DIFFERENTIABLE FUNCTIONS DEFINED IN CLOSED SETS. I†

BY

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1. **Introduction.** In a recent paper§ the author has shown that if a function $f(x)$ defined in a closed set A in n-space E satisfies certain conditions involving Taylor's formula (in finite form), i.e. if it is "of class C^m in A," then its definition can be extended over E so that it will have continuous partial derivatives through the mth order. In this paper we restrict ourselves to the one-dimensional case. (For the above theorem in this case, see §4.) Let $x_0, \cdots, x_m$ be distinct points of A. If $P(x)=c_0+\cdots+c_mx^m$ is the polynomial of degree at most m such that $P(x_i)=f(x_i)(i=0,\cdots,m)$, the mth difference quotient of $f(x)$ at these points is $\Delta_{0\ldots m}f=\Delta^mf(x)=m!c_m$. The main object of this paper is to prove (see §§2 and 3 for definitions)

THEOREM I. *A necessary and sufficient condition that $f(x)$ be of class C^m in A is that $\Delta^mf(x)$ converge in A.*

This theorem furnishes a direct definition of the differentiability of a function; the former definition (see §3) involved the existence of other functions $f_1(x), \cdots, f_m(x)$.

The necessity of the condition is easily proved. The definition of $f(x)$ being extended over the x-axis E, consider any $m+1$ points $x_0, \cdots, x_m$ $(x_0<x_1<\cdots<x_m)$. Define $P(x)$ as above. As $f(x_i)-P(x_i)=0(i=0,\cdots,m)$ there is a point $x'(x_0<x'<x_m)$ such that $(d^m/dx^m)[f(x')-P(x')]=0$. But $d^mP(x)/dx^m\equiv m!c_m=\Delta_{0\ldots m}f$; hence $\Delta_{0\ldots m}f=d^mf(x')/dx^m$. Therefore if $x_0, \cdots, x_m$ are in A and are sufficiently near a point x^* of A, $\Delta_{0\ldots m}f=d^mf(x')/dx^m=d^mf(x^*)/dx^m$ approximately, and $\Delta^mf(x)$ converges in A (in fact, in E). This may be proved also from (2.6) for $s=m$.

We note that, for $f(x)=f_0(x)$ to be of class C^m in a general closed set A, it is not sufficient that there exist functions $f_s(x)$ $(s=1,\cdots,m)$ in A such that $df_s(x)/dx=f_{s+1}(x)$ there. As an example, set $f_0(0)=0$ and $f_0(x)=1/2^{2i}$ $(1/2^i\leqq x\leqq 3/2^{i+1}, i=1, 2, \cdots)$, and set $f_1(x)\equiv 0$ and $f_2(x)\equiv 0$ in the same point set A.

The majority of the paper is devoted to the proof of Theorem I. In the

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§ *Analytic extensions of differentiable functions defined in closed sets*, these Transactions, vol. 36 (1934), pp. 63–89; this paper will be referred to as A.E.

last section we study Taylor's formula in finite form, when it holds in closed sets, and when its validity implies differentiability of the given function.

2. **Difference quotients.**† If $x_0, \cdots, x_m$ are distinct numbers, set‡

$$(2.1)\quad u_{ij} = x_j - x_i, \quad r_{ij} = |u_{ij}|, \quad \overset{i}{\alpha}_{01\cdots m} = \frac{1}{u_{0i}\cdots u_{i-1,i}u_{i+1,i}\cdots u_{mi}}.$$

Given a function $f(x)$, we define the mth difference quotient by the formula

$$(2.2)\quad \Delta^m f(x) = \Delta(x_0, x_1, \cdots, x_m; f) = \Delta_{01\cdots m} f = m! \sum_{i=0}^{m} \overset{i}{\alpha}_{01\cdots m} f(x_i).$$

In particular, $\Delta_0 f = f(x_0)$, $\Delta_{01} f = [f(x_1) - f(x_0)]/(x_1 - x_0)$. $\Delta_{0\cdots m}$ is symmetric in the points $x_0, \cdots, x_m$.

If $i \geqq 2$,

$$\frac{1}{u_{01}}(\overset{i}{\alpha}_{12\cdots s} - \overset{i}{\alpha}_{02\cdots s}) = \frac{1}{u_{01}}\left(\frac{1}{u_{1i}\cdots} - \frac{1}{u_{0i}\cdots}\right) = \frac{1}{u_{01}}\,\frac{u_{0i} - u_{1i}}{u_{0i}u_{1i}\cdots} = \overset{i}{\alpha}_{012\cdots s};$$

hence

$$(2.3)\quad \begin{aligned} \frac{s}{u_{01}}(\Delta_{12\cdots s} - \Delta_{02\cdots s}) &= \frac{s!}{u_{01}}\Big[-\overset{0}{\alpha}_{02\cdots s} f(x_0) + \overset{1}{\alpha}_{12\cdots s} f(x_1) \\ &\qquad + \sum_{i\geqq 2} (\overset{i}{\alpha}_{12\cdots s} - \overset{i}{\alpha}_{02\cdots s}) f(x_i)\Big] \\ &= s! \sum_{i=0}^{s} \overset{i}{\alpha}_{012\cdots s} f(x_i) = \Delta_{012\cdots s}. \end{aligned}$$

Suppose $*$ is a set of subscripts containing neither 0, 1, nor 2; then for some m,

$$\Delta_{012*} = \frac{m}{u_{01}}(\Delta_{12*} - \Delta_{02*}) = \frac{m}{u_{02}}(\Delta_{12*} - \Delta_{01*}).$$

Solving for Δ_{01*}, we find

$$(2.4)\quad \Delta_{01*} = \frac{u_{02}}{u_{01}}\Delta_{02*} + \frac{u_{21}}{u_{01}}\Delta_{21*},$$

which may be written as follows: $u_{01}\Delta_{01*} + u_{12}\Delta_{12*} + u_{20}\Delta_{20*} = 0$.

Let $x_0, \cdots, x_s$ be distinct numbers. If we solve the equations $\sum_{i=0}^{s}$

† Compare Nörlund, *Differenzenrechnung*, Berlin, 1924, pp. 8–9. It is seen that $\Delta_{01\cdots m} = m! \cdot [x_0 x_1 \cdots x_m]$.

‡ In the equations below, the numbers $0, 1, \cdots$, when appearing as subscripts, are to be considered as variables. Thus, as a particular case of (2.1), $\alpha^0_{023} = 1/(u_{20}u_{30})$; in the second equation of §6, $\Sigma_i 1/u_{i'i} = 1/u_{0'i} + \cdots$. Without this notation, the equations would often get quite cumbersome.

$(x_i-x)^j z_i=\delta_{js}(j=0,\cdots,s)$, x being any fixed number, we find $z_i=\alpha^i_{0\cdots s}$. Hence

$$\sum_{i=0}^{s} \alpha^i_{0\cdots s}(x_i - x)^j = 0 \qquad (j = 0,\cdots, s-1),$$
$$\sum_{i=0}^{s} \alpha^i_{0\cdots s}(x_i - x)^s = 1. \tag{2.5}$$

Suppose $f(x)=f_0(x),\cdots,f_m(x)$, $R(x',x)=R_0(x',x)$ satisfy (3.1) below for $s=0$. Then (2.2) and (2.5) give

$$\Delta_{0\cdots s}f = s!\sum_{i=0}^{s}\alpha^i_{0\cdots s}\left[\sum_{j=0}^{m}\frac{f_j(x)}{j!}(x_i-x)^j + R(x_i,x)\right]$$
$$= f_s(x) + s!\sum_{j=s+1}^{m}\frac{f_j(x)}{j!}\sum_{i=0}^{s}\alpha^i_{0\cdots s}(x_i-x)^j + s!\sum_{i=0}^{s}\alpha^i_{0\cdots s}R(x_i,x). \tag{2.6}$$

If $f(x)=c_0+\cdots+c_mx^m$ is a polynomial of degree at most m, then (3.1) is satisfied with $f_m(x)\equiv m!c_m$ and $R_s(x',x)\equiv 0$. Setting $s=m$ in (2.6) gives

$$\Delta_{0\cdots m}f \equiv m!c_m. \tag{2.7}$$

We say $\Delta^m f(x)$ *converges in the set* A if for each point x of A and every $\epsilon>0$ there is a $\delta>0$ such that if $x_0,\cdots,x_m$, $x_{0'},\cdots,x_{m'}$ are any two sets of distinct points of A, all within δ of x, then

$$\left|\Delta_{0\cdots m} - \Delta_{0'\cdots m'}\right| < \epsilon.$$

$\Delta^m f(x)$ of course converges at all isolated points of A. We say $\Delta^m f(x)\to f_m(x)$ in A if $|\Delta_{0\cdots m}-f_m(x)|<\epsilon$ whenever $x_0,\cdots,x_m$ are in A and within δ of x. Evidently *if* $\Delta^m f(x)\to f_m(x)$ *in* A, *then* $f_m(x)$ *is continuous in the set of limit points of* A.

