Conformal Field Theory (WIMP)

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1 Introduction

1.1 Physics

Conformal field theory (CFT) is a type of quantum field theory which is invariant under conformal transformations whose famous applications are string theory and statistical mechanics. To explain the precise definition, there are some concepts to be introduced and the choice of the convention of notations to be specified. First of all, we only consider flat Minkowsky space $\mathbb{R}^d$.

Definition 1.1. A *metric tensor*, $g_{\mu\nu}$ is generalisation of Pythagorean theorem defined such that the infinitesimal arc length ds given by

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{1}$$

where $x \in \mathbb{R}^d$. The metric of Minkowsky space $\mathbb{R}^{p+q}$ is $\eta_{\mu\nu} \equiv diag(-1, ..., -1, 1, ..., 1)$ with p (-1)'s and q (+1)'s.

Note that the usual dot product, $\mathbf{V} \cdot \mathbf{W}$ can be represent in in tensor notation as $g_{\mu\nu}V^\mu W^\nu$.

Definition 1.2. An *action* S in physics is a scalar functional of a trajectory $q(x)$ in spacetime such that the "correct" equation of motion is the extremising function $q(x)$ of $S[q(x)]$. The *Lagrangian*, L is given by

$$S = \int dtL. \tag{2}$$

For a field theory of d+1 dimension with n fields, we instead defined *Lagrangian density* $\mathcal{L}$ such that

$$S[\varphi_i(x)] = \int dtd^dx\mathcal{L}(\varphi_i(x), \nabla\varphi_i(x), \partial_t\varphi_i(x)), \quad i \in [\![1, n]\!]. \tag{3}$$

Hence,

$$L = \int d^dx\mathcal{L}. \tag{4}$$

Definition 1.3. A physical system is said to have a *symmetry* if the action S is invariant up to the boundary terms under a coordinate transformation $x \to x'(x)$.

Definition 1.4. A *conformal group* is the subgroup of coordinate transformations that leaves the metric invariant up to a scale change i.e.

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \tag{5}$$

for some $\Omega(x)$. Such symmetry transformation $x \to x'(x)$ is called *conformal transformation*

Proposition 1. Conformal transformation preserves angle between vectors.

Proof. Let $V, W \in \mathbb{R}^d$. We know that $V \cdot W \equiv g_{\mu\nu}(x) V^\mu(x) W^\nu(x)$ and the angle $\theta \equiv V \cdot W / \sqrt{V^2 W^2}$. Under a conformal transformation $x \to x'(x)$,

$$\theta = \frac{V \cdot W}{\sqrt{V^2 W^2}} = \frac{g_{\mu\nu} V^\mu W^\nu}{\sqrt{g_{\alpha\beta} V^\alpha V^\beta g_{\rho\sigma} W^\rho W^\sigma}} \times \frac{\Omega}{\sqrt{\Omega^2}} = \theta'$$

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1.2 Lie Theory

I talked about "flat Minkowsky space" but more generally, what we are studying is a *differentiable manifold* that is not necessarily a flat space but, loosely speaking, a space that is locally flat *enough* to be considered as a vector space (flat Euclidean) which allow calculus possible (locally differentiable). One then can quite naturally define the notion of the globally differentiable tangent space about the point on a manifold as the set of all vectors lying tangentially on the point which is a vector space. Therefore, moving along a curve $\gamma(t)$ on a manifold $\mathcal{M}$ from a point x can be considered as a series of infinitely many infinitesimal transformation $x^\mu \to x^\mu + \varepsilon X^\mu$ where $\varepsilon \ll 1$ and $X^\mu \in T_x\mathcal{M}$, the tangent space of $\mathcal{M}$ about x. X is the tangent vector along $\gamma(t)$ at time t which generates the transformation.

Definition 1.5. A *Lie group* is a group of a set $\mathcal{M}$ equipped with a binary operation f$(\cdot,\cdot)$: $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ where $\mathcal{M}$ is a manifold and $f(x, y) = xy$ is a smooth map.

