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Robust games: theory and application to a Cournot duopoly model

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Abstract In this paper, the *robust game* model proposed by Aghassi and Bertsimas (Math Program Ser B 107:231–273, 2006) for matrix games is extended to games with a broader class of payoff functions. This is a distribution-free model of incomplete information for finite games where players adopt a robust-optimization approach to contend with payoff uncertainty. They are called *robust players* and seek the maximum guaranteed payoff given the strategy of the others. Consistently with this decision criterion, a set of strategies is an equilibrium, *robust-optimization equilibrium*, if each player's strategy is a best response to the other player's strategies, under the worst-case scenarios. The aim of the paper is twofold. In the first part, we provide robust-optimization equilibrium's existence result for a quite general class of games and we prove that it exists a suitable value ϵ such that robust-optimization equilibria are a subset of ϵ-Nash equilibria of the nominal version, i.e., without uncertainty, of the robust game. This provides a theoretical motivation for the robust approach, as it provides new insight and a rational agent motivation for ϵ-Nash equilibrium. In the last part, we propose an application of the theory to a classical Cournot duopoly model which shows significant differences between the robust game and its nominal version.

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1 Introduction

The notion of Nash equilibrium and existence theorem were proposed in Nash (1950) for non-cooperative, simultaneous-move, one-shot, finite games with complete information, i.e., all parameters of the game including players' payoff functions are common knowledge. Despite being the cornerstone in the field of game theory, Nash's equilibrium needs to be generalized to answer many real-world game-theoretic situations where often some aspects of the structure of the game, such as payoff functions, are uncertain.

To meet this practical need, the assumption of complete information is relaxed in Harsanyi (1968a, b), where the so-called Bayesian games and the notion of Bayesian equilibrium are proposed. According to this model, a player shares a common knowledge prior probability distribution over the uncertainty set of the game. Then, adopting a stochastic optimization approach, each player maximizes his/her expected payoff based on the prior probability information he/she has.

Yet, a priori knowledge of probability distribution could be too strong assumption for real-world models. Therefore, some distribution-free equilibrium concepts have been proposed. An example is the ex-post equilibrium, which is a Nash equilibrium under all possible realizations of the uncertainty set. Although it is often used in auction theory, such equilibrium cannot be applied to all games with incomplete information, because it need not exist.

An alternative to the ex-post equilibrium is the *robust-optimization* equilibrium proposed in Aghassi and Bertsimas (2006). In this seminal paper, Aghassi and Bertsimas introduce a distribution-free model of incomplete-information games where the players, called *robust players*, use a robust-optimization approach to contend with payoff uncertainty. According to robust optimization, see, e.g., Ben-Tal and Nemirovski (1998), a robust player builds its best-reply function by maximizing its guaranteed payoff, or equivalently, maximizing its *worst-case expected payoff function*. Such a robust game requires no knowledge of the probability of realization of the uncertain parameters. Then, it relaxes the assumption of Harsanyi's Bayesian games and has a broader range of applicability. The only assumption required in a robust game is the knowledge of the shape and the size of the uncertainty set, i.e., all the possible realizations of the uncertain parameters.

Considering matrix games, Aghassi and Bertsimas (2006) prove the existence of an equilibrium, the so-called robust-optimization equilibrium, for any robust game with bounded uncertainty set. In this way, robust games offer a distribution-free notion of equilibrium, alternative to and more general than the ex-post equilibrium, whose existence is guaranteed. In particular, an ex-post equilibrium is a robust-optimization equilibrium but the contrary is not true. Here, we extend the existence result for robust-optimization equilibria to robust games with a more general class of payoff functions than those considered in Aghassi and Bertsimas (2006). In addition, we show that the notion of robust-optimization equilibrium is related to that of ϵ-Nash equilibrium.

The so-called ϵ-Nash equilibrium is an alternative notion of equilibrium for non-cooperative games, less stringent than the Nash equilibrium, that has been introduced in Bubelis (1979), and subsequently popularized in various papers, see, e.g., Osborne and Rubinstein (1994) and Marinacci (1997). The existence of an ϵ-Nash equilibrium implies that players are satisfied with a less-than-optimal response to the strategies of other players. Despite the possible applications of this notion, the motivation for such a behavior is still a matter of discussion. In Radner (1980), as a plausible rational motivation for neglecting a possible gain, the likely costs of discovering and using alternative strategies are proposed. As the author proposes, the unwillingness to sustain additional costs makes a nearly optimal strategy more appealing than the truly optimal one.

In this paper, we introduce uncertainty in the parameters of the payoff functions of a nominal game. Then, we obtain an incomplete-information game and we prove that a robust-optimization equilibrium of this game is an ϵ-Nash equilibrium of the nominal one as long as payoff functions are concave w.r.t. the uncertain parameters. In doing so, we suggest that the less-than-optimal solution provides indeed a measure of the price for risk adversity of the player. Therefore, to choose an ϵ-Nash equilibrium would be the rational behavior of an agent reacting to some degree of payoff uncertainty. More generally, we introduce the concept of ϵ-*robust-optimization* equilibrium and we prove that, whenever the payoff functions are concave w.r.t. the uncertain parameters, a robust-optimization equilibrium is an ϵ-robust-optimization equilibrium of the same game but with a lower degree of uncertainty.

In the last part of the paper, we propose an application of the robust model to a Cournot duopoly game where firms (players) are uncertain about the slope of the inverse demand function. We show that the robust Cournot duopoly game is characterized by properties that are significantly different to the ones of the nominal version of the game. For example, robust-optimization equilibria can be infinitely many while the nominal version of the game has a unique Nash equilibrium. Moreover, by assuming the expected level of production of the rivals to be constant, we introduce the best-reply dynamics for the robust Cournot duopoly game. The map that describes the best-reply dynamics of the duopoly is piecewise smooth and show a larger variety of possible limit points of the quantity dynamics.

The structure of the paper is as follows. Section 2 introduces the basic concepts of a robust game and provides the existence result for the robust-optimization equilibrium. Section 3 shows that, under the concavity condition of the payoff functions w.r.t. the uncertain parameters, a robust-optimization equilibrium is an ϵ-Nash equilibrium of the nominal version of the game and, more generally, shows that a robust-optimization equilibrium is an ϵ-robust-optimization equilibrium of the same game but with a lower degree of uncertainty, where ϵ-robust-optimization equilibrium is defined as the analogous for robust games of the ϵ-Nash equilibrium for nominal games. Section 4 presents an application of the theory, specifically a robust Cournot duopoly game. Section 5 concludes. All proofs are in “Appendix”.

2 Strategic games with payoff uncertainty and robust players: robust-optimization equilibrium and its existence

As specified in Harsanyi (1967), an incomplete-information game is a game where some or all of the players lack full information about the *rules* of the game, for example strategies available to other players or even to themselves, amount of information the other players have about various aspects of the game situation, other players' or even their own payoff functions. Here, we focus on payoff functions uncertainty and we consider incomplete-information games where payoff functions depend on some parameters not known in advance. Players are aware of this uncertainty, and they have perfect knowledge of the set of all possible realizations of these parameters, the so-called *uncertainty set*. We begin this section with the notation that is used throughout the paper and the standing assumptions for our results.

More formally, we assume a finite set N of players and each player i can perform infinitely many actions $A_i \subset \mathbb{R}$, $i \in N = \{1, \dots, n\}$, affecting simultaneously his/her own payoff function and those of other players. The set of all possible strategies is denoted by $A := \Pi_{i \in N} A_i$.[1]

Assumption 1 A_i is a compact and convex set, for all $i \in N$.

In perfect-information games, we usually denote by $f_i : A \subset \mathbb{R}^n \to \mathbb{R}$, with $i \in N$, each player payoff function, recording the payoffs to the player i under all possible action profiles for the players. To comprise uncertainty in our model, we further assume that each f_i depends on a vector of parameters $\boldsymbol{\alpha}_i = \left(\alpha_i^1, \dots, \alpha_i^{v_i}\right)$. Then, we can express the argument of the payoff function as $f_i\left(\boldsymbol{\alpha}_i; x_i, \mathbf{x}_{-i}\right)$ denoting the payoff to player i, when he/she plays $x_i \in A_i$ given other players' actions $\mathbf{x}_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in A_{-i} \subset \mathbb{R}^{n-1}$ and the entries of the vector parameter $\boldsymbol{\alpha}_i \in U_i \subset \mathbb{R}^{v_i}$ are uncertain. In particular, U_i denotes the set of all possible values of parameters in f_i, while $U = \Pi_{i \in N} U_i$ denotes the entire uncertainty set of the game.

Assumption 2 $U_i \subset \mathbb{R}^{v_i}$ is a compact and convex set, for all $i \in N$.

