Brachistochrone through the Earth

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Abstract

In this study, we explore the historical context of the brachistochrone problem and extend it to address the concept of a "gravity train," examining potential tunnel paths through the Earth. We establish that a hypocycloid serves as the terrestrial brachistochrone within a uniform average density model of the Earth. Employing the Preliminary Reference Earth Model (PREM). We numerically compute the brachistochrone, revealing a root-mean-square error of just 4.2 minutes when comparing journey times between the uniform density model and PREM, with the uniform density model consistently predicting shorter durations. Additionally, we analyse the discomfort associated with these paths, employing Anderson's measure of discomfort [10]. Our findings show that a straight path is more "comfortable" than a hypocycloid, although we conclude that this metric is inadequate for identifying the least uncomfortable journey.

Key words : Calculus of variations, Gravity train, Non-uniform density Earth, Discomfort

1 Introduction

The brachistochrone problem, first posed by Johann Bernoulli in 1696, was to find the curve of fastest descent under the influence of gravity. This classical conundrum has not only laid the foundation for variational calculus and optimisation theory but also spurred groundbreaking advancements in theoretical physics.

This problem naturally extends to the concept of a "gravity train", representing the fastest path between two points on the surface of a planet, achieved by tunnelling through its core. Precise calculations by Cooper [1] indicate that travel time between any two points on Earth takes approximately 42 minutes, with the shortest time curve being a hypocycloid. These findings were obtained under the assumption of uniform density for the Earth and negligible frictional forces. The objective of this paper is to extend and generalise the concept more realistically by introducing non-uniform density and to examine the level of discomfort for different paths.

2 History & Cycloid

To answer the proposed brachistochone problem, Bernoulli and Newton took completely different approaches which led to the same result: cycloid. A cycloid is a type of curve traced by a point on the rim of a rolling circle as it moves along a straight line without slipping. A cycloid curve, γ, constructed with a circle of radius r can be expressed parametrically as

$$\gamma = \begin{cases} x(\eta) = r(\eta - \sin(\eta)) \\ y(\eta) = r(1 - \cos(\eta)) \end{cases} \tag{1}$$

where η is a real parameter corresponding to the angle through which the rolling circle has rotated. γ necessarily satisfies the following differential equation:

$$\frac{dy}{dx} = \frac{\sin(\eta)}{1-\cos(\eta)} = \cot(\eta/2) = \sqrt{\mathrm{cosec}^2(\eta/2) - 1} = \sqrt{\frac{2}{1-\cos(\eta)} - 1} = \sqrt{\frac{2r}{y} - 1}$$

$$\left[\left(\frac{dy}{dx}\right)^2 + 1\right] y = 2r = \text{const.} \tag{2}$$

2.1 Bernoulli's Solution

Fermat's principle states the path between 2 points by light is one that takes the least time, enabling Johann Bernoulli to use an optical analogy to solve the problem where he compared the path of the particle moving under the influence of gravity to the path taken by light rays in a medium with varying refractive index. Bernoulli stated [2] that in the varying layer setting, the total travel time is extremal if and only if

$$\sin(\psi)/v = \text{const.} \tag{3}$$

where v is velocity and ψ is angle of velocity from the vertical. As we know, from the conservation of energy, $v(y) = \sqrt{2gy}$ for an object falling freely under gravity. Hence, it is concluded that

$$\sin(\psi)/\sqrt{y} = \text{const.} \tag{4}$$

This is a differential equation for a cycloid. Let $\overline{\psi}$ be the angle from the horizontal. (i.e. $\overline{\psi} = \pi/2 - \psi$) Then, we can express $\sin(\psi)$ as

$$\sin(\psi) = \cos(\overline{\psi}) = \frac{1}{\sqrt{1+\tan^2(\overline{\psi})}} = \frac{1}{\sqrt{1+(\frac{dy}{dx})^2}}. \tag{5}$$

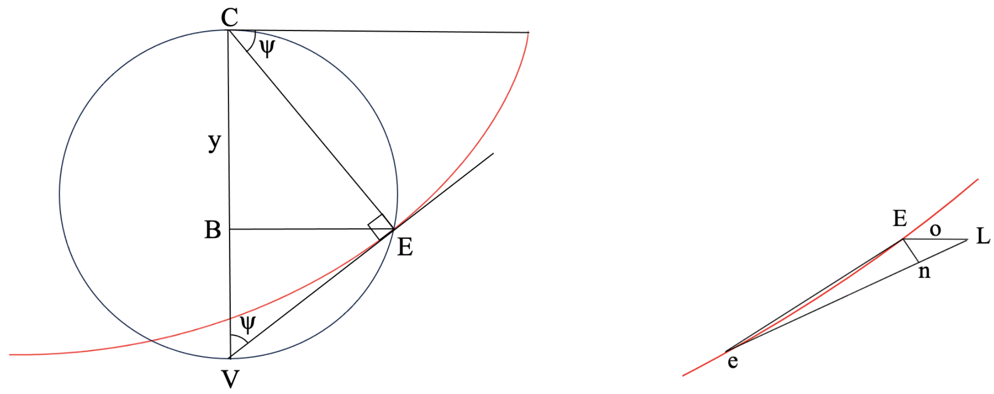
This Eq(5) is substituted into Eq(4) which can then be squared to form the differential equation for a cycloid we got in Eq(2).