Definition 1.6. A *Lie algebra*, is a vector space $\mathfrak{g}$ over $\mathbb{F}$ with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following axioms

- Bilinearity
- Alternating property: $[X, X] = 0 \quad \forall X \in \mathfrak{g}$
- Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \forall X, Y, Z \in \mathfrak{g}$.

Any Lie group G gives a rise to a corresponding Lie algebra $\mathfrak{g}$ as a tangent space of the manifold G about the identity element $\equiv 1$ of G with the Lie bracket as a commutator $[X, Y] \equiv XY - YX$. Hence $\mathfrak{g}$ is the set of infinitesimal generators of G. Therefore, to move along from $x \in G$, we repeat infinitesimal action of the group $x \to x + \varepsilon Xx = (1 + \varepsilon X)x$ on x where $X \in \mathfrak{g}$ and $\varepsilon \in \mathbb{F}$. Then x will become

$$x \to \lim_{\varepsilon \to 0}(1 + \varepsilon X)^{\varepsilon^{-1}} x = e^X x = \left(\sum_{n=0}^{\infty} \frac{X^n}{n!}\right) x.$$

Another way (more formal) to think is under a infinitesimal diffeomorphism $x \to x' = x + \delta t X x$ where t is the parameter, we have

$$\begin{aligned} \delta x &= \delta t X x \\ \implies \frac{\delta x}{\delta t} &= X x \\ \implies x(t+0) &= e^{tX} x(0). \end{aligned}$$

Therefore exponentiating a element in the algebra results in a element in the group.

Theorem 1.1. (Baker–Campbell–Hausdorff formula) Let $\mathfrak{g}$ be a Lie algebra and $X, Y \in \mathfrak{g}$. Then the solution $Z \in \mathfrak{g}$ of the equation $e^Z = e^X e^Y$ is given by

$$Z \equiv \ln(e^X e^Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + ... \tag{6}$$

It is a series of nested Lie brackets. Therefore the Lie bracket determines the structure of the Lie group. Also, if $[X, Y] = 0$, then $e^{X+Y} = e^X e^Y$.

2 Conformal Theories in d-dimension

2.1 Conformal Groups in d > 2

Let the infinitesimal generators of the conformal group is a diffeomorphism $x^\mu \to x^\mu + \varepsilon^\mu(x)$ then the passive transformation in $\mathbb{R}^{p+q}$ where $p + q = d > 2$, is $x^\mu = x'^\mu + \varepsilon^\mu$.

$$\begin{aligned} &\implies \frac{\partial x^\alpha}{\partial x'^\mu} = \delta^\alpha_\mu + \frac{\partial \varepsilon^\alpha}{\partial x'^\mu} = \delta^\alpha_\mu + \frac{\partial \varepsilon^\alpha}{\partial x^i}\frac{\partial x^i}{\partial x'^\mu} = \delta^\alpha_\mu + \partial_i \varepsilon^\alpha(\delta^i_\mu + O(\varepsilon)) \\ &\implies g'_{\mu\nu} = (\delta^\alpha_\mu \delta^\beta_\nu + \delta^\beta_\nu \partial_\mu \varepsilon^\alpha + \delta^\alpha_\mu \partial_\nu \varepsilon^\beta) g_{\alpha\beta} + O(\varepsilon^2) \\ &\implies g'_{\mu\nu} = g_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu + O(\varepsilon^2) = \Omega g_{\mu\nu} \\ &\implies \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu \propto \eta_{\mu\nu} \end{aligned}$$

and

$$\begin{aligned} (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)\eta^{\mu\nu} &= 2\partial \cdot \varepsilon, \\ \eta^{\mu\nu}\eta_{\mu\nu} &= \delta^\mu_\mu = d \end{aligned}$$

$$\implies \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d}\partial \cdot \varepsilon \eta_{\mu\nu}. \tag{7}$$

From this equation alone, which is derived solely by the condition on the transformation of the metric (Eq.5), one can extract surprisingly large amount of information. Most importantly, taking $\eta_{\mu\nu}\partial_\mu$ on both sides, one gets

$$(\eta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu)\partial \cdot \varepsilon = 0 \tag{8}$$

where d'Alembertian, $\Box$ is defined as $\eta_{\mu\nu}\partial_\mu\partial_\nu$. Therefore we can conclude that $\varepsilon(x)$ is at most quadratic in x.

