $\begin{pmatrix}-1.645161301\\ 0.688172052\end{pmatrix}$ | $\begin{pmatrix}-1.645161281\\ 0.688172025\end{pmatrix}$ | $\begin{pmatrix}-1.645161263\\ 0.688171999\end{pmatrix}$ |
| $x^{*,\mathrm{III}}$ | $\begin{pmatrix}-0.314917128\\ -0.596685085\end{pmatrix}$ | $\begin{pmatrix}-0.314917129\\ -0.596685087\end{pmatrix}$ | $\begin{pmatrix}-0.314917128\\ -0.596685084\end{pmatrix}$ | $\begin{pmatrix}-0.314917127\\ -0.596685084\end{pmatrix}$ |
| ValL I | 10.956476239 | 13.675477296 | 19.113479434 | 24.551481591 |
| ValL II | 10.792694931 | 13.565012752 | 19.109648271 | 24.654283735 |
| ValL III | 10.897573095 | 12.457109088 | 15.576180955 | 18.695252881 |
| $y^{*,1,\mathrm{I}}$ | $\begin{pmatrix}1.123556738\\ -0.664585191\\ -1.620136328\end{pmatrix}$ | $\begin{pmatrix}1.298957295\\ -0.880362760\\ -1.570757746\end{pmatrix}$ | $\begin{pmatrix}1.649758431\\ -1.311917891\\ -1.472000608\end{pmatrix}$ | $\begin{pmatrix}2.000559562\\ -1.743473028\\ -1.373243461\end{pmatrix}$ |
| $y^{*,1,\mathrm{II}}$ | $\begin{pmatrix}1.123556738\\ -0.664585191\\ -1.620136328\end{pmatrix}$ | $\begin{pmatrix}1.051962380\\ -0.420817536\\ -1.785604363\end{pmatrix}$ | $\begin{pmatrix}0.908773690\\ 0.066717777\\ -2.116540457\end{pmatrix}$ | $\begin{pmatrix}0.765585010\\ 0.554253084\\ -2.447476555\end{pmatrix}$ |
| $y^{*,1,\mathrm{III}}$ | $\begin{pmatrix}1.123556738\\ -0.664585191\\ -1.620136328\end{pmatrix}$ | $\begin{pmatrix}1.259291555\\ -0.899775936\\ -1.518261808\end{pmatrix}$ | $\begin{pmatrix}1.530761203\\ -1.370157422\\ -1.314512782\end{pmatrix}$ | $\begin{pmatrix}1.802230848\\ -1.840538914\\ -1.110763749\end{pmatrix}$ |
| $y^{*,2,\mathrm{I}}$ | $\begin{pmatrix}-0.367571540\\ 0.549850216\\ -0.929750305\end{pmatrix}$ | $\begin{pmatrix}-0.151643060\\ 0.306359239\\ -0.992049559\end{pmatrix}$ | $\begin{pmatrix}0.280213914\\ -0.180622739\\ -1.116648066\end{pmatrix}$ | $\begin{pmatrix}0.712070889\\ -0.667604724\\ -1.241246572\end{pmatrix}$ |
| $y^{*,2,\mathrm{II}}$ | $\begin{pmatrix}1.123556738\\ -0.664585191\\ -1.620136328\end{pmatrix}$ | $\begin{pmatrix}1.051962380\\ -0.420817536\\ -1.785604363\end{pmatrix}$ | $\begin{pmatrix}0.908773690\\ 0.066717777\\ -2.116540457\end{pmatrix}$ | $\begin{pmatrix}0.765585010\\ 0.554253084\\ -2.447476555\end{pmatrix}$ |
| $y^{*,2,\mathrm{III}}$ | $\begin{pmatrix}-0.367571540\\ 0.549850216\\ -0.929750305\end{pmatrix}$ | $\begin{pmatrix}-0.574052503\\ 0.899218058\\ -0.935433810\end{pmatrix}$ | $\begin{pmatrix}-0.987014417\\ 1.597953705\\ -0.946800811\end{pmatrix}$ | $\begin{pmatrix}-1.399976334\\ 2.296689340\\ -0.958167804\end{pmatrix}$ |
| $\Vert y^{*,1,\mathrm{I}} - \bar{y}^{*,1}\Vert$ | 0 | 0.282424431 | 0.847273295 | 1.412122163 |
| $\Vert y^{*,1,\mathrm{II}} - \bar{y}^{*,1}\Vert$ | 0 | 0.303196458 | 0.909589384 | 1.515982305 |
| $\Vert y^{*,1,\mathrm{III}} - \bar{y}^{*,1}\Vert$ | 0 | 00.290029386 | 0.870088157 | 1.450146934 |
| $\Vert y^{*,2,\mathrm{I}} - \bar{y}^{*,2}\Vert$ | 0 | 0.331352020 | 0.994056088 | 1.656760161 |
| $\Vert y^{*,2,\mathrm{II}} - \bar{y}^{*,2}\Vert$ | 0 | 0.405862759 | 1.217588240 | 2.029313711 |
| $\Vert y^{*,2,\mathrm{III}} - \bar{y}^{*,2}\Vert$ | 0 | 0.415197933 | 1.245593832 | 2.075989737 |
| $\Vert y^{*,1,\mathrm{I}} - y^{*,1,\mathrm{II}}\Vert$ | 0 | 0.564222802 | 1.692668399 | 2.821113994 |
| $\Vert y^{*,1,\mathrm{II}} - y^{*,1,\mathrm{III}}\Vert$ | 0 | 0.586394558 | 1.759183673 | 2.931972788 |
| $\Vert y^{*,1,\mathrm{III}} - y^{*,1,\mathrm{I}}\Vert$ | 0 | 0.068600772 | 0.205802329 | 0.343003884 |
| $\Vert y^{*,2,\mathrm{I}} - y^{*,2,\mathrm{II}}\Vert$ | 0 | 0.532684599 | 1.59805300 | 2.663423004 |
| $\Vert y^{*,2,\mathrm{II}} - y^{*,2,\mathrm{III}}\Vert$ | 0 | 0.547440871 | 1.642322612 | 2.737204351 |
| $\Vert y^{*,2,\mathrm{III}} - y^{*,2,\mathrm{I}}\Vert$ | 0 | 0.187637243 | 0.562911736 | 0.938186227 |
| Iter | 4 | 4 | 3 | 3 |