2.2 Newton's Solution & Calculus of Variations

When Newton was given this challenge by Bernoulli, he already had solved similar problem which is to determine the solid of minimum resistance, solution shown in Principia. [3] Even though his model for the fluid was wrong as per our current understanding [4], the methodology he used is considered to be the beginning of the development of calculus of variations [5] which will be explained later this chapter.

Fig(1) shows the diagram used for the proof of brachistochrone problem that Newton explained in his letter to David Gregory. [6] Consider in the Fig(1) a particle ascending in the straight line eL instead of the arc eE. The horizontal displacement of L from E, o is very small (i.e. $o \ll eE$). VE is tangent of and CEn is normal to the curve at at E. Given that o As the arc eE approaches zero, eL becomes parallel to VE, resulting in the triangle EnL being similar to the triangle CEV. Therefore the additional time, δt for the alternative path is

$$\delta t = \frac{\delta s}{v} \propto \frac{nL}{\sqrt{y}} = \frac{\sin(\psi)\cdot o}{\sqrt{y}}. \tag{6}$$



(a) (b)

Figure 1: The red curve shows a cycloid with a circle forming it where y is the vertical component of displacement of falling object. Fig(b) is a zoomed-in version of the Fig(a) at the point Ed. The line nE is parallel to the line CE.

As it was proved earlier in Eq(4), $\sin(\psi)/\sqrt{y}$ is constant for a cycloid. Therefore δt is constant under arbitrarily small variation of path, meaning the cycloid path is the stationary (minimum in this case) path.

Modern solution involves calculus of variations which was purely analytically formalised by Lagrange. In modern form, the total time travelled from point A to B, which is the functional to be minimised, is

$$T = \int_A^B \frac{ds}{v} = \int_{x_1}^{x_2} \sqrt{\frac{1+y_x^2}{2gy}}dx = \int_{x_1}^{x_2} f(y, y_x)dx \tag{7}$$

where y is vertical, x is horizontal displacement and $y_x = \frac{dy}{dx}$. As this functional is not explicitly dependent on x, in other words, $\frac{\partial f}{\partial x} = 0$, Beltrami's identity can be used to simplify the Euler-Lagrange equation:

$$f - y_x \frac{\partial f}{\partial y_x} = \text{const.} \tag{8}$$

The left hand side of the Eq(8) can be readly shown to be equivalent to the Eq(2).

3 Gravity Train for Uniform Density Planet

The earth is assumed to be a sphere with uniform density as it was done in Cooper's paper [1] throughout chapter 3. Due to Newton's shell theorem, the gravitational field of this model is

$$\boldsymbol{g} = -\frac{GM}{R^3}\boldsymbol{r} \tag{9}$$

where $\boldsymbol{r}$ is the displacement from the centre, M and R are the mass and the radius of Earth respectively.

3.1 Straight Line Path

Consider a straight line through the earth, AB in Fig(2a). $\boldsymbol{x}$ is the displacement from the midpoint of AB in the direction parallel to AB. As the motion of a particle is constrained on the straight line, gravitational field is

$$g_x = \boldsymbol{g} \cdot \hat{\boldsymbol{x}} = -\frac{GM}{R^3} r \cdot \sin(\theta) = -\frac{GM}{R^3} r \cdot \frac{x}{r} = -\frac{GM}{R^3} x. \tag{10}$$

It is therefore concluded that the motion is simple harmonic with $\omega = \sqrt{\frac{GM}{R^3}}$ Therefore the solution is given by

$$x = x_0 \cos(wt), \quad x_0 = R \sin\left(\frac{\theta_{AB}}{2}\right) \tag{11}$$

where θ_{AB} is the angle AOB, x_0 is the length BC, amplitude of the oscillation. Note that the total time travelled, T is half the period of the oscillation which is about 42.2 minutes regardless of θ_{AB}.

