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Minimax theorems for Φ-convex functions: sufficient and necessary conditions

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We prove minimax theorems for Φ-convex functions. The novelty of the theorems relies on the use of the weak Φ-intersection property which allows to relax the compactness assumption of the underlying spaces. The weak Φ-intersection property appears to be sufficient and necessary conditions for the minimax equality to hold.

Keywords: minimax theorems; Φ-convexity; abstract convexity; Φ-convexlikeness; Φ-intersection property; weak Φ-intersection property

AMS Subject Classifications: 32F17; 49K27; 49K35; 52A01

1. Introduction

We start with some definitions related to abstract convexity. Consider a set X and a set Φ of real-valued functions $\varphi : X \to \mathbb{R}$.

For any $f, g : X \to \mathbb{R}$

$$f \leq g \;\Leftrightarrow\; f(x) \leq g(x) \quad \forall x \in X.$$

Let $f : X \to \mathbb{R}$. The set

$$\mathrm{supp}(f, \Phi) := \{\varphi \in \Phi \;:\; \varphi \leq f\}$$

is called the *support* of f with respect to Φ. We will use the notation $\mathrm{supp}(f)$ if the class Φ is clear from the context.

Definition 1 [1,2] A function $f : X \to \mathbb{R}$ is called Φ-*convex* if

$$f(x) = \sup\{\varphi(x) \;:\; \varphi \in \mathrm{supp}(f)\} \quad \forall\, x \in X.$$

By $H(\Phi, X)$, we denote the set of all Φ-convex functions $f : X \to \mathbb{R}$ defined on X.

Minimax theorems provide sufficient conditions for the equality

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$$\sup_{y\in Y}\inf_{x\in X} a(y,x) = \inf_{x\in X}\sup_{y\in Y} a(y,x) \tag{1}$$

to hold, where X, Y are nonempty sets and $a : Y \times X :\rightarrow \mathbb{R}$ is a function.

In this paper, our aim is to provide sufficient and necessary conditions for the equality (1) for Φ-convex functions, i.e. we assume that for any $y \in Y$, the function $a(y,\cdot)$ is Φ-convex as a function of x only. Our main tool is the weak Φ-intersection property, the property we introduce in Section 2 of the present paper. Although technically involved, this property is not only sufficient but also a necessary condition for the minimax equality. It is worth noticing that the weak Φ-intersection property is expressed in terms of functions $\varphi \in \Phi$ only and not in terms of the function a itself. This feature is particularity important since, as shown the examples of Section 6, the functions $\varphi \in \Phi$ are usually of much simpler structure than the functions $a(y,\cdot)$, $y \in Y$.

In Section 2, we introduce the weak Φ-intersection property and we discuss its relationships with the Φ-intersection property introduced in [3].

In Section 3, we provide our main result which is a minimax theorem for Φ-convex functions. It is worth observing that in our minimax theorem, we avoided any topological assumptions, like e.g. compactness of Y and upper semicontinuity of $a(\cdot,x)$ for $x \in X$ (c.f. [4–8]).

In Section 4, we prove the necessity of the conditions appearing in our main result of Section 3, in particular we prove the necessity of the weak Φ-intersection property.

In order to relate our result to the existing minimax theorems (an exhaustive survey is given e.g. in [8]) in Section 5, we discuss the relationships between the weak Φ-intersection property and the joint Φ-convexlikeness introduced in [3]. If given functions φ_1 and φ_2 are jointly convexlike and the intersection of the strict lower level sets of φ_1 and φ_2 is empty at some level $\alpha \in \mathbb{R}$, then the weak intersection property holds for φ_1 and φ_2 at the level α. On the other hand, joint Φ-convexlikeness can be easily proved for some classes of functions, e.g. for convexlike functions in the sense of Ky Fan [4], König [5], Neumann [6], Simons [7].

In Section 6, we give some examples of classes Φ possessing the properties discussed above.

2. Weak Φ-intersection property

In this section, we introduce a new concept of weak Φ-intersection property. As we have already mentioned in the Introduction, this property is the main tool for proving sufficient and necessary conditions for the minimax equality for Φ-convex functions to hold.

Let X be a set and Φ be a class of functions $\varphi : X \rightarrow \mathbb{R}$. For any function $\varphi : X \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}$, the strict lower level set of φ is defined as

$$Z_\alpha(\varphi) := \{x \in X,\ \varphi(x) < \alpha\}$$

and for any $\varphi_1, \varphi_2 : X \rightarrow \mathbb{R}$ and $t \in [0,1]$, we write

$$Z_{\alpha,t}(\varphi_1,\varphi_2) := \{x \in X\ :\ t\varphi_1(x) + (1-t)\varphi_2(x) < \alpha\}.$$

We start with the weak intersection property defined for any two functions.

Definition 2 Let $\varphi_1, \varphi_2 : X \to \mathbb{R}$ be two real-valued functions defined on X and let $\alpha \in \mathbb{R}$. We say that φ_1 and φ_2 have *the weak intersection property on X at the level* α if for every $t \in [0, 1]$

$$Z_{\alpha,t}(\varphi_1, \varphi_2) \cap Z_\alpha(\varphi_1) = \emptyset \quad \text{or} \quad Z_{\alpha,t}(\varphi_1, \varphi_2) \cap Z_\alpha(\varphi_2) = \emptyset. \tag{2}$$

Remark 1 If for given φ_1, φ_2 and $\alpha \in \mathbb{R}$ we have $Z_\alpha(\varphi_1) \cap Z_\alpha(\varphi_2) \neq \emptyset$, then (2) does not hold, and consequently φ_1, φ_2 do not posses the weak intersection property at the level α.

