Math. Meth. Oper. Res. (2006) 63: 261–285
DOI 10.1007/s00186-005-0003-y

ORIGINAL ARTICLE

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Existence of nash equilibria for constrained stochastic games

Received: March 2004 / Revised version: January 2005 / Published online: 8 October 2005

Abstract In this paper, we consider *constrained* noncooperative N-person stochastic games with discounted cost criteria. The state space is assumed to be countable and the action sets are compact metric spaces. We present three main results. The first concerns the *sensitivity* or *approximation* of constrained games. The second shows the existence of Nash equilibria for constrained games with a *finite* state space (and compact actions space), and, finally, in the third one we extend that existence result to a class of constrained games which can be "approximated" by constrained games with finitely many states and compact action spaces. Our results are illustrated with two examples on queueing systems, which clearly show some important differences between constrained and unconstrained games.

Keywords Constrained Markov games · Nash equilibria · Constrained Markov control processes

Mathematics Subject Classification (2000) Primary: 91A15 · 91A10; Secondary: 90C40

1 Introduction

In a standard (or unconstrained) noncooperative N-person dynamic stochastic game each player $i = 1, 2, \ldots, N$ has an objective function, say $V_i^0(\boldsymbol{\pi})$, on a given

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set of multistrategies $\boldsymbol{\pi}$, and one wishes to determine the existence of Nash equilibria. In this paper, we study a class of N-player *constrained* dynamic stochastic, or Markov, games in which we have additional objective functions $V_i^1(\boldsymbol{\pi}), \ldots, V_i^q(\boldsymbol{\pi})$ for each player i, and look for Nash equilibria that satisfy constraints of the form

$$V_i^l(\boldsymbol{\pi}) \leq k_i^l \;\; \forall\, l = 1, \ldots, q; \;\; i = 1, \ldots, N,$$

where the k_i^l are given constants.

From the viewpoint of applications, constrained games (CGs) have been studied for some telecommunications systems (Hsiao and Lazar, 1991; Korilis and Lazar, 1995). Another source of CGs, albeit still unexplored, is the case of commonly owned resources, such as deep-sea fisheries. These resources are often overused or overexploited thus leading to the phenomenon known as the *tragedy of the commons* (Dutta and Sundaram, 1993). For instance, some fisheries can be modelled as noncooperative games in which the players are the different fleets competing for the fish stock, the common resource. As each fleet wishes to maximize its reward by increasing the harvest, the fish stock can reach a severely degraded state (Clark, 1990, 1980; Gordon, 1954; Levhary and Mirman, 1980; Mckelvey, 1997, 1999), which could be avoided by imposing suitable constraints. In fact, a "fishing treaty" is, roughly speaking, an agreement on each side to limit its harvest in some respect.

On the theoretical side, conditions for the existence of Nash equilibria for *unconstrained* dynamic Markov games are known when the state space is a countable set (Altman et al., 1997; Borkar and Ghosh, 1993; Federgruen, 1978; Sennott, 1993, 1994), and also for uncountable state spaces when the game has a so-called "additive" (or "separable" or "ARAT") structure – see (Ghosh and Bagchi, 1998; Nowak, 1987; Parthasarathy, 1982) or Example 5.3 below. However, to the best of our knowledge, the *constrained* case has been considered only in Shimkin (1994) for zero-sum games with average cost, and in Altman and Shwartz (2000) for N-person games where each player can have either a discounted or an average criterion. Both of these papers deal with CGs with *finite* state and *finite* action sets. Finally, on the numerical side, the problem of computing Nash equilibria is considered in Godoy-Alcantar et al. (2002a,b) for constrained *static* games with *finite* state and *finite* action sets.

Our main results generalize (Altman and Shwartz, 2000) to discounted CGs with a countable state space and compact action sets (see Remark 3.7). In order to obtain this, we first prove the existence of Nash equilibria for CGs with finite state space and *compact* action space. It should be noted that the latter generalization, in the *constrained* case, is not trivial. Indeed, the standard method to get existence of Nash equilibria consists in characterizing those equilibria as fixed points of certain multifunction from the space of stationary multistrategies into itself, and to use some fixed-point theorem. It turns out that this approach requires, on the one hand, some continuity arguments which are trivial for CGs with *finite* state-action spaces, but do not carry over directly to CGs with finite state set and (nonfinite) *compact* action spaces. On the other hand, that approach also requires convexity conditions. For *unconstrained* games these convexity conditions are easily verified via the *optimality* or *dynamic programming equation* for Markov control processes. However, there is no such optimality equation for *constrained* Markov control processes. Hence, the fixed-point/convexity approach is not directly applicable to CGs. To get around this difficulty we use the formulation of constrained Markov control processes as linear programs on suitable sets of so-called occupation measures. In the second stage, we generalize our existence result to countable-state CGs which can be "approximated" in a suitable sense by CGs with finitely many states.

The contributions in this paper, summarized in Theorem 3.6, consist of *several* results on the existence of Nash equilibria for CGs. The most straightforward essentially states that, under standard hypotheses, a CG with a *finite* state space and compact metric action spaces has a Nash equilibrium. On the other hand, to extend this existence result to CGs with a countable state space, we first study the sensitivity (or approximation) of CGs, establishing that *if* a CG can be approximated (in the sense of Assumption 3.5) by CGs each of which has a Nash equilibrium, then the CG has a Nash equilibrium. Finally, it is established that under suitable hypotheses (see Assumption 3.4), the *countable-state* CG has a Nash equilibrium and, in addition, it can be approximated by *finite-state* CGs (with compact actions sets) each having a Nash equilibrium.

The remainder of this paper is organized as follows. In section 2 we introduce the game model we are interested in. In section 3 we state our main result, Theorem 3.6. In section 4 alternative assumptions for Theorem 3.6 are given. In section 5, our results are ilustrated with two examples on queueing systems. After some technical preliminaries in section 6, Theorem 3.6 is proved in section 7. Finally, we conclude in section 8 with some general remarks.

2 The CG model

In this section we introduce the (discrete-time, time-homogeneous) constrained game (CG) model we are interested in, and the corresponding CG problem.

The CG model. We shall consider the N-person nonzero-sum *constrained stochastic game model*

$$\{X, (A_i, \{A_i(x)|x \in X\}, \mathbf{c}_i, \mathbf{k}_i)_{i\in\mathcal{N}}, Q, \gamma\}, \tag{2.1}$$

in which $\mathcal{N} := \{1, 2, \dots, N\}$ is the set of players. A generic player is indexed by i. Moreover, X denotes the state space, which is assumed to be a *countable* set with the discrete topology. Each player i is characterized by

$$(A_i, \{A_i(x)|x \in X\}, \mathbf{c}_i, \mathbf{k}_i), \tag{2.2}$$

where A_i is the action (or control) space, a *metric space* endowed with the corresponding Borel σ-algebra $\mathcal{B}(A_i)$. For each $x \in X$, the nonempty set $A_i(x) \in \mathcal{B}(A_i)$ in (2.2) stands for the set of admissible actions in x and we shall suppose that it is *compact*. Thus the set $\mathbb{A}(x) := \times_{i\in\mathcal{N}} A_i(x)$ is also *compact*. Note that the set

$$\mathbb{K}_i := \{(x, a) \mid x \in \mathrm{X},\ a \in A_i(x)\}$$

of feasible state-action pairs for player i is a closed (hence a Borel-measurable) subset of $X \times A_i$, and, similarly, the set

$$\mathbb{K} := \{(x, \boldsymbol{a})|x \in X \text{ and } \boldsymbol{a} \in \mathbb{A}(x)\}$$

of feasible state-action vectors is a closed subset of $X \times \mathbb{A}$, with $\mathbb{A} := \times_{i\in\mathcal{N}} A_i$.

The last two components in (2.2) are a measurable function $\mathbf{c}_i := (c_i^0, \ldots, c_i^q)$ from $\mathbb{K}$ to $\mathbb{R}^{q+1}$, and a vector $\mathbf{k}_i = (k_i^1, \ldots, k_i^q)$. The function c_i^0 denotes the *cost-per-stage*, while the other functions $c_i^1, \ldots, c_i^q$ and the vector $\mathbf{k}_i$ are used to formulate the CG problem; see (2.3), (2.4) below. The game's *transition law* (or law of motion) Q is a stochastic kernel from $\mathbb{K}$ to X, i.e., $Q(\cdot|x, \boldsymbol{a})$ is a probability measure on X for each fixed $(x, \boldsymbol{a}) \in \mathbb{K}$, and $Q(x'|\cdot)$ is measurable function on $\mathbb{K}$ for each fixed $x' \in X$. Finally, γ is the initial state distribution.

To state the CG problem let us first recall some standard concepts.

Strategies. Let $H_0 := X$, and $H_t := \mathbb{K} \times H_{t-1}$ for $t = 1, 2, \ldots$ For each t, an element $h_t := (x_0, \boldsymbol{a}_0, x_1, \boldsymbol{a}_1, \ldots, x_{t-1}, \boldsymbol{a}_{t-1}, x_t)$ of H_t represents a "history" of the game up to time t. A *strategy for player i* is then defined as a sequence $\pi^i := \{\pi_t^i\}$ of stochastic kernels π_t^i from H_t to A_i such that $\pi_t^i(A_i(x_t)|h_t) = 1$ for all $h_t \in H_t$ and $t = 0, 1, \ldots$. We denote by Π_i the family of all strategies for player i. A *multistrategy* is a vector $\boldsymbol{\pi} := (\pi^1, \ldots, \pi^N)$ in $\boldsymbol{Pi} := \times_{i \in \mathcal{N}} \Pi_i$.

Let Φ_i be the class of stochastic kernels φ from X to A_i such that $\varphi(A_i(x)|x) = 1$ for all $x \in X$. Then a strategy $\pi^i = \{\pi_t^i\}$ for player i, is called *randomized stationary* (or simply *stationary*) if there exists $\varphi \in \Phi_i$ such that $\pi_t^i(\cdot|h_t) = \varphi(\cdot|x_t)$ for all $h_t \in H_t$, $t = 0, 1, \ldots$ We will identify Φ_i with the family of stationary strategies for player i, so that $\boldsymbol{\Phi} := \times_{i \in \mathcal{N}} \Phi_i$ will stand for the set of stationary multistrategies.

Let $\mathbb{P}(X)$ be the family of probability measures on $\mathcal{B}(X)$. Let $(\Omega, \mathcal{F})$ be the (canonical) measurable space that consists of the sample space $\Omega := (X \times \mathbb{A})^\infty$ and its product σ-algebra $\mathcal{F}$. Then for each multistrategy $\boldsymbol{\pi} = (\pi^1, \ldots, \pi^N) \in \boldsymbol{Pi}$ and each initial distribution $\gamma \in \mathbb{P}(X)$, there exists a probability measure $P_\gamma^{\boldsymbol{\pi}}$ and a stochastic process $\{(x_t, \boldsymbol{a}_t), t = 0, 1, \ldots\}$ defined on $(\Omega, \mathcal{F})$ in a canonical way, where x_t and $\boldsymbol{a}_t$ represent the state and the action vector at stage t. The expectation operator with respect to $P_\gamma^{\boldsymbol{\pi}}$ is denoted by $E_\gamma^{\boldsymbol{\pi}}$.

For each $\boldsymbol{\pi} = (\pi^1, \ldots, \pi^N) \in \boldsymbol{Pi}$ and $\boldsymbol{a} = (a^1, \ldots, a^N) \in \mathbb{A}$, we denote by $[\boldsymbol{\pi}^{-i}|a^i]$ the N-vector whose kth component is π^k if $k \neq i$, and the ith component is a^i (for instance, $[\boldsymbol{\pi}^{-1}|a^1] := (a^1, \pi^2, \ldots, \pi^N)$). The N-vector $[\boldsymbol{a}^{-i}|\pi^i]$ is defined similarly (e.g. $[\boldsymbol{a}^{-1}|\pi^1] = (\pi^1, a^2, \ldots, a^N)$). Finally, taking π' in Π_i, it is clear that $[\boldsymbol{\pi}^{-i}|\pi']$ is the multistrategy according to which player k uses π^k if $k \neq i$, while player i uses π'.

The CG problem. Throughout the following we consider a fixed discount factor $\alpha \in (0, 1)$. Let $\mathbf{c}_i$ and $\mathbf{k}_i$ be as in (2.1), (2.2). For each player i the expected discounted cost functions are defined, for each $l = 0, 1, \ldots, q$ and each multistrategy $\boldsymbol{\pi} \in \boldsymbol{Pi}$, as

$$V_i^l(\boldsymbol{\pi}) := (1 - \alpha) E_\gamma^{\boldsymbol{\pi}} \left[\sum_{t=0}^{\infty} \alpha^t c_i^l(x_t, \boldsymbol{a}_t) \right]. \tag{2.3}$$

The *objective function* for player i is V_i^0. On the other hand, a multistrategy $\boldsymbol{\pi} \in \boldsymbol{Pi}$ is said to be *feasible for player i* if it satisfies the constraints

$$V_i^l(\boldsymbol{\pi}) \leq k_i^l \quad \forall\, l = 1, \ldots, q, \tag{2.4}$$

and $\boldsymbol{\pi}$ is said to be *feasible* for the CG if it is feasible for each player. We denote by Δ the set of all feasible multistrategies for the CG.