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Here, we use the following problem data:

$$H_{\mathrm{I}} = \begin{pmatrix} 2.5 & 1.6 \\ 1.6 & 3.8 \end{pmatrix}, \quad H_{\mathrm{II}} = \begin{pmatrix} 2.9 & 1.3 \\ 1.3 & 1.8 \end{pmatrix}, \quad H_{\mathrm{III}} = \begin{pmatrix} 3.2 & 2.3 \\ 2.3 & 2.6 \end{pmatrix},$$

$$D^1_{\mathrm{I}} = \begin{pmatrix} 0.8 & 2.1 & 1.3 \\ 1.5 & 2.3 & 0.7 \end{pmatrix}, \quad D^1_{\mathrm{II}} = \begin{pmatrix} 1.5 & 0.9 & 2.4 \\ 1.8 & 2.3 & 3.6 \end{pmatrix}, \quad D^1_{\mathrm{III}} = \begin{pmatrix} 1.3 & 1.7 & 1.7 \\ 1.1 & 2.6 & 1.6 \end{pmatrix},$$

$$D^2_{\mathrm{I}} = \begin{pmatrix} 0.8 & 2.1 & 1.3 \\ 1.5 & 2.3 & 0.7 \end{pmatrix}, \quad D^2_{\mathrm{II}} = \begin{pmatrix} 1.5 & 0.9 & 2.4 \\ 1.8 & 2.3 & 3.6 \end{pmatrix}, \quad D^2_{\mathrm{III}} = \begin{pmatrix} 0.5 & 1.1 & 2.1 \\ 1.2 & 1.5 & 1.8 \end{pmatrix},$$

$$E_{\mathrm{I,II}} = \begin{pmatrix} 1.2 & 1.5 \\ 0.4 & 1.3 \end{pmatrix}, \quad E_{\mathrm{I,III}} = \begin{pmatrix} 0.8 & 0.4 \\ 1.5 & 0.7 \end{pmatrix}, \quad E_{\mathrm{II,I}} = \begin{pmatrix} 1.3 & 0.6 \\ 1.7 & 2.5 \end{pmatrix},$$

$$E_{\mathrm{II,III}} = \begin{pmatrix} 2.1 & 3.2 \\ 2.2 & 2.0 \end{pmatrix}, \quad E_{\mathrm{III,I}} = \begin{pmatrix} 2.4 & 1.2 \\ 2.5 & 2.6 \end{pmatrix}, \quad E_{\mathrm{III,II}} = \begin{pmatrix} 1.8 & 2.4 \\ 3.5 & 2.7 \end{pmatrix},$$

$$R_{\mathrm{I}} = \begin{pmatrix} 1.7 & 2.8 \\ 0.6 & 0.7 \end{pmatrix}, \quad R_{\mathrm{II}} = \begin{pmatrix} 1.8 & 2.7 \\ 1.9 & 1.4 \end{pmatrix}, \quad R_{\mathrm{III}} = \begin{pmatrix} 2.3 & 1.8 \\ 2.3 & 0.7 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 3.7 & 1.4 & 1.2 \\ 1.4 & 2.6 & 0.6 \\ 1.2 & 0.6 & 0.7 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2.7 & 2.4 & 0.2 \\ 2.4 & 3.6 & 2.1 \\ 0.2 & 2.1 & 2.7 \end{pmatrix}, \quad c^1 = \begin{pmatrix} 0.4 \\ 1.6 \\ 2.6 \end{pmatrix}, \quad c^2 = \begin{pmatrix} 1.5 \\ 0.6 \\ 1.1 \end{pmatrix},$$