PROPOSITION 2.1 *If φ_1, φ_2 have the weak intersection property on X at the level $\alpha \in \mathbb{R}$, then φ_1, φ_2 have the weak intersection property on X at any level $\beta < \alpha$, $\beta \in \mathbb{R}$.*

Proof By contradiction, let us suppose that there exists $\beta < \alpha$ such that the weak intersection property does not hold for φ_1, φ_2 at the level β. Then there exist $t \in [0, 1]$ and $x_1, x_2 \in X$ such that

$$x_1 \in Z_{\beta,t}(\varphi_1, \varphi_2) \cap Z_\beta(\varphi_1), \quad x_2 \in Z_{\beta,t}(\varphi_1, \varphi_2) \cap Z_\beta(\varphi_2).$$

Consequently,

$$t\varphi_1(x_1) + (1 - t)\varphi_2(x_1) < \beta, \quad \varphi_1(x_1) < \beta,$$

and

$$t\varphi_1(x_2) + (1 - t)\varphi_2(x_2) < \beta, \quad \varphi_2(x_2) < \beta.$$

By the inequality $\beta < \alpha$, we get $x_1 \in Z_{\alpha,t}(\varphi_1, \varphi_2) \cap Z_\alpha(\varphi_1)$ and $x_2 \in Z_{\alpha,t}(\varphi_1, \varphi_2) \cap Z_\alpha(\varphi_2)$, which means that the weak intersection property does not hold for φ_1, φ_2 at the level α, a contradiction. □

The weak intersection property generalizes the concept of the intersection property introduced in [3].

Definition 3 (Definition 6, [3]) Let $\varphi_1, \varphi_2 : X \to \mathbb{R}$ be two real-valued functions defined on X and let $\alpha \in \mathbb{R}$. We say that *the intersection property* holds for φ_1 and φ_2 at the level α if $Z_\alpha(\varphi_1) \neq \emptyset$, $Z_\alpha(\varphi_2) \neq \emptyset$, $Z_\alpha(\varphi_1) \cap Z_\alpha(\varphi_2) = \emptyset$ and for every $x_1 \in Z_\alpha(\varphi_1)$ and $x_2 \in Z_\alpha(\varphi_2)$, we have

$$(\alpha - \varphi_1(x_2))(\alpha - \varphi_2(x_1)) \geq (\alpha - \varphi_1(x_1))(\alpha - \varphi_2(x_2)). \tag{3}$$

The following proposition shows that the intersection property implies the weak intersection property.

PROPOSITION 2.2 *Let $\varphi_1, \varphi_2 : X \to \mathbb{R}$, $\alpha \in \mathbb{R}$. If φ_1 and φ_2 posses the intersection property at the level α, then φ_1 and φ_2 posses the weak intersection property at the level α.*

Proof By contradiction, let us suppose that the weak intersection property at the level α does not hold for φ_1 and φ_2. There exist $t \in [0, 1]$ and $x_1, x_2 \in X$ such that

$$x_1 \in Z_{\alpha,t}(\varphi_1, \varphi_2) \cap Z_\alpha(\varphi_1), \quad x_2 \in Z_{\alpha,t}(\varphi_1, \varphi_2) \cap Z_\alpha(\varphi_2).$$

Hence, $Z_\alpha(\varphi_1) \neq \emptyset$ and $Z_\alpha(\varphi_2) \neq \emptyset$. According to Remark 1, it is enough to consider the case $Z_\alpha(\varphi_1) \cap Z_\alpha(\varphi_2) = \emptyset$. Then

$$t\varphi_1(x_1) + (1-t)\varphi_2(x_1) < \alpha, \quad \varphi_1(x_1) < \alpha, \quad \varphi_2(x_1) \geq \alpha$$

and

$$t\varphi_1(x_2) + (1-t)\varphi_2(x_2) < \alpha, \quad \varphi_1(x_2) \geq \alpha, \quad \varphi_2(x_2) < \alpha.$$

Hence,

$$\frac{\alpha - \varphi_2(x_1)}{\varphi_1(x_1) - \alpha} < \frac{t}{1-t} < \frac{\alpha - \varphi_2(x_2)}{\varphi_1(x_2) - \alpha},$$

which is equivalent to

$$(\alpha - \varphi_1(x_2))(\alpha - \varphi_2(x_1)) < (\alpha - \varphi_1(x_1))(\alpha - \varphi_2(x_2))$$

and we get a contradiction with (3) which shows that the intersection property does not hold for φ_1 and φ_2 at the level α. □

The following proposition shows that if the strict lower level sets of given functions φ_1 and φ_2 are nonvoid at some level $\alpha \in \mathbb{R}$, then the weak intersection property and the intersection property at the level $\alpha \in \mathbb{R}$ are equivalent.

Proposition 2.3 *Let $\varphi_1, \varphi_2 : X \to \mathbb{R}$, $\alpha \in \mathbb{R}$ and $Z_\alpha(\varphi_1) \neq \emptyset$, $Z_\alpha(\varphi_2) \neq \emptyset$. If φ_1 and φ_2 posses the weak intersection property at the level α, then φ_1 and φ_2 posses the intersection property at the level α.*

Proof By contradiction, if the intersection property does not hold at the level α, then at least one of the condition holds:

(a) $Z_\alpha(\varphi_1) \cap Z_\alpha(\varphi_2) \neq \emptyset$,
(b) there exist $x_1 \in Z_\alpha(\varphi_1)$ and $x_2 \in Z_\alpha(\varphi_2)$ such that

$$(\alpha - \varphi_1(x_2))(\alpha - \varphi_2(x_1)) < (\alpha - \varphi_1(x_1))(\alpha - \varphi_2(x_2)). \tag{4}$$

If (a) holds, then φ_1 and φ_2 do not possess the weak intersection property at the level α (see the Remark 1 above).