For each $\boldsymbol{\pi} \in \boldsymbol{Pi}$, we define the set of *feasible strategies* for player i *against* $\boldsymbol{\pi}$ as

$$\Delta^i(\boldsymbol{\pi}) := \{\pi' \in \Pi_i \mid V_i^l([\boldsymbol{\pi}^{-i}|\pi']) \leq k_i^l \quad \forall\, l = 1, \dots, q\}. \tag{2.5}$$

In particular, $\boldsymbol{\pi} = (\pi^1, \dots, \pi^N)$ is in Δ if and only if π^i is in $\Delta^i(\boldsymbol{\pi})$ for all $i \in \mathcal{N}$.

Definition 2.1 (Nash equilibrium) *Let $\boldsymbol{\pi} \in \boldsymbol{Pi}$. A strategy $\pi \in \Pi_i$ is called an optimal response of player i against $\boldsymbol{\pi}$ if it is feasible for player i, i.e., π is in $\Delta^i(\boldsymbol{\pi})$, and it also satisfies that*

$$V_i^0([\boldsymbol{\pi}^{-i}|\pi]) = \inf_{\pi' \in \Delta^i(\boldsymbol{\pi})} V_i^0([\boldsymbol{\pi}^{-i}|\pi']). \tag{2.6}$$

A feasible multistrategy $\boldsymbol{\pi} = (\pi^1, \pi^2, \dots, \pi^N) \in \Delta$ is called a Nash equilibrium for the CG (or a constrained Nash equilibrium) if, for each $i \in \mathcal{N}$, the strategy π^i is an optimal response of player i against $\boldsymbol{\pi}$, i.e., if it satisfies

$$V_i^0(\boldsymbol{\pi}) = \inf_{\pi' \in \Delta^i(\boldsymbol{\pi})} V_i^0([\boldsymbol{\pi}^{-i}|\pi']) \quad \forall\, i \in \mathcal{N}. \tag{2.7}$$

The problem we are concerned with is to give conditions ensuring the existence of a constrained Nash equilibrium.

Remark 2.2 The stochastic game is said to be *unconstrained* if $c_i^l(\cdot) \leq k_i^l$ for all $i \in \mathcal{N}$ and $l = 1, \dots, q$. In this case, from (2.3)–(2.5) we see that $\Delta^i(\boldsymbol{\pi}) = \Pi_i$ for all $\boldsymbol{\pi} \in \boldsymbol{Pi}$ and $i \in \mathcal{N}$, and so $\Delta = \boldsymbol{Pi}$. Hence the unconstrained game reduces to the standard noncooperative stochastic game, as in Altman et al. (1997); Borkar and Ghosh (1993); Federgruen (1978); Ghosh and Bagchi (1998); Nowak (1987); Parthasarathy (1982) and Sennott (1994), for instance.

3 Main results

We begin this section with a remark on the topologies on Φ_i and $\boldsymbol{\Phi}$.

Remark 3.1 (a) (*Notation*) Let $f : \mathbb{K} \to \mathbb{R}$ be a measurable function. Then for each $x \in X$, $\boldsymbol{a} = (a^1, \dots, a^N) \in \mathbb{A}(x)$, $\boldsymbol{\varphi} = (\varphi^1, \dots, \varphi^N) \in \boldsymbol{\Phi}$, and $i \in \mathcal{N}$, we define

$$f(x, [\boldsymbol{a}^{-i}|\varphi^i]) := \int_{A_i(x)} f(x, a^1, \dots, a^{i-1}, \xi, a^{i+1}, \dots, a^N)\varphi_i(d\xi|x).$$

This notation is extended in the obvious way to $f(x, [\boldsymbol{\varphi}^{-i}|a^i])$, $f(x, [\boldsymbol{\varphi}^{-i}|\varphi'])$ for $\varphi' \in \Phi_i$, $f(x, [\boldsymbol{a}^{-i}|a'])$ for $a' \in A_i(x)$, and $f(x, \boldsymbol{\varphi})$.

(b) (*Weak topology on* $\mathbb{P}(S)$) Let S be a metric space (with the Borel σ-algebra). Let $\mathbb{P}(S)$ the set of probability measures on S, and $C_b(S)$ the space of continuous bounded functions on S. A sequence $\{\nu_n\}$ in $\mathbb{P}(S)$ is said to *converge weakly* to $\nu \in \mathbb{P}(S)$ if

$$\int_S u\, d\nu_n \to \int_S u\, d\nu \quad \forall\, u \in C_b(S).$$

If S is compact, then so is $\mathbb{P}(S)$ endowed with the topology of weak convergence.

(c) (*Weak topology on* Φ_i) As $A_i(x)$ is compact, so is $\mathbb{P}(A_i(x))$. Hence, by the Tychonoff theorem, the spaces

$$\Phi_i = \times_{x \in X} \mathbb{P}(A_i(x)) \tag{3.1}$$

and

$$\boldsymbol{\Phi} = \times_{i \in \mathcal{N}} \Phi_i \tag{3.2}$$

are compact. A sequence $\{\varphi_n\}$ in Φ_i is said to *converge weakly* to $\varphi \in \Phi_i$ if the sequence $\{\varphi_n(\cdot|x)\}$ in $\mathbb{P}(A_i(x))$ converges weakly to $\varphi(\cdot|x)$ for each $x \in X$.

(d) (*Weak topology on* $\boldsymbol{\Phi}$) In view of (3.2), a sequence of stationary multistrategies $\boldsymbol{\varphi}_n = (\varphi_n^1, \ldots, \varphi_n^N) \in \boldsymbol{\Phi}$ is said to converge weakly to $\boldsymbol{\varphi} = (\varphi^1, \ldots, \varphi^N)$ if $\varphi_n^i \to \varphi^i$ weakly for each $i \in \mathcal{N}$. Using Fubini's theorem it can be seen that $\{\boldsymbol{\varphi}_n\}$ converges weakly to $\boldsymbol{\varphi}$ if and only if, for each function $f : \mathbb{K} \to \mathbb{R}$ such that $f(x, \cdot)$ is in $C_b(\mathbb{A}(x))$ for all $x \in X$, it holds that

$$f(x, \boldsymbol{\varphi}_n) \to f(x, \boldsymbol{\varphi}) \quad \forall\, x \in X.$$

We first consider CGs with a finite state space, that is CGs for which the following holds.

Assumption 3.2 *There exists a finite set* $\widetilde{\mathcal{X}} \subset X$ *such that*

(a) $Q(\widetilde{\mathcal{X}}|x, \boldsymbol{a}) = 1$ *for all* $x \in \widetilde{\mathcal{X}}$ *and* $\boldsymbol{a} \in \mathbb{A}(x)$.
(b) *The initial state distribution* γ *is concentrated on* $\widetilde{\mathcal{X}}$.

A CG that satisfies Assumption 3.2 will be called a *finite* CG, and, of course, it can be seen as a CG with finite state space and compact action space.

Assumption 3.3 (a) *For each* $x, y \in X$, *the transition law* $Q(y|x, \cdot)$ *is continuous on* $\mathbb{A}(x)$.
(b) *For each* $l = 0, \ldots, q$ *and* $i \in \mathcal{N}$, *the cost* c_i^l *is bounded on* $\mathbb{K}$ *and, moreover,* $c_i^l(x, \cdot)$ *is continuous on* $\mathbb{A}(x)$ *for each* $x \in X$.
(c) *(Slater condition) For each stationary multistrategy* $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ *and each player* i, *there exist* $\pi \in \Pi_i$ *such that*

$$V_i^l([\boldsymbol{\varphi}^{-i}|\pi]) < k_i^l \ \ \forall\, l = 1, \ldots, q. \tag{3.3}$$

Assumption 3.3 will be used, in particular, to study CGs with a *finite* state space. However, for the general CG in (2.1) we need to consider additional hypotheses. The next one, Assumption 3.4, essentially means that in one transition the state of the system remains "close" (in probability) to the previous state. In other words, if the state x_t of the game is known to be in a finite set $\mathcal{X}$ at time t, then there exists another finite set $\mathcal{X}' \supset \mathcal{X}$ such that x_{t+1} is in $\mathcal{X}'$ with probability close to 1. This is certainly the case for birth-and-death-like systems, which include many queueing and population models. Moreover, such a behavior is always true if the action sets $A_i(x)$ are finite for each $i \in \mathcal{N}$ and $x \in X$ – see Proposition 4.3(2)–(3).

Assumption 3.4 *There exists an increasing sequence $\{\mathcal{X}_n\}$ of finite subsets of X and a decreasing sequence $\{\epsilon_n\}$ of nonnegative numbers, such that $\epsilon_n \downarrow 0$, $\mathcal{X}_n \uparrow X$, and for each $n \geq 2$*

$$\sup_{x\in\mathcal{X}_{n-1}} \sup_{\boldsymbol{a}\in\mathbb{A}(x)} Q(\mathcal{X}_n^c|x, \boldsymbol{a}) \leq \epsilon_n, \tag{3.4}$$

where $\mathcal{X}_n^c$ stands for the complement of $\mathcal{X}_n$ with respect to X, and $\mathcal{X}_0$ is defined as the empty set.

Assumption 3.4 will be used to construct a sequence of CGs with finite state space that approximate the original CG. That is why we will first study the *sensitivity* (or approximation) problem of CGs.

Sensitivity. For each $n \geq 1$ consider the CG (CG_n)

$$\{X, (A_i, \{A_i(x)|x \in X\}, \mathbf{c}_{i,n}, \mathbf{k}_{i,n})_{i\in\mathcal{N}}, Q_n, \gamma_n\}, \tag{3.5}$$

with components as in (2.1). The original game (2.1) sometimes will be denoted by CG_∞, and we shall suppose that $\mathrm{CG}_n \ \rightarrow \mathrm{CG}_\infty$ in the following sense (see Remark 3.7).

Assumption 3.5 (a) *CG_∞ and each CG_n satisfies Assumption 3.3.*
(b) *For each $x, y \in X$ and $\{\boldsymbol{\varphi}_n\} \subset \boldsymbol{\Phi}$ such that $\boldsymbol{\varphi}_n \rightarrow \boldsymbol{\varphi}_\infty$, we have*

$$\lim_{n\to\infty} Q_n(y|x, \boldsymbol{\varphi}_n) = Q(y|x, \boldsymbol{\varphi}_\infty).$$

(c) *For each $\{\boldsymbol{\varphi}_n\} \subset \boldsymbol{\Phi}$ such that $\boldsymbol{\varphi}_n \rightarrow \boldsymbol{\varphi}_\infty$, and for each $i \in \mathcal{N}$ and $\{\mu_n\} \subset \mathbb{P}(\mathbb{K}_i)$ such that $\mu_n \rightarrow \mu_\infty$, we have*

$$\lim_{n\to\infty} \int_{\mathbb{K}_i} c_{i,n}^l(x, [\boldsymbol{\varphi}_n^{-i}|a])\mu_n(d(x,a)) = \int_{\mathbb{K}_i} c_i^l(x, [\boldsymbol{\varphi}_\infty^{-i}|a])\mu_\infty(d(x,a))$$

for $l = 0, 1, \ldots, q$.
(d) *For each $x \in X$, $\gamma_n(x) \rightarrow \gamma(x)$ as $n \rightarrow \infty$.*
(e) $\mathbf{k}_{i,n} \rightarrow \mathbf{k}$ *as $n \rightarrow \infty$.*

We now state our main results.

Theorem 3.6 (a) *Suppose that Assumption* 3.5 *holds. If each CG_n has a Nash equilibrium $\boldsymbol{\varphi}_n$, then so does CG_∞. In fact, any accumulation point of the sequence $\{\boldsymbol{\varphi}_n\}$* (*whose existence is ensured by the compactness of $\boldsymbol{\Phi}$ – see Remark* 3.1(c)) *is a Nash equilibrium for CG_∞.*
(b) *Suppose that the CG* (2.1) *satisfies Assumptions* 3.2 *and* 3.3 *(in particular, CG has a* finite *state set). Then the CG has a Nash equilibrium.*
(c) *If the CG* (2.1) *satisfies Assumptions* 3.3 *and* 3.4*, then it has a Nash equilibrium. Moreover, there exists a sequence CG_n of CGs with* finite *state space such that CG_n "converges" (in the sense of Assumption* 3.5*) to the CG* (2.1).