2.1.1 Constant $\varepsilon(x)$

Let $\varepsilon^\mu(x) = a^\mu$ where a is a constant vector. Then the generator is $a^\mu\partial_\mu$

$$x \to x' = e^{a^\mu \partial_\mu} x = x + a$$

This is a translation.

2.1.2 Linear $\varepsilon(x)$

Let $\varepsilon^\mu(x) = w^\mu{}_\nu x^\nu$. Then the generator is $w^\mu{}_\nu x^\nu \partial_\mu$

$$x^\mu \to x'^\mu = e^{w^\mu{}_\nu x^\nu \partial_\mu} x^\mu = x^\mu + w^\mu{}_\nu x^\nu$$

Substitute $\varepsilon(x)$ in the the Killing vector equation (Eq.7), if $\mu \neq \nu$

$$\begin{aligned} & w^\mu{}_\nu + w^\nu{}_\mu = 0 \\ & \implies w^\mu{}_\nu \quad \text{antisymmetric}, \\ & \implies x' = e^w x = \Lambda x, \quad \Lambda^\mu{}_\nu \in SO(p,q) \end{aligned}$$

This is a rotation in Euclidean or a Lorentz transformation in Minkowsky space.

Or let $\varepsilon^\mu(x) = l x^\mu$ where $l \in \mathbb{R}$. So the generator is $l x^\mu \partial_\mu$.

$$x' = e^{l x^\mu \partial_\mu} x = x + l x \equiv \lambda x, \quad l \in \mathbb{R}$$

This is a dilation.

2.1.3 Quadratic $\varepsilon(x)$

Let $\varepsilon^\mu(x) = b^\mu x^2 - 2x^\mu(b \cdot x)$. We then can easily check if $\varepsilon^\mu(x)$ is a Killing vector by (Eq.7). The generator is $b^\mu(x^2\partial_\mu - 2x^\mu x^\nu \partial_\nu)$.

$$\begin{aligned} x^\mu \to x'^\mu &= e^{b^\mu(x^2\partial_\mu - 2x^\mu x^\nu \partial_\nu)} x^\mu = x^\mu + x^2 b^\mu - 2(b\cdot x)x^\mu \\ \implies \frac{x'^\mu}{x'^2} &= \frac{x^\mu(1-2b\cdot x) + x^2 b^\mu}{x^2(1-2b\cdot x)^2}, \quad |b|\ |\varepsilon| \ll 1 \\ &= \frac{x^\mu(1-2b\cdot x + (b\cdot x)^2)}{x^2(1-2b\cdot x)^2} + b^\mu(1-b\cdot x)^{-2} + O(b^2) \\ &= \frac{x^\mu}{x^2} + b^\mu + O(b^2) \end{aligned}$$

This is an inversion + translation. Or inversion about the surface $1 + 2b \cdot x + b^2 x^2 = 1$. This is called a special conformal transformation (SCT).

Remark. Liouville's theorem states that these four types of transformation are the only possible action of the conformal group in $d > 2$ and the corresponding conformal algebra (i.e. commutation relation of the generators) is isomorphic to $\mathfrak{so}(p+1, q+1)$

2.2 Conformal Algebra in 2-dimensions

2.2.1 Complex Representation

Consider $\mathbb{R}^{0+2}$. Hence $g_{\mu\nu} = \delta_{\mu\nu}$ if $d = 2$ then the Killing equation (Eq.7) implies

$$\partial_1 \varepsilon_1 = \partial_2 \varepsilon_2, \quad \partial_1 \varepsilon_2 = -\partial_2 \varepsilon_1. \tag{9}$$

