

# Explanation of Nash's Theorem and Proof with Examples

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In this tutorial, we'll assume that you know the basics of game theory, such as the definitions of pure-strategy and mixed-strategy Nash equilibrium. We'll use the following example throughout.

**Hawk-dove game:** This game has 2 players and each player has two actions—dove and hawk. According to the following payoff matrix, there are two pure-strategy Nash equilibria, namely (dove, hawk) and (hawk, dove). Remember that every pure-strategy Nash equilibrium is also a mixed-strategy Nash equilibrium. In this game, there is another mixed-strategy Nash equilibrium, namely  $p = 1/3$  and  $q = 1/3$ .

		Column player	
		Dove ( $q$ )	Hawk ( $1-q$ )
Row player	Dove ( $p$ )	3, 3	1, 5
	Hawk ( $1-p$ )	5, 1	0, 0

**Nash's Theorem** (Nash, 1950). Any game with a finite number of players and a finite number of actions has a mixed-strategy Nash equilibrium.

Before explaining Nash's proof, we'll start with a review of some game-theoretic terms. Consider any game with  $n$  players (Alice, Bob, Cindy, ...,  $n$ -th player) and  $m$  actions (1, 2, ...,  $m$ ).

## Expected payoffs

Alice's expected payoff for playing a particular action  $k$  =

$$\sum_{\text{Each payoff-matrix cell } C \text{ where Alice plays } k} \text{Alice's payoff in cell } C \times \text{Prob}(\text{other players are also in } C)$$

Alice's expected payoff for playing a mixed strategy  $p$  =

$$p(\text{Alice plays 1}) \times \text{Alice's expected payoff for playing 1} + \\ p(\text{Alice plays 2}) \times \text{Alice's expected payoff for playing 2} + \dots + \\ p(\text{Alice plays } m) \times \text{Alice's expected payoff for playing } m$$

Above,  $\text{Prob}(\text{Other players are also in } C)$  is a multiplication of Bob's probability of being in  $C$ , Cindy's probability of being in  $C$ , etc. Alice's expected payoff is *multilinear*, i.e., linear in each player's probability separately. Such functions are guaranteed to be *continuous*, which roughly means that if you plot that function you'll not see any disconnected parts.

## Example (hawk-dove)

Player 1's  $E[\text{Dove}] = q \times 3 + (1 - q) \times 1 = 1 + 2q$ .

Player 1's  $E[\text{Hawk}] = 5q$ .

Player 1's expected payoff =  $p(1 + 2q) + (1 - p)5q = p + 5q - 3pq$ . This is plotted in Fig. 1.

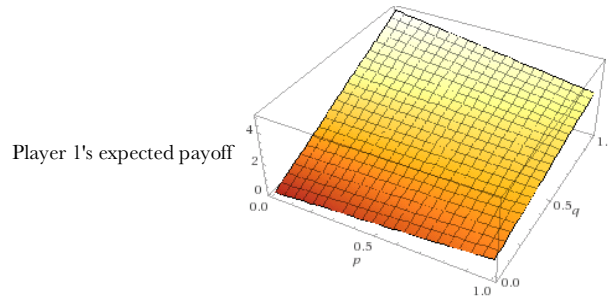


Figure 1: Player 1's expected payoff in the hawk-dove game is multilinear in  $p$  and  $q$  and is continuous. There are no holes or sudden jumps and therefore no disconnected parts.

### Example (hawk-dove)

Player 1's  $E[D] \in [1, 3]$ , since you can vary  $q$  from 0 to 1.

### Joint strategy

A joint strategy is an  $n$ -tuple consisting of the following  $n$  mixed strategies:

- Alice's mixed strategy, which is her probability distribution over actions 1, 2, ...,  $m$  (in other words, her probability of playing 1, probability of playing 2, ..., probability of playing  $m$ , where the sum of these probabilities is 1).
- Bob's probabilities of playing 1, 2, 3, ...
- Cindy's probabilities of playing 1, 2, 3, ...
- ... and the  $n$ -th player's probabilities of playing 1, 2, 3, ...

### Example (hawk-dove)

The 2-tuple  $(0.1, 0.6)$  is a joint strategy, where player 1 plays dove with probability 0.1 and player 2 plays dove with probability 0.6. (It is not a Nash equilibrium, but is still a valid joint strategy).