Remark 3.7 Suppose that in the game model (2.1) the set of actions A_i is finite for each player $i \in \mathcal{N}$. If in addition the game satisfies Assumption 3.2, then it is in fact a CG with *finite* state *and finite* action spaces. In this case, part (b) of Theorem 3.6 reduces to results obtained in Altman and Shwartz (2000) for discounted CGs.

Theorem 3.6 is proved in section 7, after some technical preliminaries in section 6.

4 Alternative assumptions

In this section we give alternative assumptions to obtain the results in section 3. In fact, we give conditions that imply Assumption 3.4 and the hypotheses on the cost functions and the transition laws in Assumption 3.5.

Assumption 4.1 (Uniform convergence)

(a) *CG_∞ and each CG_n satisfies parts (a) and (b) of Assumption 3.3.*
(b) *$Q_n(y|x, \boldsymbol{a}) \to Q(y|x, \boldsymbol{a})$, for each x, $y \in X$, $\boldsymbol{a} \in \mathbb{A}(x)$, and the convergence is uniform on $\mathbb{A}(x)$.*
(c) *$c^l_{i,n}(x, \boldsymbol{a}) \to c^l_i(x, \boldsymbol{a})$ for each $x \in X$, $\boldsymbol{a} \in \mathbb{A}(x)$ $i \in \mathcal{N}$ and $l = 0, 1, \ldots, q$, and the convergence is uniform on $\mathbb{A}(x)$.*

Assumption 4.2 *For each $x \in X$, $u_x(\cdot) := \sup_{\boldsymbol{a}\in\mathbb{A}(x)} Q(\cdot|x, \boldsymbol{a})$ satisfies*

$$\sum_{y\in X} u_x(y) < \infty. \tag{4.1}$$

Proposition 4.3 (1) *Assumption* 4.1 *implies parts* (a)–(c) *of Assumption* 3.5.
(2) *Assumption* 4.2 *implies Assumption* 3.4.
(3) *If for each $x \in X$ and $i \in \mathcal{N}$ the sets $A_i(x)$ of admissible actions in x are finite, then Assumption* 3.4 *holds.*

Proof (1) Part (b) of Assumption 3.5 follows from Assumption 4.1(a) and (b) and Lemma 4.5(b) given in Alvarez-Mena and Hernández-Lerma (2002). The proof of part (c) of Assumption 3.5 is the same as the proof of part (b) of Lemma 6.6, except that to justify the convergence (6.15), which in this case becomes

$$c^l_{i,n}(x, [\boldsymbol{\varphi}^{-i}_{m_r}|\varphi'_{m_r}]) \to c^l_i(x, [\boldsymbol{\varphi}^{-i}_\infty|\varphi']),$$

it is necessary to invoke Lemma 4.5(b) in Alvarez-Mena and Hernández-Lerma (2002), instead of our Remark 3.1(d).

To prove (2) we fix an arbitrary $x \in X$ and $\epsilon > 0$. By (4.1), there exists a finite set $B_x \subset X$ such that $\sum_{y\notin B_x} u_x(y) < \epsilon$. Thus

$$\sup_{\boldsymbol{a}\in\mathbb{A}(x)} Q(B^c_x|x, \boldsymbol{a}) < \epsilon \tag{4.2}$$

Now let $\mathcal{X}_1$ be a finite subset of X and let $\epsilon_n := (\frac{1}{2})^n$ for $n \geq 1$. By (4.2), for each $x \in \mathcal{X}_1$ there exists a finite subset B_x of X such that $B_x \supset \mathcal{X}_1$, and $\sup_{\boldsymbol{a}\in\mathbb{A}(x)} Q(B^c_x|x, \boldsymbol{a}) < \epsilon_2$. Hence, taking $\mathcal{X}_2 := \cup_{x\in\mathcal{X}_1} B_x$, for each $x \in \mathcal{X}_1$ we see that

$$\sup_{\boldsymbol{a}\in\mathbb{A}(x)} Q(\mathcal{X}^c_2|x, \boldsymbol{a}) \leq \sup_{\boldsymbol{a}\in\mathbb{A}(x)} Q(B^c_x|x, \boldsymbol{a}) < \epsilon_2.$$

Hence, we have found $\mathcal{X}_1$ and $\mathcal{X}_2$ that satisfy (3.4). Proceeding in the same way we can find a sequence $\{\mathcal{X}_n\}$ that satisfies (3.4), and therefore Assumption 3.4 holds.

Finally, to prove (3) we shall verify (4.1), and then the desired conclusion follows from (2). As the action sets $A_i(x)$ are *finite*, so is $\mathbb{A}(x)$. Therefore,

$$\sum_{y\in X} \sup_{\boldsymbol{a}\in\mathbb{A}(x)} Q(y|x, \boldsymbol{a}) \leq \sum_{\boldsymbol{a}\in\mathbb{A}(x)} \sum_{y\in X} Q(y|x, \boldsymbol{a}) = \sum_{\boldsymbol{a}\in\mathbb{A}(x)} 1 < \infty,$$

for each $x \in X$. □

5 Examples

In this section, we present two examples to illustrate Theorem 3.6. In each of them we consider two cases, *constrained* and *unconstrained* games, and we show their differences.

Example 5.1 (Two competitive servers) Consider a discrete-time, two-server queueing system in which the state variable x_t denotes the total number of customers in the system at each time $t = 0, 1, \ldots$. Arriving customers form a single waiting line, and a customer can be served by server 1 or server 2 which are the players 1 and 2, respectively. We assume that the potential departures occur in $(t-, t)$ and the potential arrivals in $(t, t+)$, which following a standard convention in discrete-time queues means that departures occur "a little" before t (for $t > 0$), whereas arrivals occur "a little" after t (for $t \geq 0$). The state variable x_t includes both situations, what occurs in $(t-, t)$ and $(t, t+)$. The dynamics of the system in the slot $[t-1, t)$ is as follows. At time $t-1$, players 1 and 2 observe the state x_{t-1} and choose their service rates $a^1 \in A_1(x_{t-1})$ and $a^2 \in A_2(x_{t-1})$, respectively, with $A_i(x) \subset [0, 1]$ as specified below. If player i chooses the service rate a^i, the service will be completed (and the customer will leave the system in $(t-, t)$) with probability a^i. If the service is not completed, then in the next slot, $[t, t+1)$, the service starts again and player i has to choose a new service rate.

When $x_t = 1$, we assume that the single customer is served by player 1 and so player 2 is idle, i.e.,

- If $x_{t-1} = 0$, and only one customer arrives at the system in $(t, t+)$, it will be served by player 1.
- If $x_{t-1} = 2$, and in $(t-, t)$ only one customer leaves the system and none arrives in $(t, t+)$, we assume that the remaining custumer will be served by player 1 (in particular, if that customer is with player 2, it is transferred to player 1 to be served).

In this way, $x_t = 1$ means that server 2 is idle. It is then obvious that $A_1(0) = A_2(0) = A_2(1) := \{0\}$. Otherwise, $A_1(1) = A_1(x) := [\alpha_1, \beta_1]$, with $0 \leq \alpha_1 < \beta_1 \leq 1$, and $A_2(x) := [\alpha_2, \beta_2]$ for all $x \geq 2$, with $0 \leq \alpha_2 < \beta_2 \leq 1$.

In each slot the probability of $j \geq 0$ customers arriving to the system is P_j, with $P_0 < 1$. Let γ be the initial distribution. The state space is $X := \{0, 1, \ldots\}$, and the transition law for $(x, a^1, a^2) \in \mathbb{K}$ and $y \in X$ is given by

$$Q(y|x, a^1, a^2) := a^1 a^2 P_{y-x+2} + [a^1(1-a^2) + a^2(1-a^1)]P_{y-x+1} + (1-a^1)(1-a^2)P_{y-x} \tag{5.1}$$

where $P_{-z} := 0$ for all $z \geq 1$. There are two types of costs for each server i ($i = 1, 2$): a service cost $c_i^0(x, a^1, a^2) := s_i(a^i)$ and a penalization cost $c_i^1(x, a^1, a^2) := H_i(x, a^i)$ for all $(x, a^1, a^2) \in \mathbb{K}$, with $s_i(0) = H_i(0, 0) := 0$. The penalization cost can be thought of as a holding cost plus a cost for the server's inefficiency.

The servers behave as players of a *noncooperative* CG in which each server i ($i = 1, 2$) wishes to minimize the expected discounted service cost V_i^0, while maintaining the expected discounted penalization cost V_i^1 bounded above by a given number $k_i \geq 0$. We may represent this queueing CG as

$$\mathrm{QCG} := (X, \{[\alpha_i, \beta_i], s_i, H_i, k_i\}_{i=1,2}, \{P_x\}, \gamma).$$

It is clear that the transition law Q satisfies the Assumption 3.3(a). On the other hand, we may verify the Assumption 3.4 with $\mathcal{X}_n := \{1, \ldots, 2^n\}$ for each $n \geq 1$. Indeed, for $y \in \mathcal{X}_{n+1}^c$ and $x \in \mathcal{X}_n$, we have $y - x > 2^n$, whereas $Q(y|x, \boldsymbol{a}) \leq P_{y-x+2} + P_{y-x+1} + P_{y-x}$ for all $\boldsymbol{a} \in \mathbb{A}(x)$, by (5.1). Therefore

$$
\begin{aligned}
\sup_{x \in \mathcal{X}_n} \sup_{\boldsymbol{a} \in \mathbb{A}(x)} Q(\mathcal{X}_{n+1}^c | x, \boldsymbol{a}) &\leq \sup_{x \in \mathcal{X}_n} \sum_{y > 2^{n+1}} (P_{y-x+2} + P_{y-x+1} + P_{y-x}) \\
&\leq \sup_{x \leq 2^n} \sum_{z > 2^{n+1} - x} (P_{z+2} + P_{z+1} + P_z) \\
&\leq 3 \sum_{z > 2^n} P_z := \epsilon_{n+1} \ \downarrow \ 0 \ \text{ as } \ n \to \infty.
\end{aligned}
$$

Finally, we may obtain Assumptions 3.3(b) and (c) from the following.

Assumption 5.2 *For each $i = 1, 2$:*

(a) *The service cost s_i is continuous on $[\alpha_i, \beta_i]$.*
(b) *The penalization cost H_i is bounded on $X \times [\alpha_i, \beta_i]$, and $H_i(x, \cdot)$ is continuous on $[\alpha_i, \beta_i]$ for each $x \geq i$.*
(c) *For each stationary multistrategy $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ we have*

$$V_i^1([\boldsymbol{\varphi}^{-i} | \beta_i]) < k_i,$$

where $[\boldsymbol{\varphi}^{-i} | \beta_i]$ is the multistrategy according to which player j uses φ^j if $j \neq i$, while player i uses the deterministic strategy that always chooses the maximum service rate *$a = \beta_i$ for $x \geq i$, and $a = 0$ for $x < i$.*

Under Assumption 5.2, it is now clear that QCG satisfies Assumptions 3.3 and 3.4. Hence Theorem 3.6(c) yields:

There exists a Nash equilibrium for the QCG.

We next examine a particular QCG with the following parameters, for $i = 1, 2$: $\alpha_i = 0, \beta_i = 1, s_i(a) := a$ for all $a \in [0, 1]$, and $H_i(x, a) := h(x) + (1-a) I_{[i,\infty)}(x)$ for all $(x, a) \in X \times [0, 1]$. (As usual, I_B denotes the indicator function of the set B.) Here, h is a holding cost given by

$$
h(x) := \begin{cases} 0 & \text{if } \ x \leq 2, \\ (x-2)/(x-1) & \text{if } \ x > 2, \end{cases} \tag{5.2}
$$

and the term $(1 - a) I_{[i,\infty)}(x)$ corresponds to the inefficiency cost of player i for selecting the action a in the state x. We take the initial distribution γ as the Dirac measure at the point $x = 3$, and, finally, the constraint constants $k_1 = k_2 := 1$. It is clear that Assumptions 5.2(a),(b) hold. Thus, to ensure the existence of a Nash equilibrium, we only need to verify the Slater condition 5.2(c).