Note that this set of equations is the Cauchy-Riemann condition for holomorphic functions if we complexify the coordinates $z(x) = x^1 + ix^2, \quad \bar{z}(x) = x^1 - ix^2$ and $\varepsilon(z(x)) \equiv \varepsilon^1(x) + i\varepsilon^2(x), \quad \bar{\varepsilon}(\bar{z}(x)) \equiv \varepsilon^1(x) - i\varepsilon^2(x). \quad \Longrightarrow \quad \partial \equiv \partial_z = (\partial_1 - i\partial_2)/2, \quad \bar{\partial} \equiv \partial_{\bar{z}} = (\partial_1 + i\partial_2)/2.$

2.2.2 Global Conformal Transformation

A global conformal transformation corresponds to an entire holomorphic function:

$$z \mapsto z' = f(z), \quad \bar{z} \mapsto \bar{z}' = \bar{f}(\bar{z})$$

On a compactified Riemann sphere $S^2 = \mathbb{C} \cup \infty$,

$$\mathrm{Conf}(S^2) = \{f(z) | f(z) = \frac{az+b}{cz+d}, (a,b,c,d) \in \mathbb{C}^4, ad - bc = 1\} \tag{10}$$

2.2.3 Local Algebra

From now on for everything I write in z coordinates, there is a notion with the same form in $\bar{z}$ coordinates We take for the basis (Laurent series) of infinitesimal transformation,

$$z \mapsto z' = z + \varepsilon_n(z) \equiv z - \varepsilon z^{n+1}, \quad n \in \mathbb{Z}, \quad \varepsilon \ll 1$$

then with a generator l_n,

$$\begin{aligned} z' = e^{\varepsilon l_n} z &= z + \varepsilon l_n z + O(\varepsilon^2) \\ &\Longrightarrow l_n = -z^{n+1}\partial \end{aligned}$$

Proposition 2. (Witt algebra) the local conformal algebra with generators $l_n = -z^{n+1}\partial$, $n \in \mathbb{Z}$ is an infinite dimensional algebra with following commutation relations

$$\begin{aligned} [l_m, l_n] &= (m-n) l_{m+n}, \\ [\bar{l}_m, \bar{l}_n] &= (m-n) \bar{l}_{m+n}, \\ [l_m, \bar{l}_n] &= 0, \quad \forall (m,n) \in \mathbb{Z}^2 \end{aligned} \tag{11}$$

Proof. The last relation is trivial as $[\partial, \bar{\partial}] = 0$ and showing the first relation automatically implies the second.

$$\begin{aligned}
[l_m, l_n] &= z^{m+1}\partial(z^{n+1}\partial) - z^{n+1}\partial(z^{m+1}\partial) \\
&= z^{m+n+1}(n + 1 - m - 1)\partial
\end{aligned}$$

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They are clearly closed under the commutator. But also l_m , $\bar{l}_m$ are closed separately. Let $\mathcal{A}$, $\bar{\mathcal{A}}$ be the algebras generated by each set of generators respectively, because of the third relation, the full local conformal algebra or *Witt algebra* is a direct sum, $\mathcal{A} \oplus \bar{\mathcal{A}}$. Since the action of the conformal group naturally factorises into independent coordinates z, $\bar{z}$, we may analytically continue the functions in 2d CFT into $\mathbb{C}^2$ treating z, $\bar{z}$ as independent variables where the 'real' 'physical' domain is the surface defined as $\bar{z} = z^*$.

2.2.4 Global Algebra and Möbius Transformation

The generators of the local algebra are not all well-defined globally on the Riemann sphere S^2. A holomorphic conformal transformation is generated by vector fields

$$v(z) = -\sum_{n \in \mathbb{Z}} a_n l_n$$

As $z \to 0$, $v(z)$ is non-singular only if $a_n = 0$ for $n < -1$. As $z \to \infty$, $v(z)$ is non-singular only if $a_n = 0$ for $n > 1$. The global algebra is generated by $l_0, l_{\pm 1}$ and its anti-holomorphic versions. Note that $l_{-1}, \bar{l}_{-1}$ generate translations, $l_0 + \bar{l}_0$ and $i(l_0 - \bar{l}_0)$ generates dilations and rotations (translation of r, θ in polar $z = re^{i\theta}$), and lastly $l_1, \bar{l}_1$ generates special conformal transformations.