Differentiable functions

3. Definition of differentiable functions. Let $f(x)=f_0(x)$ be defined in the closed set A. We say $f(x)$ *is of class* C^m *in* A (see A. E.) if there exist functions $f_1(x),\cdots,f_m(x)$, $R(x',x)=R_0(x',x),\cdots,R_m(x',x)$ in A such that

$$f_s(x') = \sum_{i=s}^{m}\frac{f_i(x)}{(i-s)!}(x'-x)^{i-s} + R_s(x',x) \qquad (s=0,\cdots,m), \tag{3.1}$$

and for each s, each point x of A, and every $\epsilon>0$ there is a $\delta>0$ such that

$$\left|\frac{R_s(x'',x')}{(x''-x')^{m-s}}\right| < \epsilon \qquad (x', x'' \text{ in } A;\ |x'-x|,\ |x''-x| < \delta). \tag{3.2}$$

If $f_i(x),\cdots,f_m(x)$, $R_i(x',x)$ satisfy (3.1) and (3.2) for $s=i$, we say $f_i(x)$

can be expanded in a Taylor's formula to the $(m-i)$*th order locally uniformly in terms of* $f_i(x), \cdots, f_m(x)$. If $f(x)$ is defined throughout an open interval and has a continuous mth derivative there, then it is of class C^m, by Taylor's theorem.

4. **Extension of differentiable functions.** *If* $f_0(x)$ *is of class* C^m *in terms of* $f_0(x), \cdots, f_m(x)$ *in* A, *then the definitions of these functions can be extended throughout* E *so they will be continuous and so that* $df_s(x)/dx = f_{s+1}(x)$ *there* $(s=0, \cdots, m-1)$ (see A. E., Lemma 2). As the proof can be given more simply in the one-dimensional case, we give it here. We can assume A is unbounded on both sides; otherwise, take a point a beyond A on either side, and set $f_s(x) \equiv 0$ $(s=0, \cdots, m)$ beyond a.

For each interval (a, b) of $E-A$, let $P(x)$ be the polynomial of degree at most $2m+1$ such that

$$\frac{d^s}{dx^s}P(a) = f_s(a), \qquad \frac{d^s}{dx^s}P(b) = f_s(b) \qquad (s = 0, \cdots, m); \tag{4.1}$$

we set

$$f_s(x) = \frac{d^s}{dx^s}P(x) \text{ in } (a, b). \tag{4.2}$$

$df_s(x)/dx = f_{s+1}(x)$ $(s=0, \cdots, m-1)$ in $E-A$; we must show that this holds also at any point x_0 of A.

Suppose each $f_{s+1}(x)$ is continuous in E. Then given x_0 in A and $\epsilon > 0$, take $\delta > 0$ so small that

$$|f_{s+1}(x') - f_{s+1}(x_0)| < \frac{\epsilon}{2} \qquad (|x' - x_0| < \delta).$$

By (3.1) and (3.2), we can also take δ so small that if a is in A, $|a-x_0| < \delta$, and

$$f_s(a) = f_s(x_0) + f_{s+1}(x_0)(a - x_0) + R'(a, x_0),$$

then $|R'(a, x_0)/(a-x_0)| < \epsilon/2$. Now take any point x within δ of x_0. If x is in A, set $a=x$; otherwise, let a be the end point nearest x_0 of the interval of $E-A$ containing x. Now for some x', $a \leqq x' \leqq x$,

$$f_s(x) = f_s(a) + f_{s+1}(x')(x - a).$$

Adding this to the last equation and dividing by $x - x_0$, we find

$$\frac{f_s(x) - f_s(x_0)}{x - x_0} = f_{s+1}(x_0) + [f_{s+1}(x') - f_{s+1}(x_0)]\frac{x - a}{x - x_0} + \frac{R'(a, x_0)}{x - x_0}.$$

As $|x'-x_0|<\delta$, $|x-a|\leqq|x-x_0|$ and $|x-x_0|\geqq|a-x_0|$,

$$\left|\frac{f_s(x)-f_s(x_0)}{x-x_0}-f_{s+1}(x_0)\right|<\epsilon \qquad (|x-x_0|<\delta),$$

as required. (We have given here the details of A. E., Lemma 1.)

We must prove still that each $f_s(x)$ is continuous at each point x_0 of A; it is of course true in $E-A$. As $f_s(x)$ is continuous in A, it is sufficient to prove that for every $\epsilon>0$ there is a $\delta>0$ such that if (a, b) is any interval of $E-A$ lying within δ of x_0, then

$$|f_s(x)-f_s(a)|<\epsilon \qquad (a\leqq x\leqq b).$$

Take $\epsilon'<\epsilon/[2(m+1)^2K]$, where K is a number to be determined later. Let M be the maximum of $|f_i(x)|$ in A ($|x-x_0|\leqq 1$, $i=0,\cdots,m$). Take $\delta<\epsilon/(2mM)$ and <1 so small that (3.2) holds with ϵ replaced by ϵ' for any x, x' within δ of x_0. Now take any interval (a, b) of $E-A$ lying within δ of x_0. In (a, b), $f(x)$ equals

$$P(x)=\sum_{i=0}^{m}\frac{f_i(a)}{i!}(x-a)^i+\sum_{i=m+1}^{2m+1}\frac{\gamma_i}{i!}(x-a)^i,$$

where the γ_i are determined by the relations

$$\frac{d^s}{dx^s}P(b)=\sum_{i=s}^{m}\frac{f_i(a)}{(i-s)!}(b-a)^{i-s}+\sum_{i=m+1}^{2m+1}\frac{\gamma_i}{(i-s)!}(b-a)^{i-s}=f_s(b);$$

hence

$$\sum_{i=m+1}^{2m+1}\frac{\gamma_i}{(i-s)!}(b-a)^{i-s}=f_s(b)-\sum_{i=s}^{m}\frac{f_i(a)}{(i-s)!}(b-a)^{i-s}=R_s(b, a).$$

Solving for the γ_i, we find

$$\gamma_i=\sum_{j=0}^{m}K_{ij}\frac{R_j(b, a)}{(b-a)^{i-j}},$$

where the K_{ij} depend on m alone. Set $K=\max|K_{ij}|$; then

$$|\gamma_i|\leqq\sum_{j=0}^{m}\frac{K}{|b-a|^{i-m}}\left|\frac{R_j(b, a)}{(b-a)^{m-j}}\right|<\frac{(m+1)K}{|b-a|^{i-m}}\epsilon'.$$

Now if x is any point in (a, b), then $|x-a|\leqq|b-a|$, and

$$
\begin{aligned}
|f_s(x) - f_s(a)| &= \left| \frac{d^s}{dx^s} P(x) - f_s(a) \right| \\
&= \left| \sum_{i=s+1}^{m} \frac{f_i(a)}{(i-s)!}(x-a)^{i-s} + \sum_{i=m+1}^{2m+1} \frac{\gamma_i}{(i-s)!}(x-a)^{i-s} \right| \\
&< mM|x-a| + (m+1)K\epsilon' \sum_{i=m+1}^{2m+1} \frac{|x-a|^{i-s}}{|b-a|^{i-m}} \\
&< mM\delta + (m+1)^2K|b-a|^{m-s}\epsilon' < \epsilon,
\end{aligned}
$$

as required.