If (b) holds, then it must be $\varphi_2(x_1) \geq \alpha$ and $\varphi_1(x_2) \geq \alpha$. From (4), we get

$$(\varphi_2(x_2) - \alpha)(\varphi_2(x_1) - \varphi_1(x_1)) < (\varphi_2(x_1) - \alpha)(\varphi_2(x_2) - \varphi_1(x_2)),$$

which is equivalent to

$$0 \leq \frac{\varphi_2(x_1) - \alpha}{\varphi_2(x_1) - \varphi_1(x_1)} < \frac{\varphi_2(x_2) - \alpha}{\varphi_2(x_2) - \varphi_1(x_2)} \leq 1.$$

Hence, there exists $t \in [0, 1]$ such that

$$\frac{\varphi_2(x_1) - \alpha}{\varphi_2(x_1) - \varphi_1(x_1)} < t < \frac{\varphi_2(x_2) - \alpha}{\varphi_2(x_2) - \varphi_1(x_2)}.$$

So, we get following inequalities

$$t\varphi_1(x_1) + (1-t)\varphi_2(x_1) < \alpha, \quad t\varphi_1(x_2) + (1-t)\varphi_2(x_2) < \alpha.$$

Hence,

$$x_1 \in Z_{\alpha,t}(\varphi_1, \varphi_2) \cap Z_\alpha(\varphi_1), \quad x_2 \in Z_{\alpha,t}(\varphi_1, \varphi_2) \cap Z_\alpha(\varphi_2)$$

which is in contradiction to the weak intersection property of φ_1 and φ_2 at the level α. □

Now, using the weak intersection property, we introduce the weak Φ-intersection property for Φ-convex functions.

Definition 4 Let $f, g \in H(\Phi, X)$ and $\alpha \in \mathbb{R}$. We say that *the weak Φ-intersection property* holds for f and g at the level α if there exist $\varphi_1 \in \text{supp}(f)$ and $\varphi_2 \in \text{supp}(g)$ such that the weak intersection property holds for φ_1 and φ_2 at the level α.

In a similar way, the intersection property (Definition 3) has been used in [3] to define the Φ-intersection property.

Definition 5 ([3], Definition 6) Let $f, g \in H(\Phi, X)$ and $\alpha \in \mathbb{R}$. We say that *Φ-intersection property* holds for f and g at the level $\alpha \in \mathbb{R}$ if there exist $\varphi_f \in \text{supp}(f)$, $\varphi_g \in \text{supp}(g)$ such that the intersection property holds for φ_f and φ_g at the level $\alpha \in \mathbb{R}$.

In view of Propositions 2.2 and 2.3, for given Φ-convex functions f and g if there exist $\varphi_f \in \text{supp}(f)$, $\varphi_g \in \text{supp}(g)$ such that $Z_\alpha(\varphi_f) \neq \emptyset$ and $Z_\alpha(\varphi_g) \neq \emptyset$ at some level α and the weak intersection property holds for φ_f and φ_g, then the Φ-intersection property and the weak Φ-intersection property are equivalent at the level α.

3. Main result

Let X and Y be given sets and let $a : Y \times X \to \mathbb{R}$ be a function.

We use the following notation:

$$a_* := \sup_{y \in Y} \inf_{x \in X} a(y, x), \quad a^* := \inf_{x \in X} \sup_{y \in Y} a(y, x),$$

for any $y_1, \ y_2 \in Y$, we write

$$\text{supp}_1 := \text{supp}\,(a(y_1, \cdot)), \quad \text{supp}_2 := \text{supp}\,(a(y_2, \cdot)),$$

if y_1 and y_2 are clear from the context.

The inequality $a_* \leq a^*$ always holds. To formulate our minimax theorem, we need the following definition introduced in [9].

Definition 6 (Definition 1, [9]) We say that the family $(a(\cdot, x))_{x \in X}$ is *weakly concavelike on* Y if for every $y_1, y_2 \in Y$ and $t \in [0, 1]$, we have

$$\sup_{y \in Y} \inf_{x \in X} a(y, x) \geq \inf_{x \in X} \{ta(y_1, x) + (1 - t)a(y_2, x)\}.$$

Let us note that if for all $x \in X$, the functions $a(\cdot, x) : Y \to \mathbb{R}$ are concave on Y, then the family $(a(\cdot, x))_{x \in X}$ is weakly concavelike on Y.

Now, we are in a position to give our main result.