The general form of the transformation generated by the global algebra, therefore, is the conformal group (Eq.10) which is also known as the *Möbius transformations*. In matrix representations, we can define a homomorphism $\phi : Conf(\mathbb{C} \cup \infty) \to GL(2, \mathbb{C})$ as

$$f(z) = \frac{az + b}{cz + d} \mapsto \phi(f) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where the determinant $= ad - bc = 1$. It can easily be shown that composition of functions $f(z)$ is equivalent to matrix multiplication. We know that this group is $SL(2, \mathbb{C})/Z_2 \cong SO(3, 1)$ the quotient is due to the equivalence relation $(a, b, c, d) \sim (-a, -b, -c, -d)$.

3 Conformal Theories in 2-dimensions

3.1 Correlation Functions of Primary Fields

Definition 3.1. A *primary field* $\Phi(z, \bar{z})$ of *conformal weight* $(h, \bar{h}) \in \mathbb{R}^2$ is a field that transforms as

$$\Phi(z, \bar{z}) \to (\partial f)^h (\bar{\partial} \bar{f})^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})) \tag{12}$$

under $z \to f(z)$ which implies

$$\delta_{\varepsilon,\bar{\varepsilon}} \Phi(z, \bar{z}) = ((h\partial\varepsilon + \varepsilon\partial) + a.h.)\Phi(z, \bar{z}) \tag{13}$$

where $a.h.$ denotes the anti-holomorphic form of the preceding expression. Any other fields are known as *secondary fields.*

Proposition 3. Let $\Phi(z, \bar{z})$ be a primary field of conformal weight $(h, \bar{h})$. Then $\Phi(z, \bar{z})dz^h d\bar{z}^{\bar{h}}$ is invariant under a conformal transformation $z \to f(z)$. Since $ds^2 \equiv dzd\bar{z} \to (\partial f)(\bar{\partial}\bar{f})ds^2$.

Definition 3.2. A *correlation function* or an *N-point function* is a Green function that gives the probability amplitude for finding $N - n$ particles at $z_{n+1}, ..., z_N$ given n initial values particle at $z_1, ..., z_n$ where z_i is the spacetime coordinate of the ith particle (field) ϕ_i. It is denoted as $\langle \phi_1(z_1)...\phi_N(z_N) \rangle$

3.1.1 Constraints of Conformal Invariance of Correlators

- translation symmetry $\implies$ only $z_i - z_j$ dependence
- rotational symmetry $\implies$ only $(N - 1)N/2$ distances $r_{ij} \equiv |z_i - z_j|$
- dilation symmetry $\implies$ only ratios of distances $\frac{r_{ij}}{r_{jk}}$ dependence
- inversion symmetry $\implies$ only cross ratios $\frac{r_{ij}r_{kl}}{r_{ik}r_{jl}}$ dependence

3.2 Radial Quantisation

In Minkowsky "space" and "time" coordinates are σ^1 and σ^0, the standard light cone coordinates would be $\sigma^0 \pm \sigma^1$. If we do the Wick rotation $\sigma^1 \to i\sigma^1$, in Euclidean they would be the complex coordinates $\zeta, \bar{\zeta} = \sigma^0 \pm i\sigma^1$. Compactifying the space coordinate as $\sigma^1 \equiv \sigma^1 + 2\pi$ allows us to think the quotient space to be a cylinder. Let the conformal map $\zeta \to z = exp(\zeta)$. We are back to the familiar complex plane and the Hilbert space of the states are quantised as the surfaces of constant radius.

3.2.1 Noether's Theorem

Simply put, physics is the solution of $\delta S = 0$ for some functional $S[\phi_i]$ and the dynamics is the result of varying the fields ϕ_i's. Loosely speaking, Noether stated that if we define a 'current' j^μ and stress tensor $T^{\mu\nu}$ such that

$$\delta S = \int d^d x j^\mu \delta_\mu \varepsilon = \int d^d x T^{\mu\nu} \delta g_{\mu\nu} \tag{14}$$

continuous symmetry ($\delta S = 0$ under diffeomorphism) $\implies$ $\partial_\mu j^\mu = 0$. j^μ is therefore conserved. From this very easily on can show that the $T^{\mu\nu}$ is traceless if $\delta g_{\mu\nu} \propto g_{\mu\nu}$. In z coordinates of the 2d CFT, we can show that only the diagonal components of $T^{\mu\nu}$ are non-zero using this property. We hence denote

$$T(z) \equiv T_{zz}(z), \quad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}).$$