Theorem I, A perfect

5. A succession of lemmas culminates in Lemma 7, which is the sufficiency part of Theorem I for perfect sets.

Lemma 1. *Let A be a closed set, and let $\Delta^s f(x)$ converge on A. Then we can define $f_s(x)$ on the set of limit points A^* of A so that the following is true. Given x in A^* and $\epsilon>0$, we can choose a $\delta>0$ so that if $x_0, \cdots, x_s$ is any set of distinct points of A lying within δ of x, then $|\Delta_{0\ldots s}-f_s(x)|<\epsilon$.*

The proof is simple.

Lemma 2. *If $\Delta^s f_0(x)$ converges in the perfect set A, then for each point x of A and every $\epsilon>0$ there is a $\delta>0$ such that*

$$
|\Delta_{0\cdots t-1,t\cdots s-1} - \Delta_{0'\cdots(t-1)',t\cdots s-1}| < \epsilon \tag{5.1}
$$

($0 \leqq t \leqq s$) whenever all the points concerned lie within δ of x.

This is trivial if $t=0$. We assume it holds for numbers $0, \cdots, t-1$, and shall prove it for t. Given a point x_s distinct from all former points, the equations

$$
\Delta_{0\cdots t-1,t\cdots s-1,s} = \frac{s}{u_{0s}}(\Delta_{1\cdots t-1,t\cdots s} - \Delta_{0\cdots t-1,t\cdots s-1}),
$$

$$
\Delta_{0'\cdots(t-1)',t\cdots s-1,s} = \frac{s}{u_{0's}}(\Delta_{1'\cdots(t-1)',t\cdots s} - \Delta_{0'\cdots(t-1)',t\cdots s-1})
$$

give

$$
\begin{aligned}
\Delta_{0\cdots t-1,t\cdots s-1} - \Delta_{0'\cdots(t-1)',t\cdots s-1} &= (\Delta_{1\cdots t-1,t\cdots s} - \Delta_{1'\cdots(t-1)',t\cdots s}) \\
&\quad - \frac{1}{s}(u_{0s}\Delta_{0\cdots t-1,t\cdots s} - u_{0's}\Delta_{0'\cdots(t-1)',t\cdots s}).
\end{aligned} \tag{5.2}
$$

As $\Delta_{0\cdots s}$ converges, we can take $M>0$ and $\delta'<\epsilon/(4M)$ so that $|\Delta_{0\cdots s}|<M$ whenever $x_0, \cdots, x_s$ are within δ' of x. By induction, we can take $\delta<\delta'$ so small that the first term on the right in (5.2) is in absolute value $<\epsilon/2$ whenever all points concerned are within δ of x. Now given the points $x_0, \cdots, x_{s-1}$,

$x_{0'}, \cdots, x_{(t-1)'}$ within δ of x, let x_s be another such point; then (5.2) gives (5.1).

The lemma with $t=s$ shows that $\Delta^{s-1}f_0(x)$ converges in A.

LEMMA 3. *If $\Delta^m f_0(x)$ converges in the perfect set A, then there are continuous functions $f_1(x), \cdots, f_m(x)$ in A such that $\Delta^s f_0(x) \to f_s(x)$ in A $(s=1, \cdots, m)$.*

We prove this successively for $s=m, m-1, \cdots, 1$ with the help of Lemmas 1 and 2.

6. We proceed to the following lemma.

LEMMA 4. *If $\Delta^p g(x) \to g_p(x)$ and $\Delta^1 g(x) \to g_1(x)$ in the perfect set A, then $\Delta^{p-1}g_1(x) \to g_p(x)$ in A.*

Set $q=p-1$. If we apply the relation $\Delta^1\phi(x_i) \to d\phi(x_i)/dx_i$ to $\overset{i}{\alpha}_{0\cdots i\cdots q}$ as a differentiable function of x_i, we find

$$\frac{\overset{i'}{\alpha}_{0\cdots i'\cdots q} - \overset{i}{\alpha}_{0\cdots q}}{u_{ii'}} = -\overset{i}{\alpha}_{0\cdots q} \sum_{j\neq i} \frac{1}{u_{ji}} + \epsilon(x_{i'}),$$

where $\epsilon(x_{i'}) \to 0$ as $x_{i'} \to x_i$. Hence

$$\begin{aligned} \overset{i}{\alpha}_{0\cdots q} \sum_{j} \frac{1}{u_{j'i}} &= -\frac{\overset{i'}{\alpha}_{0\cdots i'\cdots q}}{u_{ii'}} + \overset{i}{\alpha}_{0\cdots q} \sum_{j\neq i}\left(\frac{1}{u_{j'i}} - \frac{1}{u_{ji}}\right) + \epsilon(x_{i'}) \\ &= -\frac{\overset{i'}{\alpha}_{0\cdots i'\cdots q}}{u_{ii'}} + \zeta_i(x_{0'}, \cdots, x_{q'}), \end{aligned}$$

where $\zeta_i(x_{0'}, \cdots, x_{q'}) \to 0$ as $x_{j'} \to x_j (j=0, \cdots, q)$. Consider the $2q$ points $x_0, x_{0'}, \cdots, x_q, x_{q'}$. We have

$$\Delta_{0\cdots jj'\cdots q}g = p!\left[\sum_{i=0}^{q} \frac{\overset{i}{\alpha}_{0\cdots q}}{u_{j'i}} g(x_i) + \frac{\overset{j'}{\alpha}_{0\cdots j'\cdots q}}{u_{jj'}} g(x_{j'})\right],$$

$$\begin{aligned} \frac{1}{p}\sum_{j=0}^{q} \Delta_{0\cdots jj'\cdots q}g &= (p-1)!\left[\sum_{i=0}^{q} \overset{i}{\alpha}_{0\cdots q} g(x_i) \sum_{j=0}^{q} \frac{1}{u_{j'i}} + \sum_{i=0}^{q} \frac{\overset{i'}{\alpha}_{0\cdots i'\cdots q}}{u_{ii'}} g(x_{i'})\right] \\ &= q! \sum_{i=0}^{q}\left[\overset{i'}{\alpha}_{0\cdots i'\cdots q} \frac{g(x_{i'}) - g(x_i)}{u_{ii'}} + g(x_i)\zeta_i(x_{0'}, \cdots, x_{q'})\right]. \end{aligned}$$

As $\Delta^1 g(x) \to g_1(x)$, this gives, letting $x_{j'} \to x_j$ $(j=0, \cdots, q)$,

$$\frac{1}{p}\lim \sum_{j=0}^{q} \Delta_{0\cdots jj'\cdots q}g = q! \sum_{i=0}^{q} \overset{i}{\alpha}_{0\cdots q} g_1(x_i) = \Delta_{0\cdots q}g_1. \tag{6.1}$$

Now given a point x of A and an $\epsilon>0$, take $\delta>0$ so that if $x_0, \cdots, x_j, x_{j'}, \cdots, x_q$ are within δ of x, then

$$\left|\Delta_{0\cdots jj'\cdots q}g - g_p(x)\right| < \epsilon.$$

Then if $x_0, \cdots, x_q$ are within δ of x, we find by adding points $x_{0'}, \cdots, x_{q'}$ within δ of x and letting $x_{i'}\to x_i$ $(i=0, \cdots, q)$ that

$$\left|\Delta_{0\cdots q}g_1 - g_p(x)\right| < \epsilon,$$

as required.