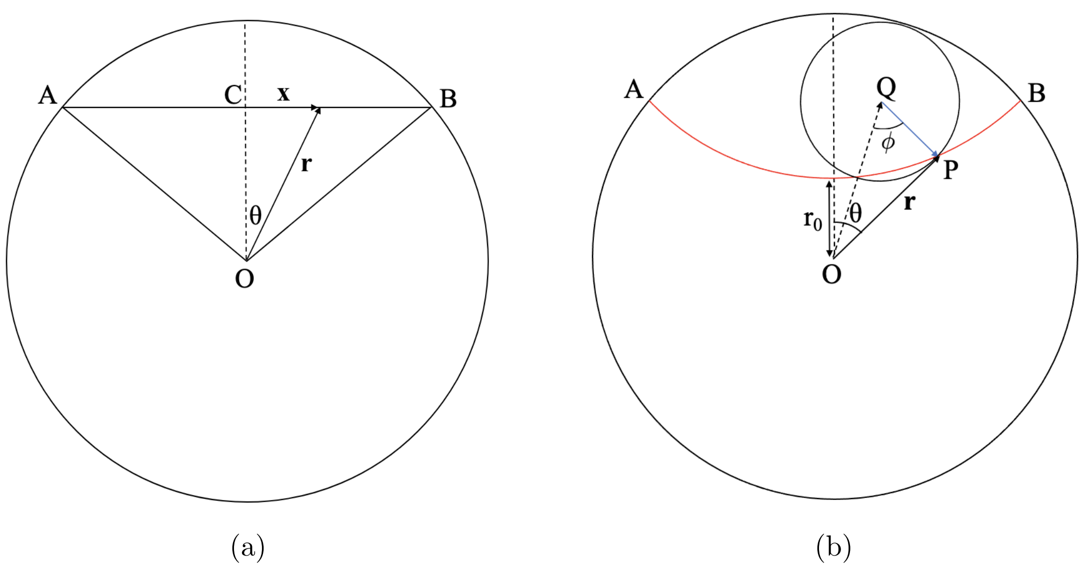


Figure 2: Cross section of uniform density earth of the plane containing point A, B and the centre of the sphere. The red curve is a hypocycloid generated with a circle of radius $(R - r_0)/2$.

3.2 Shortest Time Path

Calculus of variations approach can be used to determine the shortest time curve for transit tunnel through Earth. The potential energy, V of a particle is

$$V(r) = -\int_0^r \boldsymbol{F} \cdot d\bar{\boldsymbol{r}} = \frac{GMm}{2R^3} r^2 \tag{12}$$

where $\bar{\boldsymbol{r}}$ is a dummy variable. The speed, v of the particle, therefore, is

$$v(r) = \sqrt{\frac{GM}{R^3}(R^2 - r^2)}. \tag{13}$$

Total time travelled, T in polar coordinates is

$$T = \int_0^T dt = \int_A^B \frac{ds}{v} = \int_{\theta_A}^{\theta_B} \sqrt{\frac{r'^2 + r^2}{\frac{GM}{R^3}(R^2 - r^2)}} d\theta = \int_{\theta_A}^{\theta_B} F(r, r') d\theta \tag{14}$$

where $r' = \frac{dr}{d\theta}$. Beltrami's identity, $F - r'\frac{\partial F}{\partial r'} =$ const. can be used again here as $\frac{\partial F}{\partial \theta} = 0$:

$$\sqrt{\frac{r'^2 + r^2}{\frac{GM}{R^3}(R^2 - r^2)}} - r' \cdot \frac{r'}{\sqrt{(r'^2 + r^2)\frac{GM}{R^3}(R^2 - r^2)}} = C \tag{15}$$

where C is a constant. The boundary conditions $r(\theta = 0) = r_0$, $r'(\theta = 0) = 0$ imply

$$C = \frac{r_0}{\sqrt{\frac{GM}{R^3}(R^2 - r_0^2)}}, \quad r_0(\theta_{AB}) = R\left(1 - \frac{\theta_{AB}}{\pi}\right), \tag{16}$$

$$\frac{dr}{d\theta} = \frac{Rr}{r_0}\sqrt{\frac{r^2 - r_0^2}{R^2 - r^2}}. \tag{17}$$

We can solve the above differential equation by u-substitution as

$$\theta(r) = \frac{r_0}{R}\int_{r_0}^{r} \frac{1}{\bar{r}u(\bar{r})} d\bar{r}, \quad u(r) = \sqrt{\frac{r^2 - r_0^2}{R^2 - r^2}} \tag{18}$$

$$\implies \theta(u(r)) = \arctan\left(\frac{R}{r_0}u(r)\right) - \frac{r_0}{R}\arctan(u(r)). \tag{19}$$

By introducing new variable $\phi = 2\arctan(u(r))$, which will later be shown as the angle inside the rolling circle as in Fig(2b), for simplicity and squaring $\tan(\phi/2)$, we get $r(\phi)$. After applying the trigonometric addition formula for cosine and rearranging $r(\phi)$, the parametric solution to the differential Eq(17) in terms of ϕ is given by

$$\begin{cases} r^2(\phi) = \frac{1}{2}(R^2 + r_0^2) - \frac{1}{2}(R^2 - r_0^2)\cos(\phi) \\ \\ \theta(\phi) = \arctan\left(\frac{R}{r_0}\tan\left(\frac{\phi}{2}\right)\right) - \frac{\phi r_0}{2R} \end{cases} \tag{20}$$

which can be shown to be a hypocycloid [7], i.e. a curve traced by a point on a circle of radius $(R - r_0)/2$, rolling inside another circle of radius R, as depicted in the Fig(2b).