When U is a singleton, i.e., no payoff uncertainty (games with *complete payoff information*), player's i best response to the other players' strategies $\mathbf{x}_{-i} \in A_{-i}$ belongs, by definition, to

$$\arg\max_{x_i \in A_i} f_i\left(x_i, \mathbf{x}_{-i}\right) \tag{1}$$

where, for the sake of notational simplicity, the dependence of f on the vector of parameters $\boldsymbol{\alpha}_i$ is dropped for games with complete payoff information.

Thus, a player's strategy is called a best response to other players' strategies when, given the latter, the former provides no incentive to unilaterally deviate from his/her aforementioned strategy. Consequently, a set of strategies $\left(x_1^*, \dots, x_n^*\right) \in A$ is said to be a Nash equilibrium of such nominal game if and only if, $\forall i \in N$:

$$x_i^* \in \arg\max_{x_i \in A_i} f_i\left(x_i, \mathbf{x}_{-i}^*\right) \qquad \text{(NE)} \tag{2}$$

[1] For the sake of notational simplicity, we consider $A_i \subset \mathbb{R}$. Nevertheless, the results that follow can be extended in straightforward fashion to the general case $A_i \subset \mathbb{R}^m$.

where $\mathbf{x}^*_{-i} = \left(x^*_1, \ldots, x^*_{i-1}, x^*_{i+1}, \ldots, x^*_n\right) \in A_{-i}$.

When U is not a singleton, the payoff functions f_i are subject to uncertainty (games with *incomplete payoff information*). Aghassi and Bertsimas (2006) introduced a *parametric distribution-free decision mechanism* for these games. According to this seminal paper, player's i best response to other players' strategies $\mathbf{x}_{-i} \in A_{-i}$ belongs to

$$\arg\max_{x_i \in A_i} \left[\min_{\boldsymbol{\alpha}_i \in U_i} f_i\left(\boldsymbol{\alpha}_i; x_i, \mathbf{x}_{-i}\right)\right] \tag{3}$$

Such a strategy is known as *robust* and the player is a *robust player*, see again Aghassi and Bertsimas (2006). This approach defines a conservative behavior of all players, since they act to maximize the payoff in the worst-case scenario. This *robust game* is denoted in the sequel by

$$G = \{A_i, f_i, U_i : i \in N\} \tag{4}$$

In particular, G can be defined a finite-person, non-cooperative, simultaneous-move, one-shot robust game.

Definition 1 (*Robust-optimization equilibrium*) A set of strategies $\left(x^*_1, \ldots, x^*_n\right) \in A$ is said to be a *robust-optimization equilibrium* (hereafter ROE) of strategic game G, when:

$$x^*_i \in \arg\max_{x_i \in A_i} \left[\min_{\boldsymbol{\alpha}_i \in U_i} f_i\left(\boldsymbol{\alpha}_i; x_i, \mathbf{x}^*_{-i}\right)\right] \quad \forall i \in N \qquad \text{(ROE)} \tag{5}$$

Remark 1 ROE provides an alternative distribution-free equilibrium concept to that of the ex-post equilibrium introduced in Crémer and McLean (1985). For details see, e.g., Aghassi and Bertsimas (2006).

The proof of the existence of a ROE requires to introduce the *worst-case expected payoff functions*, given by

$$\rho_i\left(x_i, \mathbf{x}_{-i}\right) \triangleq \min_{\boldsymbol{\alpha}_i \in U_i} f_i\left(\boldsymbol{\alpha}_i; x_i, \mathbf{x}_{-i}\right) \tag{6}$$

with $i \in N$.

Remark 2 If we assume f_i are continuous w.r.t. $\boldsymbol{\alpha}_i \in U_i\ \forall\left(x_i, \mathbf{x}_{-i}\right) \in \mathbb{R}^n$, then by Weierstrass' Theorem and Assumption 2 the existence of a global minimizer of f_i w.r.t. $\boldsymbol{\alpha}_i$ is ensured $\forall\left(x_i, \mathbf{x}_{-i}\right) \in \mathbb{R}^n$. Thus, ρ_i is completely defined and with finite values.

Nevertheless, continuity of f_i w.r.t. α_i is only a sufficient condition to guarantee that ρ_i is defined. In fact, in Remark 2 the assumption of continuity can be weakened to lower semicontinuity that ρ_i are still defined.

Assumption 3 $\rho_i : A \to \mathbb{R}$ is continuous, and $\rho_i\left(\cdot, \mathbf{x}_{-i}\right) \in \mathbb{R}^n$ is concave for every $\mathbf{x}_{-i} \in A_{-i}$.

It is well known that the concavity of ρ_i w.r.t. $x_i \in A_i$ can be easily ensured by assuming the concavity of f_i w.r.t. $x_i \in A_i$.

Lemma 1 *f_i concave in x_i, $\forall \boldsymbol{\alpha}_i \in U_i$, is a sufficient condition to have the concavity of ρ_i w.r.t. $x_i \in A_i$.*

Since a ROE of the robust game G is a Nash equilibrium of a nominal game, i.e., a game without payoff uncertainty, with the same number of players, the same action space of G and payoff functions ρ_i, we can prove (see "Appendix") the following existence result.

Theorem 1 *Under Assumptions 1, 2 and 3, any finite-person, non-cooperative, simultaneous-move, one-shot robust game G, in which there is no private information, has a ROE.*

To complete the section, we illustrate ROE in the following example. One can easily notice that existence of solution is guaranteed by Theorem 1.

Example 1 Let us consider the following robust game in matrix form:

		Player 2	
	Strategies	s_1^2	s_2^2
Player 1	s_1^1	(α_1, α_2)	(1,0)
	s_2^1	(2,0)	(0,1)

in which the payoff uncertainty set for players 1 and 2 is, respectively

$$U_1 = \left\{ \begin{bmatrix} \alpha_1 & 1 \\ 2 & 0 \end{bmatrix} \middle| \alpha_1 \in [1, 2] \right\} \quad \text{and} \quad U_2 = \left\{ \begin{bmatrix} \alpha_2 & 0 \\ 0 & 1 \end{bmatrix} \middle| \alpha_2 \in [1, 3] \right\} \tag{7}$$

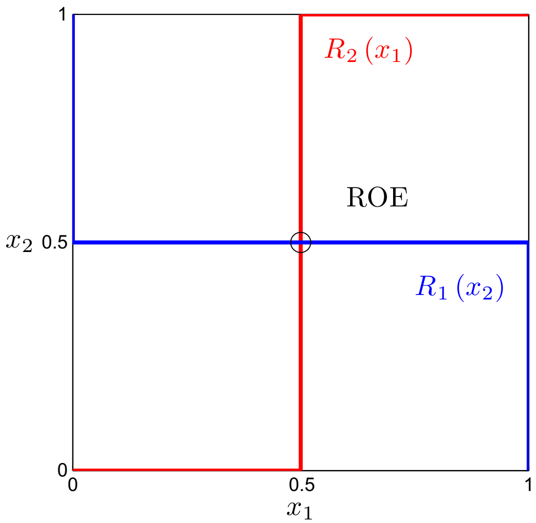
Let us consider mixed strategies and let us denote by x_i the probability that player i plays strategy s_1^i. Then player i has as strategy space the set of real numbers $A_i = [0, 1]$, with $i = 1, 2$, and the expected payoff functions of the two players are, respectively,

$$\begin{aligned} f_1(\alpha_1; x_1, x_2) &= [(\alpha_1 - 3)\, x_2 + 1]\, x_1 + 2x_2 \quad \text{and} \\ f_2(\alpha_2; x_1, x_2) &= [(\alpha_2 + 1)\, x_1 - 1]\, x_2 + 1 - x_1 \end{aligned} \tag{8}$$

Let us denote this robust game by $G = \{A_i, f_i, U_i : i \in \{1, 2\}\}$. Assuming that the players behave as robust optimizers, they maximize their worst-case expected payoff functions:

$$\rho_1(x_1, x_2) = \min_{\alpha_1 \in [1,2]} f_1(\alpha_1; x_1, x_2) = [1 - 2x_2]\, x_1 + 2x_2 \tag{9}$$

Fig. 1 Worst-case best-reply function of player 1, $R_1(x_2)$, in blue and worst-case best-reply function of player 2, $R_2(x_1)$, in red. The ROE is marked with an empty dot and is the unique intersection point of the two worst-case best-reply functions



and

$$\rho_2(x_1, x_2) = \min_{\alpha_2 \in [1,3]} f_2(\alpha_2; x_1, x_2) = [2x_1 - 1]\, x_2 + 1 - x_1 \tag{10}$$

from which we obtain the following *worst-case best-reply functions*, $R_1(x_2) = \arg\max_{x_1 \in A_1} \rho_1(x_1, x_2)$ and $R_2(x_1) = \arg\max_{x_2 \in A_2} \rho_2(x_1, x_2)$, given by

$$R_1(x_2) \begin{cases} 0 & \text{if } \frac{1}{2} < x_2 \leq 1 \\ [0, 1] & \text{if } x_2 = \frac{1}{2} \\ 1 & \text{if } 0 \leq x_2 < \frac{1}{2} \end{cases} \quad \text{and} \quad R_2(x_1) = \begin{cases} 0 & \text{if } 0 \leq x_1 < \frac{1}{2} \\ [0, 1] & \text{if } x_1 = \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x_1 \leq 1 \end{cases} \tag{11}$$

Hence, the unique ROE of the game G is $\left(\frac{1}{2}, \frac{1}{2}\right)$, see the intersection point of the two worst-case best-reply functions in Fig. 1.