THEOREM 3.1 *Let X and Y be nonempty sets and $a : Y \times X \to \mathbb{R}$ be a function. Let Φ be a class of functions and for any $y \in Y$ the function $a(y, \cdot) : X \to \mathbb{R}$ be Φ-convex on X. Assume that*

(i) *for every $\alpha \in \mathbb{R}$, $\alpha < a^*$ there exist $y_1, y_2 \in Y$ such that the functions $a(y_1, \cdot)$ and $a(y_2, \cdot)$ have the weak Φ-intersection property on X at the level α,*
(ii) *the family $(a(\cdot, x))_{x\in X}$ is weakly concavelike on Y.*

Then

$$a^* = a_*.$$

Proof Take any $\alpha \in \mathbb{R}$ such that $\alpha < a^*$. By (i), there exist y_1, y_2 such that the functions $a(y_1, \cdot)$ and $a(y_2, \cdot)$ have the weak Φ-intersection property on X at the level α. We show that there exists $t_0 \in [0, 1]$ such that

$$t_0 a(y_1, x) + (1 - t_0)a(y_2, x) \geq \alpha \quad \text{for all} \quad x \in X. \tag{5}$$

By the weak Φ-intersection property of $a(y_1, \cdot)$ and $a(y_2, \cdot)$, there exist $\varphi_1 \in \text{supp}_1$ and $\varphi_2 \in \text{supp}_2$ such that for every $t \in [0, 1]$

$$Z_{\alpha,t}(\varphi_1, \varphi_2) \cap Z_\alpha(\varphi_1) = \emptyset \quad \text{or} \quad Z_{\alpha,t}(\varphi_1, \varphi_2) \cap Z_\alpha(\varphi_2) = \emptyset. \tag{6}$$

Obviously, $Z_{\alpha,t}(\varphi_1, \varphi_2) \subseteq Z_\alpha(\varphi_1) \cup Z_\alpha(\varphi_2)$ for all $t \in [0, 1]$.

If $Z_\alpha(\varphi_1) = \emptyset$, then $\varphi_1(x) \geq \alpha$ for all $x \in X$ and $a(y_1, x) \geq \varphi_1(x) \geq \alpha$ for all $x \in X$ and (5) holds with $t_0 = 1$.

If $Z_\alpha(\varphi_2) = \emptyset$, then $\varphi_2(x) \geq \alpha$ for all $x \in X$ and $a(y_2, x) \geq \varphi_2(x) \geq \alpha$ for all $x \in X$ and (5) holds with $t_0 = 0$.

Denote

$$T_i := \{t \in [0, 1] \ : \exists x \in Z_\alpha(\varphi_i) \text{ s.t. } t\varphi_1(x) + (1 - t)\varphi_2(x) < \alpha\}, \quad i = 1, 2.$$

Suppose that both $Z_\alpha(\varphi_1)$ and $Z_\alpha(\varphi_2)$ are nonempty. Then, T_1 and T_2 are nonempty sets. Since they are open in $[0, 1]$ and disjoint (by (6)), we have $[0, 1] \setminus (T_1 \cup T_2) \neq \emptyset$. For any $t_0 \in [0, 1] \setminus (T_1 \cup T_2)$, we have

$$t_0\varphi_1(x) + (1 - t_0)\varphi_2(x) \geq \alpha \quad \text{for all} \quad x \in X. \tag{7}$$

Since $\varphi_1 \in \text{supp}_1$ and $\varphi_2 \in \text{supp}_2$, we have

$$t_0 a(y_1, x) + (1 - t_0)a(y_2, x) \geq t_0\varphi_1(x) + (1 - t_0)\varphi_2(x) \geq \alpha \quad \text{for all} \quad x \in X$$

which proves (5).

Hence, the following inequality holds

$$\inf_{x\in X}\{t_0 a(y_1, x) + (1 - t_0)a(y_2, x)\} \geq \alpha$$

and by weak concavelikeness of the family $(a(\cdot, x))_{x\in X}$, we get

$$\sup_{y\in Y} \inf_{x\in X} a(y, x) \geq \inf_{x\in X}\{t_0 a(y_1, x) + (1 - t_0)a(y_2, x)\} \geq \alpha.$$

Then, the inequality

$$a_* = \sup_{y \in Y} \inf_{x \in X} a(y, x) \geq \alpha.$$

holds for every $\alpha < a^*$. Then $a_* \geq a^*$. □

Let us note that in the above theorem, we do not need any compactness assumption, neither of X nor of Y. We also do not require any kind of continuity of function $a(y, x)$.

One can observe that the assumptions (i) and (ii) are not of the same nature (like e.g. (i) convexlikeness of $a(y, \cdot)$, $y \in Y$ and (ii) concavelikeness of $a(\cdot, x)$, $x \in X$ in the theorem by Fan [4] or (i) t-convexlikeness of $a(y, \cdot)$, $y \in Y$ and (ii) t-concavelikeness of $a(\cdot, x)$, $x \in X$ in the theorem by König [5]). This follows from fact that the Φ-convexity is imposed only on the functions $a(y, \cdot)$, $y \in Y$.

Let us note that condition (i) of Theorem 3.1 is in fact a requirement on function $a(y, \cdot)$, $y \in Y$. In view of this, we can make the following remark.

Remark 2 We can say that $a : Y \times X \to \mathbb{R}$ has the weak Φ-intersection property on X if

(1) for any $y \in Y$, the function $a(y, \cdot) : X \to \mathbb{R}$ is Φ-convex on X,
(2) for every $\alpha < a^*$, there exist $y_1, y_2 \in Y$ such that the functions $a(y_1, \cdot)$ and $a(y_2, \cdot)$ have the weak Φ-intersection property on X at the level α.

4. Necessary conditions

In order to prove the necessity of conditions (i) and (ii) of Theorem 3.1, we use the following well-known relations.