3.3 Hypocycloid

We start from the geometric properties of hypocycloids and prove that they leads to the parametric equations(20) by investigating the triangle OPQ shown in Fig(3).

Let a be the radius of the inner circle, $(R - r_0)/2$. Then we know that QP $= a$, OQ $= R - a$. Applying the cosine rule for ϕ, we obtain

$$r^2 = (R - a)^2 + a^2 - 2a(R - a)\cos(\phi) \tag{21}$$

which is equivalent to the r equation in Eq(20).

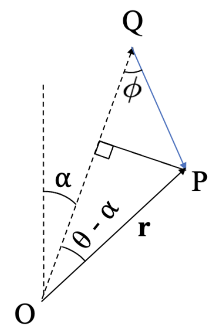


Figure 3: Triangle OPQ from the Fig(2b).

The expression for $\tan(\theta-\alpha)$ is derived using trigonometric addition formulae again as

$$\tan(\theta-\alpha)=\frac{a\sin(\phi)}{(R-a)-a\cos(\phi)}=\frac{(R-r_0)\tan\left(\frac{\phi}{2}\right)}{R\tan^2\left(\frac{\phi}{2}\right)+r_0} \tag{22}$$

which can be substituted into $\tan(\theta-\alpha+\phi/2)$ and be rearranged as

$$\tan\left(\theta-\alpha+\frac{\phi}{2}\right)=\frac{\tan(\theta-\alpha)+\tan\left(\frac{\phi}{2}\right)}{1-\tan(\theta-\alpha)\tan\left(\frac{\phi}{2}\right)}=\frac{R}{r_0}\tan\left(\frac{\phi}{2}\right). \tag{23}$$

Using the fact that $\alpha R=a\phi$, the θ equation(20) is derived. Hence it is proved that the parametric equations represent a hypocycloid with a large circle of radius R, and a smaller inner circle of radius a where the variable ϕ represents the angle OQP ($\phi=-\pi,\pi$ when $r=R$ and $\phi=0$ when $r=r_0$).

3.4 Position as a Function of Time

Substitution of r' in Eq(17) into the Euler-Lagrange equation(14) followed by substitution of the variable ϕ and integration, results in time as a function of ϕ:

$$t(\phi)=\frac{\phi}{2}\sqrt{\frac{1-(\frac{r_0}{R})^2}{\frac{GM}{R^3}}}=\frac{\phi}{2}\sqrt{\frac{1-(1-\frac{\theta_{AB}}{\pi})^2}{\frac{GM}{R^3}}}. \tag{24}$$

The linear relationship between the angle inside the rolling circle of a hypocycloid, ϕ and time, t means the angular speed of the particle in the inner circle's frame of reference, $\frac{d\phi}{dt}$ is constant. Let $\omega t=\alpha$, the angle of the line joining the centres of the two circles from the vertical:

$$\omega t=\frac{\phi(t)\cdot a}{R}=\frac{\frac{d\phi}{dt}t\cdot(R-r_0)/2}{R}. \tag{25}$$

Knowing that the curve is hypocycloid, the parametric equation for the position as a function of time is given by

$$\begin{cases} x(t)=(R-a)\cos(\omega t)+\cos\left(\dfrac{R-a}{a}\omega t\right) \\ \\ y(t)=(R-a)\sin(\omega t)+\sin\left(\dfrac{R-a}{a}\omega t\right) \end{cases}, \quad \omega(\theta_{AB})=\frac{\theta_{AB}}{\pi}\sqrt{\frac{\frac{GM}{R^3}}{1-(1-\frac{\theta_{AB}}{\pi})^2}} \tag{26}$$

where $w = \frac{d\phi}{dt}$ and a is the radius of the inner circle. As the inner circle is rotated by 2π rad to complete a journey from A to B, the total time taken, T is

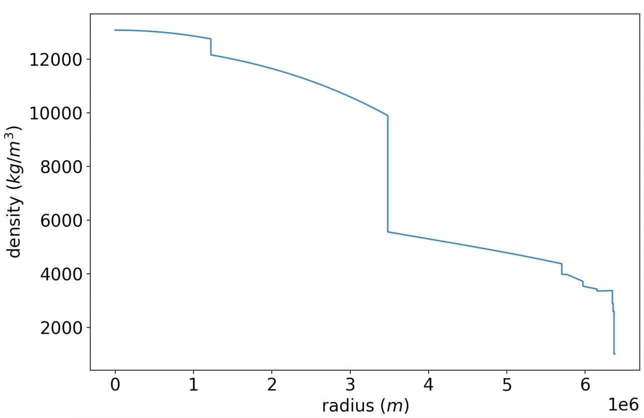
$$T = \pi\sqrt{\frac{1-(1-\frac{\theta_{AB}}{\pi})^2}{\frac{GM}{R^3}}}. \tag{27}$$