LEMMA 5. *If $\Delta^m f_0(x)$ converges in the perfect set A, then there are continuous functions $f_1(x), \cdots, f_m(x)$ such that $\Delta_p f_q(x)\to f_{p+q}(x)$ in A.*

This follows from Lemmas 3 and 4.

7. We now present the two final lemmas needed for the proof of Theorem I when A is perfect.

LEMMA 6. *Let $g(x)=g_0(x), \cdots, g_s(x)$ be defined in the perfect set A, and suppose $\Delta^s g(x)\to g_s(x)$. If $g(x)$ can be expanded in a Taylor's formula to the $(s-1)$th order in terms of $g_0(x), \cdots, g_{s-1}(x)$, then it can be expanded in a Taylor's formula to the sth order locally uniformly in terms of $g_0(x), \cdots, g_s(x)$.*

Given a point x of A and an $\epsilon>0$, take $\delta>0$ so that

$$(7.1)\qquad \left|\Delta_{0\cdots s} - g_s(x_0)\right| < \frac{s!}{2^s}\frac{\epsilon}{3}$$

whenever $x_0, \cdots, x_s$ are within δ of x (recall that $g_s(x)$ is continuous, by §2). Take any two points x_0 and x_s of A within δ of x; we must show that $\left|R^{(s)}(x_s, x_0)\right|/r_{0s}^s<\epsilon$.

Take δ' so small that if $|x'-x_0|<\delta'$, then

$$\left|\frac{R^{(s-1)}(x', x_0)}{(x' - x_0)^{s-1}}\right| < \frac{r_{0s}}{2^{2s}s}\frac{\epsilon}{3},$$

where $R^{(s-1)}(x', x_0)=g(x')-\sum_{j=0}^{s-1} g_j(x_0)(x'-x_0)^j/j!$. Take $M>|g_s(x_0)|$. Take a point x_{s-1} in A within δ' of x_0 and so close to x_0 that

$$\frac{r_{0,s-1}}{r_{0s}} < \frac{s!}{2^sM}\frac{\epsilon}{3} \quad\text{and}\quad < \frac{1}{2},$$

and (if $s>2$) take in succession points $x_{s-2}, \cdots, x_1$ in A so that

$$(7.2)\qquad r_{0,t-1} < \tfrac{1}{2}r_{0t} \qquad (t = 2. \cdots, s - 1);$$

let these points lie within δ of x. Then if $i<s$,

$$\left|\overset{i}{\alpha}_{0\cdots s}R^{(s-1)}(x_i,x_0)\right| < \frac{1}{r_{si}}\,\frac{r_{0i}^{s-1}}{r_{0i}\cdots r_{i-1,i}r_{i+1,i}\cdots r_{s-1,i}}\,\frac{r_{0s}}{2^{2s}s}\,\frac{\epsilon}{3} < \frac{1}{2^s s}\,\frac{\epsilon}{3}.$$

Now

$$\begin{aligned}\frac{1}{s!}\Delta_{0\cdots s} &= \alpha^s g(x_s) + \sum_{i=0}^{s-1}\alpha^i\left[\sum_{j=0}^{s-1}\frac{g_j(x_0)}{j!}u_{0i}^{j} + R^{(s-1)}(x_i,x_0)\right]\\ &= \alpha^s g(x_s) - \alpha^s\sum_{j=0}^{s-1}\frac{g_j(x_0)}{j!}u_{0s}^{j} + \sum_{i=0}^{s-1}\alpha^i R^{(s-1)}(x_i,x_0),\end{aligned}$$

on account of (2.5). Therefore

$$\begin{aligned}R^{(s)}(x_s,x_0) &= g(x_s) - \sum_{j=0}^{s}\frac{g_j(x_0)}{j!}u_{0s}^{j}\\ &= \frac{\Delta_{0\cdots s}}{s!\alpha^s} - \frac{g_s(x_0)u_{0s}^{s}}{s!} - \sum_{i=0}^{s-1}\frac{\alpha^i}{\alpha^s}R^{(s-1)}(x_i,x_0),\end{aligned}\tag{7.3}$$

and as $r_{is}/r_{0s} \leqq (r_{0s}+r_{0i})/r_{0s} = 1 + r_{0i}/r_{0s}$,

$$\begin{aligned}\frac{\left|R^{(s)}(x_s,x_0)\right|}{r_{0s}^{s}} &\leqq \frac{r_{0s}\cdots r_{s-1,s}}{s!r_{0s}^{s}}\left|\Delta_{0\cdots s} - g_s(x_0)\right| + \frac{|g_s(x_0)|}{s!}\left|\frac{r_{0s}\cdots r_{s-1,s}}{r_{0s}} - 1\right|\\ &\quad + \sum_{i=0}^{s-1}\frac{r_{0s}\cdots r_{s-1,s}}{r_{0s}}\left|\alpha^i R^{(s-1)}(x_i,x_0)\right|\\ &< \frac{2^s}{s!}\,\frac{s!}{2^s}\,\frac{\epsilon}{3} + \frac{M}{s!}\left[\left(1+\frac{r_{00}}{r_{0s}}\right)\cdots\left(1+\frac{r_{0,s-1}}{r_{0s}}\right) - 1\right] + s\cdot 2^s\frac{1}{2^s s}\,\frac{\epsilon}{3} < \epsilon,\end{aligned}$$

as required.

LEMMA 7. *If $\Delta^m f(x)$ converges in the perfect set A, then $f_0(x)=f(x)$, $f_1(x),\cdots,f_m(x)$ can be defined in A so that $f(x)$ is of class C^m in A in terms of the $f_s(x)$ $(s=0,\cdots,m)$.*

We define $f_1(x),\cdots,f_m(x)$ by means of Lemma 5. Taylor's formula for each $f_s(x)$ holds to the 0th order, as $f_s(x)$ is continuous (see §2). We prove in succession that it holds to the kth order for $k=1,\cdots,m-s$. This completes the proof of the lemma, and therefore of Theorem I for the case that A is perfect.

P-SETS AND *Q*-SETS

8. We shall prove a lemma which will be needed in the next part. Let $A'=a_1, a_2, \cdots$ be a set of isolated points, at least $m+1$ in number. With each point a_i we shall associate m other points $a_{i_1},\cdots,a_{i_m}$; these m points

together with a_i we say form the *Q-set* $Q(a_i)$. Take a Q-set Q_i, and let $a_{j_1}, \cdots, a_{j_s}$ be all those points such that $Q(a_{j_t}) = Q_i$; these points form the *P-set* P_i corresponding to Q_i. Each point of P_i is in Q_i. Each point a_i lies in just one P-set $P(a_i)$, as a_i is associated with just one Q-set $Q(a_i)$; however, a_i may lie in several Q-sets. Let $\delta(Q_i)$ be the greatest distance between pairs of points of Q_i.

LEMMA 8. *The P-sets and Q-sets may be so chosen that for any two points a_i and a_j,*

$$(8.1)\qquad \text{if}\quad \frac{\delta(Q(a_i)) + \delta(Q(a_j))}{|a_j - a_i|} > 2m, \quad \text{then } P(a_i) = P(a_j).$$

We first associate sets of points with certain of the limit points of the points $a_1, a_2, \cdots$ as follows. Let c_i be a point such that there is a sequence of points of A' approaching it from one side, say the left, while there is a nearest point of A' to c_i on the other side of c_i. Let μ equal $m+1$, or the number of points a_j between c_i and the next limit point c_k to the right of c_i if that number is smaller, and let $a_{j_1}, \cdots, a_{j_\mu}$ be the points nearest c_i on the right (counting from left to right). Let τ be the smallest of the numbers $|a_{j_t} - a_{j_s}|$ $(s, t = 1, \cdots, \mu)$ which are $> |a_{j_1} - c_i|$, if there are such. Let $a_1(c_i)$ be a point of A' to the left of c_i such that

$$(8.2)\qquad |c_i - a_1(c_i)| < |a_{j_1} - c_i|, \quad \text{and} \quad |a_{j_1} - a_1(c_i)| < \tau$$

if τ is defined. Let $a_2(c_i), \cdots, a_m(c_i)$ be points of A' lying between c_i and $a_1(c_i)$.