Integrating j^μ over space at fixed time gives conserved 'charge' Q:

$$Q = \int j_r(\theta) d\theta = \frac{1}{2\pi i} \oint [dz T(z)\varepsilon(z) + a.h.]$$

where the orientation of the contour integral is anticlockwise.

Proposition 4. Let A, Q be square matrices and Q be hermitian and let $U = e^{-\varepsilon Q}$ then under $A \to U^{-1} A U$,

$$\delta_\varepsilon A = \varepsilon [Q, A]. \tag{15}$$

Proof.

$$\begin{aligned}
Q \quad &\text{hermitian} \implies U \quad \text{unitary} \\
\implies U^{-1} A U &= e^{\varepsilon Q} A e^{-\varepsilon Q} = (1 + \varepsilon Q + O(\varepsilon^2)) A (1 - \varepsilon Q + O(\varepsilon^2)) \\
&= A + \varepsilon [Q, A] + O(\varepsilon^2)
\end{aligned}$$

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In quantum mechanics, usually Q is hermitian. Therefore the variation of any field is given by

$$\delta_{\varepsilon,\bar{\varepsilon}} \Phi(w, \bar{w}) = \frac{1}{2\pi i} \oint ([dz T(z)\varepsilon(z), \Phi(w, \bar{w})] + [a.h., \Phi(w, \bar{w})]) \tag{16}$$

3.2.2 Operator Product Expansion (OPE)

The product of operators $A(z)B(w)$ are only defined if $|z| > |w|$.

Definition 3.3. A *radial ordering operation* R is defined as

$$R(A(z)B(w)) = \begin{cases} A(z)B(w), & \text{if } |z| > |w| \\ B(w)A(z), & \text{if } |z| < |w|. \end{cases}$$

Now we can evaluate the commutators in $\delta_{\varepsilon,\bar{\varepsilon}}\Phi(w,\bar{w})$ (Eq.16) as

$$\begin{aligned}\delta_{\varepsilon,\bar{\varepsilon}}\Phi(w,\bar{w}) = \frac{1}{2\pi i}\left(\oint_{|z|>|w|} - \oint_{|z|<|w|}\right)(dz\varepsilon(z)R(T(z)\Phi(w,\bar{w})) + a.h.) \\ = \frac{1}{2\pi i}\oint_{w}(dz\varepsilon(z)R(T(z)\Phi(w,\bar{w})) + a.h.) \\ = ((h\partial\varepsilon(w) + \varepsilon(w)\partial) + a.h.)\Phi(w,\bar{w})\end{aligned}$$

where the last line is desired if the infinitesimal variation is conformal. Note that it is in the same form as the variation of primary fields in CFT (Eq.16). Therefore we can drop the R for primary fields and the short distance singularities of T and $\bar{T}$ are

$$T(z)\Phi(w,\bar{w}) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{(z-w)}\partial\Phi(w,\bar{w}) + ... \tag{17}$$

3.3 Conformal Ward Identities

By global conformal invariance, these correlation functions satisfy with conformal transformation $w = f(z)$:

$$\begin{aligned}&\langle\oint_{w_i\forall i}\frac{dz}{2\pi i}\varepsilon(z)T(z)\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle \\ &= \sum_{j=1}^{n}\langle\phi_1(w_1,\bar{w}_1)...\left(\oint_{w_j}\frac{dz}{2\pi i}\varepsilon(z)T(z)\phi_j(w_j,\bar{w}_j)\right)...\phi_n(w_n,\bar{w}_n)\rangle\end{aligned}$$

which is true for arbitrary $\varepsilon(z)T(z)$ so the differential version (unintegrated) with the operator product expansion with $T(z)$ substituted in is

$$\begin{aligned}&\langle T(z)\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle = \\ &\sum_{j=1}^{n}\left(\frac{h_j}{(z-w_j)^2} + \frac{1}{(z-w_j)}\partial_{w_j}\right)\langle\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle\end{aligned} \tag{18}$$

4 Virasoro Algebra

4.1 The Central Charge

T is not a primary field. Its OPE with itself is

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial T(w) \tag{19}$$

which shows T's conformal weight is (2,0) and where c is a constant which depends on the theory.