3 Robust-optimization equilibria as a subset of ϵ-Nash equilibria

In complete payoff information games, the notion of ϵ-Nash equilibrium has been proposed as a generalization of Nash equilibrium, see, e.g., Bubelis (1979), Osborne and Rubinstein (1994) and Marinacci (1997):

Definition 2 (ϵ-*Nash equilibrium*) A set of strategies $\left(x_1^*, \ldots, x_n^*\right) \in A$ is said to be an ϵ-Nash equilibrium of strategic game G defined in (4) without uncertainty (i.e., when U is a singleton) if and only if:

$$f_i\left(x_i^*, \mathbf{x}_{-i}^*\right) \geq f_i\left(x_i, \mathbf{x}_{-i}^*\right) - \epsilon \quad \forall x_i \in A_i, \forall i \in N \qquad (\epsilon\text{-NE}) \tag{12}$$

The basic idea behind the notion of ϵ-Nash equilibrium is that a player accepts to play a strategy that is not optimal with respect to Nash definition, yet he/she will not deviate unless the payoff improvement is greater than ϵ. Clearly, an ϵ-Nash equilibrium provides a weaker concept of equilibrium in games and its existence can be proved also when a Nash equilibrium does not exist. In particular, a Nash equilibrium is an ϵ-Nash equilibrium but the contrary does not hold true and, under continuity of the payoff functions, a set of strategies is a Nash equilibrium if and only if it is an ϵ-Nash equilibrium for every $\epsilon > 0$. Despite this, the payoffs in an ϵ-Nash equilibrium can be substantially lower than in a Nash equilibrium as observable in the following classical example:

		Player 2	
	Strategies	s_1^2	s_2^2
Player 1	s_1^1	(1,1)	(0,0)
	s_2^1	$(1+\epsilon, 1)$	(10,10)

Assuming $\epsilon > 0$, the set of strategies $\left(s_2^1, s_2^2\right)$ is a Nash (and so even ϵ-Nash) equilibrium of the game, while the set of strategies $\left(s_1^1, s_1^2\right)$ is an ϵ-Nash equilibrium without being a Nash equilibrium. Analyzing players' payoffs, we observe that at the Nash equilibrium they are ten times higher than at the equilibrium which is only ϵ-Nash.

At the same time, the payoffs in an ϵ-Nash equilibrium can be substantially higher than in a Nash equilibrium as observable in another classical example, i.e., the following *prisoner's dilemma*:

		Player 2	
	Strategies	s_1^2	s_2^2
Player 2	s_1^1	(1,1)	$(10+\epsilon, 0)$
	s_2^1	$(0, 10+\epsilon)$	(10,10)

Here, assuming $\epsilon > 0$, the set of strategies $\left(s_1^1, s_1^2\right)$ is a Nash (and so even ϵ-Nash) equilibrium of the game, while the set of strategies $\left(s_2^1, s_2^2\right)$ is an ϵ-Nash equilibrium without being a Nash equilibrium. Analyzing players' payoffs, we observe that at the Nash equilibrium they are ten times lower than at the equilibrium which is only ϵ-Nash.

These examples point out that an ϵ-Nash equilibrium is not necessarily close to any Nash equilibrium. This undermines the interpretation of ϵ-Nash equilibria as approximations of Nash equilibria. In this respect, ϵ-Nash equilibria need a motivation that justifies their applicability in economics. In particular, some authors point out a lack of rational motivations to renounce a gain, although minimal, a behavior which is equivalent to a decrease in the evaluation of its own payoff, see, e.g., Radner (1980). We propose an answer to this open question in terms of *uncertainty aversion* of players. Indeed, assuming payoff uncertainty, the maximum guaranteed payoff reduces with the increase of uncertainty, see (6). Then, a robust player accepts a decrease on its payoff evaluation when there is uncertainty, a behavior that finds economic explanation in the *uncertainty aversion* of players.

To deal with uncertainty aversion, we first need to introduce some generalizations of the robust-optimization equilibrium introduced in the previous section. In particular, we define ϵ-robust-optimization equilibria (ϵ-ROE).

Definition 3 (*ϵ-robust-optimization equilibrium*) A set of strategies $\left(x_1^*, \ldots, x_n^*\right)$ such that

$$\rho_i\left(x_i^*, \mathbf{x}_{-i}^*\right) \geq \rho_i\left(x_i, \mathbf{x}_{-i}^*\right) - \epsilon, \forall x_i \in A_i, \forall i \in N \quad (\epsilon\text{-ROE}) \tag{13}$$

where ρ_i are worst-case expected payoff functions defined in (6), is an ϵ-robust-optimization equilibrium.

Remark 3 When U_i are singletons, i.e., G is a game with complete payoff information, then $f_i\left(\cdot; x_i, \mathbf{x}_{-i}\right) = \rho_i\left(x_i, \mathbf{x}_{-i}\right)$ and the ϵ-ROE coincides with the ϵ-Nash equilibrium.

Then, we need to model possible changes in the degree of uncertainty. Parameters uncertainty can, indeed, be steered somehow, e.g., using better measurement procedures or techniques, increasing the information applied to model specification and so on. Therefore, it seems reasonable to assume that we can define two convex and compact uncertainty sets U_i^0 and U_i, with $U_i^0 \subseteq U_i$, for all $i \in N$, where U_i^0 represents the least degree of uncertainty achievable and U_i the greatest. Possibly, $U_i^0 = \left\{\boldsymbol{\alpha}_i^0\right\}$ depicting absence of uncertainty. By acting on the available leverages to "measure" the parameters, we can possibly achieve any intermediate level between U_i and U_i^0, namely bounding the parameters in the uncertainty set

$$W_i^\delta = (1-\delta)\,U_i^0 + \delta U_i, \;\; \delta \in [0, 1]\,, \;\; i \in N \tag{14}$$

clearly $W_i^1 = U_i$ and $W_i^0 = U_i^0$.

For simplicity sake, we assume that $f_i\left(\cdot; x_i, \mathbf{x}_{-i}\right)$, with $i \in N$, are continuous functions $\forall x_i \in A_i$. Moreover, we denote the worst-case expected payoff with respect to uncertainty set W_i^δ as $\rho_i^\delta\left(x_i, \mathbf{x}_{-i}\right)$ and by G the related robust game, i.e., $G = \left\{A_i, f_i, W_i^\delta : i \in N\right\}$. Likewise ROE can be defined for each δ-uncertainty set and denoted by ROE$^\delta$. Namely ROE$^\delta$ is any set of strategies $\left(x_1^*, \ldots, x_n^*\right)_\delta \in A$ such that

$$\rho_i^\delta\left(x_i^*, \mathbf{x}_{-i}^*\right) \geq \rho_i^\delta\left(x_i, \mathbf{x}_{-i}^*\right), \forall x_i \in A_i \quad \forall i \in N \tag{15}$$

In the following, we show that a ROE$^\delta$ is an ϵ-ROE, or an ϵ-Nash equilibrium when U_i^0 are singletons, of game $G^0 = \left\{A_i, f_i, U_i^0 : i \in N\right\}$. To this aim, we define

$$E_{f_i}\left(x_i, \mathbf{x}_{-i}\right) := \min_{\boldsymbol{\alpha}_i^0 \in U_i^0} f_i\left(\boldsymbol{\alpha}_i^0; x_i, \mathbf{x}_{-i}\right) - \min_{\boldsymbol{\alpha}_i \in U_i} f_i\left(\boldsymbol{\alpha}_i; x_i, \mathbf{x}_{-i}\right), i = 1, \ldots, n \tag{16}$$

and we assume

$$\bar{E}_{f_i}(\mathbf{x}_{-i}) := \max_{x_i \in A_i} E_{f_i}(x_i, \mathbf{x}_{-i}), \, i = 1, \dots, n \tag{17}$$

are finite.