We now define the Q-sets. Given a point a_i, we associate another point with it as follows. Suppose, Case I, there is a point a_j whose distance from a_i is less than or equal to the distance from any other a_k to a_i; then we associate a_j with a_i, or that one of the pair a_j, a_k which lies to the left of a_i, if their distances from a_i are the same. Suppose, Case II, there is no such point. Then there is a limit point c_i nearer a_i than any point a_j. If there are two such points, we consider that one c_i on the left. The point we associate with a_i is then $a_1(c_i)$.

Suppose now we have associated a number of points with a_i, forming the set of points S. We associate the next point in a fashion much the same as above. If Case II has not occurred in associating the other points of S with a_i, we again have two cases to consider. Case I, there is a nearest point a_j to the set S; we then associate this point with S (or the point a_k, as above). Case II, there is none; then take the point c_i as above, and associate $a_1(c_i)$ with S. At any time we employ Case II, we immediately associate also the points $a_2(c_i)$, $a_3(c_i)$, $\cdots$ with S, till we have the required $m+1$ points $Q(a_i)$.

Note that the point we associate with S does not depend on which point a_i of S we started with. Also if Case I has occurred each time in forming the subset S of $Q(a_i)$, then there is no point a_k not in S which lies between two points of S.

9. To prove that (8.1) holds take any two points a_i and a_j; set $r_{ij}=|a_j-a_i|$.

(1) Suppose there are at most a finite number of points of A' between a_i and a_j. If $\delta(Q(a_i))+\delta(Q(a_j))>2mr_{ij}$, then either $\delta(Q(a_i))>mr_{ij}$ or $\delta(Q(a_j))>mr_{ij}$, say the former. Then there is a first time when, on adding a point a_k to a set S in forming $Q(a_i)$, the distance from a_k to S is $>r_{ij}$.

(a) In forming S from a_i, Case I has occurred each time. For if Case II had occurred, say in adding the point $a_1(c_i)$ to the subset S_1 of S, then a_k would be some $a_s(c_i)$; but the distance from a_k to S is then at most the distance from $a_s(c_i)$ to $a_1(c_i)$ which is less than the distance from $a_1(c_i)$ to S_1 which is by hypothesis $\leqq r_{ij}$.

(b) There is no point a_s whose distance from S is $\leqq r_{ij}$. For suppose there were; then Case II must occur in adding $a_k=a_1(c_i)$ to S, and c_i is nearer S than any point a_r. (If Case I occurred, a_s or a nearer point, not a_k, would be added to S.) Say c_i lies to the left of S. Let a_p and a_q be the left and right-hand end points of S respectively. As there is a point a_s distant $\leqq r_{ij}$ from S, $|a_p-c_i|<r_{ij}$. Suppose a_j is not in S. As there are no limit points between a_i and a_j, a_j lies to the right of S, and hence there is a first point a_r to the right of S. Then as a_i is in S, $|a_r-a_q|\leqq r_{ij}$. But as a_q and a_r are among the first $m+1$ (or μ) points to the right of c_i, and $|a_p-c_i|<|a_r-a_q|$, (8.2) gives $|a_p-a_k|<|a_r-a_q|\leqq r_{ij}$, a contradiction; therefore a_j is in S. As a_j is in S and $|a_p-c_i|<r_{ij}$, (8.2) gives $|a_p-a_k|<r_{ij}$, again a contradiction.

(c) S contains a_j. For otherwise (b) would be contradicted.

(d) In forming $Q(a_j)$, the points of S are chosen first. For suppose not. Then after perhaps adding some points of S to a_j, forming the set S', we choose a point a_l not in S. By (b), the distance from a_l to S is $>r_{ij}$. As there is a point in S whose distance from S' is at most r_{ij}, a_l must have been chosen under Case II; then the distance from some point c_j to S' is $<r_{ij}$. But then as c_j is a limit point of points a_s, there is a point a_s whose distance from S is $<r_{ij}$, a contradiction.

Now in forming both $Q(a_i)$ and $Q(a_j)$, the points of S are chosen first. As the remaining points chosen depend only on S, $Q(a_i)$ and $Q(a_j)$ must coincide; hence a_i and a_j lie in the same P-set.

(2) Suppose there is a limit point of isolated points b between a_i and a_j. In forming $Q(a_i)$, the set S at any step is at a distance $\leqq|b-a_i|$ from b; hence in adding the next point a_k to S, its distance from S is $\leqq|b-a_i|$ if Case I

occurs, and is $<2|b-a_i|$ if Case II occurs, by (8.2). Therefore $\delta(Q(a_i))<2m|b-a_i|$. Similarly $\delta(Q(a_j))<2m|b-a_j|$. Adding,

$$\delta(Q(a_i)) + \delta(Q(a_j)) < 2m(|b - a_i| + |b - a_j|) = 2mr_{ij},$$

completing the proof.

Remark. Given a point a_i, if there exist m points $a_{i_1}, \cdots, a_{i_m}$ such that the m intervals between $a_i, a_{i_1}, \cdots, a_{i_m}$ are all $\leqq \rho$, or if there exists a point a not in $Q(a_i)$ within ρ of a_i, then $\delta(Q(a_i))<2m\rho$. This follows from the proof in (2).

Theorem I, A closed

Each isolated point of A is enclosed in an interval; this gives a perfect set B. The definition of $f(x)$ is extended over B. With the help of Lemma 8 it is shown that $\Delta^m f(x)$ now converges over B. By Lemma 7, $f(x)$ is of class C^m in B; hence the same is true in A.

10. **The sets A' and B.** Let A_1 be the set of isolated points of the closed set A, let A_2 be the set of limit points of isolated points, and let A_3 be the remaining points of A. Let A' consist of A_1, together with certain other points as follows. A_1+A_2 being closed, let I be any open interval of $E-(A_1+A_2)$ containing points of A_3. If an end point a_i of I is in A_1, then there is, in I, a nearest point $a_1(a_i)$ of A_3 to a_i. We associate this point with a_i, and also points $a_2(a_i), \cdots, a_m(a_i)$ of A_3 in I, chosen so that

$$(10.1) \qquad |a_s(a_i) - a_1(a_i)| < |a_1(a_i) - a_i| \qquad (s = 2, \cdots, m).$$

A' is a set of isolated points; we may name them $a_1, a_2, \cdots$. A' is contained in A_1+A_3.

For each point a_i of A_1, let $d(a_i)$ be its distance from the rest of A, and let B_i be a closed interval of length $d(a_i)/2$, with a_i as center. Let the perfect set B be A plus all of these intervals. Arrange the points of A' into P-sets and Q-sets so as to obey Lemma 8. For each P-set P_i, let the corresponding P'-set P_i' contain the points of P_i, together with the points of any intervals B_j there may be which enclose points of P_i.

Given any set S of $m+1$ points in B, we shall define its *complexity* $\sigma(S)$ as follows. If all the points of S are in A, set $\sigma(S)=0$. If S contains $p>0$ points in $B-A$, and all these points lie in a single P'-set P_i', let q be the number of remaining points of S which do not lie in the corresponding Q-set Q_i, and set $\sigma(S)=pq$. The complexity of S is in this case certainly $\leqq m^2$. If S contains p points in $B-A$, and these points do not all lie in the same P'-set, set $\sigma(S)=m^2+p-1$. The complexity of any set S is $\leqq m^2+m$.