Definition 4.1. A *central charge* is the constant c that appears in $T(z)T(w)$.

4.2 Mode Expansions

Consider the Laurent's series of T and $\bar{T}$:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \quad \implies \quad L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z). \tag{20}$$

To evaluate the commutators of the operator modes, use the same technique used in varying a field with the conserved charge.

$$\left[\oint \frac{dz}{2\pi i}, \oint \frac{dw}{2\pi i}\right] = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} = \oint_w \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i}.$$

Using the above relation, substituting $T(z)T(w)$ (Eq.19), integration by parts and residue formula results in the *Virasoro algebra*.

4.3 Virasoro Algebra

It is the algebra of L_n as generators given by

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} \frac{\bar{c}}{12}(n^3 - n)\delta_{n+m,0} \\ [L_n, \bar{L}_m] &= 0 \end{aligned} \tag{21}$$

Remark. Virasoro algebra with $c = \bar{c} = 0$ is isomorphic to the classical Witt algebra (Eq.11).

4.4 In- and Out-States

Definition 4.2. *Adjoint* of the operator A is defined as

$$(A(z, \bar{z}))^\dagger = A(1/\bar{z}, 1/z) \frac{1}{\bar{z}^{2h} z^{2\bar{h}}}. \tag{22}$$

Definition 4.3. *Vacuum state* $|0\rangle$ is the state in CFT that is invariant under conformal transformations.

Definition 4.4. *In-state* is the operator acting on the vacuum at time $\sigma^0 = -\infty$.

$$|A_{in}\rangle = \lim_{z,\bar{z}\to 0} A(z, \bar{z}) |0\rangle . \tag{23}$$

This is also known as the *operator-state correspondence.*

Definition 4.5. A *Out-state* is asymptotically defined in the similar way to in-states at time $\sigma^0 = \infty$. Using the conformal map $w = 1/z$,

$$\begin{aligned} \langle A_{out}| &= \lim_{w,\bar{w}\to 0} \langle 0| \tilde{A}(w, \bar{w}) \quad \text{definition} \\ &= \lim_{z,\bar{z}\to 0} \langle 0| A(1/z, 1/\bar{z}) \frac{1}{z^{2h} \bar{z}^{2\bar{h}}} \quad \text{conformal map} \\ &= |A_{in}\rangle^\dagger \end{aligned} \tag{24}$$

Definition 4.6. A correlation function is

$$\langle \phi_1 ... \phi_n \rangle = \langle 0 | T \phi_1 ... \phi_n | 0 \rangle \tag{25}$$

From these definitions, correlators can be interpreted as the usual probability amplitude $\langle A_{out} | U(t_\infty - t_{-\infty}) | A_{in} \rangle$.

4.4.1 Conditions in Stress Tensor

Demanding our quantum theory to have T as real observable i.e. hermitian ($T = T^\dagger$), we get

$$L_m^\dagger = L_{-m}. \tag{26}$$

Also non-singular $T(z)\,|0\rangle$

$$\begin{aligned} \implies L_m\,|0\rangle = 0 \quad \text{for } m \geq -1 \\ \implies \langle 0|\,L_m = 0 \quad \text{for } m \leq 1. \end{aligned} \tag{27}$$