Remark 4 Assume $U_i^0 = \left\{\boldsymbol{\alpha}_i^0\right\}$, with $i = 1, \dots, n$:

(i) If $f_i(\boldsymbol{\alpha}_i; x_i, \mathbf{x}_{-i})$ are superadditive w.r.t. $\boldsymbol{\alpha}_i$, it holds:

$$\begin{aligned} E_{f_i}(x_i, \mathbf{x}_{-i}) &= f_i\left(\boldsymbol{\alpha}_i^0; x_i, \mathbf{x}_{-i}\right) - \min_{\boldsymbol{\alpha}_i' \in U_i - \boldsymbol{\alpha}_i^0} f_i\left(\boldsymbol{\alpha}_i' + \boldsymbol{\alpha}_i^0; x_i, \mathbf{x}_{-i}\right) \\ &\leq f_i\left(\boldsymbol{\alpha}_i^0; x_i, \mathbf{x}_{-i}\right) - \min_{\boldsymbol{\alpha}_i' \in U_i - \boldsymbol{\alpha}_i^0} \left[f_i\left(\boldsymbol{\alpha}_i'; x_i, \mathbf{x}_{-i}\right) + f_i\left(\boldsymbol{\alpha}_i^0; x_i, \mathbf{x}_{-i}\right)\right] \\ &= -\min_{\boldsymbol{\alpha}_i' \in U_i - \boldsymbol{\alpha}_i^0} f_i\left(\boldsymbol{\alpha}_i'; x_i, \mathbf{x}_{-i}\right) \end{aligned} \tag{18}$$

Moreover, when $U_i = B\left(\boldsymbol{\alpha}_i^0\right)$ (closed ball of radius one centered at $\boldsymbol{\alpha}_i^0$), we have

$$E_{f_i}(x_i, \mathbf{x}_{-i}) \leq -\min_{\boldsymbol{\alpha}_i' \in B(0)} f_i\left(\boldsymbol{\alpha}_i'; x_i, \mathbf{x}_{-i}\right) \tag{19}$$

(ii) If, in particular, $f_i(\boldsymbol{\alpha}_i; x_i, \mathbf{x}_{-i})$ is linear w.r.t. $\boldsymbol{\alpha}_i$, i.e., $f_i(\boldsymbol{\alpha}_i; x_i, \mathbf{x}_{-i}) = \langle \tilde{f}_i(x_i, \mathbf{x}_{-i}), \boldsymbol{\alpha}_i\rangle$, with $\tilde{f}_i : A \to \mathbb{R}$, it holds:

$$E_{f_i}(x_i, \mathbf{x}_{-i}) = -\min_{\boldsymbol{\alpha}_i' \in U_i - \boldsymbol{\alpha}_i^0} \langle \tilde{f}_i(x_i, \mathbf{x}_{-i}), \boldsymbol{\alpha}_i'\rangle \tag{20}$$

which for $U_i = B\left(\boldsymbol{\alpha}_i^0\right)$ entails:

$$\begin{aligned} E_{f_i}(x_i, \mathbf{x}_{-i}) &= -\min_{\boldsymbol{\alpha}_i' \in B(0)} \langle \tilde{f}_i(x_i, \mathbf{x}_{-i}), \boldsymbol{\alpha}_i'\rangle \\ &= \max_{\boldsymbol{\alpha}_i' \in B(0)} \langle -\tilde{f}_i(x_i, \mathbf{x}_{-i}), \boldsymbol{\alpha}_i'\rangle = \left\| \tilde{f}_i(x_i, \mathbf{x}_{-i}) \right\| \end{aligned} \tag{21}$$

In the particular case $f_i(\boldsymbol{\alpha}_i; x_i, \mathbf{x}_{-i}) = \langle\langle x_i, M\mathbf{x}_{-i}\rangle, \boldsymbol{\alpha}_i\rangle$, where M is the matrix payoff, and $U_i = B\left(\boldsymbol{\alpha}_i^0\right)$, it holds:

$$E_{f_i}(x_i, \mathbf{x}_{-i}) = \| < x_i, M\mathbf{x}_{-i} > \| \tag{22}$$

Then, it is possible to prove (see "Appendix") the following result.

Theorem 2 *Let $f_i(\boldsymbol{\alpha}_i; x_i, \mathbf{x}_{-i})$ be concave w.r.t. $\boldsymbol{\alpha}_i$, $\forall i \in N$:*

(i) *If $\left(x_1^*, \dots, x_n^*\right)_\delta \in A$ is a ROE$^\delta$, then $\left(x_1^*, \dots, x_n^*\right)_\delta$ is an ϵ-ROE, with $\epsilon = \max\left\{\delta \bar{E}_{f_1}\left(\mathbf{x}_{-1}^*\right), \dots, \delta \bar{E}_{f_n}\left(\mathbf{x}_{-n}^*\right)\right\}$, for the robust game with worst-case expected payoff functions*

$$\rho_i^0(x_i, \mathbf{x}_{-i}) = \min_{\boldsymbol{\alpha}_i^0 \in U_i^0} f_i\left(\boldsymbol{\alpha}_i^0; x_i, \mathbf{x}_{-i}\right), \, i = 1, \dots, n \tag{23}$$

(ii) *If in particular* $U_i^0 = \{\boldsymbol{\alpha}_i^0\}$, $i = 1, \ldots, n$ *and* $(x_1^*, \ldots, x_n^*)_\delta \in A$ *is a ROE*$^\delta$, *then* $(x_1^*, \ldots, x_n^*)_\delta$ *is an* ϵ*-Nash equilibrium, with* $\epsilon = \max\{\delta \bar{E}_{f_1}(\mathbf{x}_{-1}^*), \ldots, \delta \bar{E}_{f_n}(\mathbf{x}_{-n}^*)\}$, *for the game with payoff functions:*

$$\rho_i^0(x_i, \mathbf{x}_{-i}) = f_i\left(\boldsymbol{\alpha}_i^0; x_i, \mathbf{x}_{-i}\right), \quad i = 1, \ldots, n \tag{24}$$

Theorem 2 (ii) allows to describe the robust-optimization equilibrium of a game under a degree δ of uncertainty as the proper ϵ-Nash equilibrium of a nominal game without uncertainty. Hence, for each game there exists a *non-empty* subset of ϵ-Nash equilibria that has a counterpart in terms of ROE$^\delta$ of the original game but with a certain uncertainty set. For these ϵ-Nash equilibria, and only for them, we can explain the ϵ tolerance as the price a risk-averse player (indeed the player will act so to maximize a worst-case scenario) is willing to pay for reducing the degree of uncertainty in the value of the parameters. In particular, given a nominal game and an incomplete-information counterpart, where the set of all possible realizations of the uncertain payoff parameters is given by a convex combination of an uncertainty set and the value of the payoff parameters in the nominal version, ROEs of the incomplete-information game are a subset of the ϵ-Nash equilibria of the nominal game where ϵ is chosen in a suitable way, see Theorem 2.

Moreover, from Theorem 2 follows (proof in "Appendix") the following result.

Corollary 1 *Assume*

$$\rho_i^0(x_i, \mathbf{x}_{-i}) = \min_{\boldsymbol{\alpha}_i^0 \in U_i^0} f_i\left(\boldsymbol{\alpha}_i^0; x_i, \mathbf{x}_{-i}\right), \quad i = 1, \ldots, n \tag{25}$$

are continuous. Let $\delta_k \to 0$. *If* $(x_1^*, \ldots, x_n^*)_{\delta_k} := (x_1^*, \ldots, x_n^*)_k$ *is a sequence of ROE*$^{\delta_k}$, *converging to* $(x_1^*, \ldots, x_n^*) \in A$, *then* $(x_1^*, \ldots, x_n^*)$ *is a ROE for the game with payoff functions* $\rho_i^0(x_i, \mathbf{x}_{-i})$.

Corollary 1 indicates that, whenever U_i^0 is a singleton and the payoff functions are continuous, if a ROE$^\delta$ of a game with uncertainty converges to a set of strategies when the degree of uncertainty δ goes to zero, then such a ROE$^\delta$ converges to a Nash equilibrium of the game without uncertainty, i.e., of the original game with payoff parameters U_i^0. As for ϵ-Nash equilibrium, this may lead to think to ROEs$^\delta$, or more generally to ROEs, as approximations of Nash equilibria of the game without uncertainty. Nevertheless, as it is for ϵ-Nash equilibria when we consider complete information game, also ROEs$^\delta$ of a game with uncertainty can offer a certain payoff that is substantially lower/higher for all players involved w.r.t. the payoffs offered by the Nash equilibria of the nominal version of the game. We conclude this section with an example to illustrate Theorem 2 and for which all players are substantially better off at the ROE$^\delta$ of the game with uncertainty w.r.t. the Nash equilibrium of a nominal counterpart of the game.