11. The following lemma together with Lemma 7 gives Theorem I.

LEMMA 9. *Let $f(x)$ be defined in the closed set A so that $\Delta^m f(x)$ converges in A. Then A can be enclosed in a perfect set B, the definition of $f(x)$ can be extended over B, and $f_m(x)$ can be defined in B, so that $\Delta^m f(x) \to f_m(x)$ in B.*

Define the sets A', B etc. as above. We may assume there are at least $m+1$ points in A'. Define $f_m(x)$ at each point of A_2+A_3 as in Lemma 1. Take a fixed interval B_i with center a_i; we define $f(x)$ and $f_m(x)$ over B_i as follows. Let $Q_i = Q(a_i)$ be the corresponding Q-set. Let

$$R_i(x) = \gamma_0 + \cdots + \gamma_m x^m \tag{11.1}$$

be the polynomial of degree at most m such that $R_i(x) = f(x)$ at each point of Q_i; then $\Delta(Q_i) = m!\gamma_m$, by (2.7). Set

$$f(x) = R_i(x), \quad f_m(x) = m!\gamma_m \text{ in } B_i. \tag{11.2}$$

The same polynomial $R_i(x)$ is used in defining $f(x)$ and $f_m(x)$ over each interval of the P'-set P_i' corresponding to Q_i; hence if S is any set of $m+1$ points such that all of its points in $B-A$ lie in P_i', and all remaining points lie in Q_i, then $f(x) = R_i(x)$ at each point of S, and therefore, by (2.7), $\Delta(S) = m!\gamma_m = \Delta(Q_i)$.

Each point x of A_3 is at a positive distance from $B-A$; by the definition of $f_m(x)$, $\Delta^m f(x) \to f_m(x)$ at such points. Each point x of $B-(A_2+A_3)$ is in an interval B_i; hence near x, $f(x)$ is a polynomial, and $\Delta^m f(x) \to f_m(x)$ there also. It remains to show that for each point x of A_2 and every $\epsilon > 0$ there is a $\delta > 0$ such that if S is any set of $m+1$ points of B within δ of x, then

$$|\Delta(S) - f_m(x)| < \epsilon. \tag{11.3}$$

By Lemma 1, we can take $\delta' > 0$ so that

$$|\Delta(S_0) - f_m(x)| < \frac{\epsilon}{(8m+8)^{m^2+m}} \tag{11.4}$$

for any set S_0 of $m+1$ points of A lying within δ' of x. Set $\delta = \delta'/(4m+2)$. We shall prove the following:

(A) *If S is any set in B, of complexity $\sigma(S) = \sigma$, composed of sets of points S_1 in $B-A$ and S_2 in A, and if S_1 lies within δ of x and S_2 lies within δ' of x, then*

$$|\Delta(S) - f_m(x)| < \epsilon_\sigma = \frac{\epsilon}{(8m+8)^{m^2+m-\sigma}}. \tag{11.5}$$

As $\sigma \leqq m^2+m$, (11.3) follows.

12. We note first that *if b_i is in some interval of the P'-set P_i', and b_i lies within δ of x, then Q_i lies within δ' of x*. Say a_i is the center of B_i; then a_i lies

within 2δ of x, a limit point of points of A'. Hence $\delta(Q(a_i))<4m\delta$, by the remark at the end of §9, and $Q_i=Q(a_i)$ lies within $(4m+2)\delta=\delta'$ of x.

We shall prove (A) first for $\sigma=0$, then for $\sigma>0$, using induction. Suppose $\sigma=0$. If S is in A, the fact follows from (11.4). If S contains points of $B-A$, then all these points lie in a single P'-set P_i', and the rest of S lies in the corresponding Q-set Q_i; hence $\Delta(S)=\Delta(Q_i)$. Q_i lies within δ' of x; hence (11.4) holds with S_0 replaced by Q_i or by S, and therefore (11.5) holds.

Now suppose (11.5) is proved for all sets S' with $\sigma(S')<\sigma$; we shall prove it for any set S with $\sigma(S)=\sigma$. Suppose first $\sigma>m^2$; then the points of S in $B-A$ lie in at least two P'-sets. Let P_i' and P_j' be two of these sets, let b_i and b_j be points of S (in $B-A$) in P_i' and P_j' respectively, and let a_i and a_j be the centers of the corresponding intervals. Let a_k be a point of $Q(a_i)$ not lying in S. If $S'=S-b_i-b_j$, then, by (2.4),

$$(12.1)\quad \Delta(S) = \Delta(b_i, b_j, S') = \frac{a_k - b_i}{b_j - b_i}\Delta(b_i, a_k, S') + \frac{b_j - a_k}{b_j - b_i}\Delta(a_k, b_j, S').$$

The sets $S'+b_i+a_k$ and $S'+b_j+a_k$ each contain fewer points of $B-A$ than S; hence their complexities are each $<\sigma$. Also $Q(a_i)$ and therefore a_k lie within δ' of x. Therefore, by induction,

$$(12.2)\quad |\Delta(b_i, a_k, S') - f_m(x)| < \epsilon_{\sigma-1}, \quad |\Delta(a_k, b_j, S') - f_m(x)| < \epsilon_{\sigma-1}.$$

As a_i and a_j lie in distinct P-sets, $\delta(Q(a_i))+\delta(Q(a_j))\leqq 2mr_{ij}$, by (8.1). As $|b_i-a_i|\leqq r_{ij}/4$ and $|b_j-a_j|\leqq r_{ij}/4$, $|b_j-b_i|\geqq r_{ij}/2$. As a_k and a_j lie in $Q(a_i)$ and $Q(a_j)$ respectively, $|a_j-a_k|\leqq\delta(Q(a_i))+\delta(Q(a_j))+r_{ij}\leqq(2m+1)r_{ij}$; hence $|b_j-a_k|<(2m+2)r_{ij}$. Also $|a_k-b_i|\leqq\delta(Q(a_i))+|a_i-b_i|<(2m+2)r_{ij}$; hence

$$(12.3)\qquad \left|\frac{a_k - b_i}{b_j - b_i}\right| < 4m + 4, \qquad \left|\frac{b_j - a_k}{b_j - b_i}\right| < 4m + 4.$$

This with (12.2) and (12.1) gives

$$\begin{aligned} |\Delta(S) - f_m(x)| &< \left|\frac{a_k - b_i}{b_j - b_i}\right| |\Delta(b_i, a_k, S') - f_m(x)| \\ &\quad + \left|\frac{b_j - a_k}{b_j - b_i}\right| |\Delta(a_k, b_j, S') - f_m(x)| \\ &< (8m + 8)\epsilon_{\sigma-1} = \epsilon_\sigma, \end{aligned}$$

as required.

Suppose now $0<\sigma\leqq m^2$; then the points of S in $B-A$ lie in a single P'-set P_i', and there are points of S not in $P_i'+Q_i$. Let b_i be a point of S in $B-A$, let a be a point of S not in $P_i'+Q_i$, and let a_k be a point of Q_i which is not in S. If $S'=S-b_i-a$, the sets $S'+b_i+a_k$ and $S'+a+a_k$ each have a smaller complexity than S. a_k lies within δ' of x, and hence, by induction, (12.2) holds with b_j replaced by a. Let a_i be the center of the interval B_i containing b_i.

Suppose, (1), $a=a_j$ is in A'. Then $|a_j-b_i|>r_{ij}/2$. As a_k is in $Q_i=Q(a_i)$ while a_j is not, $|a_k-b_i|<|a_k-a_i|+r_{ij}<(2m+1)r_{ij}$, by the remark, and $|a_j-a_k|<(2m+1)r_{ij}$. Hence (12.3) holds with b_j replaced by $a_j=a$, and (11.5) follows just as before. Suppose, (2), a is in $A-(A'+A_2)$. From a, move toward a_i to the first point a' in A_1+A_2. If a' is in A_1, move back to the first point $a_1(a')$ in A_3. Then $|a_1(a')-a'|\leqq|a-a_i|$ and $|a_s(a')-a_1(a')|<|a_1(a')-a'|\leqq|a-a_i|$ $(s=2,\cdots,m)$, by (10.1). Hence $\delta(Q_i)<2m|a-a_i|$, by the remark, and $|a_k-b_i|<(2m+1)|a-a_i|$, and $|a-a_k|<(2m+1)|a-a_i|$. As $|a-b_i|>|a-a_i|/2$, (12.3) and (11.5) follow, as before. If a' is in A_2, there are m points of A' nearer a_i than a, and again $\delta(Q_i)<2m|a-a_i|$ and (11.5) follows. Suppose finally, (3), a is in A_2. Again we must have $\delta(Q_i)<2m|a-a_i|$ and (11.5) follows. This completes the proof of (A), therefore of Lemma 9, and therefore of Theorem I.