4 Non-uniform Density Earth

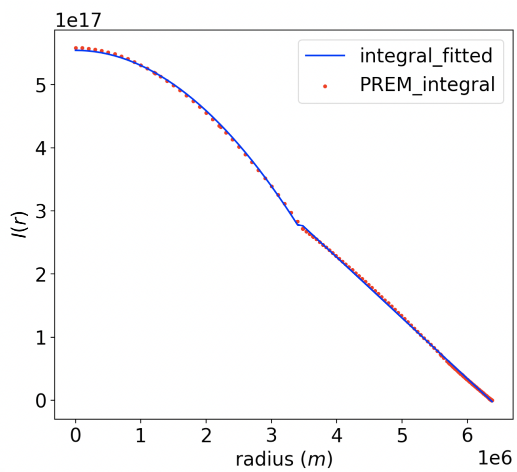
To achieve more precise model, Klotz discarded the assumption of uniform density. [8] He stated, in his paper, the time taken for a straight path is no longer constant and the brachistochrone path also differ slightly. In this chapter, the data he presented will be replicated using Python, presenting more detailed intermediate steps.

4.1 Preliminary Reference Earth Model

Preliminary Reference Earth Model (PREM) provides various physical quantity of spherically symmetric model of Earth. [9] Radial density data from PREM will be used in this chapter to investigate the effect of varying density.



(a)



(b)

Figure 4: PREM density against radial position data is shown in Fig(4a). The red dots in Fig(4b) are numerically calculated $I(r)$ and the blue curve is the fitted result.

4.2 Numerical Solution

With the assumption that the density of the earth is dependant only on the radial position, the exactly same logic of Beltrami's identity can be used here to obtain the differential equation:

$$T = \int_A^B \frac{ds}{v} = \int_{\theta_A}^{\theta_B} \sqrt{\frac{(\frac{dr}{d\theta})^2 + r^2}{2GI(r)}} d\theta \tag{28}$$

$$\implies \frac{dr}{d\theta} = \frac{r}{r_0}\sqrt{\frac{r^2 I(r_0) - r_0^2 I(r)}{I(r)}} \tag{29}$$

where r_0 is the maximum depth ($\theta = 0$) and $\frac{dr}{d\theta} = 0$ when $r = r_0$. Defining I(r) as

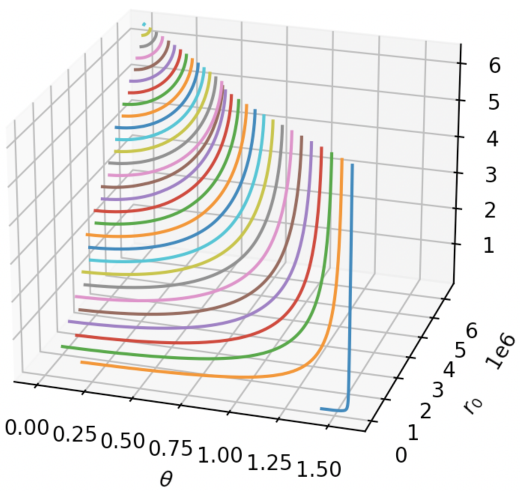
$$I(r) \equiv \int_r^R \frac{M(r')}{r'^2} dr' = \int_r^R \frac{4\pi}{3} r' \rho(r') dr' \tag{30}$$

where r' is a dummy variable, the values of $I(r)$ were numerically calculated with scipy.integrate.cumulative_trapezoid() function using the density, $\rho(r)$, data in Fig(4a) from PREM [9].

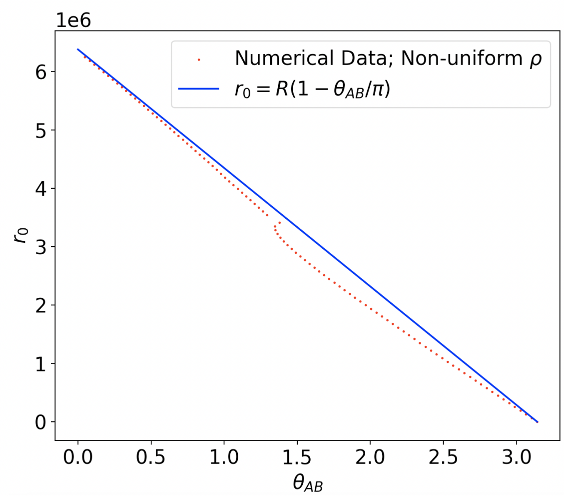
As one can clearly see the significant drop in density at $r = 3480{,}000$ m, we defined it to be the point joining two curves when modelling $I(r)$. Inside that radius is modelled as quadratic, and the rest is done as linear. As a result of fitting the $I(r)$ data, the red dots in Fig(4b), with scipy.optimize.curve_fit() function, we obtained the coefficients of the polymonials as

$$I(r) = \begin{cases} -23924r^2 + 5.5440 \cdot 10^{17} & \text{if} \quad r < 3480000 \\ -9.6153 \cdot 10^{10} r + 6.1107 \cdot 10^{17} & \text{if} \quad r \geq 3480000. \end{cases} \tag{31}$$

This equation is substituted into Eq(29) to obtain the brachistochrone curve, $\theta(r)$ by numerical integration with scipy.integrate.quad() function. The Fig(5a) shows the graphs for different r_0. Note that the relationship between r_0 and θ_{AB} is not the same as that in the Eq(16) so it is calculated numerically in Fig(5b) using the fact that the maximum values of θ for a specific r_0 in Fig(5a) is $\theta_{AB}/2$ as they only depict half the journey.