Example 2 Let us consider the following robust game in matrix form:

		Player 2	
	Strategies	s_1^2	s_2^2
Player 1	s_1^1	(2,2)	(1,0)
	s_2^1	$(\alpha, 0)$	(0,1)

in which the payoff uncertainty set for players 1 and 2 is, respectively

$$U_1 = \left\{ \begin{bmatrix} 2 & 1 \\ \alpha & 0 \end{bmatrix} \middle| \alpha \in [1,5] \right\} \quad \text{and} \quad U_2 = \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right\} \tag{26}$$

Denoting by x_i the probability that player i plays strategy s_1^i, then player i has as strategy space the set of real numbers $A_i = [0,1]$, with $i = 1, 2$, and the payoff functions of the two players are, respectively,

$$\begin{aligned} f_1(\alpha; x_1, x_2) &= [1 + x_2]\, x_1 + \alpha\, [1 - x_1]\, x_2 \quad \text{and} \\ f_2(x_1, x_2) &= [3x_1 - 1]\, x_2 + 1 - x_1 \end{aligned} \tag{27}$$

We assume the least degree of uncertainty is represented by

$$U_1^0 = \left\{ \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \right\} \quad \text{and} \quad U_2^0 = U_2, \tag{28}$$

i.e., U_1^0 and U_2^0 are singletons, which means that players 1 and 2 would not be affected by uncertainty. The nominal version of the game can be solved in mixed strategies. Let us set $\rho_1^0(x_1, x_2) = f_1(3; x_1, x_2)$ $\rho_2^0(x_1, x_2) = f_2(x_1, x_2)$, then the best-reply functions are $R_1(x_2) = \arg\max_{x_1 \in A_1} \rho_1^0(x_1, x_2)$ and $R_2(x_1) = \arg\max_{x_2 \in A_2} \rho_2^0(x_1, x_2)$, given by

$$R_1(x_2) = \begin{cases} 0 & \text{if } \frac{1}{2} < x_2 \le 1 \\ [0,1] & \text{if } x_2 = \frac{1}{2} \\ 1 & \text{if } 0 \le x_2 < \frac{1}{2} \end{cases} \quad \text{and} \quad R_2(x_1) = \begin{cases} 0 & \text{if } 0 \le x_1 < \frac{1}{3} \\ [0,1] & \text{if } x_1 = \frac{1}{3} \\ 1 & \text{if } \frac{1}{3} < x_1 \le 1 \end{cases} \tag{29}$$

Hence, the unique Nash equilibrium of the game $G^0 = \left\{ A_i, f_i, U_i^0 : i \in \{1,2\} \right\}$ is $\left(\frac{1}{3}, \frac{1}{2}\right)$, see the intersection point of the best-reply functions in panel (a) of Fig. 2. Let us now consider $G = \left\{ A_i, f_i, W_i^\delta : i \in \{1,2\} \right\}$, a variant of game G^0, where the payoff uncertainty set of the game is $W^\delta = W_1^\delta \times W_2^\delta$ with $W_2^\delta = U_2$ and W_1^δ which is a convex combination of parameter $\delta \in [0,1]$ of two sets U_1 defined in (7) and U_1^0 defined in (28), i.e.,

$$W_1^\delta = \delta U_1 + (1 - \delta)\, U_1^0 = \left\{ \begin{bmatrix} 2 & 1 \\ \alpha & 0 \end{bmatrix} \middle| \alpha \in [3 - 2\delta, 3 + 2\delta] \right\} \tag{30}$$

Let us consider, for example, $\delta = \frac{1}{2}$. Assuming that the players behave as robust optimizers, they maximize their worst-case expected payoff functions:

$$\rho_1^{\frac{1}{2}}(x_1, x_2) = \min_{\alpha \in [2,4]} f_1(\alpha; x_1, x_2) = [1 - x_2] x_1 + 2x_2$$
$$\text{and} \quad \rho_2^{\frac{1}{2}}(x_1, x_2) = f_2(x_1, x_2) \tag{31}$$

from which we obtain the following worst-case best-reply functions w.r.t. $W_1^{\frac{1}{2}}$ and $W_2^{\frac{1}{2}}$:

$$R_1(x_2) = \begin{cases} 1 & \text{if } 0 \le x_2 < 1 \\ [0, 1] & \text{if } x_2 = 1 \end{cases} \quad \text{and} \quad R_2(x_1) = \begin{cases} 0 & \text{if } 0 \le x_1 < \frac{1}{3} \\ [0, 1] & \text{if } x_1 = \frac{1}{3} \\ 1 & \text{if } \frac{1}{3} < x_1 \le 1 \end{cases} \tag{32}$$

from which it follows that $(1, 1)$ is a ROE$^\delta$, with $\delta = \frac{1}{2}$, of game G, see the intersection points of the worst-case best-reply functions in panel (b) of Fig. 2. Since $f(\alpha; x_1, x_2)$ is concave w.r.t. α and

$$E_{f_1}(x_1, x_2) = \min_{\alpha \in U_1^0} f_1(\alpha; x_1, x_2) - \min_{\alpha \in U_1} f_1(\alpha; x_1, x_2) = 2x_2(1 - x_1) \tag{33}$$

with

$$\bar{E}_{f_1}(x_2) = \max_{x_1 \in [0,1]} E_{f_1}(x_1, x_2) = 2x_2 \tag{34}$$

Therefore for $\delta = \frac{1}{2}$ $\bar{E}_{f_1}(1) = 2$ at the ROE$^\delta$. Then, from Theorem 2 follows that $(1, 1)$ is also an ϵ-Nash equilibrium, with $\epsilon = 1$, of game G^0. Indeed, considering G^0 the strategy profile $(1, 1)$ satisfies the following conditions when $\epsilon = 1$

$$\begin{cases} \rho_1^0(1, 1) \ge \rho_1^0(x_1, 1) - \epsilon & \forall x_1 \in [0, 1] \\ \rho_2^0(1, 1) \ge \rho_2^0(1, x_2) - \epsilon & \forall x_2 \in [0, 1] \end{cases} \Leftrightarrow \begin{cases} x_1 \ge 0 & \forall x_1 \in [0, 1] \\ 2x_2 \le 3 & \forall x_2 \in [0, 1] \end{cases} \tag{35}$$

required for being an ϵ-Nash equilibrium.

It is worth pointing out that the strategic profile $(1, 1)$ is both an ϵ-Nash equilibrium, with $\epsilon = 1$, and a ROE$^\delta$ of game G^0. However, as specified in Theorem 2 there are ϵ-Nash equilibria that are not ROE$^\delta$. It is the case of the strategic profile $\left(\frac{1}{3}, \frac{1}{2}\right)$ which, being a Nash equilibrium for G^0, is an ϵ-Nash equilibrium, but not a ROE$^\delta$, of game G.

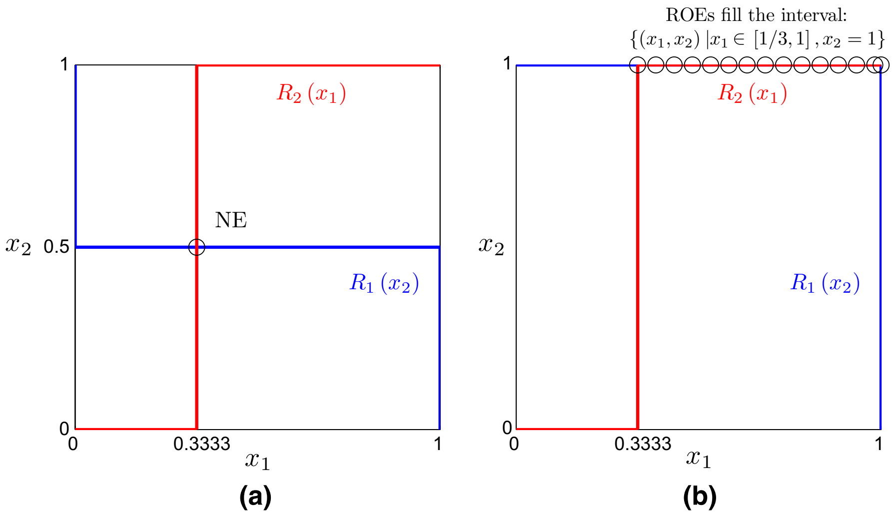


Fig. 2 Panel **a** refers to the game without uncertainty, where the best-reply function of player 1, i.e., $R_1(x_2)$ in Eq. (29), is in blue and the best-reply function of player 2, i.e., $R_2(x_1)$ in equation (29), is in red. Panel **b** refers to the game with uncertainty, where the worst-case best-reply function of player 1, i.e., $R_1(x_2)$ in (32), is in blue and the worst-case best-reply function of player 2, i.e., $R_2(x_1)$ in (32), is in red. The ROEs are marked with empty dots (color figure online)

4 Application to Cournot duopoly games

Among the most classical applications of Nash equilibrium, we acknowledge Cournot duopoly, see Cournot (1838). In this section, we exploit the model under uncertainty, that, to the best of our knowledge, has never been considered both in games and economic theory.

Two firms produce the same product sharing the same cost function and the same information set, i.e., they are identical players. Let $i = 1, 2$, q_i be the output quantity of firm i and $Q = q_1 + q_2$ the total supply on the market. We assume a downward-sloping inverse demand (or price) function that depends on the total level of production of the industry defined as follows:

$$P(a, b; q_1, q_2) = a - b(q_1 + q_2) = a - bQ \tag{36}$$

where $a > 0$ is the reservation or choke price and $b > 0$ is the slope of the price function. This price function can be obtained by assuming a quadratic utility function of a representative consumer, see, among others, Vives (2001) and Bischi et al. (2010). The cost function of each firm i is linear: $C(q_i) = cq_i$, where $c > 0$ is the unitary cost of raw materials.