We first note some simple facts. (Recall that $c_i^0(x, a^1, a^2) = s_i(a^i)$, and $c_i^1(x, a^1, a^2) = H_i(x, a^i)$.) For each $i = 1, 2$

$$c_i^1(x, 1, 1) < 1 \quad \text{for all } \ x \geq i, \tag{5.3}$$

$$c_i^1(x, 0, 0) > 1 \quad \text{for all } \ x > 3, \tag{5.4}$$

$$c_i^0(x, 0, 0) = 0 \quad \text{for all } \ x \in X, \tag{5.5}$$

$$Q(y|3, 0, 0) = P_{y-3} \quad \text{for all } \ y \in X. \tag{5.6}$$

Let $\mathbb{I}$ be the deterministic strategy under which the maximum service rate $\beta_i = 1$ is chosen for $x \geq i$, and 0 for $x < i$. Then from (5.3) we get

$$V_1^1(\mathbb{I}, \pi) < V_1^1(\mathbb{I}, \mathbb{I}) < 1 \;\; \forall \pi \in \Pi_2,$$
$$V_2^1(\pi, \mathbb{I}) < V_2^1(\mathbb{I}, \mathbb{I}) < 1 \;\; \forall \pi \in \Pi_1.$$

Hence, Assumption 5.2(c) holds and so there exists a Nash equilibrium for this particular QCG.

To conclude the example, we next consider the (*constrained*) QCG and the *unconstrained* version of it, and we compare the results.

Unconstrained case: Consider the game QCG *without* constraints. The pair of strategies (θ, θ), according to which the players always choose the minimum service rate $a = 0$, is a Nash equilibrium for the unconstrained version of QCG. Indeed, by (5.5), for each $i = 1, 2$

$$V_i^0(\theta, \theta) = 0.$$

On the other hand, as the service cost is nonnegative, for each $i = 1, 2$,

$$V_i^0(\pi^1, \pi^2) \geq 0 \;\; \forall \, (\pi^1, \pi^2) \in \Pi_1 \times \Pi_2.$$

Constrained case: We already know that there exists a Nash equilibrium for the *constrained* QCG, but it cannot be (θ, θ). In fact, (θ, θ) is not even feasible for QCG because, as the system starts with $x_0 = 3$, by (5.6), $x_t \geq 3$ for all $t \geq 0$. Therefore (5.4) yields

$$V_i^1(\theta, \theta) > 1.$$

To conclude, note that in the *unconstrained* version of QCG, (θ, θ) is certainly an "undesired" Nash equilibrium because it implies that both servers should always be idle! Hence the example shows that the constraints can avoid undesired equilibria.

Example 5.3 (N queues competing for service) Consider a discrete-time, single-server queueing system with N infinite-capacity buffers. The state of the system at time t is given by a vector $\boldsymbol{x}_t := (x_t^1, \ldots, x_t^N)$, where x_t^i is the number of customers currently in buffer i at time t. Departures occur in $(t-, t)$; the service time of a customer is (exactly) one slot. The (potential) arrivals occur in $(t, t+)$; the probability of x customers requesting admission to the system at buffer i is P_x^i for each $x \geq 0$ (with $P_0^i < 1$) and all $i = 1, \ldots, N$. Thus the state space is $X := \times_{i \in \mathcal{N}} \mathbb{Z}^+$, where $\mathbb{Z}^+ := \{0, 1, \ldots\}$, and $\mathcal{N} = \{1, \ldots, N\}$.

At each decision time $t = 0, 1, \ldots$, the server chooses at random a nonempty buffer, from which he takes a customer to serve. At the entrance of each buffer $i = 1, \ldots, N$ there is a player (or entrance controller) who decides if arriving customers are admitted or not. At time t the players observe the state x_t, and each player i takes an action from the action set $A_i = A := \{1, 0\}$, where $a^i = 1$ means that he accepts the arriving customers to buffer i, and $a^i = 0$ means that he rejects them. There are two costs for each player i ($i \in \mathcal{N}$): a bounded holding cost $c_i^0(\boldsymbol{x}, \boldsymbol{a}) := h_i(x_i, a^i)$, and a rejection cost $c_i^1(\boldsymbol{x}, \boldsymbol{a}) := r_i(a_i)$ for all $(\boldsymbol{x}, \boldsymbol{a}) \in \mathbb{K}$. Naturally we assume that $h_i(0, 0) = r_i(1) := 0$. Let γ be the initial state distribution.

To get the full description of the game, we next give an expression for the transition law. For each $\boldsymbol{x}$, $\boldsymbol{y} \in X$ and $\boldsymbol{a} = (a^1, \ldots, a^N) \in \mathbb{A}$ the transition law is given by

$$Q(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{a}) = \frac{1}{N} \sum_{i \in \mathcal{N}} Q_i(\boldsymbol{y}|\boldsymbol{x}, a^i) \tag{5.7}$$

where Q_i is a stochastic kernel from $\mathbb{K}_i$ to X defined as follows. Let I_D be the indicator function of a set D, and $\delta_x(y) \equiv \delta_{xy}$ the Kronecker function (i.e., $\delta_y(x) := 1$ if $y = x$, and $:= 0$ otherwise). Moreover, let $|\boldsymbol{x}| := x^1 + \cdots + x^N$, and

$$L^i_{\boldsymbol{x}} := \begin{cases} 1/|\boldsymbol{x}| & \text{if} \quad x^i \geq 1, \\ 0 & \text{if} \quad x^i = 0. \end{cases} \tag{5.8}$$

The stochastic kernel $Q_i(\boldsymbol{y}|\boldsymbol{x}, a^i)$ for $\boldsymbol{x} = (x^1, \ldots, x^N)$ and$\boldsymbol{y} = (y^1, \ldots, y^N)$ in X and $a^i \in \{1, 0\}$ is given by

$$\begin{aligned} Q_i(\boldsymbol{y}|\boldsymbol{x}, a^i) := {} & a^i[(1 - L^i_{\boldsymbol{x}})P^i_{y^i - x^i} + L^i_{\boldsymbol{x}} P^i_{y^i - x^i + 1}] \\ & +(1 - a^i)[(1 - L^i_{\boldsymbol{x}})\delta_0(y^i - x^i) + L^i_{\boldsymbol{x}}\delta_0(y^i - x^i + 1)] \end{aligned}$$

where $P_{-z} := 0$ for all $z \geq 1$.

By (5.7), the game's transition law has an *additive* (or separable) structure, in the sense that Q is a convex combination of stochastic kernels Q_i which only depend on the action of player i ($i \in \mathcal{N}$), and similarly for the costs c^l_i. Games with this kind of structure are sometimes called separable or additive-rewards and additive-transition (ARAT) games.

Each player $i \in \mathcal{N}$ wishes to minimize the expected discounted holding cost V^0_i, while keeping the expected discounted rejection cost V^1_i bounded above by a given number $k_i > 0$.

We may represent this constrained noncooperative game as

$$\text{ARATG} = (X, \{1, 0\}, \{\{P^i_x\}_{x \in \mathbb{Z}}, h_i, r_i, k_i\}_{i \in \mathcal{N}}, \gamma).$$

The *Slater condition* (3.3) trivially holds. Indeed, for each $i \in \mathcal{N}$ and $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$

$$V^1_i([\boldsymbol{\varphi}^{-i}|\mathbb{I}]) = 0 < k_i,$$

where $[\boldsymbol{\varphi}^{-i}|\mathbb{I}]$ is the multistrategy according to which player j uses φ^j if $j \neq i$, while player i uses the deterministic strategy that *always accepts* the customers arriving to buffer i.

It is also clear that Q satisfies Assumptions 3.3(a),(b) and, by Proposition 4.3(3), also the Assumption 3.4. We thus have the following.

Suppose that, for each $i \in \mathcal{N}$, the holding cost function h_i is bounded. Then, by Theorem 3.6(c), there exists a constrained Nash equilibrium for ARATG.

Finally we consider a particular ARATG with the following parameters. For each $i \in \mathcal{N}$, let $k_i = 1/2$, $r_i(1) = 0$, $r_i(0) = 1$, $h_i(x, a) := x/(x + 1)$ for all $(x, a) \in \mathbb{Z}^+ \times \{0, 1\}$, and the initial state distribution γ is the Dirac measure at $\mathbf{0} := (0, \ldots, 0) \in X$.

As in the Example 5.1 we can see fundamental differences between the (*constrained*) ARATG and the *unconstrained* version of ARATG.

Unconstrained case: Consider the game ARATG *without* constraints. Then the pair of strategies (θ, θ), according to which the players always reject the arriving customers is a Nash equilibrium. Indeed, $Q(\boldsymbol{y}|\mathbf{0}, \theta, \theta) = 0$ if $y^i > 0$ for some $i \in \mathcal{N}$. Thus, under the multistrategy (θ, θ), $x_t = 0$ for all $t \geq 0$. Hence

$$V_i^0(\theta, \theta) = 0 \text{ for } i = 1, 2.$$

On the other hand, as the holding cost is nonnegative, for each $i = 1, 2$ we have

$$V_i^0(\pi^1, \pi^2) \geq 0 \ \forall\, (\pi^1, \pi^2) \in \Pi_1 \times \Pi_2.$$

Constrained case: We already know that there exists a Nash equilibrium for the (*constrained*) ARATG, but it cannot be (θ, θ). Actually, (θ, θ) is not even feasible for ARATG because

$$V_i^1(\theta, \theta) = 1 > k_i \ \ \forall\, i \in \mathcal{N}.$$

6 Technical preliminaries

In this section we introduce some preliminary facts needed in the proof of Theorem 3.6. In particular, we associate a *constrained* (Markov) *control process* (CCP) to each stationary multistrategy and show that the results in Alvarez-Mena and Hernández-Lerma (2002) hold for these CCPs.

Since the state space X is *countable*, a sequence $\{\nu_n\}$ in $\mathbb{P}(X)$ *converges weakly* to $\nu \in \mathbb{P}(X)$ iff (if and only if) $\nu_n(x) \to \nu(x)$ for all $x \in X$. Equivalently, by Scheffé's Theorem [Billingsley, p. 233], $\nu_n \to \nu$ weakly iff $\sum_x |\nu_n(x) - \nu(x)| \to 0$. This fact yields the following.

Lemma 6.1 *Let $\{u_n\}$ be a bounded sequence of functions from X to $\mathbb{R}$ such that $u_n(x) \to u(x)$ for all $x \in X$, and let $\{\nu_n\}$ be a sequence in $\mathbb{P}(X)$ such that $\nu_n \to \nu$ weakly. Then*

$$\sum_{x \in X} u_n(x)\nu_n(x) \to \sum_{x \in X} u(x)\nu(x).$$

For each $\varphi \in \Phi_i$ and $\nu \in \mathbb{P}(X)$ we define a probability measure $\nu \circ \varphi$ in $\mathbb{P}(\mathbb{K}_i)$ by $(\nu \circ \varphi)(\{x\} \times C) := \varphi(C|x)\nu(x)$ for all $x \in X$ and $C \in \mathcal{B}(A_i(x))$.

Lemma 6.2 *Let $i \in \mathcal{N}$. Let $\{\varphi_n\}$ be a sequence in Φ_i and $\{\nu_n\}$ a sequence in $\mathbb{P}(X)$ such that $\varphi_n \to \varphi$ weakly in Φ_i, and $\nu_n \to \nu$ weakly in $\mathbb{P}(X)$. Then the sequence of measures $\mu_n := \nu_n \circ \varphi_n$ in $\mathbb{P}(\mathbb{K}_i)$ satisfies that*

$$\mu_n \to \mu \ \ \textit{weakly},$$

where $\mu := \nu \circ \varphi$.

Proof See Alvarez-Mena and Hernández-Lerma (2002), Lemma 4.6. □

The CCP associated to a stationary multistrategy. Fix an arbitrary $\boldsymbol{\varphi}$ in $\boldsymbol{\Phi}$. Suppose that all the players except one of them, say player i, selects the corresponding strategy given by the stationary multistrategy $\boldsymbol{\varphi}$. Then the constrained game model (2.1) reduces to the *constrained control model*

$$(X, A_i, \{A_i(x)|x \in X\}, \mathbf{c}_i^{\boldsymbol{\varphi}}, \mathbf{k}_i, Q_i^{\boldsymbol{\varphi}}, \gamma) \tag{6.1}$$

where the elements in (6.1) are defined exactly as in section 2, except that the transition law $Q_i^{\boldsymbol{\varphi}}$, a stochastic kernel from $\mathbb{K}_i$ to X, and the cost $\mathbf{c}_i^{\boldsymbol{\varphi}} = (c_i^{0,\boldsymbol{\varphi}}, c_i^{1,\boldsymbol{\varphi}}, \ldots, c_i^{q,\boldsymbol{\varphi}}) : \mathbb{K}_i \to \mathbb{R}^{q+1}$ are given by

$$Q_i^{\boldsymbol{\varphi}}(y|x,a) := Q(y|x,[\boldsymbol{\varphi}^{-i}|a]) \;\; \forall\, y \in X,\;\; (x,a) \in \mathbb{K}_i, \tag{6.2}$$

and for each $l = 0, 1, \ldots, q$,

$$c_i^{l,\boldsymbol{\varphi}}(x,a) := c_i^l(x,[\boldsymbol{\varphi}^{-i}|a]) \;\; \forall\, (x,a) \in \mathbb{K}_i. \tag{6.3}$$

Thus, for each $l = 0, 1, \ldots, q$, the expected α-discounted cost function $V_i^{l,\boldsymbol{\varphi}}$ for the constrained control model (6.1) is given (in analogy with (2.3)) for each $\pi \in \Pi_i$ by

$$V_i^{l,\boldsymbol{\varphi}}(\pi) = (1-\alpha)E_\gamma^\pi\left[\sum_{t=0}^{\infty}\alpha^t c_i^{l,\boldsymbol{\varphi}}(x_t,a_t)\right],$$

which of course satisfies

$$V_i^{l,\boldsymbol{\varphi}}(\pi) = V_i^l([\boldsymbol{\varphi}^{-i}|\pi]) \;\; \forall \pi \in \Pi_i. \tag{6.4}$$

The corresponding constraints are given by (2.4) replacing $V_i^l(\boldsymbol{\pi})$ with $V_i^{l,\boldsymbol{\varphi}}(\pi)$. We thus obtain a CCP that will be denoted by $\mathrm{CCP}_i^{\boldsymbol{\varphi}}$. Moreover, directly from (6.4) and the Definition 2.1 of optimal response, we get the following.