### Best response (BR)

BR is a correspondence (AKA multivalued function) mapping a joint strategy  $\mathbf{x}$  to one or more joint strategies. For example, Alice may have multiple best responses to a joint strategy  $\mathbf{x}$  where Bob plays 2 and Cindy plays 4 and so on.  $BR(\mathbf{x})$  is the set of joint strategies where each player maximizes her payoff in response to  $\mathbf{x}$ .

### Examples (hawk-dove)

- $p = 0$  is player 1's best response to  $q = 1$  (i.e., playing hawk is player 1's best response when player 2 plays dove). Therefore, if  $\mathbf{x}$  is the joint strategy  $(p = 1, q = 1)$ , then  $BR(\mathbf{x})$  will be  $\{(p = 0, q = 0)\}$ . Similarly, for  $\mathbf{x} = (p = 0, q = 1)$ ,  $BR(\mathbf{x}) = \{(p = 0, q = 1)\}$ .
- Any  $p$  (e.g.,  $p = 0, p = 1/2, p = 1/4, p = 1$ , etc.) is player 1's BR to  $q = 1/3$ , because when player 2 plays dove with probability  $1/3$ , player 1 becomes indifferent between its two actions. Therefore,  $BR(\mathbf{x})$  will be an infinite set when  $q = 1/3$  in  $\mathbf{x}$ .

### Nash equilibrium (NE)

A joint strategy is a NE if it represents simultaneous best responses of all the players. Let  $\mathbf{x}_{\text{Alice}}$  denote Alice's mixed strategy (her probabilities of playing 1, 2, etc.). A joint strategy  $(\mathbf{x}_{\text{Alice}}, \mathbf{x}_{\text{Bob}}, \mathbf{x}_{\text{Cindy}}, \dots)$  is a NE iff  $(\mathbf{x}_{\text{Alice}}, \mathbf{x}_{\text{Bob}}, \mathbf{x}_{\text{Cindy}}, \dots) \in BR(\mathbf{x}_{\text{Alice}}, \mathbf{x}_{\text{Bob}}, \mathbf{x}_{\text{Cindy}}, \dots)$ , which basically says that  $(\mathbf{x}_{\text{Alice}}, \mathbf{x}_{\text{Bob}}, \mathbf{x}_{\text{Cindy}}, \dots)$  is a fixed point of the BR correspondence. In other words, if you feed

$(\mathbf{x}_{\text{Alice}}, \mathbf{x}_{\text{Bob}}, \mathbf{x}_{\text{Cindy}}, \dots)$  into the BR correspondence, one of the joint strategies it will spit out is exactly the same  $(\mathbf{x}_{\text{Alice}}, \mathbf{x}_{\text{Bob}}, \mathbf{x}_{\text{Cindy}}, \dots)$ . Any such fixed point like  $(\mathbf{x}_{\text{Alice}}, \mathbf{x}_{\text{Bob}}, \mathbf{x}_{\text{Cindy}}, \dots)$  is a NE.

### Examples (hawk-dove)

- Player 1 playing hawk and player 2 playing dove ( $p = 0, q = 1$ ) is a NE, because it's a fixed point of the BR correspondence. I.e., for  $\mathbf{x} = (p = 0, q = 1)$ ,  $\text{BR}(\mathbf{x}) = \{ (p = 0, q = 1) \}$ .
- $(p = 1/3, q = 1/3)$  is another NE of the game, because it's also a fixed point of the BR correspondence. To see this,  $p = 1/3$  is one of the infinitely many best responses of player 1 to player 2's  $q = 1/3$  and vice versa. In other words,  $(1/3, 1/3) \in \text{BR}(1/3, 1/3)$ . Note that  $(0.1, 0.9)$  is also in  $\text{BR}(1/3, 1/3)$ , because  $p = 0.1$  (or any other probability for that matter) is one of player 1's best responses to player 2's  $q = 1/3$ . Similarly,  $q = 0.9$  is one of player 2's best responses to player 1's  $p = 1/3$ . However,  $(0.1, 0.9)$  is NOT a NE, because  $(0.1, 0.9) \notin \text{BR}(0.1, 0.9)$ . In fact,  $\text{BR}(0.1, 0.9) = \{ (p = 0, q = 1) \}$ . In other words,  $(0.1, 0.9)$  is not a NE, because it's not a fixed point of the BR correspondence.