Taylor's formula

13. **Conditions under which Taylor's formula is valid.** Taylor's formula for $f(x)$ may hold to the mth order in certain closed sets even if $f(x)$ is not of class C^m (see §14). We find here a difference quotient condition equivalent to the validity of Taylor's formula, at least for perfect sets.

Lemma 10. *If $f(x)=f_0(x)$ can be expanded in a Taylor's formula to the mth order locally uniformly in terms of $f_0(x),\cdots,f_m(x)$ in the closed set A, then these functions are continuous in A.*

It is apparent from (3.1) and (3.2) with $s=0$ that $f_0(x)$ is continuous. Take any s, $0<s\leqq m$. We shall assume $f_j(x)$ is continuous for $s<j\leqq m$, if there are such values of j, and shall prove that $f_s(x)$ is continuous.

Let $x_0,\cdots,x_s$ be distinct points of A. If we subtract (2.6) with x replaced by x_0 from the same equation with x replaced by x_1, we find

$$(13.1)\qquad \begin{aligned} f_s(x_1)-f_s(x_0) &= s!\sum_{j=s+1}^{m}\left[\frac{f_j(x_0)}{j!}\sum_{i=0}^{s}\overset{i}{\alpha}\,\overset{j}{u}_{0i}-\frac{f_j(x_1)}{j!}\sum_{i=0}^{s}\overset{i}{\alpha}\,\overset{j}{u}_{1i}\right]\\ &\quad + s!\sum_{i=0}^{s}\alpha^i[R(x_i,x_0)-R(x_i,x_1)]. \end{aligned}$$

Given any limit point x_0 of A and any $\epsilon>0$, take $\delta<\epsilon/[2^{s+3}(s+1)mM]$ (if $s<m$) and $<1/2$ so small that (3.2) holds with x and ϵ replaced by x_0 and $\epsilon/[2^{s+2}(s+1)!]$ respectively, where $M=\max|f_j(x')|$ $(|x'-x_0|\leqq 1,\ s<j\leqq m)$. If $s>1$, take a point x_s of A within δ of x_0, and take points $x_{s-1},\cdots,x_2$ of A so that $r_{0i}<r_{0,i+1}/3$ $(i=2,\cdots,s-1)$. Now take any point x_1 within δ

of x_0, so that $r_{01}<r_{02}/3$ if $s>1$. From (13.1) we see that $|f_s(x_1)-f_s(x_0)|<\epsilon$, as required (see the proof of Lemma 6).

Let $x_0, \cdots, x_s$ be an ordered set of points. We say they form an (x_0, ρ)-set $(\rho>1)$, if

$$r_{0,i-1} < \frac{r_{0i}}{\rho} \qquad (i = 1, \cdots, s). \tag{13.2}$$

THEOREM II. *Let $f(x)=f_0(x), \cdots, f_m(x)$ be defined in the closed set A. A necessary condition that a Taylor's expansion for $f(x)$ should hold to the mth order locally uniformly in terms of $f_0(x), \cdots, f_m(x)$ is that for each (or some) $\rho>1$, each s $(0\leqq s\leqq m)$, each point x of A, and each $\epsilon>0$, there exist a $\delta>0$, such that if $x_0, \cdots, x_s$ is any (x_0, ρ)-set of points lying within δ of x, then*

$$|\Delta_{0\cdots s}f - f_s(x)| < \epsilon.$$

By the last lemma, the $f_j(x)$ are continuous. Take M so that $|f_j(x')|<M$ for $|x'-x|<1$. Take $\delta<\epsilon(\rho-1)^s/[2(s+1)mM\rho^s]$ and <1 so that $|f_s(x')-f_s(x)|<\epsilon/2$ $(|x'-x|<\delta)$, and so that (3.2) holds with ϵ replaced by $\epsilon(\rho-1)^s/[2(s+1)!\rho^s]$. Now take any (x_0, ρ)-set of points $x_0, \cdots, x_s$ lying within δ of x. Then

$$\frac{r_{0i}}{r_{ki}} < \frac{\rho}{\rho - 1}$$

for $k\neq i$. For if $k<i$, then $r_{0k}\leqq r_{0,i-1}<r_{0i}/\rho$, hence $r_{ki}\geqq r_{0i}-r_{0k}>r_{0i}(1-1/\rho)$, and $r_{0i}/r_{ki}<1/(1-1/\rho)=\rho/(\rho-1)$; if $k>i$, then $r_{0k}\geqq r_{0,i+1}>\rho r_{0i}$, hence $r_{ki}\geqq r_{0k}-r_{0i}>r_{0i}(\rho-1)$, and $r_{0i}/r_{ki}<1/(\rho-1)<\rho/(\rho-1)$. Replacing x by x_0 in (2.6) gives immediately $|\Delta_{0\cdots s}f-f_s(x_0)|<\epsilon/2$; hence $|\Delta_{0\cdots s}f-f_s(x)|<\epsilon$.

THEOREM III. *If A is perfect, then the condition in Theorem* II *is also sufficient.*

We shall prove successively for $s=0, \cdots, m$ that $f(x)$ can be expanded in a Taylor's formula to the sth order locally uniformly in terms of $f_0(x), \cdots, f_s(x)$. Evidently $f_0(x)$ is continuous; hence this is true for $s=0$. The proof for a general s follows the proof of Lemma 6; we need merely be careful to choose $x_{s-1}, \cdots, x_1$ so that $r_{0,t-1}<r_{0t}/\rho$ $(t=2, \cdots, s)$.

14. **Taylor's formula and differentiability.** We shall say the set A has the property Z_ρ at the point $x(\rho>1)$ if there is an $\eta>0$ such that corresponding to any two points x_0 and x_1 of A within η of x, points $x_2, \cdots, x_s$ of A can be found such that

$$\frac{1}{\rho} < \left| \frac{x_i - x_j}{x_1 - x_0} \right| < \rho \qquad (i, j = 0, \cdots, s;\ i \neq j); \tag{14.1}$$

then $r_{ij}/r_{kl}<\rho^2$ for $i\neq j$, $k\neq l$. This condition is satisfied for instance by Cantor's set. s is any number $\leqq m$, m fixed.

THEOREM IV.* *Let A be a closed set having the property Z_ρ for some $\rho=\rho(x)$ at each point x, and let $f(x)=f_0(x), \cdots, f_m(x)$ be defined in A. A necessary and sufficient condition that $f(x)$ be of class C^m in terms of $f_0(x), \cdots, f_m(x)$ is that Taylor's formula for $f(x)$ should hold to the mth order locally uniformly in terms of $f_0(x), \cdots, f_m(x)$.*

In short, in this case, Taylor's formula for $f_0(x)$ implies Taylor's formula for each $f_s(x)$.

The necessity of the condition being trivial, we turn to the sufficiency. By Lemma 10, $f_m(x)$ is continuous. It remains to prove that for any s, $0<s<m, f_s(x)$ may be expanded in a Taylor's formula to the $(m-s)$th order locally uniformly in terms of $f_s(x), \cdots, f_m(x)$. We shall prove this for s, assuming it for numbers $s+1, \cdots, m$.