Let us assume that the value of parameter b is known with a certain degree of uncertainty. In particular, parameter b, by any reasonable consideration/measurement, takes values in $\left[\underline{b}, \bar{b}\right]$, and each firm hedges against selling price fluctuations subscribing a commodity futures contract that links the unit cost of raw materials, i.e., parameter c, to b, i.e., to the market unit price of the output. Specifically, we assume that

$\boldsymbol{\alpha} \in U_i = U$, with $i = 1, 2$, where

$$U = \left\{ \boldsymbol{\alpha} = (b, c) \mid \bar{b} \geq b \geq \underline{b} \text{ and } c = \frac{\underline{c} - \bar{c}}{\bar{b} - \underline{b}} b + \frac{\bar{c}\bar{b} - \underline{b}\underline{c}}{\bar{b} - \underline{b}} \right\} \tag{37}$$

The shape of this uncertainty set implies that the value of c is linearly and negatively related to the value of b, and it varies between $\bar{c}$ and $\underline{c}$, where $\underline{c} < \bar{c}$, according to the value of b. When $b \approx \bar{b}$, the unit price of the production output is low and, by the hedging assumed, the cost of raw materials is low $c \approx \underline{c}$. This ensures that the profits do not decrease dramatically when the market price reduces due to exogenous price variations. Given the uncertainty set U, in order for the duopoly model to be meaningful, we further assume the *profitability condition* $a > \bar{c}$. Then, $\boldsymbol{\alpha} = (b, c)$ is the vector of the uncertain parameters that affects the profit (payoff) function of each firm (player) i, given by

$$f_i\left(\boldsymbol{\alpha}; q_i, q_{-i}\right) = P\left(a, b; q_i, q_{-i}\right) q_i - c q_i, \tag{38}$$

where q_{-i} is the level of production of the other firm.

The *worst-case expected payoff (profit) function* [see (6)] of firm (player) i is given by

$$\rho_i\left(q_i, q_{-i}\right) = \begin{cases} \left(a - \bar{b}\left(q_i + q_{-i}\right)\right) q_i - \underline{c} q_i & \text{if } q_i \geq \max\left[\frac{\bar{c}-\underline{c}}{\bar{b}-\underline{b}} - q_{-i}, 0\right] \\ \left(a - \underline{b}\left(q_i + q_{-i}\right)\right) q_i - \bar{c} q_i & \text{if } q_i < \max\left[\frac{\bar{c}-\underline{c}}{\bar{b}-\underline{b}} - q_{-i}, 0\right] \end{cases} \tag{39}$$

and the *worst-case best response* or *worst-case best-reply mapping* of firm (player) i is given by

$$q_i = R_i\left(q_{-i}\right) = \begin{cases} \frac{a-\underline{c}}{2\bar{b}} - \frac{1}{2} q_{-i} & \text{if } q_{-i} \geq \bar{q} \\ \frac{\bar{c}-\underline{c}}{\bar{b}-\underline{b}} - q_{-i} & \text{if } \underline{q} \leq q_{-i} < \bar{q} \\ \frac{a-\bar{c}}{2\underline{b}} - \frac{1}{2} q_{-i} & \text{if } q_{-i} < \underline{q} \end{cases} \tag{40}$$

where

$$\begin{aligned} \bar{q} &= \min\left[\frac{\bar{c} - \underline{c}}{\bar{b} - \underline{b}}, \frac{2\bar{b}\left(\bar{c} - \underline{c}\right) - \left(a - \underline{c}\right)\left(\bar{b} - \underline{b}\right)}{\bar{b}\left(\bar{b} - \underline{b}\right)}\right] \geq \underline{q} \\ &= \min\left[\frac{\bar{c} - \underline{c}}{\bar{b} - \underline{b}}, \frac{2\underline{b}\left(\bar{c} - \underline{c}\right) - \left(a - \bar{c}\right)\left(\bar{b} - \underline{b}\right)}{\underline{b}\left(\bar{b} - \underline{b}\right)}\right] \end{aligned} \tag{41}$$

Being the *worst-case best responses* downward sloping and imposing the nonnegativity constraints of the level of output production for both firms, the action space is bounded, in particular $(q_1, q_2) \in [0, q_{\max}]^2$, where $q_{\max}$ is the maximum level of production that ensures that both firms produce a nonnegative level of output. It is worth pointing out that to determine the *worst-case best responses* a nonnegative constraint on the level of production output is required as well as a nonnegative price constraint when the price function is defined. Nevertheless, we are interested in the equilibria of the model, around of which, both the nonnegative price constraint and the nonnegative output quantity constraint are largely satisfied. Then, for the sake of simplicity, we omit to impose such constrains.

Concerning ROE of the game, i.e., $\left(q_1^*, q_2^*\right)$ such that $q_i^* = R_i\left(q_{-i}^*\right)$ with $i = 1, 2$, which by analogy with the theory on Cournot oligopoly games in the following are named *robust-optimization Cournot–Nash equilibria*, it is possible to prove (see "Appendix") the following existence result.

Theorem 3 *Let us consider the robust Cournot duopoly game $G = \{f_i, A_i, U_i : i \in \{1, 2\}\}$, where f_i are defined in (38), $A_i = [0, q_{\max}]$ and $U_i = U$, where U is defined in (37), and let us define:*

$$q^- = \max\left[\frac{\bar{b}\left(a-\bar{c}\right)-\underline{b}\left(a-\underline{c}\right)}{\bar{b}\left(\bar{b}-\underline{b}\right)}, \frac{2\underline{b}\left(\bar{c}-\underline{c}\right)-\left(a-\bar{c}\right)\left(\bar{b}-\underline{b}\right)}{\underline{b}\left(\bar{b}-\underline{b}\right)}\right],$$

$$\bar{b}^- = \underline{b}\left(\frac{a-\underline{c}}{a-\bar{c}}+\frac{\bar{c}-\underline{c}}{2\left(a-\bar{c}\right)}\right), \quad \bar{c}^* = \frac{2a+\underline{c}}{3} \tag{42}$$

$$q^+ = \min\left[\frac{\bar{b}\left(a-\bar{c}\right)-\underline{b}\left(a-\underline{c}\right)}{\underline{b}\left(\bar{b}-\underline{b}\right)}, \frac{2\bar{b}\left(\bar{c}-\underline{c}\right)-\left(a-\underline{c}\right)\left(\bar{b}-\underline{b}\right)}{\bar{b}\left(\bar{b}-\underline{b}\right)}\right] \quad \text{and} \quad \bar{b}^+ = \underline{b}\frac{2\left(a-\underline{c}\right)}{2\left(a-\bar{c}\right)-\left(\bar{c}-\underline{c}\right)} \tag{43}$$

where $\bar{b}^+ > \bar{b}^-$. The robust-optimization Cournot–Nash equilibria of the game are:

- $\left(\frac{a-\underline{c}}{3\bar{b}}, \frac{a-\underline{c}}{3\bar{b}}\right)$*, when $\bar{b} > \bar{b}^+$ and $\bar{c} < \bar{c}^*$,*
- $\left(\frac{a-\bar{c}}{3\underline{b}}, \frac{a-\bar{c}}{3\underline{b}}\right)$*, when $\bar{b} < \bar{b}^-$, and*
- $\left(q^*, \frac{\bar{c}-\underline{c}}{\bar{b}-\underline{b}} - q^*\right)$*, $\forall q^*$ s.t. $q^- \leq q^* \leq q^+$, otherwise.*

No other robust-optimization Cournot–Nash equilibria exist.

Theorem 3 proves that a symmetric robust Cournot duopoly game (where symmetric games are games where the payoffs for playing a particular strategy depend only on the other strategies employed, not on who is playing them) can have infinitely many robust-optimization Cournot–Nash equilibria which are not necessarily symmetric. This means that at robust-optimization Cournot–Nash equilibria, the level of production, and thus the profit, of a firm can be different from the ones of its competitor, despite the firm and its competitors are identical players. Such a situation cannot occur in a nominal version of the game, i.e., a duopoly game without payoff uncertainty. For the sake of comparison, we consider the following nominal version of the game where

$$U = \left\{\left(\hat{b}, \hat{c}\right) | \hat{b} = \frac{\bar{b}+\underline{b}}{2} \text{ and } \hat{c} = \frac{\underline{c}-\bar{c}}{\bar{b}-\underline{b}}\hat{b} + \frac{\bar{c}\bar{b}-\underline{b}\underline{c}}{\bar{b}-\underline{b}}\right\} \tag{44}$$

In this special case, the *best responses* reduce to $\hat{R}_i = \frac{a-\hat{c}}{2\hat{b}} - \frac{1}{2}q_{-i}$ and the unique Cournot–Nash equilibrium is given by $\left(\frac{a-\hat{c}}{3\hat{b}}, \frac{a-\hat{c}}{3\hat{b}}\right)$, see red circle in Panel (a) of Fig. 3, while the robust duopoly game's equilibria are any point on the red segment in Panel (a) of Fig. 3.