(a)



(b)

Figure 5: $r(\theta)$ of non-uniform density brachistochrone for different r_0 in (a); The blue line in (b) is $r_0(\theta_{AB})$ for uniform density whereas the red dots are $r_0(\theta_{AB})$ for PREM density.

Time taken for the journey is calculated by substituting Eq(29) and Eq(31) into Eq(28) and changing variable as

$$T(r_0) = 2\int_0^{\theta_{AB}} \sqrt{\frac{(\frac{dr}{d\theta})^2 + r^2}{2GI(r)}} d\theta = 2\int_{r_0}^R \sqrt{\frac{(\frac{dr}{d\theta})^2 + r^2}{2GI(r)}} \frac{d\theta}{dr} dr \tag{32}$$

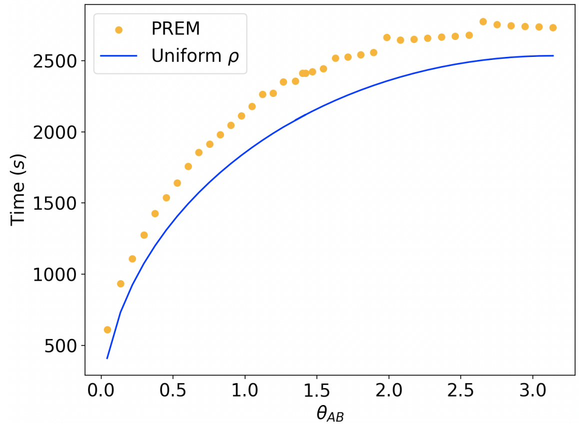


Figure 6: Time taken for the whole journey as a function of θ_{AB}

which can be evaluated with scipy.integrate.cumulative_trapezoid() function as shown in Fig(6).

The root-mean-sqaure error between the time taken for PREM model and the uniform density model is 252.9 s which is not very significant. Furthermore, in Fig(7), the numerical paths are not drastically different from uniform density model.

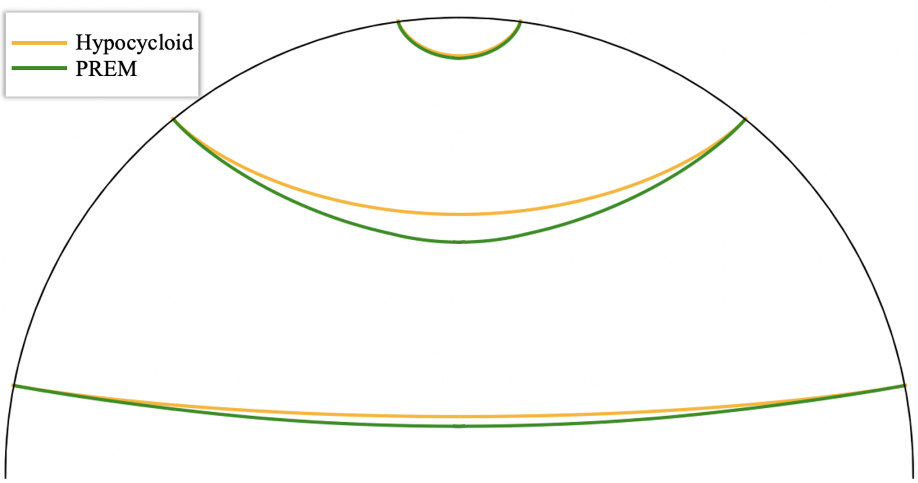


Figure 7: Terrestrial Brachistochrone

5 Discomfort

In this chapter, **the effect of varying density is ignored** for simplicity as it has not shown to be significant in the preceding chapter.

Calculus of variations can also be used in analysis of the level of discomfort for a journey. Anderson, Desaix and Nyqvist discussed a problem of the "least uncomfortable" linear trajectory from point A to point B. [10] In their paper, "discomfort", J was defined as

$$J \equiv \int_0^T a^2 dt = \int_0^T \left(\frac{GM}{R^3}\right)^2 r^2(t) dt \tag{33}$$

where a is acceleration and T is the total time travelled. They stated that acceleration squared was chosen to avoid cancellation effects. Although the solution was later generalised to account for relativistic effects by Klotz and Antonelli by redefining the discomfort functional with proper time τ [11], the Newtonian limit is taken in this paper as the maximum speed that can be achieved is approximately 7900 ms^{-1}, resulting in γ factor being 1.0000... up to 9th decimal places.

5.1 Linear and Hypocycloidal Path

The levels of discomfort between linear path and hypocycloid are to be compared.