(1) $a_* = a^*$ if and only if for every $\alpha < a^*$ the set

$$Y_\alpha(X) := \bigcap_{x \in X} Y_\alpha(x) = \bigcap_{x \in X} \{y \in Y \ : \ a(y, x) \geq \alpha\} \neq \emptyset.$$

(2) $a_* = a^*$ if and only if for every $\alpha > a_*$ the set

$$X_\alpha(Y) := \bigcap_{y \in Y} X_\alpha(y) = \bigcap_{y \in Y} \{x \in X \ : \ a(y, x) \leq \alpha\} \neq \emptyset.$$

Let Φ be a set of functions $\varphi : X \to \mathbb{R}$. In the sequel, we assume that all constant functions with values less than a^* are in Φ, i.e.

$$\text{if } \varphi := \alpha, \ \text{ where } \ \alpha < a^*, \ \text{ then } \ \varphi \in \Phi. \tag{8}$$

Clearly, condition (8) is satisfied if Φ contains all the constant functions defined on X.

THEOREM 4.1 *Let X and Y be nonempty sets and $a : Y \times X \to \mathbb{R}$ be a function. Let Φ be a set of functions $\varphi : X \to \mathbb{R}$. Assume that Φ satisfies condition (8) and for any $y \in Y$, the function $a(y, \cdot) : X \to \mathbb{R}$ is Φ-convex on X. If*

$$a^* = a_*.$$

Then

(i) *for every $\alpha < a^*$, there exist $y_1, y_2 \in Y$ such that the functions $a(y_1, \cdot)$ and $a(y_2, \cdot)$ have the weak Φ-intersection property on X at the level α,*
(ii) *the family $(a(\cdot, x))_{x \in X}$ is weakly concavelike on Y.*

Proof

(i) Pick $\alpha < a^*$. By the equality $a_* = a^*$, the set $Y_\alpha(X)$ is nonempty. $a(\bar{y}, x) \geq \alpha$. By condition (8), $\bar{\varphi} := \alpha$ belongs to supp $(a(\bar{y}, \cdot))$. Since $Z_\alpha(\bar{\varphi}) = \{x \in X \ : \ \bar{\varphi}(x) < \alpha\} = \emptyset$, we have

$$Z_{\alpha,t}(\bar{\varphi}, \varphi) \cap Z_\alpha(\bar{\varphi}) = \emptyset \text{ for every } \varphi \in \Phi.$$

Hence, for any $y \in Y \backslash \{\bar{y}\}$, the functions $a(\bar{y}, \cdot)$ and $a(y, \cdot)$ have the weak Φ-intersection property at the level α.

(ii) Pick $y_1, y_2 \in Y$. By the equality $a_* = a^*$, for any $\alpha > a_*$, the set $X_\alpha(Y)$ is nonempty, i.e. there exists $\bar{x} \in X$ such that for every $y \in Y$, we have $a(y, \bar{x}) \leq \alpha$. Hence,

$$ta(y_1, \bar{x}) + (1 - t)a(y_2, \bar{x}) \leq \alpha,$$

for every $t \in [0, 1]$, and

$$\inf_{x \in X_\alpha(Y)} \{ta(y_1, x) + (1 - t)a(y_2, x)\} \leq \alpha.$$

Obviously $X_\alpha(Y) \subseteq X$, therefore

$$\inf_{x \in X} \{ta(y_1, x) + (1 - t)a(y_2, x)\} \leq \inf_{x \in X_\alpha(Y)} \{ta(y_1, x) + (1 - t)a(y_2, x)\} \leq \alpha.$$

Hence

$$\inf_{x \in X} \{ta(y_1, x) + (1 - t)a(y_2, x)\} \leq a_* = \sup_{y \in Y} \inf_{x \in X} a(y, x),$$

which means, that the family $(a(\cdot, x))_{x \in X}$ is weakly concavelike on Y. □

Remark 3 By Theorem 4.1 and Remark 2 if the minimax equality holds for the function a such that $a(y, \cdot)$ is Φ_0-convex for any $y \in Y$, then the function $a(y, \cdot)$, $y \in Y$ has the weak Φ-intersection property for any class Φ containing constant functions and such that $a(y, \cdot)$, $y \in Y$ is Φ-convex.

5. Joint Φ-convexlikeness

As we have shown, the weak Φ-intersection property plays a crucial role in the results of Sections 3 and 4. According to Theorems 3.1 and 4.1, the weak Φ-intersection property introduced in Section 2 is not only sufficient but also a necessary condition for the minimax inequality to hold.

However, this concept seems to be technically involved. To investigate this idea more closely, we now consider a stronger condition, namely joint Φ-convexlikeness (Definition 5, [3]), which is a kind of convexlikeness for Φ-convex functions. If the class Φ is convexlike on X in the sense of [4], then functions from Φ are jointly convexlike on X.

In the present section, we recall the definition of joint Φ-convexlikeness of functions and investigate relationships between the joint Φ-convexlikeness and the weak Φ-intersection property.

Definition 7 Let $\varphi_1, \varphi_2 : X \to \mathbb{R}$ be two real-valued functions defined on X. We say that φ_1 and φ_2 are *jointly convexlike on* X if for every $x_1, x_2 \in X$ and $t \in [0, 1]$ there exists $x_0 \in X$ such that

$$\max\{\varphi_1(x_0),\ \varphi_2(x_0)\} \le \max\{t\varphi_1(x_1) + (1-t)\varphi_1(x_2),\ t\varphi_2(x_1) + (1-t)\varphi_2(x_2)\}. \quad (9)$$

The following proposition provides a relationship between the weak intersection property and the joint convexlikeness.