### Proof Nash's Theorem

The proof strategy is to show that the BR correspondence has a fixed point. Nash did it by using Kakutani's fixed-point theorem.

### Kakutani's fixed-point theorem

A correspondence  $f: X \rightarrow X$  has a fixed point (i.e.,  $\mathbf{x} \in f(\mathbf{x})$  for some  $\mathbf{x} \in X$ ) if all of the following conditions hold.

- (1)  $X$  is a non-empty, closed, bounded, and convex set.
- (2)  $f(\mathbf{x})$  is non-empty for any  $\mathbf{x}$ .
- (3)  $f(\mathbf{x})$  is convex for any  $\mathbf{x}$ .
- (4) The set  $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x}) \}$  is closed.

Next is Nash's proof that the BR correspondence satisfies Kakutani's fixed-point theorem. The BR correspondence maps a joint strategy  $\mathbf{x}$  to potentially multiple joint strategies, each of which constitutes of the best responses of the players to  $\mathbf{x}$ . Let  $X$  be the set of *all* (i.e., infinite number of) joint strategies. Mimicking Kakutani's theorem, we can say that  $\text{BR}: X \rightarrow X$ . The following four points prove that BR has a fixed point (i.e., the game has NE).

**(1)  $X$  is a non-empty, closed, bounded, and convex set.** First, let's visualize  $X$ . If there are two players and each has two actions (e.g., hawk-dove game) then  $X$  is a square in a 2D plane as shown in Fig. 2. One dimension represents player 1's probability  $p$  of playing action 1 and the other dimension represents player 2's probability  $q$  of playing action 1.

If there are three players and each has two actions, then  $X$  will be a cube in three dimensions. It's hard to visualize examples with four or more players. However, it's easy to check the followings.

- The set  $X$  of joint strategies is **non-empty**.
- $X$  is a **closed** set, because  $X$  contains its boundary, AKA its limiting points in technical terms (and that's the definition of a "closed set"). See the black (square-shaped) boundary of  $X$  in the hawk-dove visualization in Fig. 2.

- Note: the opposite of any closed set is an open set. Example:  $[0, 1)$  is an open set, because it doesn't contain its boundary 1, even though it contains 0.99999999999.

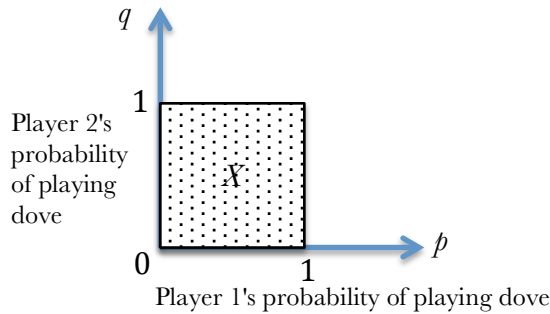


Figure 2: 2D joint strategy space of the hawk-dove game.

- $X$  is a **bounded** set, because any probability  $p$  has a lower bound of 0 and upper bound of 1 (i.e.,  $0 \leq p \leq 1$ ). (The definition of a "bounded set" is that the set can be bounded from "all sides.")
  - Note: Closed and bounded are not the same. The airy space strictly inside a soccer ball is bounded but not closed. It's bounded all around by the covering, but it's not closed since it doesn't contain the covering. On the other hand, the set of all integer numbers is closed but unbounded. It's closed because it contains its boundary integer numbers—whatever they are. It's unbounded because there's no way we can specify its upper and lower bounds.
- Last,  $X$  is **convex**, because if we take any two points  $\mathbf{x}_1, \mathbf{x}_2 \in X$  and connect  $\mathbf{x}_1$  and  $\mathbf{x}_2$  by a line, then each point on that line is also in  $X$  (and that's the definition of "convexity"). Verify this using Fig. 2.

(2) **BR( $\mathbf{x}$ ) is non-empty for any joint strategy  $\mathbf{x}$ .** This is because every player has a best response to the other players' strategies—whatever those strategies are.