Firstly, substituting $r(t)$ for linear path into Eq(33) in cartesian coordinates, the discomfort is given by

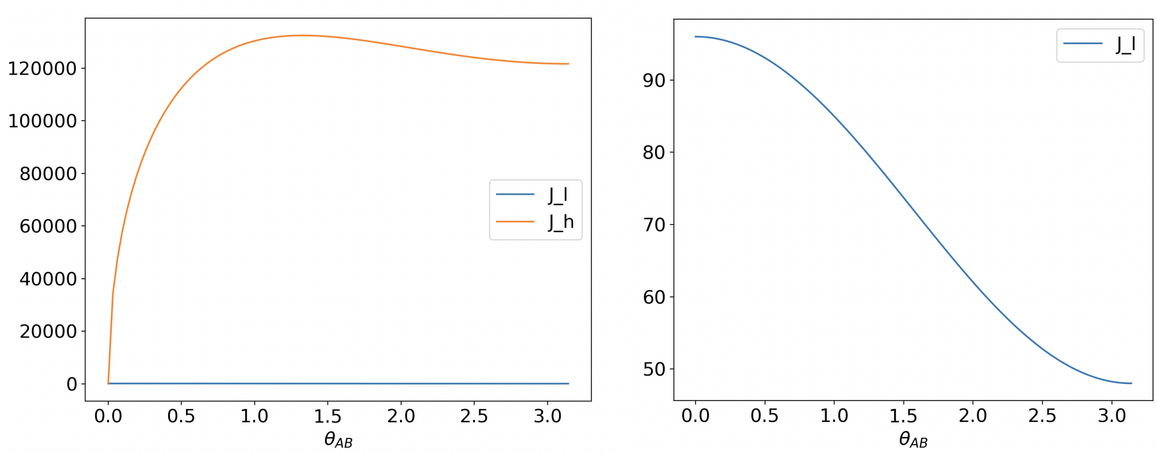
$$J_l = \int_0^{T_l} \left(\frac{GM}{R^3}\right)^2 [x_0^2 \cos^2(\omega_l t) + y_0^2] dt = \left(\frac{GM}{R^3}\right)^2 \left(\frac{x_0^2}{2} + y_0^2\right) T_l \tag{34}$$

where $\omega_l = \sqrt{\frac{GM}{R^3}}$, $T_l = \dfrac{\pi}{\omega_l}$, $x_0 = R\sin(\theta_{AB}/2)$ and $y_0 = R\cos(\theta_{AB}/2)$.

Secondly, substituting $r(t)$ for hypocycloid, Eq(26) into Eq(33) results in

$$\begin{aligned} J_h &= \int_0^{T_h} \left(\frac{GM}{R^3}\right)^2 \left[(R-a)^2 + a^2 - 2a(R-a)\cos\left(\frac{R}{a}\omega_h t\right)\right] dt \\ &= \left(\frac{GM}{R^3}\right)^2 \left[[(R-a)^2 + a^2]T_h - \frac{2a^2(R-a)}{R\omega_h}\sin\left(\frac{R}{a}\omega_h T_h\right)\right] \end{aligned}$$

where $\omega_h = \omega(\theta_{AB})$ in Eq(26), $T_h = T(\theta_{AB})$ in Eq(27) and $a(\theta_{AB}) = \dfrac{R\theta_{AB}}{2\pi}$.



(a) (b)

Figure 8: Graphs of J_l and J_h against θ_{AB}

As shown in Fig(8), linear path is more "comfortable" than hypocycloid. The reason for the shapes of the graphs will be discussed later this chapter.

5.2 Optimised Path

In this problem of optimisation, the minimiser is the discomfort J which can be expressed in cartesian coordinates as

$$J = \int_0^T \left(\frac{GM}{R^3}\right)^{3/2} (x^2+y^2)\sqrt{\frac{\dot{x}^2+\dot{y}^2}{R^2-(x^2+y^2)}}dt = \int_0^T f(x,y,\dot{x},\dot{y})dt \tag{35}$$

by substituting Eq(9) and Eq(13) where $\dot{x}$ and $\dot{y}$ are $\frac{dx}{dt}$, $\frac{dy}{dt}$ respectively. In order that this 2D integral be stationary, 2 Euler-Lagrange equations

$$\frac{\partial f}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) \quad i = 1, 2 \tag{36}$$

where $q_1 = x$, $q_2 = y$ must be satisfied. [12] Using the Beltrami's identity again separately, we obtain

$$f - \dot{x}\frac{\partial f}{\partial \dot{x}} = \left(\frac{GM}{R^3}\right)^{3/2} \frac{(x^2+y^2)}{\sqrt{R^2-(x^2+y^2)}} \cdot \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = C_x, \tag{37}$$

$$f - \dot{y}\frac{\partial f}{\partial \dot{y}} = \left(\frac{GM}{R^3}\right)^{3/2} \frac{(x^2+y^2)}{\sqrt{R^2-(x^2+y^2)}} \cdot \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = C_y. \tag{38}$$

Adding these two equations gives

$$\left(\frac{GM}{R^3}\right)^{3/2} \frac{(x^2+y^2)\sqrt{\dot{x}^2+\dot{y}^2}}{\sqrt{R^2-(x^2+y^2)}} = C_x + C_y = C \tag{39}$$

where C is just another constant. Because we know

$$\frac{ds}{dt} = \sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{\frac{GM}{R^3}(R^2-(x^2+y^2))}, \tag{40}$$

we can substitute it into Eq(39) and the path is turned out to be represented by a circular function. In other words, the radial position of a particle is constant which imply constant acceleration (i.e. jerk = 0) as it was also shown in Anderson's paper [10].