Proposition 5.1 *Let $\varphi_1, \varphi_2 : X \to \mathbb{R}$, $\alpha \in \mathbb{R}$ and $Z_\alpha(\varphi_1) \cap Z_\alpha(\varphi_2) = \emptyset$. If φ_1 and φ_2 are jointly convexlike on X, they have the weak intersection property on X at the level α.*

Proof We proceed by contradiction. Suppose $Z_\alpha(\varphi_1) \cap Z_\alpha(\varphi_2) = \emptyset$ but φ_1 and φ_2 do not have intersection property at the level α on X. Then there exists $t \in [0, 1]$ such that there exist $x_1, x_2 \in X$ $x_1 \in Z_{\alpha,t}(\varphi_1, \varphi_2) \cap Z_\alpha(\varphi_1)$ and $x_2 \in Z_{\alpha,t}(\varphi_1, \varphi_2) \cap Z_\alpha(\varphi_2)$.

Then, we have the following situation

$$t\varphi_1(x_1) + (1-t)\varphi_2(x_1) < \alpha,\ \ \varphi_1(x_1) < \alpha,\ \varphi_2(x_1) \ge \alpha$$

and

$$t\varphi_1(x_2) + (1-t)\varphi_2(x_2) < \alpha,\ \ \varphi_2(x_2) < \alpha,\ \varphi_1(x_2) \ge \alpha.$$

Hence,

$$\frac{\varphi_1(x_2) - \alpha}{\varphi_1(x_2) - \varphi_1(x_1)} < \frac{\varphi_2(x_2) - \alpha}{\varphi_2(x_2) - \varphi_2(x_1)}. \quad (10)$$

So, there exists $t \in \left(\frac{\varphi_1(x_2)-\alpha}{\varphi_1(x_2)-\varphi_1(x_1)}, \frac{\varphi_2(x_2)-\alpha}{\varphi_2(x_2)-\varphi_2(x_1)}\right)$. Then (10) implies that

$$t\varphi_1(x_1) + (1-t)\varphi_1(x_2) < \alpha,$$

and

$$t\varphi_2(x_1) + (1-t)\varphi_2(x_2) < \alpha.$$

From the assumption that φ_1, φ_2 are jointly convexlike on X, there exists $x_0 \in X$ such that

$$\max\{\varphi_1(x_0),\ \varphi_2(x_0)\} \le \max\{t_0\varphi_1(x_1) + (1-t_0)\varphi_1(x_2),\ t_0\varphi_2(x_1) + (1-t_0)\varphi_2(x_2)\}.$$

Hence,

$$\max\{\varphi_1(x_0),\ \varphi_2(x_0)\} < \alpha$$

which means that $\varphi_1(x_0) < \alpha$ and $\varphi_2(x_0) < \alpha$. Hence, $x_0 \in Z_\alpha(\varphi_1) \cap Z_\alpha(\varphi_2)$ which is in contradiction to our assumption that $Z_\alpha(\varphi_1) \cap Z_\alpha(\varphi_2) = \emptyset$. □

Now, we recall from [3] the concept of joint Φ-convexlikeness which is shaped for Φ-convex functions.

Definition 8 (Definition 5, [3]) Let $f, g \in H(\Phi, X)$. We say that f and g are *jointly Φ-convexlike on X* if every two $\varphi_1,\ \varphi_2 \in \Phi$, $\varphi_1 \in \operatorname{supp}(f)$, $\varphi_2 \in \operatorname{supp}(g)$ are jointly convexlike on X.

It is an immediate consequence of Proposition 5.1 that if Φ-convex functions f and g are jointly Φ-convexlike on X and for given level $\alpha \in \mathbb{R}$, there exist $\varphi_f \in \operatorname{supp}(f)$, $\varphi_g \in \operatorname{supp}(g)$ such that

$$Z_\alpha(\varphi_f) \cap Z_\alpha(\varphi_g) = \emptyset,$$

then the weak Φ-intersection property holds for f and g at the level α.

In view of this, we can formulate sufficient conditions for the minimax equality for jointly Φ-convexlike functions.

Theorem 5.2 *Let X and Y be nonempty sets and $a : Y \times X \to \mathbb{R}$ be a function. Let Φ be a class of functions and for any $y \in Y$ the function $a(y, \cdot) : X \to \mathbb{R}$ be Φ-convex on X. Assume that*

(i) *for every $\alpha \in \mathbb{R}$, $\alpha < a^*$, there exist $y_1, y_2 \in Y$ such that $a(y_1, \cdot)$ and $a(y_2, \cdot)$ are jointly Φ-convexlike on X and there exist $\varphi_1 \in supp_1$ and $\varphi_2 \in supp_2$ with the property $Z_\alpha(\varphi_1) \cap Z_\alpha(\varphi_2) = \emptyset$,*
(ii) *the family $(a(\cdot, x))_{x \in X}$ is weakly concavelike on Y.*

Then

$$a^* = a_*.$$

Proof Follows from Proposition 5.1 and Theorem 3.1. □

6. Examples

Following [3], we say that the class Φ is *jointly convexlike on X* if every $\varphi_1, \varphi_2 \in \Phi$ are jointly convexlike on X. To illustrate Theorem 5.2 in this section, we give example of jointly convexlike class Φ of functions. We also present a class Φ which is not jointly convexlike.

To present our examples, we use the concept of supremal generators, introduced in Definition 6.1 of [2] (supremal generators where discussed also in [10]).

Definition 9 Let Q be a set of functions $f : X \to \mathbb{R}$. The class Φ is called a *supremal generator* of the set Q if every $f \in Q$ is Φ-convex.

6.1. *Convex lower semicontinuous functions*

As a first example, we present a supremal generator of convex (in the usual sense) lower semicontinuous functions.