Proposition 6.3 *Let $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ and $i \in \mathcal{N}$. A strategy $\pi \in \Pi_i$ is an optimal response of player i against $\boldsymbol{\varphi}$ if and only if π is an optimal strategy for* $\mathrm{CCP}_i^{\boldsymbol{\varphi}}$.

A formulation of $\mathrm{CCP}_i^{\boldsymbol{\varphi}}$ on a set of measures. There is a formulation of $\mathrm{CCP}_i^{\boldsymbol{\varphi}}$ using the so-called occupation measures – see, for instance, Alvarez-Mena and Hernández-Lerma (2002) or Hernández-Lerma and González-Hernández (2001). We introduce this formulation because we will refer to it later on. Let $M(\mathbb{K}_i)$ be the set of finite signed measures on $X \times A_i$ concentrated on $\mathbb{K}_i$, and $M^+(\mathbb{K}_i)$ the cone of nonnegative measures in $M(\mathbb{K}_i)$. Consider de problem

$$\underline{\mathrm{CCP}_i^{\boldsymbol{\varphi}}}: \qquad \text{Minimize } \int_{\mathbb{K}_i} c_i^{0,\boldsymbol{\varphi}}(x,a)\mu(d(x,a))$$

$$\text{subject to: } \quad \int_{\mathbb{K}_i} c_i^{l,\boldsymbol{\varphi}}(x,a)\mu(d(x,a)) \;\le\; k_i^l \;\; \forall\, l = 1, \ldots, q, \tag{6.5}$$

$$\widehat{\mu}(x) = (1-\alpha)\gamma(x) + \alpha\int_{\mathbb{K}_i} Q_i^{\boldsymbol{\varphi}}(x|y,a)\mu(d(y,a)) \;\forall\, x \in \mathrm{X}, \tag{6.6}$$

$$\mu \in M^+(\mathbb{K}_i), \tag{6.7}$$

where $\widehat{\mu}$ stands for the marginal (or projection) of μ on X, defined as $\widehat{\mu}(x) := \mu(x \times A_i)$. A measure that satisfies (6.6) and (6.7) is called an *occupation measure* for $\mathrm{CCP}_i^{\boldsymbol{\varphi}}$ – cf. Definition 3.5 in Alvarez-Mena and Hernández-Lerma (2002). Note that an occupation measure is in fact a probability measure.

Definition 6.4 *Fix an arbitrary $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ and $i \in \mathcal{N}$. Let $\varphi' \in \Phi_i$ be a stationary strategy for player i. A measure $\mu \in M^+(\mathbb{K}_i)$ is said to be $(\boldsymbol{\varphi}, i)$–equivalent (or simply $\boldsymbol{\varphi}$-equivalent) to φ' if*

(1) *μ is an occupation measure for $CCP_i^{\boldsymbol{\varphi}}$, and*
(2) *φ' and μ satisfy that*

$$\mu = \widehat{\mu} \circ \varphi' \tag{6.8}$$

$$V_i^{l,\boldsymbol{\varphi}}(\varphi') = \int_{\mathbb{K}_i} c_i^{l,\boldsymbol{\varphi}}(x, a)\mu(d(x, a)) \ \ \forall\, l = 0, 1, \ldots, q. \tag{6.9}$$

On the other hand, if $\mu \in M^+(\mathbb{K}_i)$ is an occupation measure for $\mathrm{CCP}_i^{\boldsymbol{\varphi}}$*, then a stationary policy $\varphi' \in \Phi_i$ is said to be $(\boldsymbol{\varphi}, i)$-equivalent (or $\boldsymbol{\varphi}$-equivalent) to μ if (2) holds.*

For each $\varphi' \in \Phi_i$, the existence of a measure $\mu \in M^+(\mathbb{K}_i)$ which is $\boldsymbol{\varphi}$-equivalent to φ', and also, for each occupation measure $\mu \in M^+(\mathbb{K}_i)$, the existence of a strategy $\varphi \in \Phi_i$ which is $\boldsymbol{\varphi}$-equivalent to μ, are guaranteed by Proposition 4.2 in Alvarez-Mena and Hernández-Lerma (2002).

Remark 6.5 (a) Note that the occupation measures in (6.6)–(6.7) and the objective functions (2.3) only depend on the transition law, the initial distribution and the multistrategy being used. That is why, to avoid confusions and to show explicitly this dependence, sometimes we write "$(\boldsymbol{\varphi}, Q, \gamma, i)$–equivalent" or "$V_i^l(\boldsymbol{\varphi}, Q, \gamma)$" instead of simply "$(\boldsymbol{\varphi}, i)$–equivalent" or "$V_i^l(\boldsymbol{\varphi})$", respectively.

(b) If $\mu \in M^+(\mathbb{K}_i)$ is $(\boldsymbol{\varphi}, i)$*–equivalent* to $\varphi' \in \Phi_i$, then, from (6.8), we can write (6.6) as

$$\widehat{\mu}(x) = (1-\alpha)\gamma(x) + \alpha \sum_{y \in X} Q(x|y, [\boldsymbol{\varphi}^{-i}|\varphi'])\widehat{\mu}(y) \ \ \forall\, x \in \mathrm{X}, \tag{6.10}$$

and (6.9) as

$$V_i^{l,\boldsymbol{\varphi}}(\varphi') = \sum_{x \in X} c_i^l(x, [\boldsymbol{\varphi}^{-i}|\varphi'])\widehat{\mu}(x) \ \ \forall\, l = 0, 1, \ldots, q. \tag{6.11}$$

Next, we shall establish the connection between a sequence of constrained control process $\mathrm{CCP}_i^{\boldsymbol{\varphi}_n}$ and the convergence results in Alvarez-Mena and Hernández-Lerma (2002).

Lemma 6.6 *Suppose that the Assumption* 3.3 *holds. Let $\{\boldsymbol{\varphi}_n\} \subset \boldsymbol{\Phi}$ be a sequence of multistrategies such that $\boldsymbol{\varphi}_n \to \boldsymbol{\varphi}_\infty$ weakly in $\boldsymbol{\Phi}$. Then for each $i \in \mathcal{N}$ the following holds.*

(a) *For each sequence $\{\varphi'_n\} \subset \Phi_i$ such that $\varphi'_n \to \varphi'_\infty$ weakly in Φ_i we have*

$$\lim_{n\to\infty} \int_{A_i(x)} Q_i^{\boldsymbol{\varphi}_n}(y|x, a)\varphi'_n(da|x) = \int_{A_i(x)} Q_i^{\boldsymbol{\varphi}_\infty}(y|x, a)\varphi'_\infty(da|x). \tag{6.12}$$

(b) *For each sequence* $\{\mu_n\} \subset \mathbb{P}(\mathbb{K}_i)$ *such that* $\mu_n \to \mu$ *weakly in* $\mathbb{P}(\mathbb{K}_i)$ *we have*

$$\lim_{n\to\infty} \int_{\mathbb{K}_i} c_i^{l,\boldsymbol{\varphi}_n} d\mu_n = \int_{\mathbb{K}_i} c_i^{l,\boldsymbol{\varphi}_\infty} d\mu \;\; \forall\, l = 0, 1, \ldots, q. \tag{6.13}$$

(c) *Theorem* 3.9 *and Corollary* 3.10 *in Alvarez-Mena and Hernández-Lerma (2002) hold for the sequence* $\{\mathrm{CCP}_i^{\boldsymbol{\varphi}_n}\}$ *and the limit* $\mathrm{CCP}_i^{\boldsymbol{\varphi}_\infty}$. *This implies, in particular, the following: If* $\mu_n^* \in M^+(\mathbb{K}_i)$ *is an optimal solution for* $\underline{\mathrm{CCP}}_i^{\boldsymbol{\varphi}_n}$ *for each* $n \geq 1$, *then* $\{\mu_n^*\}$ *is relatively compact and, moreover, any accumulation point of* $\{\mu_n^*\}$ *is an optimal solution for* $\underline{\mathrm{CCP}}_i^{\boldsymbol{\varphi}_\infty}$. *A similar result holds for the optimal strategies for* $\mathrm{CCP}_i^{\boldsymbol{\varphi}_n}$.

Proof (a) By Assumption 3.3(a), $Q(y|x, \cdot)$ is a continuous bounded function on $\mathbb{A}(x)$ for each $x, y \in X$. Then (by Remark 3.1(d)), as $[\boldsymbol{\varphi}_n^{-i}|\varphi_n'] \to [\boldsymbol{\varphi}_\infty^{-i}|\varphi_\infty']$ weakly in $\mathbf{\Phi}$, we have

$$Q(y|x, [\boldsymbol{\varphi}_n^{-i}|\varphi_n']) \to Q(y|x, [\boldsymbol{\varphi}_\infty^{-i}|\varphi_\infty']) \;\; \forall\, x, y \in X. \tag{6.14}$$

Thus part (a) follows from (6.14) and the following equality that results from Fubini's Theorem:

$$\int_{A_i(x)} Q^{\boldsymbol{\varphi}_n}(y|x, a)\varphi_n'(da|x) = Q(y|x, [\boldsymbol{\varphi}_n^{-i}|\varphi_n']) \;\; \forall\, n \in \overline{\mathbf{N}},$$

where $\overline{\mathbf{N}} := \{1, 2, \ldots\} \cup \{\infty\}$.

(b) Let $\{\mu_n\} \subset \mathbb{P}(\mathbb{K}_i)$ be such that $\mu_n \to \mu$ weakly. To prove (6.13), choose an arbitrary l with $0 \leq l \leq q$, and let $\{\int_{\mathbb{K}_i} c_i^{l,\boldsymbol{\varphi}_m} d\mu_m\}_m$ be a subsequence of the bounded sequence of real numbers $\{\int_{\mathbb{K}_i} c_i^{l,\boldsymbol{\varphi}_n} d\mu_n\}_n$. Let $\varphi_m' \in \Phi_i$ be such that $\mu_m = \widehat{\mu}_m \circ \varphi_m'$, where $\widehat{\mu}_m$ denotes the marginal of μ_m on X; the existence of φ_m' is well known, see page 88 in Dynkin and Yushkevich (1979), for instance. By the compactness of Φ_i (see Remark 3.1(c)) there exists a subsequence $\{\varphi_{m_r}'\}$ of $\{\varphi_m'\}$ such that $\varphi_{m_r}' \to \varphi'$ weakly for some φ' in Φ_i. As the marginals $\widehat{\mu}_{m_r}$ also converge to $\widehat{\mu}$, Lemma 6.2 implies that $\mu = \widehat{\mu} \circ \varphi'$. Moreover $[\boldsymbol{\varphi}_{m_r}^{-i}|\varphi_{m_r}'] \to [\boldsymbol{\varphi}_\infty^{-i}|\varphi']$ weakly in $\mathbf{\Phi}$ and, on the other hand, by Assumption 3.3(b), $c_i^l(x, \cdot)$ is continuous and bounded on $\mathbb{A}(x)$ for each $x \in X$. Therefore (by Remark 3.1(d))

$$c_i^l(x, [\boldsymbol{\varphi}_{m_r}^{-i}|\varphi_{m_r}']) \to c_i^l(x, [\boldsymbol{\varphi}_\infty^{-i}|\varphi']). \tag{6.15}$$

Hence, as in the proof of part (a), from (6.15) and Fubini's Theorem it follows that

$$\widetilde{c}_{m_r} \to \widetilde{c}_\infty \quad \text{pointwise on } X, \tag{6.16}$$

where

$$\widetilde{c}_{m_r}(x) := \int_{A_i(x)} c_i^{l,\boldsymbol{\varphi}_{m_r}}(x, a)\varphi_{m_r}'(da|x) = c_i^l(x, [\boldsymbol{\varphi}_{m_r}^{-i}|\varphi_{m_r}']).$$

Note that, by Assumption 3.3(b), the sequence $\{\widetilde{c}_{m_r}\}$ is bounded. Hence (6.16) and Lemma 6.1 yield

$$\sum_{x\in X}\widetilde{c}_{m_r}(x)\widehat{\mu}_{m_r}(x) \to \sum_{x\in X}\widetilde{c}_{\infty}(x)\widehat{\mu}(x). \tag{6.17}$$

Finally, since $\int_{\mathbb{K}_i} c_i^{l,\boldsymbol{\varphi}_{m_r}} d\mu_{m_r} = \sum_{x\in X}\widetilde{c}_{m_r}(x)\widehat{\mu}_{m_r}(x)$, from (6.17) we obtain

$$\lim_{r\to\infty}\int_{\mathbb{K}_i} c_i^{l,\boldsymbol{\varphi}_{m_r}} d\mu_{m_r} = \int_{\mathbb{K}_i} c_i^{l,\boldsymbol{\varphi}_{\infty}} d\mu.$$

Therefore, as the subsequence $\{\int_{\mathbb{K}_i} c_i^{l,\boldsymbol{\varphi}_m} d\mu_m\}_m$ of $\{\int_{\mathbb{K}_i} c_i^{l,\boldsymbol{\varphi}_n} d\mu_n\}_n$ was arbitrary, (6.13) follows.