The boundary condition $r(t=0) = R$ implies the particle remains on the planet's surface throughout its motion. Additional boundary conditions, $v(t=0) = 0$ and $v(t = T) = 0$, result in the particle having no tangential acceleration, making $v(t) = 0$ for all times. Consequently, $T \to \infty$ and $J \to \infty$ as a remains a finite, non-zero constant, indicating the path is not the "least uncomfortable," but the "most uncomfortable" one.

Even if we were to introduce finite initial and final speeds, the maximum acceleration occurs at $r = R$ and linearly decreases as r decreases. Therefore, any path deviating below the surface, such as a hypocycloid, would be even "more uncomfortable" than the circular path. This outcome arises from the problem's setup, where external energy is not applied to the gravity train, and acceleration is solely influenced by radial position, with no normal reaction forces experienced due to assumption that the particle do not make any physical contact. Therefore, weather or not the definition of discomfort in Eq(33) is a good measure of the actual discomfort is questionable.

The graphs of discomfort, J for linear and hypocycloid path can be explained as follows. J_l in Fig(8b) decreases as θ_{AB} increases because the straight path gets deeper. However J_h increases at a decreasing rate until $\theta_{AB} \approx 1.3$ rad as the effect of decrease in time (Fig.6) is greater than that of increase in acceleration and the difference in these effects gradually decreases. Beyond the point of 1.3 rad, the binary relation between the effects gets inverted. Hence the decrease. Lastly the variation of radial position for a hypocycloid is clearly greater than that for a linear path which results in the difference shown in Fig(8a).

6 Conclusion

Analysing the PREM data revealed that the deviation between the uniform density model and actual observations is relatively minor. The root-mean-square error in the time required for the brachistochrone journey between the uniform density model and reality is only 4.2 minutes, with the uniform density model consistently predicting shorter journey times.

Also, it is important to note that the discomfort metric defined in Eq(33) does not efficiently quantify the discomfort level in the context of the gravity train problem. A more effective approach would involve modeling both the tunnel and the particle as rigid bodies, accounting for all external forces like friction and normal reactions and equipping the train with mechanisms to control acceleration such as fuel and brakes.

References

[1] Paul W. Cooper, "Through the Earth in Forty Minutes", American Journal of Physics 34, p. 68-70, Jan (1966).

[2] Henk W. Broer, "Bernoulli's Light Ray Solution of the Brachistochrone Problem Through Hamilton's Eyes", International Journal of Bifurcation and Chaos, Vol. 24, No. 8 , 1440009, February (2014).

[3] S. Chandrasekhar, "Newton's 'Principia' for the Common Reader", Clarendon Press; Reprint edition, June (2003).

[4] Frank H. Berkshire, Simon J. A. Malham, J. Trevor Stuart, "Introductory Incompressible Fluid Mechanics", Cambridge University Press; New edition, February, (2022).

[5] Thiele, Rüdiger, "Euler and the Calculus of Variations", In Bradley, Robert E.; Sandifer, C. Edward (eds.). Leonhard Euler: Life, Work and Legacy. Elsevier. p. 249. ISBN 9780080471297, (2007).

[6] Dubois, Jacques, "Chute d'une bille le long d'une gouttière cycloïdale; Tautochrone et brachistochrone; Propriétés et historique", Bulletin de l'Union des Physiciens. 85 (737), 1251–1289, (1991).

[7] Weisstein, Eric W. "Hypocycloid." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Hypocycloid.html

[8] Alexander R. Klotz, "The gravity tunnel in a non-uniform Earth", American Journal of Physics 83, 231–237 (2015).

[9] Dziewonski, A.M., and D.L. Anderson, "Preliminary reference Earth model", Phys. Earth Plan. Int. 25:297-356, (1981).

[10] D. Anderson, M. Desaix and R. Nyqvist, "The least uncomfortable journey from A to B", American Journal of Physics 84, 690–695, (2016).

[11] Riccardo Antonelli, Alexander R. Klotz, "A smooth trip to Alpha Centauri: Comment on 'The least uncomfortable journey from A to B' [Am. J. Phys. 84(9) 690–695]", American Journal of Physics 85, 469–472 (2017).

[12] T. W. B. Kibble, F. H. Berkshire, "Classical Mechanics", Imperial College Press; 5th edition, June, (2004).