Let $x_0, \cdots, x_s$ be distinct points of A. Set

$$(14.2)\quad H_j = \sum_{i=0}^{s} \overset{i}{\alpha}\, \overset{j}{u}_{0i},$$

$$(14.3)\quad H_j' = \sum_{i=0}^{s} \overset{i}{\alpha}\, \overset{j}{u}_{1i} = \sum_{i=0}^{s} \overset{i}{\alpha}(u_{0i}-u_{01})^{j} = \sum_{i=0}^{s} \overset{i}{\alpha} \sum_{l} (-1)^{j-l}\binom{j}{l} \overset{l}{u}_{0i}\overset{j-l}{u}_{01}$$
$$= \sum\nolimits_l (-1)^{j-l}\binom{j}{l} H_l \overset{j-l}{u}_{01},$$

where $\sum_l$ means summation over all values of l. We can write (if $s<m$)

$$\sum_{j=s+1}^{m} \frac{f_j(x_1)}{j!} H_j' = \sum_{j=s+1}^{m} \frac{1}{j!} \sum_{k=j}^{m} \frac{f_k(x_0)}{(k-j)!} \overset{k-j}{u}_{01} \sum\nolimits_l (-1)^{j-l}\binom{j}{l} H_l \overset{j-l}{u}_{01} + R$$
$$= \sum_{k=s+1}^{m} \frac{f_k(x_0)}{k!} \sum\nolimits_l \overset{k-l}{u}_{01} H_l \sum_{j=s+1}^{k} (-1)^{j-l}\binom{k}{j}\binom{j}{l} + R,$$

where

$$R = \sum_{j=s+1}^{m} \frac{1}{j!} H_j' R_j(x_1, x_0).$$

Now if $k\geqq l>s$, then on replacing j by $k-j$ we find

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* For the special case that A is a closed interval, see a paper by the author, *Derivatives, difference quotients and Taylor's formula*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 89–94; Theorem III.

$$\sum_{j=s+1}^{k} (-1)^{j-l}\binom{k}{j}\binom{j}{l} = \sum_{j} (-1)^{k-j-l}\binom{k}{j}\binom{k-j}{l}$$
$$= (-1)^{k-l}\binom{0}{k-l} = \delta_{kl},*$$

and if $k>l=s$,

$$\sum_{j=s+1}^{k} (-1)^{j-l}\binom{k}{j}\binom{j}{l} = \delta_{ks} - (-1)^{s-s}\binom{k}{s}\binom{s}{s} = -\binom{k}{s}.$$

Therefore, as $H_l=0$ $(l<s)$ and $H_s=1$,

$$\sum_{j=s+1}^{m} \frac{f_j(x_1)}{j!} H_j' = \sum_{k=s+1}^{m} \frac{f_k(x_0)}{k!}\left[-\binom{k}{s} u_{01}^{k-s} + H_k\right] + R. \tag{14.4}$$

Putting this in (13.1) gives

$$\begin{aligned} f_s(x_1) = & \sum_{k=s}^{m} \frac{f_k(x_0)}{(k-s)!} u_{01}^{k-s} - \sum_{j=s+1}^{m} \frac{s!}{j!} \sum_{i=0}^{s} \alpha^i u_{1i}^j R_j(x_1, x_0) \\ & + s! \sum_{i=0}^{s} \alpha^i [R(x_i, x_0) - R(x_i, x_1)]. \end{aligned} \tag{14.5}$$

Given a point x of A and an $\epsilon>0$, take ρ and η corresponding to x, and take $\delta'<\eta$ so that (3.2) holds with δ and ϵ replaced by δ' and $\epsilon/[3m(s+1)!\rho^{2m}]$ and with s taking on the values $0, s+1, \cdots, m$. Set $\delta=\delta'/(2\rho)$. Now if x_0 and x_1 are points of A within δ of x, we can add points $x_2, \cdots, x_s$ of A so that (14.1) holds and these points will lie within δ' of x. Then

$$\left| \frac{\alpha^i u_{1i}^j}{u_{01}^{m-s}} R_j(x_1, x_0) \right| = \frac{r_{1i}^j}{r_{0i} \cdots r_{i-1,i}\, r_{i+1,i} \cdots r_{si}\, r_{01}^{j-s}} \frac{|R_j(x_1, x_0)|}{r_{01}^{m-j}} < \frac{\epsilon}{3m(s+1)!},$$

and similarly for the other remainder terms. Therefore $|R_s(x_1, x_0)|/r_{01}^{m-s}<\epsilon$, as required.

Corollary. *If $m\leqq 2$, Theorem* IV *holds for all closed sets.*

The only value of s we may need in the above proof is $s=1$; the condition Z_ρ is satisfied trivially if $s=1$.

Example. Theorem IV does not hold for all closed sets, as we now show, using $m=3$. Set $a_i=1/2^i$, $b_i=1/2^{2i}$, $c_i=1/2^{3i}$; $b_i'=a_i+b_i$, $c_i'=a_i+c_i$, $d_i=a_i+b_i-c_i (i=1, 2, \cdots)$. Let A be the set of points $0, a_i, c_i', d_i, b_i'$. Set $f_0(0)=f_1(0)=f_2(0)=f_3(0)=0$,

* See Netto, *Lehrbuch der Combinatorik*, Leipzig, 1927, §158, (27).

$$
\begin{array}{llll}
f_0(a_i) = 0, & f_0(c_i') = 0, & f_0(d_i) = 0, & f_0(b_i') = b_i^2 c_i, \\
f_1(a_i) = 0, & f_1(c_i') = 0, & f_1(d_i) = b_i^2 - b_i c_i, & f_1(b_i') = b_i^2 + b_i c_i, \\
f_2(a_i) = 0, & f_2(c_i') = 0, & f_2(d_i) = 2b_i, & f_2(b_i') = 2b_i, \\
f_3(a_i) = 0, & f_3(c_i') = 0, & f_3(d_i) = 0, & f_3(b_i') = 0.
\end{array}
$$

As $\Delta(0, a_i, c_i', d_i) = 0$, while $\Delta(a_i, c_i', d_i, b_i') = 3!\, b_i^2 c_i / [b_i(b_i - c_i)c_i] \to 6$ as $i \to \infty$, $\Delta^3 f_0(x)$ does not converge at $x = 0$, and hence $f_0(x)$ is not of class C^3, by Theorem I. However, Taylor's formula holds for $f_0(x)$ to the third order locally uniformly. For a calculation shows that $R(x, y) = 0$ whenever x and y are chosen from the points a_i, c_i', d_i, b_i', except that $R(b_i', a_i) = R(b_i', c_i') = b_i^2 c_i$, $R(c_i', d_i) = R(c_i', b_i') = b_i c_i (b_i - 2c_i)$; hence if x and y are chosen in any manner from the points a_i, c_i', d_i, b_i', $R(y, x)/(y-x)^3 \to 0$ as $i \to \infty$. Suppose now x_i and y_j are chosen from a_i, c_i', d_i, b_i', and from a_j, c_j', d_j, b_j' respectively, $j \neq i$ (or $x_i = 0$ or $y_j = 0$). If k is the larger of the numbers i, j, then

$$
|R(y_j, x_i)| < 2b_k^2 c_k + (b_k^2 + b_k c_k)(a_k + b_k) + b_k(a_k + b_k)^2,
$$

and as $|y_j - x_i|^3 \geqq a_k^3/8$, $R(y_j, x_i)/(y_j - x_i)^3 \to 0$ as $i, j \to \infty$ $(j \neq i)$. Hence for some $\delta > 0$, if x and y are any two points of A within δ of 0, $|R(y, x)/(y-x)^3| < \epsilon$. This is true also at each isolated point of A; hence Taylor's formula is valid.

Note that we may increase A to a perfect set by adding the intervals between a_i and c_i' and between d_i and b_i', and giving the obvious definitions of $f_0(x), \cdots, f_3(x)$ there. In this example, Taylor's formula holds to the required order for neither $f_1(x)$ nor $f_2(x)$.

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