Let X be a locally convex Hausdorff topological vector space. Let Φ_{conv} be the set of all continuous affine functions defined on X. It is a well-known fact (see, for example, Proposition 3.1 of [11]) that the set Φ_{conv} is a supremal generator of convex lower semicontinuous functions $f : X \to \mathbb{R}$.

It is easy to see that the set Φ_{conv} is jointly convexlike on X; therefore, we can formulate the following corollary.

Corollary 6.1 *Let X be a locally convex Hausdorff topological vector space, Y be a nonempty set, $a : Y \times X \to \mathbb{R}$ be a function and $\alpha \in \mathbb{R}$. Assume that for any $y \in Y$, the function $a(y, \cdot) : X \to \mathbb{R}$ is convex lower semicontinuous on X. If*

(i) *for every $\alpha < a^*$, there exist $y_1, y_2 \in Y$ such that there exist $\varphi_1 \in supp_1$ and $\varphi_2 \in supp_2$ such that $Z_\alpha(\varphi_1) \cap Z_\alpha(\varphi_2) = \emptyset$,*
(ii) *the family $(a(\cdot, x))_{x\in X}$ is weakly concavelike on Y.*

Then

$$a^* = a_*.$$

We show that condition (i) of Corollary 6.1 holds if X is a compact subset of some topological space $a(\cdot, x)$ is lower semicontinuous convex on X for every $y \in Y$.

Theorem 6.2 *Let X be a nonempty convex compact subset of a vector topological space and let Y be a nonempty convex subset of a vector space. Let $a : Y \times X \to \mathbb{R}$ be such that*

(i) *$a(y, \cdot)$ is convex and lower semicontinuous on X for all $y \in Y$,*
(ii) *$a(\cdot, x)$ is concave on Y for all $x \in X$.*

Then for every $\alpha < a^*$, there exist $y_1, y_2 \in Y$ such that there exist $\varphi_1 \in supp_1$ and $\varphi_2 \in supp_2$ such that

$$Z_\alpha(\varphi_1) \cap Z_\alpha(\varphi_2) = \emptyset.$$

Proof The proof consists of two steps.

Step 1 Let us notice that all the assumptions of the Theorem 2 of [4] are satisfied, so we can use the following fact proved in part (iii) of the proof of Theorem 2 in [4]: let $\alpha \in \mathbb{R}$, for all $n \in \mathbb{N}$ and arbitrary finite subset $Y_n = \{y_1, ..., y_n\}$ of Y such that

$$\bigcap_{i=1}^{n} [a(y_i, \cdot) \leq \alpha] = \emptyset,$$

there exists $y_0 \in Y$ such that

$$a(y_0, x) > \alpha \quad \text{for all} \quad x \in X. \tag{11}$$

Let $\bar{\varphi} \equiv \alpha$. By (11), $\bar{\varphi} \in \mathrm{supp}(a(y_0, \cdot), \Phi_{conv})$ and

$$Z_\alpha(\bar{\varphi}) = \emptyset.$$

Hence,

$$Z_\alpha(\bar{\varphi}) \cap Z_\alpha(\varphi) = \emptyset \quad \text{for any} \quad \varphi \in \Phi.$$

Step 2 Condition $\alpha < a^*$ is equivalent to

$$\bigcap_{y \in Y} \{x \in X \,:\, a(y, x) \leq \alpha\} = \emptyset.$$

By the compactness of X and the lower semicontinuity of $a(y, \cdot)$ for all $y \in Y$, there exists a finite subset $\{y_1, ..., y_n\} \subset Y$ such that

$$\bigcap_{i=1}^{n} \{x \in X \,:\, a(y_i, x) \leq \alpha\} = \emptyset.$$

By Step (1), we get that there exist $\bar{y}, \tilde{y} \in Y$ and $\bar{\varphi} \in \operatorname{supp}(a(\bar{y}, \cdot), \Phi_{conv})$, $\tilde{\varphi} \in \operatorname{supp}(a(\tilde{y}, \cdot), \Phi_{conv})$ such that

$$Z_\alpha(\bar{\varphi}) \cap Z_\alpha(\tilde{\varphi}) = \emptyset.$$

□

Taking into account Corollary 6.1 and Theorem 6.2, we can formulate the following theorem.

THEOREM 6.3 *Let X be a nonempty convex compact subset of a vector topological space and let Y be a nonempty convex subset of a vector space. Let $a : Y \times X \to \mathbb{R}$ be such that*

(i) *$a(y, \cdot)$ is convex and lower semicontinuous on X for all $y \in Y$,*
(ii) *$a(\cdot, x)$ is concave on Y for all $x \in X$.*

Then

$$a^* = a_*.$$

6.2. *Nonnegative quasiconvex lower semicontinuous functions*

It may happen that the given set of functions has more than one supremal generator. Now, we give an example of two different supremal generators of the same set, such that one is jointly convexlike and other is not.

Let $X = \mathbb{R}$. Following [12], we recall the definition of quasiconvex function.

Definition 10 A function $f : \mathbb{R} \to \mathbb{R}$ is called *quasiconvex* if for every $x_1, x_2 \in \mathbb{R}$ and $t \in [0, 1]$

$$f(tx_1 + (1-t)x_2) \leq \max\{f(x_1), f(x_2)\}.$$

Below, we present a supremal generator of nonnegative quasiconvex lower semicontinuous functions.

Let $X = \mathbb{R}$ and Φ_{quasi_1} be a class of functions $\varphi : \mathbb{R} \to [0, +\infty)$ given by the formula

$$\varphi(x) := \begin{cases} c & vx \geq d \\ 0 & vx < d \end{cases},$$

where $v \in \{1, -1\}$, $c \geq 0$, $d \in \mathbb{R}$.