(c) We need to guarantee that Theorem 3.9 and Corollary 3.10 in Alvarez-Mena and Hernández-Lerma (2002) can be applied. Assumptions 3.1, 3.2 and 3.8 of Alvarez-Mena and Hernández-Lerma (2002), except 3.1(c) and 3.2(b), hold. However, by Remark 6.1(a) in Alvarez-Mena and Hernández-Lerma (2002), 3.1(c) and 3.2(b) admit alternative assumptions which follow from parts (a) and (b) above. □

7 Proof of Theorem 3.6

Proof of Theorem 3.6(a) (Sensitivity of CGs) Let $\{\boldsymbol{\varphi}_n\} \subset \boldsymbol{\Phi}$ be such that $\boldsymbol{\varphi}_n = (\varphi_n^1, \ldots, \varphi_n^N)$ converges to some $\boldsymbol{\varphi} = (\varphi^1, \ldots, \varphi^N) \in \boldsymbol{\Phi}$. Fix an arbitrary $i \in \mathcal{N}$. Each CG$_n$ in (3.5) defines a constrained control problem $\mathrm{CCP}_i^{\boldsymbol{\varphi}_n}$ associated to the multistrategy $\boldsymbol{\varphi}_n$ as in (6.1)–(6.3), i.e.

$$(X, A_i, \{A_i(x)|x \in X\}, Q_n, \gamma_n, \mathbf{c}_n, \mathbf{k}_n,).$$

where the law transition Q_n, the cost vector $\mathbf{c}_n = (c_n^0, c_n^1, \ldots, c_n^q)$, and $\mathbf{k}_n$ are given by

$$\begin{aligned} Q_n(y|x,a) &:= Q_n(y|x,[\boldsymbol{\varphi}_n^{-i}|a]) \ \forall\, y \in X,\ (x,a) \in \mathbb{K}_i, \\ c_n^l(x,a) &:= c_{i,n}^l(x,[\boldsymbol{\varphi}_n^{-i}|a]) \ \forall\, (x,a) \in \mathbb{K}_i,\ l = 0,1,\ldots,q, \end{aligned}$$

and $\mathbf{k}_n := \mathbf{k}_{i,n}$.

As in Lemma 6.6(c) above, Theorem 3.9 and Corollary 3.10 in Alvarez-Mena and Hernández-Lerma (2002) hold for the sequence $\{\mathrm{CCP}_i^{\boldsymbol{\varphi}_n}\}$ and the limit $\mathrm{CCP}_i^{\boldsymbol{\varphi}}$.

Now, for each $n \in \mathbb{N}$, let $\boldsymbol{\varphi}_n$ be a Nash equilibrium for the CG$_n$. By the compactness of $\boldsymbol{\Phi}$ (see Remark 3.1(c)), there exists a subsequence $\{\boldsymbol{\varphi}_m\}$ of $\{\boldsymbol{\varphi}_n\}$ that converges to some $\boldsymbol{\varphi} = (\varphi^1, \ldots, \varphi^N) \in \boldsymbol{\Phi}$.

Hence, from Proposition 6.3 in Alvarez-Mena and Hernández-Lerma (2002), for each $m \in \mathbb{N}$, φ_m^i is an optimal strategy for $\mathrm{CCP}_i^{\boldsymbol{\varphi}_m}$, and so, by [Alvarez-Mena and Hernández-Lerma 2002, Corollary 3.10], the limit point φ^i of the sequence $\{\varphi_m^i\}_m$ is also an optimal strategy for $\mathrm{CCP}_i^{\boldsymbol{\varphi}}$. Therefore, by our Proposition 6.3 above, φ^i is an optimal response of player i against $\boldsymbol{\varphi}$. As $i \in \mathcal{N}$ was arbitrary, it follows that $\boldsymbol{\varphi}$ is a Nash equilibrium for CG$_\infty$. □

CGs with finite state set. Next, we develop some lemmas necesary to prove part (b) of Theorem 3.6. We assume that Assumption 3.2 holds and, furthermore, the initial distribution γ has support on $\widetilde{\mathcal{X}}$. For each $i \in \mathcal{N}$, the initial distribution γ defines an equivalence relation on Φ_i as follows: $\varphi \in \Phi_i$ is related to $\varphi' \in \Phi_i$ if $\varphi(\cdot|x) = \varphi'(\cdot|x)$ γ-a.e. (almost everywhere) on $\widetilde{\mathcal{X}}$. This equivalence relation defines a partition of Φ_i in equivalence classes. We denote by $R\varphi$ the equivalence class of φ, and by Φ_i/γ the set of all equivalence classes on Φ_i.

Let μ be in $\mathcal{M}_i := \{\mu \in \mathbb{P}(\mathbb{K}_i) : \mu$ has support on $\widetilde{\mathbb{K}}_i\}$, where $\widetilde{\mathbb{K}}_i := \{(x, a) : x \in \widetilde{\mathcal{X}},\ a \in A_i(x)\}$. As the marginal $\widehat{\mu}$ of $\mu \in \mathcal{M}_i$ has support on $\widetilde{\mathcal{X}}$, if $\mu := \widehat{\mu} \circ \varphi_1$ and $\mu := \widehat{\mu} \circ \varphi_2$, then $\varphi_1(\cdot|x) = \varphi_2(\cdot|x)$ for all $x \in \widetilde{\mathcal{X}}$, i.e., $R\varphi_1 = R\varphi_2$. Furthermore,

$$\{\varphi \in \Phi_i : \mu = \widehat{\mu} \circ \varphi\} = R\varphi_1 = R\varphi_2.$$

Thus, the map G from $\times_{i=1}^N \mathcal{M}_i$ to $\times_{i=1}^N (\Phi_i/\gamma)$ given by

$$\boldsymbol{\mu} = (\mu^1, \ldots, \mu^N) \mapsto G(\boldsymbol{\mu}),$$

where

$$G(\boldsymbol{\mu}) := (\{\varphi \in \Phi_1 : \mu^1 = \widehat{\mu^1} \circ \varphi\}, \ldots, \{\varphi \in \Phi_N : \mu^N = \widehat{\mu^N} \circ \varphi\}),$$

is well defined.

Note that if $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^N) \in \times_{i=1}^N \mathcal{M}_i$ and $\boldsymbol{\varphi} = (\varphi^1, \ldots, \varphi^N) \in \boldsymbol{\Phi}$ are such that $\mu^i = \widehat{\mu}^i \circ \varphi^i$ for each $i \in \mathcal{N}$, then $G(\boldsymbol{\mu}) = (R\varphi^1, \ldots, R\varphi^N)$. We denote by $R\boldsymbol{\varphi}$ the equivalence class $(R\varphi^1, \ldots, R\varphi^N)$ in $\times_{i=1}^N (\Phi_i/\gamma)$.

Lemma 7.1 *Suppose that Assumptions 3.3 and 3.2 hold, and that γ has support on $\widetilde{\mathcal{X}}$. Let $\boldsymbol{\varphi}_1 = (\varphi_1^1, \ldots, \varphi_1^N)$ and $\boldsymbol{\varphi}_2 = (\varphi_2^1, \ldots, \varphi_2^N)$ be in $\boldsymbol{\Phi}$. If $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ lie in the same equivalence class (that is, $R\varphi_1^i = R\varphi_2^i$ for all $i \in \mathcal{N}$), then both $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ have the same associated CCPs, that is, for each $i \in \mathcal{N}$, the set of feasible solutions and the set of optimal solutions for* $\mathrm{CCP}_i^{\boldsymbol{\varphi}_1}$ *and* $\mathrm{CCP}_i^{\boldsymbol{\varphi}_2}$ *are the same.*

Proof Fix an arbitrary $i \in \mathcal{N}$. We will use the formulations $\underline{\mathrm{CCP}}_i^{\boldsymbol{\varphi}_1}$ and $\underline{\mathrm{CCP}}_i^{\boldsymbol{\varphi}_2}$ in (6.5)–(6.7) to prove that

(a) μ is feasible for $\underline{\mathrm{CCP}}_i^{\boldsymbol{\varphi}_1}$ iff μ is feasible for $\underline{\mathrm{CCP}}_i^{\boldsymbol{\varphi}_2}$, and
(b) μ is optimal for $\underline{\mathrm{CCP}}_i^{\boldsymbol{\varphi}_1}$ iff μ is optimal for $\underline{\mathrm{CCP}}_i^{\boldsymbol{\varphi}_2}$.

Since, $\varphi_1^j(\cdot|x) = \varphi_2^j(\cdot|x)$ for all $x \in \widetilde{\mathcal{X}}$ and $j \in \mathcal{N}$, it follows that

$$c_i^l(x, [\boldsymbol{\varphi}_1^{-i}|a]) = c_i^l(x, [\boldsymbol{\varphi}_2^{-i}|a]) \ \ \forall\, (x, a) \in \widetilde{\mathbb{K}}_i,\ l = 1, \ldots, q, \tag{7.1}$$

and

$$Q(y|x, [\boldsymbol{\varphi}_1^{-i}|a]) = Q(y|x, [\boldsymbol{\varphi}_2^{-i}|a]) \ \ \forall\, y \in X,\ (x, a) \in \widetilde{\mathbb{K}}_i. \tag{7.2}$$

Let μ be a feasible solution of $\mathrm{CCP}_i^{\boldsymbol{\varphi}_1}$. From (6.6) and Assumption 3.2, μ has support on $\widetilde{\mathbb{K}}_i$, that is, μ is in $\mathcal{M}_i$. This fact together with (6.2), (6.3), (7.1) and

(7.2) yields that

$$\int_{\mathbb{K}_i} Q_i^{\boldsymbol{\varphi}_1}(y|x,a)\mu(d(x,a)) = \int_{\mathbb{K}_i} Q_i^{\boldsymbol{\varphi}_2}(y|x,a)\mu(d(x,a)) \;\; \forall\, y \in X,$$
$$\int_{\mathbb{K}_i} c_i^{l,\boldsymbol{\varphi}_1}(x,a)\mu(d(x,a)) = \int_{\mathbb{K}_i} c_i^{l,\boldsymbol{\varphi}_2}(x,a)\mu(d(x,a)) \;\; \forall\, l = 0, 1, \ldots, q.$$

The last two facts imply (a) and (b). □

By Lemma 7.1, each equivalence class $R\boldsymbol{\varphi} = (R\varphi^1, \ldots, R\varphi^N)$ defines a *unique* $\mathrm{CCP}_i^{\boldsymbol{\varphi}}$ which will be denoted by $\mathrm{CCP}_i^{R\boldsymbol{\varphi}}$. Hence, we can define without ambiguity two multifunctions $\Gamma_i : \times_{l=1}^N \mathcal{M}_l \to \mathcal{M}_i$ and $\Psi : \times_{l=1}^N \mathcal{M}_l \to \times_{l=1}^N \mathcal{M}_l$ as follows. For each $i \in \mathcal{N}$ and $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^N)$ in $\times_{i=1}^N \mathcal{M}_i$,

$$\Gamma_i(\boldsymbol{\mu}) := \{\nu \in \mathcal{M}_i \mid \nu \text{ is an optimal solution for } \underline{\mathrm{CCP}}_i^{G(\boldsymbol{\mu})}\}$$

and

$$\Psi(\boldsymbol{\mu}) := \times_{i=1}^N \Gamma_i(\boldsymbol{\mu}).$$

Next, to relate a stationary Nash equilibrium with the fixed points of Ψ it suffices to note that the formulation (6.5)–(6.7) of a CCP yields the following.