$$A_{\mathrm{I}} = \begin{pmatrix} 1.4 & 1.8 & 1.6 \\ 2.4 & 1.2 & 0.7 \end{pmatrix}, \quad A_{\mathrm{II}} = \begin{pmatrix} 1.8 & 1.9 & 1.6 \\ 2.3 & 1.2 & 1.5 \end{pmatrix}, \quad A_{\mathrm{III}} = \begin{pmatrix} 2.3 & 1.9 & 1.6 \\ 1.3 & 1.7 & 2.7 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 1.3 & 1.4 & 1.5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2.3 & 1.4 & 2.5 \end{pmatrix}, \quad a^1 = 1.9, \quad a^2 = 2.4,$$

$$b_{\mathrm{I}} = \begin{pmatrix} 1.7 \\ 1.3 \\ 2.4 \end{pmatrix}, \quad b_{\mathrm{II}} = \begin{pmatrix} 1.3 \\ 2.3 \\ 1.6 \end{pmatrix}, \quad b_{\mathrm{III}} = \begin{pmatrix} 1.5 \\ 0.3 \\ 1.8 \end{pmatrix}.$$

The computational results are shown in Table 5.5. In this table, $(x^{*,\mathrm{I}}, x^{*,\mathrm{II}}, x^{*,\mathrm{III}})$ denotes the leaders' optimal strategies and $(y^{*,\omega,\mathrm{I}}, y^{*,\omega,\mathrm{II}}, y^{*,\omega,\mathrm{III}})$ denotes the follower $\omega$'s responses estimated respectively by the three leaders at the computed equilibria for various values of the uncertainty bounds $\rho = (\rho^{\mathrm{I}}, \rho^{\mathrm{II}}, \rho^{\mathrm{III}})$, $\sigma^1 = (\sigma^{1,\mathrm{I}}, \sigma^{1,\mathrm{II}}, \sigma^{1,\mathrm{III}})$, and $\sigma^2 = (\sigma^{2,\mathrm{I}}, \sigma^{2,\mathrm{II}}, \sigma^{2,\mathrm{III}})$. Similar to the previous examples, in the particular case where there is no uncertainty ($\rho = 0, \sigma^1 = 0$, $\sigma^2 = 0$), the followers' responses anticipated by the three leaders naturally coincide, i.e., $y^{*,\omega,\mathrm{I}} = y^{*,\omega,\mathrm{II}} = y^{*,\omega,\mathrm{III}}$, which is denoted $\bar{y}^{*,\omega}$, $\omega = 1, 2$. We can observe similar behavior of computed solutions to that in the previous examples.

**6. Conclusion.** In this paper, we have considered a class of multi-leader single-follower games with uncertainty. We have defined a new concept for the multi-leader single-follower game with uncertainty, called robust L/F Nash equilibrium. We have discussed the existence and the uniqueness of a robust L/F Nash equilibrium by reformulating the game as a NEP with uncertainty and then a GVI problem. Through numerical experiments including those for the multi-follower case, we have observed the influence of uncertainty on the follower's responses estimated by the leaders.

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REFERENCES

[1] M. Aghassi and D. Bertsimas, *Robust game theory*, Math. Program., 107 (2006), pp. 231–273.
[2] J.-P. Aubin, *Mathematical Methods of Game and Economic Theory*, North-Holland, Amsterdam, 1979.