Moreover, the level of production output at the robust-optimization Cournot–Nash equilibria is higher for each firm than at the Cournot–Nash equilibrium.

Having a multiplicity of robust-optimization Cournot–Nash equilibria, a natural question is which one will be selected. In order to answer such a question, we consider the classical Cournot adjustment process based on constant rival's output expectation, i.e., $q^{e}_{-i}(t+1) = q_{-i}(t)$. In oligopoly games without parameter uncertainty, such an adjustment process is considered in Theocharis (1960) and Fisher (1961). They show that the output dynamics of the duopoly game converges to the Cournot–Nash equilibrium. Introducing the same adjustment process in case of parameter uncertainty, we obtain the following piecewise smooth map:

$$T : (q_1(t), q_2(t)) \to (R_1(q_2(t)), R_2(q_1(t))) \quad \text{with} \quad t \in \mathbb{N} \tag{45}$$

The dynamics of map T is also known as *best-reply dynamics* when U is a singleton, see, e.g., Bischi et al. (2010) and references therein. Then, by analogy, we call the dynamics of map T the *robust best-reply dynamics* whenever U is not a singleton.

The map remains the same by interchanging the firms (players) and this symmetry property implies that the diagonal (*line of equal production output*)

$$\Delta = \{(q_1, q_2) \,|q_1 = q_2\} \tag{46}$$

is an invariant submanifold for map T, i.e., $T(\Delta) \subset \Delta$. This means that two identical players, starting with identical initial strategies $q_1(0) = q_2(0)$, behave identically for each $t \geq 0$. The trajectories belonging to Δ, are governed by the one-dimensional map $q(t+1) = g(q(t))$, where $g(q(t))$ represents the restriction of the two-dimensional map T to Δ:

$$g = T|_{\Delta} : \Delta \to \Delta \tag{47}$$

The simpler model $q(t+1) = g(q(t))$ can be seen as the model of a representative player, whose dynamics summarizes the common behavior of the two identical players. In the special case of no uncertainty, i.e., U is a singleton, the Cournot–Nash equilibrium belongs to Δ, i.e., it is a symmetric equilibrium where both firms produce the same output. In case of uncertainty, the robust-optimization Cournot–Nash equilibria can be more than one and not all of them belong to Δ. Another difference between the duopoly game with and without payoff function uncertainty is the limiting point of the best-reply dynamics, which in case of uncertainty can be a 2-cycle. In particular, 2-cycles fill the square region indicated in Panel (b) of Fig. 3. The 2-cycle that lies in the borders of one of diagonal of this square, see the two blue dots in Panel (b) of Fig. 3, has a stable set of positive measures that is represented by the yellow region in Panel (b) of Fig. 3. On the other diagonal of the square, red line in Panel (b) of Fig. 3, is instead filled of fixed points, the so-called robust-optimization equilibria.

The coexistence of a 2-cycle and many robust-optimization Cournot–Nash equilibria that have attractive sets of positive measure introduces *path dependence* in the robust version of the game. This means that trajectories converge to different invari-

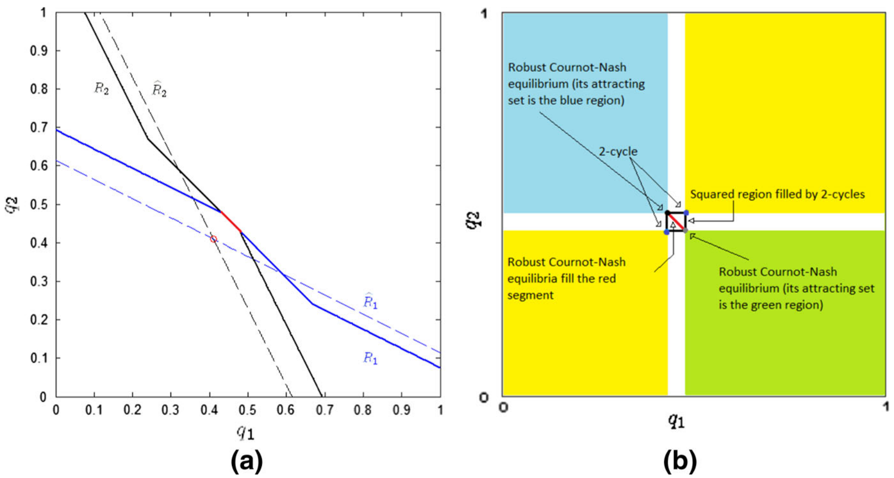


Fig. 3 Panel **a** $\hat{R}_1$ and $\hat{R}_2$ are the reaction function of the two firms when U is a singleton, while R_1 and R_2 are the reaction functions of the two firms when U is not a singleton and the two firms behave as robust players. Moreover, the red circle indicates the Cournot–Nash equilibrium of the nominal version of the robust duopoly model, while the red line indicates the set of robust-optimization Cournot–Nash equilibria. Panel **b** in yellow the basin of attraction of the 2-cycle indicated by blue dots, in light blue the basin of attraction of the robust-optimization Cournot–Nash equilibrium indicated by a black dot (firm 1 produces less than firm 2) and in dark green the basin of attraction of the robust-optimization Cournot–Nash equilibrium indicated by a gray dot (firm 1 produces more than firm 2). Values of the parameters: $a = 2.3$, $\bar{b} = 2$, $\underline{b} = 1.01$, $\bar{c} = 0.9$, $\underline{c} = 0$ (color figure online)

ant sets according to their initial conditions. In particular, trajectories starting in the blue region in Panel (b) of Fig. 3 converge to a robust-optimization Cournot–Nash equilibrium where player 2 produces more than player 1, i.e., the black dot in the blue region in Panel (b) of Fig. 3. Trajectories starting in the green region in Panel (b) of Fig. 3 converge to a robust-optimization Cournot–Nash equilibrium where player 1 produces more than player 2, i.e., the gray dot in the green region in Panel (b) of Fig. 3. Trajectories starting in the white region in Panel (b) of Fig. 3 converge to either one of the infinity many robust-optimization Cournot–Nash equilibria laying in the red line of the same figure or to one of the infinity many 2-cycles filling the square indicated in the same figure.

The stability properties of these two cycles, as well as the global dynamics of the Cournot map T, can be studied through the second iterated of map T^2, which is a decoupled map: $T^2(q_1, q_2) = T(R_1(q_2), R_2(q_1)) = (R_2(R_1(q_1)), R_1(R_2(q_2))) = (F(q_1), G(q_2))$. Then, the 2-cycle of the Cournot map T corresponds to two fixed points of the one-dimensional maps F and G, which are the same function since $R_1 = R_2$. In particular, the squared filled by two cycles is simply the Cartesian products of the fixed points of G and F. A detailed analysis of the global dynamics of the duopoly model can be performed by means of standard results in the qualitative theory of piecewise smooth maps. This investigation is out of the scope of the current paper, and the interested reader may refer to Bischi et al. (2000) and references therein.

5 Conclusions

Based on robust-optimization approach, robust games were proposed in Aghassi and Bertsimas (2006) as a distribution-free model of games with incomplete information. In matrix games, the notion of robust-optimization equilibrium allows to relax the assumptions of Harsanyi's Bayesian games, see Harsanyi (1968b) and Harsanyi (1968a), which require a prior probability information by players about the realization of the uncertain parameters of the game. According to this model, players choose the strategy that ensures the maximum guaranteed payoff by maximizing the worst-case expected payoff function. Aghassi and Bertsimas (2006) prove the existence of the so-called robust-optimization equilibrium of the model, which is a Nash equilibrium of an associated game without uncertainty where the payoff functions are the worst-case expected payoff functions.

Here, the robust-optimization model and the concept of robust-optimization equilibrium are extended to games with payoff function that cannot be represented by matrices. In particular, the existence result of the robust-optimization equilibrium is provided. In addition, we prove that a robust-optimization equilibrium is an ϵ-Nash equilibrium of the nominal version of the robust game, i.e., of the robust game when the uncertainty set is a singleton. Thus robust-optimization equilibria provide new insight into the popular notion of ϵ-Nash equilibria, allowing to motivate this concept in terms of rational behavior.