Lemma 7.2 *Suppose that Assumptions* 3.3 *and* 3.2 *hold, and that γ has support on $\widetilde{\mathcal{X}}$. Then the CG has a stationary Nash equilibrium if and only if Ψ has a fixed point.*

Proof Suppose that $\boldsymbol{\varphi} = (\varphi^1, \ldots, \varphi^N)$ is a stationary Nash equilibrium for the CG. We take $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^N)$ such that, for each $i \in \mathcal{N}$, μ^i is $(\boldsymbol{\varphi}, i)$-equivalent to φ^i. Thus $R\boldsymbol{\varphi} = G(\boldsymbol{\mu})$, and μ^i is in $\mathcal{M}_i$ for all $i \in \mathcal{N}$. Let $i \in \mathcal{N}$ be fixed. As $\boldsymbol{\varphi}$ is a Nash equilibrium, φ^i is an optimal response of player i against $\boldsymbol{\varphi}$, that is (by Proposition 6.3), φ^i is an optimal strategy for $\mathrm{CCP}_i^{\boldsymbol{\varphi}}$. Hence, μ^i is an optimal solution for $\underline{\mathrm{CCP}}_i^{\boldsymbol{\varphi}}$, that is, $\mu^i \in \Gamma_i(\boldsymbol{\mu})$. As $i \in \mathcal{N}$ was arbitrary, it follows that $\boldsymbol{\mu}$ is in $\Psi(\boldsymbol{\mu})$, that is, $\boldsymbol{\mu}$ is a fixed point of Ψ.

Now suppose that $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^N)$ is a fixed point of Ψ, and let $\boldsymbol{\varphi} = (\varphi^1, \ldots, \varphi^N)$ be such that $\mu^i = \widehat{\mu}^i \circ \varphi^i$ for each $i \in \mathcal{N}$. Thus $R\boldsymbol{\varphi} = G(\boldsymbol{\mu})$. Now, we take $\boldsymbol{\varphi}^* = (\varphi^{1*}, \ldots, \varphi^{N*})$ such that φ^{i*} is $(\boldsymbol{\varphi}, i)$-equivalent to μ^i for each $i \in \mathcal{N}$. Thus $R\boldsymbol{\varphi}^* = R\boldsymbol{\varphi} = G(\boldsymbol{\mu})$. Pick an arbitrary $i \in \mathcal{N}$. As $\boldsymbol{\mu}$ is in $\Psi(\boldsymbol{\mu})$, μ^i is an optimal solution of $\underline{\mathrm{CCP}}_i^{\boldsymbol{\varphi}}$, and as φ^{i*} is $(\boldsymbol{\varphi}, i)$-equivalent to μ^i, then φ^{i*} is an optimal strategy for $\mathrm{CCP}_i^{R\boldsymbol{\varphi}^*}$, that is, φ^{i*} is an optimal response of player i against $\boldsymbol{\varphi}^*$. As $i \in \mathcal{N}$ was arbitrary, it follows that $\boldsymbol{\varphi}^*$ is a stationary Nash equilibrium for the CG. □

In view of Lemma 7.2, to show the existence of a Nash equilibrium it suffices to show that Ψ has a fixed point. This requires to study some properties of the multifunction Ψ.

Lemma 7.3 *Suppose that Assumptions* 3.3 *and* 3.2 *hold, and that γ has support on $\widetilde{\mathcal{X}}$. Then*

(a) *for each* $i \in \mathcal{N}$, Γ_i *is upper semicontinuous and, moreover, for each* $\boldsymbol{\mu}$ *in* $\times_{l=1}^N \mathcal{M}_l$ *the set* $\Gamma_i(\boldsymbol{\mu})$ *is nonempty and convex;*
(b) Ψ *is upper semicontinuous and, for each* $\boldsymbol{\mu}$ *in* $\times_{l=1}^N \mathcal{M}_l$, *the set* $\Psi(\boldsymbol{\mu})$ *is nonempty and convex.*

Proof (a) Let $i \in \mathcal{N}$ be fixed. From Theorem 3.9(a) in Alvarez-Mena and Hernández-Lerma (2002)(see Lemma 6.6(c) above), for each $\boldsymbol{\mu} \in \times_{l=1}^N \mathcal{M}_l$, the problem $\underline{\mathrm{CCP}}_i^{G(\boldsymbol{\mu})}$ is solvable, that is, $\Gamma_i(\boldsymbol{\mu})$ is nonempty, and, on the other hand, from (6.5)–(6.7) it is clear that the set $\Gamma_i(\boldsymbol{\mu})$ of optimal solutions for $\underline{\mathrm{CCP}}_i^{G(\boldsymbol{\mu})}$ is convex. To prove that Γ_i is upper semicontinuous, let $\boldsymbol{\mu}_n \in \times_{l=1}^N \mathcal{M}_l$ and $\nu_n \in \Gamma_i(\boldsymbol{\mu}_n)$ be such that

(1) $\boldsymbol{\mu}_n \to \boldsymbol{\mu}_\infty$ in $\times_{l=1}^N \mathcal{M}_l$, and
(2) $\nu_n \to \nu_\infty$ in $\mathcal{M}_i$.

We have to show that ν_∞ is in $\Gamma_i(\boldsymbol{\mu}_\infty)$. Let $\varphi_\infty \in \Phi_i$ be such that $\nu_\infty = \widehat{\nu}_\infty \circ \varphi_\infty$. Taking $\varphi_n \in \Phi_i$ such that $\nu_n = \widehat{\nu}_n \circ \varphi_n$, it follows from (2) that $\varphi_n(\cdot|x) \to \varphi_\infty(\cdot|x)$ for each $x \in \widetilde{\mathcal{X}}$. Choosing φ_n' as

$$\varphi_n'(\cdot|x) := \begin{cases} \varphi_n(\cdot|x) & \text{if } x \in \widetilde{\mathcal{X}}, \\ \varphi_\infty(\cdot|x) & \text{if } \notin \widetilde{\mathcal{X}}, \end{cases}$$

we see that φ_n' is in $R\varphi_n$ and $\varphi_n' \to \varphi_\infty$. In this manner we can find a sequence of multistrategies $\boldsymbol{\varphi}_n = (\varphi_n^1, \ldots, \varphi_n^N)$ such that, for each $n \geq 1$, $\mu_n^i = \widehat{\mu}_n^i \circ \varphi_n^i$ ($i = 1, \ldots, N$) and $\boldsymbol{\varphi}_n \to \boldsymbol{\varphi}_\infty$. Now, as $R\boldsymbol{\varphi}_n = G(\boldsymbol{\mu}_n)$, ν_n is an optimal solution for $\underline{\mathrm{CCP}}_i^{\boldsymbol{\varphi}_n}$ ($\equiv \underline{\mathrm{CCP}}_i^{G(\boldsymbol{\mu}_n)}$). Hence, by Lemma 6.6(c) it follows that the limit point ν_∞ of the sequence $\{\nu_n\}$ of optimal solutions is also an optimal solution for $\underline{\mathrm{CCP}}_i^{\boldsymbol{\varphi}_\infty}$ ($\equiv \underline{\mathrm{CCP}}_i^{G(\boldsymbol{\mu}_\infty)}$), that is, ν_∞ is in $\Gamma_i(\boldsymbol{\mu}_\infty)$

Part (b) follows from (a) and the definition of Ψ. □

Proof of part (b) of Theorem 3.6 First we suppose that γ has support on the finite set $\widetilde{\mathcal{X}}$. The space $\mathcal{M}_i$ is compact because so is $\widetilde{\mathbb{K}}_i$ (see Remark 3.1(b)). Hence, $\times_{i=1}^N \mathcal{M}_i$ is a convex compact subset of the convex Hausdorff linear topological space $\times_{i=1}^N M(\mathbb{K}_i)$. On the other hand, by Proposition 7.3(b), the multifunction Ψ is convex and upper semicontinuous. Hence, by Glicksberg's Theorem Glicksberg (1952), Ψ has a fixed point.

Now let $\gamma \in \mathbb{P}(X)$ be concentrated on $\widetilde{\mathcal{X}}$. Taking another $\widetilde{\gamma} \in \mathbb{P}(X)$ with support on $\widetilde{\mathcal{X}}$, the probability measure

$$\gamma_n := \frac{1}{n}\widetilde{\gamma} + \frac{n-1}{n}\gamma \tag{7.3}$$

also has support on $\widetilde{\mathcal{X}}$. We now consider a CG with initial distribution and transition law γ_n and Q, respectively. Then, for any $\boldsymbol{\varphi} = (\varphi^1, \ldots, \varphi^N) \in \boldsymbol{\Phi}$ and $i \in \mathcal{N}$ (see Remark 6.5),

$$
\begin{aligned}
\left|V_i^l(\boldsymbol{\varphi},\gamma_n) - V_i^l(\boldsymbol{\varphi},\gamma)\right| \\
&= \left|\int c_i^l(x,[\boldsymbol{\varphi}^{-i}|a])\mu_n(d(x,a)) - \int c_i^l(x,[\boldsymbol{\varphi}^{-i}|a])\mu(d(x,a))\right| \\
&= \left|\sum_x c_i^l(x,\boldsymbol{\varphi})\widehat{\mu}_n(x) - \sum_x c_i^l(x,\boldsymbol{\varphi})\widehat{\mu}(x)\right| \\
&\leq L\|\widehat{\mu}_n - \widehat{\mu}\|_{TV}, \qquad (7.4)
\end{aligned}
$$

where μ is $(\boldsymbol{\varphi},\gamma)$-equivalent to φ^i, μ_n is $(\boldsymbol{\varphi},\gamma_n)$-equivalent to φ^i, L is a positive constant, and $\|\cdot\|_{TV}$ stands for the total variation norm. On the other hand, by (6.6),

$$
\begin{aligned}
\|\widehat{\mu}_n - \widehat{\mu}\|_{TV} &:= \sum_y |\widehat{\mu}_n(y) - \widehat{\mu}(y)| \\
&\leq (1-\alpha)\|\gamma_n - \gamma\| + \alpha\|\widehat{\mu}_n - \widehat{\mu}\|_{TV}.
\end{aligned}
$$

Using the latter inequality and (7.3) we get that

$$
\begin{aligned}
\|\widehat{\mu}_n - \widehat{\mu}\|_{TV} &\leq \|\gamma_n - \gamma\|_{TV} \\
&= \frac{1}{n}\|\widetilde{\gamma} - \gamma\|_{TV}. \qquad (7.5)
\end{aligned}
$$

Thus, (7.4) and (7.5) yield that, for each $l = 1,\ldots,q$ and $i = 1,\ldots,N$,

$$
\begin{aligned}
\sup_{\boldsymbol{\varphi}\in\boldsymbol{\Phi}} |V_i^l(\boldsymbol{\varphi},\gamma_n) - V_i^l(\boldsymbol{\varphi},\gamma)| &\leq \frac{L}{n}\|\widetilde{\gamma} - \gamma\|_{TV} \qquad (7.6) \\
&=: \eta_n \downarrow 0 \text{ as } n \to \infty.
\end{aligned}
$$

Once we have (7.6) we can choose suitable constraint constant vectors $\mathbf{k}_{i,n}$ to obtain the Slater condition and such that $\mathbf{k}_{i,n} \to \mathbf{k}_i$ as $n \to \infty$. Moreover, letting η_n be as in (7.6), for each $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$, it holds that $V_i^l(\boldsymbol{\varphi},\gamma_n) \leq V_i^l(\boldsymbol{\varphi},\gamma) + \eta_n$ for all $l = 1,\ldots,q$ and $i \in \mathcal{N}$.

Now, for each $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ and $i \in \mathcal{N}$, let $\pi = \varphi' \in \boldsymbol{\Phi}_i$ be as in Assumption 3.3(c), and define

$$
\mathbf{k}_{i,n} := (k_i^1 + \eta_n, \ldots, k_i^N + \eta_n). \qquad (7.7)
$$

This yields the Slater condition

$$
V_i^l([\boldsymbol{\varphi}^{-i}|\varphi'],\gamma_n) < k_{i,n}^l \quad \forall\, l = 1,\ldots,q.
$$

with $k_{i,n}^l := k_i^l + \eta_n$. Hence, for each $n \geq 1$, the constrained game CG_n

$$
\{X, (A_i, \{A_i(x)|x \in X\}, \mathbf{c}_i, \mathbf{k}_{i,n})_{i\in\mathcal{N}}, Q, \gamma_n\}. \qquad (7.8)
$$

satisfies the Asumptions 3.3 and 3.2, and in addition γ_n has support on $\widetilde{X}$. Therefore, CG_n has a Nash equilibrium.