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[3] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Basel, 1990.
[4] X. Bai, S. M. Shahidehpour, V. C. Ramesh, and E. Yu, *Transmission analysis by Nash game method*, IEEE Trans. Power Systems, 12 (1997), pp. 1046–1052.
[5] A. Ben-Tal and A. Nemirovski, *Robust convex optimization*, Math. Oper. Res., 23 (1979), pp. 769–805.
[6] A. Ben-Tal and A. Nemirovski, *Robust solutions of uncertain linear programs*, Oper. Res. Lett., 25 (1999), pp. 1–13.
[7] A. Ben-Tal and A. Nemirovski, *Selected topics in robust convex optimization*, Math. Program., 112 (2008), pp. 125–158.
[8] D. P. Bertsekas, *Nonlinear Programming*, Athena Scientific, Belmont, MA, 1999.
[9] D. P. Bertsekas, *Convex Analysis and Optimization*, Athena Scientific, Belmont, MA, 2003.
[10] V. DeMiguel and H. Xu, *A stochastic multiple-leader Stackelberg model: Analysis, computation, and application*, Oper. Res., 57 (2009), pp. 1220–1235.
[11] L. El Ghaoui and H. Lebret, *Robust solutions to least-squares problem with uncertain data*, SIAM J. Matrix Anal. Appl., 18 (1997), pp. 1035–1064.
[12] F. Facchinei and C. Kanzow, *Generlized Nash equilibrium problems*, Ann. Oper. Res., 175 (2010), pp. 177–211.
[13] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, vol. I, Springer, New York, 2003.
[14] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, vol. II, Springer, New York, 2003.
[15] S. C. Fang and E. L. Peterson, *Generalized variational inequalities*, J. Optim. Theory Appl., 38 (1982), pp. 363–383.
[16] M. C. Ferris and J. S. Pang, *Engineering and economic applications of complementarity problems*, SIAM Rev., 39 (1997), pp. 669–713.
[17] D. Gabay, *Applications of the method of multipliers to variational inequalities.* in Augmented Lagrangian Methods: Applications to the Solution of Boundary Value Problems, M. Fortin and R. Glowinski, eds., North-Holland, Amsterdam, 1983, pp. 299–340.
[18] J. C. Harsanyi, *Games with incomplete information played by "Bayesian" players, Part* I. *The basic model*, Management Sci., 14 (1967), pp. 159–182.
[19] J. C. Harsanyi, *Games with incomplete information played by "Bayesian" players, Part* II. *Bayesian equilibrium points*, Management Sci., 14 (1968), pp. 320–334.
[20] J. C. Harsanyi, *Games with incomplete information played by "Bayesian" players, Part* III. *The basic probability distribution of the game*, Management Sci., 14 (1968), pp. 486–502.
[21] S. Hayashi, N. Yamashita, and M. Fukushima, *Robust Nash equilibria and second-order cone complementarity problems*, J. Nonlinear Convex Anal., 6 (2005), pp. 283–296.
[22] M. Hu and M. Fukushima, *Variational inequality formulation of a class of multi-leader-follower games*, J. Optim. Theory Appl., 151 (2011), pp. 455–473.
[23] X. Hu and D. Ralph, *Using EPECs to model bilevel games in structured electricity markets with locational prices*, Oper. Res., 55 (2007), pp. 809–827.
[24] J. B. Krawcyzk, *Coupled constraint Nash equilibria in environmental games*, Resource Energy Economics, 27 (2005), pp. 157–181.
[25] J. B. Krawcyzk, *Numerical solutions to coupled-constraint (or generalised Nash) equilibrium problems*, Comput. Management Sci., 4 (2007), pp. 183–204.
[26] S. Leyffer and T. Munson, *Solving multi-leader-common-follower games*, Optim. Methods Softw., 25 (2010), pp. 601–623.
[27] A. Nagurney, *Network Economics: A Variational Inequality Approach*, Kluwer Academic Publishers, Boston, 1993.
[28] J. F. Nash, *Equilibrium points in n-person games*, Proc. Nat. Acad. Sci. USA, 36 (1950), pp. 48–49.
[29] J. F. Nash, *Non-cooperative games*, Ann. Math., 54 (1951), pp. 286–295.
[30] R. Nishimura, S. Hayashi, and M. Fukushima, *Robust Nash equilibria in $N$-person non-cooperative games: Uniqueness and reformulations*, Pacific J. Optim., 5 (2009), pp. 237–259.
[31] J. S. Pang and M. Fukushima, *Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games*, Comput. Management Sci., 2 (2005), pp. 21–56.
[32] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.
[33] J. B. Rosen, *Existence and uniqueness of equilibrium points for concave $N$-person games*, Econometrica, 33 (1965), pp. 520–534.
[34] U. V. Shanbhag, G. Infanger, and P. W. Glynn, *A complementarity framework for forward contracting under uncertainty*, Oper. Res., 59 (2011), pp. 810–834.

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[35] H. D. Sherali, *A multiple leader Stackelberg model and analysis*, Oper. Res., 32 (1984), pp. 390–404.

[36] H. Song, C. C. Liu, J. Lawarre, and W. A. Bellevue, *Nash equilibrium bidding strategies in a bilateral electricity market*, IEEE Trans. Power Systems, 17 (2002), pp. 73–79.

[37] J. F. Sturm, *Using SeDuMi* 1.0*x, a MATLAB Toolbox for Optimization over Symmetric Cones*, http://www.unimass.nl/sturm.

[38] C. L. Su, *Analysis on the forward market equilibrium model*, Oper. Res. Lett., 35 (2007), pp. 74–82.

[39] K. C. Toh, M. J. Todd, and R. H. Tütüncü, *On the implementation and usage of SDPT*3 *— a Matlab software package for semidenite-quadratic-linear programming, version* 4.0, http://www.math.nus.edu.sg/mattohkc/sdpt3.html.

[40] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim., 38 (2000), pp. 431–446.

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