In the last part of the paper, a quantity-setting robust Cournot duopoly game is proposed. In this game, firms, which are the players of the game, are uncertain about the slope of the inverse demand function and adopt a robust-optimization approach to determine the best-reply function. This application of the theory on robust games shows significant and intriguing differences between the robust and the nominal version of the Cournot duopoly game.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

Proof (of Theorem 1) Since a ROE is a Nash equilibrium of a suitably defined game, having the worst-case expected payoff functions as utility functions which are continuous and concave, the proof of the theorem follows by Nash' theorem, see Nash (1950). □

Proof (of Theorem 2) (i) Let $w_i^\delta \in W_i^\delta$, that is

$$w_i^\delta = (1-\delta)\,\boldsymbol{\alpha}_i^0 + \delta\boldsymbol{\alpha}_i \tag{48}$$

for some $\boldsymbol{\alpha}_i^0 \in U_i^0$ and $\boldsymbol{\alpha}_i \in U_i$. From concavity w.r.t. w_i^δ, it follows

$$f_i\left(w_i^\delta; x_i, \mathbf{x}_{-i}\right) \geq (1-\delta)\, f_i\left(\boldsymbol{\alpha}_i^0; x_i, \mathbf{x}_{-i}\right) + \delta f_i\left(\boldsymbol{\alpha}_i; x_i, \mathbf{x}_{-i}\right), \quad \forall x_i \in A_i \tag{49}$$

Hence, recalling that $U_i^0 \subseteq W_i^\delta$, for every $i \in N$ and $\delta \in [0, 1]$, we have:

$$\begin{aligned}
\rho_i^0\left(x_i, \mathbf{x}_{-i}\right) &= \min_{\boldsymbol{\alpha}_i^0 \in U_i^0} f_i\left(\boldsymbol{\alpha}_i^0; x_i, \mathbf{x}_{-i}\right) \geq \rho_i^\delta\left(x_i, \mathbf{x}_{-i}\right) = \min_{w_i^\delta \in W_i^\delta} f_i\left(w_i^\delta; x_i, \mathbf{x}_{-i}\right)\\
&= \min_{\boldsymbol{\alpha}_i^0 \in U_i^0;\, \boldsymbol{\alpha}_i \in U_i} f_i\left((1-\delta)\,\boldsymbol{\alpha}_i^0 + \delta \boldsymbol{\alpha}_i; x_i, \mathbf{x}_{-i}\right)\\
&\geq (1-\delta) \min_{\boldsymbol{\alpha}_i^0 \in U_i^0} f_i\left(\boldsymbol{\alpha}_i^0; x_i, \mathbf{x}_{-i}\right) + \delta \min_{\boldsymbol{\alpha}_i \in U_i} f_i\left(\boldsymbol{\alpha}_i; x_i, \mathbf{x}_{-i}\right)\\
&= \min_{\boldsymbol{\alpha}_i^0 \in U_i^0} f_i\left(\boldsymbol{\alpha}_i^0; x_i, \mathbf{x}_{-i}\right) - \delta E_{f_i}\left(x_i, \mathbf{x}_{-i}\right)\\
&= \rho_i^0\left(x_i, \mathbf{x}_{-i}\right) - \delta E_{f_i}\left(x_i, \mathbf{x}_{-i}\right), \forall x_i \in A_i
\end{aligned} \tag{50}$$

where $E_{f_i}\left(x_i, \mathbf{x}_{-i}\right)$ is defined in (16).

Let $\left(x_1^*, \ldots, x_n^*\right)_\delta \in A$ be a RNE$^\delta$. It holds:

$$\begin{aligned}
\rho_i^\delta\left(x_i^*, \mathbf{x}_{-i}^*\right) &= \min_{\boldsymbol{\alpha}_i \in W_i^\delta} f_i\left(\boldsymbol{\alpha}_i; x_i^*, \mathbf{x}_{-i}^*\right) \geq \rho_i^\delta\left(x_i, \mathbf{x}_{-i}^*\right)\\
&= \min_{\boldsymbol{\alpha}_i \in W_i^\delta} f_i\left(\boldsymbol{\alpha}_i; x_i, \mathbf{x}_{-i}^*\right) \quad \forall i = 1, \ldots, n
\end{aligned} \tag{51}$$

and recalling the previous inequalities, we get:

$$\begin{aligned}
\rho_i^0\left(x_i^*, \mathbf{x}_{-i}^*\right) \geq \rho_i^\delta\left(x_i^*, \mathbf{x}_{-i}^*\right) \geq \rho_i^\delta\left(x_i, \mathbf{x}_{-i}^*\right) &\geq \rho_i^0\left(x_i, \mathbf{x}_{-i}^*\right) - \delta E_{f_i}\left(x_i, \mathbf{x}_{-i}^*\right)\\
&\geq \rho_i^0\left(x_i, \mathbf{x}_{-i}^*\right) - \delta \bar{E}_{f_i}\left(\mathbf{x}_{-i}^*\right), \forall x_i \in A_i
\end{aligned} \tag{52}$$

where $\bar{E}_{f_i}\left(\mathbf{x}_{-i}\right)$ is defined in (17).

Letting $\epsilon = \max\left\{\delta \bar{E}_{f_1}\left(\mathbf{x}_{-1}^*\right), \ldots, \delta \bar{E}_{f_n}\left(\mathbf{x}_{-n}^*\right)\right\}$, we get

$$\rho_i^0\left(x_i^*, \mathbf{x}_{-i}^*\right) \geq \rho_i^0\left(x_i, \mathbf{x}_{-i}^*\right) - \epsilon, \forall x_i \in A_i \quad \text{and} \quad \forall i = 1, \ldots, N \tag{53}$$

which proves (i)
(ii) Straightforward from point (i). □

Proof (of Corollary 1) It follows directly from inequalities (52) and the continuity of $\rho_i^0\left(x_i, \mathbf{x}_{-i}\right)$. □

Proof (of Theorem 3) According to the definition of the *worst-case best responses*, see (40), we can split the action space in nine subregions, the border of which are points of non-differentiability for R_i, $i = 1, 2$. The subregions are $\Omega_i \cap [0, q_{\max}]^2$, $i = 1, \ldots, 9$, where

$$
\begin{array}{ll}
\Omega_1 := \{(q_1, q_2) \,|\, q_1, q_2 > \bar{q}\} & \Omega_2 := \left\{(q_1, q_2) \,|\, \underline{q} < q_1, q_2 < \bar{q}\right\} \\
\Omega_3 := \left\{(q_1, q_2) \,|\, q_1, q_2 < \underline{q}\right\} & \\
\Omega_4 := \left\{(q_1, q_2) \,|\, q_1 > \bar{q} \wedge \underline{q} < q_2 < \bar{q}\right\} & \Omega_5 := \left\{(q_1, q_2) \,|\, q_1 > \bar{q} \wedge q_2 < \underline{q}\right\} \\
\Omega_6 := \left\{(q_1, q_2) \,|\, \underline{q} < q_1 < \bar{q} \wedge q_2 > \bar{q}\right\} & \\
\Omega_7 := \left\{(q_1, q_2) \,|\, \underline{q} < q_1 < \bar{q} \wedge q_2 < \underline{q}\right\} & \Omega_8 := \left\{(q_1, q_2) \,|\, q_1 < \underline{q} \wedge q_2 > \bar{q}\right\} \\
\Omega_9 := \left\{(q_1, q_2) \,|\, q_1 < \underline{q} \wedge \underline{q} < q_2 < \bar{q}\right\} &
\end{array}
$$

and $\bar{q}$ and $\underline{q}$ are as in (41). Since the game is symmetric, we have $R_1(q) = R_2(q) = R(q)$. Function $R(q)$ is continuous and strictly decreasing in $[0, q_{\max}]$, then $R^{-1}(q)$ exists and is continuous and strictly decreasing. According to (5), robust Cournot–Nash equilibria (or more simply ROE) of the game are all the couples $\left(q_1^*, q_2^*\right)$ such that $q_i^* = R\left(q_{-i}^*\right)$, $i = 1, 2$, i.e., they are the intersection points of functions $R(q)$ and $R^{-1}(q)$. Excluding the kink points where R is not differentiable, $R'(q) \geq \left(R^{-1}\right)'(q)$ in $[0, q_{\max}]$, with $R'(q) = \left(R^{-1}\right)'(q)$ only in $\Omega_2 \cap [0, q_{\max}]^2$. It follows that there is at most one robust-optimization Cournot–Nash equilibrium in $[0, q_{\max}]^2 \cap \Omega_2^c$ and is of the type (q^*, q^*) which means that in $[0, q_{\max}]^2 \cap \Omega_2^c$ only a symmetric robust Cournot–Nash equilibrium can exist. On the contrary, on $[0, q_{\max}]^2 \cap \Omega_2$ infinitely many robust Cournot–Nash equilibria exist. This implies that the robust Cournot–Nash equilibria can belong only to Ω_1, Ω_2 and Ω_3, and by trivial algebra comes out that they are the following:

$$
\begin{array}{l}
\left(\frac{a-\underline{c}}{3\bar{b}}, \frac{a-\underline{c}}{3\bar{b}}\right) \in \Omega_1; \quad \left(\frac{a-\bar{c}}{3\underline{b}}, \frac{a-\bar{c}}{3\underline{b}}\right) \in \Omega_3; \\
\left(q^*, \frac{\bar{c}-\underline{c}}{\bar{b}-\underline{b}} - q^*\right) \in \Omega_2 \quad \forall q^* \text{ s. t. } \underline{q} < q^* < \bar{q}
\end{array}
\tag{54}
$$

Imposing the feasibility conditions to each of the robust Cournot–Nash equilibrium, the statement of the Theorem follows. □

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