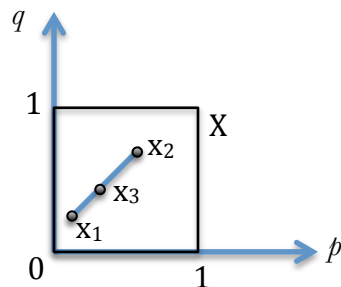


Figure 3: Two best responses  $\mathbf{x}_1$  and  $\mathbf{x}_2$  to  $\mathbf{x}$  in the joint strategy space  $X$  of a fictitious 2-player, binary action game. The joint strategy  $\mathbf{x}$  is not shown here.

(3) **BR( $\mathbf{x}$ ) is convex for any joint strategy  $\mathbf{x}$ .** To see this, let  $\mathbf{x}_1$  and  $\mathbf{x}_2$  be *any* two joint strategies in  $\text{BR}(\mathbf{x})$ . In other words,  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are two points in  $X$  (see Fig. 3 for visualization), with the additional property that in  $\mathbf{x}_1$  and  $\mathbf{x}_2$  everyone plays his/her best response to  $\mathbf{x}$ . We'll show that if we connect  $\mathbf{x}_1$  and  $\mathbf{x}_2$  by a line, then every point on that line is also a best response to  $\mathbf{x}$  (and that's the definition of convexity). Let  $\mathbf{x}_3 = \lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2$  be a point on that line, for

some  $\lambda$  between 0 and 1, both inclusive (this is a common way of representing all possible points on the line segment between  $\mathbf{x}_1$  and  $\mathbf{x}_2$ ).

By the definition of expected payoffs (page 1), every player gets the same payoff in  $\mathbf{x}_3$  as in either  $\mathbf{x}_1$  or  $\mathbf{x}_2$ . E.g., Alice's expected payoff in  $\mathbf{x}_3 = \lambda \times$  her expected payoff in  $\mathbf{x}_1 + (1 - \lambda) \times$  her expected payoff in  $\mathbf{x}_2 = (\lambda + 1 - \lambda) \times$  her expected payoff in either  $\mathbf{x}_1$  or  $\mathbf{x}_2$ .

**(4) The set  $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in \mathbf{BR}(\mathbf{x}) \}$  is closed.** Remember that the expected payoff of a player is multilinear (and therefore continuous) in the mixed-strategies of the players. This implies that if we plot the best responses of any player, the  $(\mathbf{x}, \mathbf{y})$  plot will not have any "holes" in it. See Fig. 4 for an illustration of what's possible and Fig. 5 for what's impossible. Such  $(\mathbf{x}, \mathbf{y})$  plot—AKA graph—is closed, because it contains its limiting points (and that's the definition of "closed"). In other words, if you trace any sequence of points on the graph, you'll never have to come to a screeching halt. In contrast, if you look at the horizontal red (thick) line segment at  $q = 1$  in Fig. 5 (the impossible case), it does not contain its limiting point due to the hole.

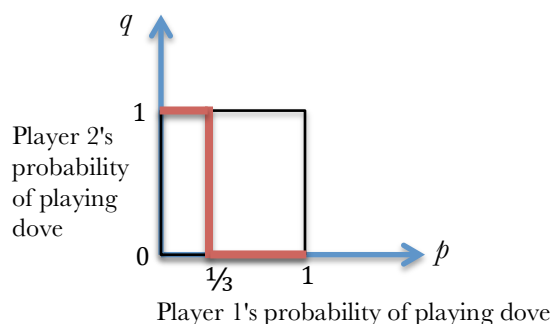


Figure 4: Player 2's best response to player 1's mixed strategy  $p$  is shown in red (thick line).

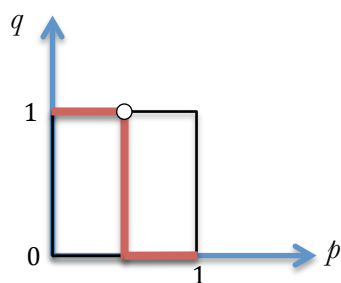


Figure 5: A best response correspondence (shown in red/thick line) with a hole (white dot) in it is impossible.

The above four points complete the proof that the BR correspondence has a fixed point. In other words, any finite game has a Nash equilibrium.

## Reference

Nash, J. F. (1950). Equilibrium points in  $n$ -person games. *Proceedings of the national academy of sciences*, 36(1), 48-49.