As was shown in [12], the set Φ_{quasi_1} is a supremal generator of nonnegative quasiconvex lower semicontinuous functions.

It was proved in [3] that the class Φ_{quasi_1} is jointly convexlike on X; therefore, we can formulate the following corollary.

COROLLARY 6.4 *Let Y be a nonempty set and $a : Y \times \mathbb{R} \to \mathbb{R}$ be a function. Assume that for any $y \in Y$ the function $a(y, \cdot) : \mathbb{R} \to \mathbb{R}$ is nonnegative quasiconvex lower semicontinuous on X. If*

(i) *for every $\alpha \in \mathbb{R}$, $\alpha < a^*$ there exist $y_1, y_2 \in Y$ such that there exist $\varphi_1 \in supp_1$ and $\varphi_2 \in supp_2$ with the property $Z_\alpha(\varphi_1) \cap Z_\alpha(\varphi_2) = \emptyset$,*
(ii) *the family $(a(\cdot, x))_{x \in \mathbb{R}}$ is weakly concavelike on Y.*

Then

$$a^* = a_*.$$

Now, we present another example of a supremal generator of nonnegative quasiconvex lower semicontinuous functions.

Let Φ_{quasi_2} be a class of functions $\varphi : \mathbb{R} \to \mathbb{R}$ given by the formula

$$\varphi := \min\{\ell(x), c\},$$

where $c \in \mathbb{R}$ and $\ell : \mathbb{R} \to \mathbb{R}$ is linear function.

The set Φ_{quasi_2} is a supremal generator of quasiconvex lower semicontinuous functions $f : \mathbb{R} \to \mathbb{R}$, so Φ_{quasi_2} is also a supremal generator of nonnegative quasiconvex lower semicontinuous functions.

Now, we show that Φ_{quasi_2} is not jointly convexlike on $\mathbb{R}$.

Let $\varphi_1 = \min\{x + 3, 2\}$ and $\varphi_2 = \min\{-x + 2, 1\}$. Let $x_1 = -4$, $x_2 = 3$ and $t = \frac{1}{2}$. We show that there is no $x_0 \in \mathbb{R}$ such that

$$\max\{\varphi_1(x_0),\ \varphi_2(x_0)\} \le \max\{t\varphi_1(x_1) + (1 - t)\varphi_1(x_2),\ t\varphi_2(x_1) + (1 - t)\varphi_2(x_2)\}.$$

By calculating the right side of above inequality, we get

$$\max\{\varphi_1(x_0),\ \varphi_2(x_0)\} \le \frac{1}{2}.$$

Therefore, x_0 has to be a solution of the inequalities

$$\min\{x + 3, 2\} \le \frac{1}{2} \qquad \min\{-x + 2, 1\} \le \frac{1}{2}.$$

But, the set of solutions of above inequalities are disjoint, so such $x_0 \in \mathbb{R}$ does not exist.

These examples show that the set of nonnegative quasiconvex lower semicontinuous functions defined on $\mathbb{R}$ has two different supremal generators, such that only one of them is jointly convexlike. Therefore, the joint Φ-convexlikness of given Φ-convex functions depends on the choice of the class Φ.

6.3. *P-functions*

Now, we give an example of a supremal generator of so-called P-functions, which might have very complicated structure and possess a supremal generator consisting of simple and easily handle-able functions.

Let $S \subset \mathbb{R}$ be a segment.

Definition 11 (Section 3, [13]) A function $p : S \to [0, +\infty)$ is called *P_S-function* if

$$p(\lambda x + (1-\lambda)y) \leq p(x) + p(y) \quad \text{for all} \quad \lambda \in (0,1) \quad \text{and} \quad x, y \in S.$$

If $S = \mathbb{R}$, then $P_S = P$. It was shown ([2], Proposition 6.13) that if g is a bounded function defined on a segment S and c is a number large enough, then function $f(x) = g(x) + c$ is a P_S-function.

Let Φ_{P_S} be the family of functions indexed by a triplet $(u; c_1, c_2)$, where $u \in S$, $c_1 \geq 0$, $c_2 \geq 0$. For a given $(u; c_1, c_2)$, function $\varphi : S \to [0, +\infty)$ is given by the formula

$$\varphi(x) := \begin{cases} c_1 & x < u \\ c_1 + c_2 & x = u \\ c_2 & x > u \end{cases}.$$

The class Φ_{P_S} is a supremal generator of the set of all P_S-functions (Proposition 6.16, [2]) and the class Φ_{P_S} is jointly convexlike on S (Example 1, [3]); therefore, we can formulate the minimax theorem for P_S-functions.

Corollary 6.5 *Let Y be a nonempty set, $S \subset \mathbb{R}$ be a segment and $a : Y \times S \to \mathbb{R}$ be a function and $\alpha \in \mathbb{R}$. Assume that for any $y \in Y$, the function $a(y, \cdot) : S \to \mathbb{R}$ is a P_S-function. If*

(i) *for every $\alpha < a^*$, there exist $y_1, y_2 \in Y$ such that there exist $\varphi_1 \in supp_1$ and $\varphi_2 \in supp_2$ such that $\{x \in S : \varphi_1(x) < \alpha\} \cap \{x \in S : \varphi_2(x) < \alpha\} = \emptyset$,*

(ii) *the family $(a(\cdot, x))_{x \in X}$ is weakly concavelike on Y.*

Then

$$a^* = a_*.$$

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