4.5 Highest Weight States

We will represent the state created by a primary field $\phi(z, \bar{z})$ of weight $(h, \bar{h})$ by

$$|h, \bar{h}\rangle = \phi(0,0)\,|0\rangle\,.$$

and applying OPE of holomorphic field $\phi(w)$, $(h, 0)$ with T,

$$\begin{aligned} [L_n, \phi(w)] &= \oint \frac{dz}{2\pi i} z^{n+1} T(z)\phi(w) = (n+1) w^n h \phi(w) + w^{n+1} \partial\phi(w) \\ \implies [L_n, \phi(w)] &= 0 \quad \text{if } n > 0, \\ L_0\,|h\rangle &= L_0 \phi(0)\,|0\rangle = [L_0, \phi(0)]\,|0\rangle + \phi(0)\,\cancelto{0}{L_0\,|0\rangle} \\ \implies L_0\,|h\rangle &= h\,|h\rangle\,, \quad L_n\,|h\rangle = 0 \quad \text{if } n > 0 \end{aligned} \tag{28}$$

Definition 4.7. A state satisfying above relations (Eq.28) $\iff$ created by a primary field is a *highest weight state*.

Definition 4.8. Let $|h\rangle$ be a highest weight state, then $L_{-n_1} ... L_{-n_k}\,|h\rangle$ for $n_i > 0$, $\forall i$ is a *descendant state* of $|h\rangle$.

4.6 Descendant Fields

The main goal of this talk was to construct the *irreducible representation* of the Virasoro algebra i.e. organizing a 2d CFT in terms of conformal families which is a set of a primary field and all of its descendants. The utility of it is that the theory may then be completely specified by the correlation functions of the primary fields.

Proposition 5. Let ϕ be a primary field with $(h, \bar{h})$. Then the conformal weight of $L_{-n}\phi$ is $(h + n, \bar{h})$

Proof. Virasoro (Eq.21) $\implies [L_0, L_n] = -nL_n$. Hence L_n is a *ladder operator* of L_0. ■

Therefore for $n > 0$, primary fields satisfy $L_n\phi = 0$. The number of descendant fields at level $n \equiv$ conformal dimension $h + n$, is the integer partition function $P(n)$ which counts the number of ways to write n as a sum of positive integers.

In terms of primary ϕ, the secondary field is

$$L_{-n}\phi(w, \bar{w}) \equiv \oint \frac{dz}{2\pi i} \frac{T(z)\phi(w, \bar{w}}{(z - w)^{n-1}} \tag{29}$$

All the correlation functions of the secondary fields are given by differential operators acting on those of primary fields.

Theorem 4.1. Let $w \to z$ for some primary ϕ, ϕ_i,

$$\begin{aligned} &\langle \phi_1(w_1, \bar{w}_1)...\phi_{n-1}(w_{n-1}, \bar{w}_{n-1})(L_{-k}\phi)(z, \bar{z})\rangle \\ &= \mathcal{L}_{-k} \langle \phi_1(w_1, \bar{w}_1)...\phi_{n-1}(w_{n-1}, \bar{w}_{n-1})\phi(z, \bar{z})\rangle \end{aligned} \tag{30}$$

where $k \in \mathbb{Z}_{>0}$ and

$$\mathcal{L}_{-k} \equiv -\sum_{j=1}^{n-1} \left(\frac{(1-k)h_j}{w_j - z} + \frac{1}{(w_j - z)^{k-1}} \partial_{w_j} \right) \tag{31}$$

Proof.

$$\begin{aligned} &\langle \phi_1(w_1, \bar{w}_1)...\phi_{n-1}(w_{n-1}, \bar{w}_{n-1})(L_{-k}\phi)(z, \bar{z})\rangle \\ &= \oint_w \frac{dz}{2\pi i} \frac{1}{(z - w)^{k-1}} \langle \phi_1(w_1, \bar{w}_1)...\phi_{n-1}(w_{n-1}, \bar{w}_{n-1})T(z)\phi(w, \bar{w})\rangle \\ &= -\oint_{w_j \forall j} \frac{dz}{2\pi i} \frac{1}{(z - w)^{k-1}} \langle \phi_1(w_1, \bar{w}_1)...\phi_{n-1}(w_{n-1}, \bar{w}_{n-1})T(z)\phi(w, \bar{w})\rangle \end{aligned}$$

Then we can complete the proof using the Ward identity. ■

There is no closed form equation for correlators with descendant fields in general order but one can use the commutation relations and annihilation to always express them in terms of primary fields only.