From the latter fact and Lemma 6.6, together with (6.2) and (6.3), the sequence CG_n and the original $\mathrm{CG} \equiv \mathrm{CG}_\infty$ satisfy Assumption 3.5; hence, by part (a) of Theorem 3.6, the CG (with finite state space) also has a Nash equilibrium. □

Proof of part (c) of Theorem 3.6. Let $\{\mathcal{X}_n\}$ be as in Assumption 3.4. Since $\epsilon_n \downarrow 0$ and $\mathcal{X}_n \uparrow X$, there exists a subsequence $\{\mathcal{X}_m\}$ of $\{\mathcal{X}_n\}$ such that $\gamma(\mathcal{X}_m^c) \leq \alpha^m$ and $\sup_{\mathcal{X}_m} \sup_{\mathbb{A}(x)} Q(\mathcal{X}_m^c|x, \boldsymbol{a}) \leq \alpha^m$ where $0 < \alpha < 1$ is the discount factor. Hence, without loss of generality we can suppose that

$$\sup_{x\in\mathcal{X}_{n-1}} \sup_{\boldsymbol{a}\in\mathbb{A}(x)} Q(\mathcal{X}_n^c|x, \boldsymbol{a}) \leq \alpha^n \text{ and } \gamma(\mathcal{X}_n^c) \leq \alpha^n. \tag{7.9}$$

Now, for each $n \geq 1$, pick $x_n \in \mathcal{X}_n \setminus \mathcal{X}_{n-1}$. For all $y \in X$ and $(x, a) \in \mathbb{K}$, define the transition law

$$Q_n(y|x, \boldsymbol{a}) := \begin{cases} Q(y|x, \boldsymbol{a}) & \text{if } y \in \mathcal{X}_n \setminus \{x_n\}, \\ Q(x_n|x, \boldsymbol{a}) + Q(\mathcal{X}_n^c|x, \boldsymbol{a}) & \text{if } y = x_n, \\ 0 & \text{if } y \notin \mathcal{X}_n, \end{cases}$$

and the initial distribution

$$\gamma_n(y) := \begin{cases} \gamma(y) & \text{if } y \in \mathcal{X}_n \setminus \{x_n\} \\ \gamma(x_n) + \gamma(\mathcal{X}_n^c) & \text{if } y = x_n \\ 0 & \text{if } y \notin \mathcal{X}_n. \end{cases}$$

Note that Q_n and γ_n satisfy the Assumption 3.2 for $\widetilde{\mathcal{X}} = \mathcal{X}_n$. Furthermore, with Q_n and γ_n we may define a constrained game CG_n as in (7.8). Then, as in (7.4), for any $\boldsymbol{\varphi} = (\varphi^1, \ldots, \varphi^N) \in \boldsymbol{\Phi}$ and $i \in \mathcal{N}$ we have

$$|V_i^l(\boldsymbol{\varphi}, Q_n, \gamma_n) - V_i^l(\boldsymbol{\varphi}, Q, \gamma)| \leq L\|\widehat{\mu}_n - \widehat{\mu}\|_{TV} \;\; \forall\, l = 1, \ldots, q. \tag{7.10}$$

Here, $\widehat{\mu}_n$ and $\widehat{\mu}$ are the marginals of μ_n and μ, respectively, which are $(\boldsymbol{\varphi}, Q_n, \gamma_n)$-equivalent to φ^i and $(\boldsymbol{\varphi}, Q, \gamma)$-equivalent to φ^i. From (6.6) and the definition of Q_n and γ_n, the measure $\widehat{\mu}_n$ is concentrated on $\mathcal{X}_n$, and, thus,

$$\|\widehat{\mu}_n - \widehat{\mu}\|_{TV} = \sum_{y\in\mathcal{X}_n} |\widehat{\mu}_n(y) - \widehat{\mu}(y)| + \widehat{\mu}(\mathcal{X}_n^c). \tag{7.11}$$

We next estimate an upper bound for the second term on the right-hand side of (7.11). From (6.6), (6.10) and (7.9), for each $n \geq 1$ we get

$$\begin{aligned} \widehat{\mu}(\mathcal{X}_n^c) &= (1-\alpha)\gamma(\mathcal{X}_n^c) + \alpha\Big(\sum_{x\in\mathcal{X}_{n-1}} + \sum_{x\in\mathcal{X}_{n-1}^c}\Big) Q(\mathcal{X}_n^c|x, \boldsymbol{\varphi})\widehat{\mu}(x) \\ &\leq (1-\alpha)\gamma(\mathcal{X}_n^c) + \alpha^{n+1}\widehat{\mu}(\mathcal{X}_{n-1}) + \alpha\widehat{\mu}(\mathcal{X}_{n-1}^c) \end{aligned}$$

Iterating n-times (recall that $\mathcal{X}_0$ is the empty set) and using (7.9) again,

$$\begin{aligned} \widehat{\mu}(\mathcal{X}_n^c) &\leq (1-\alpha)\sum_{l=0}^{n} \alpha^l \gamma(\mathcal{X}_{n-l}^c) + \alpha^{n+1}\sum_{l=1}^{n} \widehat{\mu}(\mathcal{X}_{n-l}) + \alpha^n \widehat{\mu}(\mathcal{X}_0^c) \\ &\leq (n+1)\alpha^n \downarrow 0\,, \quad \text{as } n \to \infty. \end{aligned} \tag{7.12}$$

We now consider the first term on the rigth-hand side of (7.11), which we denote by $\|\widehat{\mu}_n - \widehat{\mu}\|^{\mathcal{X}_n}$. Using once more (6.6), (6.10), (7.9) and the definition of Q_n and γ_n,

$$\begin{aligned}
\|\widehat{\mu}_n - \widehat{\mu}\|^{\mathcal{X}_n} &:= \sum_{y\in\mathcal{X}_n\setminus\{x_n\}} |\widehat{\mu}_n(y) - \widehat{\mu}(y)| + |\widehat{\mu}_n(x_n) - \widehat{\mu}(x_n)| \\
&= \sum_{y\in\mathcal{X}_n\setminus\{x_n\}} \alpha\Big|\sum_{x\in X}\big(Q_n(y|x,\boldsymbol{\varphi})\widehat{\mu}_n(x) - Q(y|x,\boldsymbol{\varphi})\widehat{\mu}(x)\big)\Big| \\
&\quad +\Big|\alpha\sum_{x\in X}\big(Q_n(x_n|x,\boldsymbol{\varphi})\widehat{\mu}_n(x) - Q(x_n|x,\boldsymbol{\varphi})\widehat{\mu}(x)\big) \\
&\quad +(1-\alpha)(\gamma_n(x_n) - \gamma(x_n))\Big| \\
&\le \alpha\Big(\sum_{x\in\mathcal{X}_n} + \sum_{x\in\mathcal{X}_n^c}\Big)Q(\mathcal{X}_n|x,\boldsymbol{\varphi})|\widehat{\mu}_n(x) - \widehat{\mu}(x)| \\
&\quad +\alpha\Big(\sum_{x\in\mathcal{X}_{n-1}} + \sum_{x\in\mathcal{X}_n\setminus\mathcal{X}_{n-1}}\Big)Q(\mathcal{X}_n^c|x,\boldsymbol{\varphi})\widehat{\mu}_n(x) + (1-\alpha)\gamma(\mathcal{X}_n^c) \\
&\le \alpha\|\widehat{\mu}_n - \widehat{\mu}\|^{\mathcal{X}_n} + \alpha\widehat{\mu}(\mathcal{X}_n^c) + \alpha^{n+1}\widehat{\mu}_n(\mathcal{X}_{n-1}) + \alpha\widehat{\mu}_n(\mathcal{X}_n\setminus\mathcal{X}_{n-1}) \\
&\quad +(1-\alpha)\gamma(\mathcal{X}_n^c).
\end{aligned}$$

Hence

$$\begin{aligned}
\|\widehat{\mu}_n - \widehat{\mu}\|^{\mathcal{X}_n} \le \frac{1}{1-\alpha}\Big[&\alpha\widehat{\mu}(\mathcal{X}_n^c) + \alpha^{n+1}\widehat{\mu}_n(\mathcal{X}_{n-1}) + \alpha\widehat{\mu}_n(\mathcal{X}_n\setminus\mathcal{X}_{n-1}) \\
&+(1-\alpha)\gamma(\mathcal{X}_n^c)\Big]. \qquad (7.13)
\end{aligned}$$

Finally, we consider the next-to-last term on the right-hand side of (7.13). By (7.12), for each $n \ge 1$ we have (recall that $\mathcal{X}_0$ is the empty set)

$$\begin{aligned}
\widehat{\mu}_n(\mathcal{X}_n\setminus\mathcal{X}_{n-l}) &\le \widehat{\mu}_n(\mathcal{X}_{n-l}^c) \\
&\le n\alpha^{n-1} \downarrow 0, \quad \text{as } n\to\infty. \qquad (7.14)
\end{aligned}$$

Hence, by (7.14), (7.12) and (7.9), the inequality (7.13) yields

$$\|\widehat{\mu}_n - \widehat{\mu}\|^{\mathcal{X}_n} < \Big(\frac{1}{1-\alpha}\Big)(n+2)\alpha^n \downarrow 0, \quad \text{as } n\to\infty. \qquad (7.15)$$

Furthermore, from (7.15), (7.12) and (7.11), we see that (7.10) yields

$$\sup_{\boldsymbol{\varphi}\in\boldsymbol{\Phi}} |V_i^l(\boldsymbol{\varphi}, Q_n, \gamma_n) - V_i^l(\boldsymbol{\varphi}, Q, \gamma)| \;\downarrow\; 0, \quad \text{as } n\to\infty$$

for all $i = 1,\dots,N$ and $l = 0,1,\dots,q$. Thus, taking $\mathbf{k}_{i,n}$ as in (7.7) for each $n \ge 1$, the constrained game CG_n given by

$$\{X, (A_i, \{A_i(x)|x\in X\}, \mathbf{c}_i, \mathbf{k}_{i,n})_{i\in\mathcal{N}}, Q_n, \gamma_n\}. \qquad (7.16)$$

is a CG with finite state space, and it satisfies the Assumptions 3.2 and 3.3. Thus, by part (b) of Theorem 3.6, each CG_n has a NE. (Note that we have constructed state approximation for the CG by CGs with *finite* state spaces.)

From the latter fact, the proof will follow from part (a) of Theorem 3.6 if we verify Assumption 3.5. To this end, observe that the definition (7.16) of CG_n and Lemma 6.6(b), together with (6.3), yield parts (c)–(e) and (a) of Assumption 3.5, respectively. Finally, to verify part (b), let $x, y \in X$ be fixed, and let $\{\boldsymbol{\varphi}_n\} \subset \boldsymbol{\Phi}$ be such that $\boldsymbol{\varphi}_n \to \boldsymbol{\varphi}_\infty$ weakly. As $\mathcal{X}_l \uparrow X$, there exists an integer l such that $x, y \in \mathcal{X}_l$ weakly. Thus

$$Q_n(y|x, \boldsymbol{a}) = Q(y|x, \boldsymbol{a}) \; \forall \, n \geq l+1, \;\; a \in \mathbb{A}(x),$$

- and so, by Remark 3.1(d), part(b) of Assumption 3.5 follows. □

8 Concluding remarks

In this paper, we considered CGs with a countable state space and compact action sets. Sufficient conditions were given to ensure the existence of Nash equilibria for those CGs. Two examples illustrate important differences between constrained and unconstrained games. These examples suggest that *some* constraints eliminate "unreasonable" Nash equilibria. The obvious question then is, is this always possible? In other words, suppose we wish to eliminate Nash strategies that give an undesirable behavior. Is it possible to introduce constraints to remove those strategies? This would avoid paradoxical or objectionable situations as in the "prisoner's dilemma" or the "tragedy of the commons" (Clark, 1990, 1980; Gordon, 1954; Levhary and Mirman, 1980; Mckelvey, 1997, 1999).

Another key issue still unexplored is the *computation* of Nash equilibria for *constrained* dynamic games (when they are known to exist). A possible line of research would be to extend to countable CGs the existing computation methods for *unconstrained* dynamic games on finite spaces (Filar and Vrieze, 1997) in combination with suitable approximation schemes as in our Theorem 3.6(a),(c), or (Alvarez-Mena and Hernández-Lerma, 2002). To this end we could use, for instance, the *finite state approximations* defined by (7.16)

Acknowledgements This research was supported partially by a CONACyT scholarship and partially by CONACyT grant 37355-3.

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