3.E Products and Quotients of Vector spaces

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$V_1, \cdots V_m$ are vector space over $\mathbb{F}$

definition: product space, $V_1 \times \cdots \times V_m := \{(v_1, \cdots, v_m) : v_1 \in V_1, \cdots, v_m \in V_m\}$

addition and scalar multiplication is defined as usual (lists of vectors as usual vectors)

$\Rightarrow V_1 \times \cdots \times V_m$ is a vector space

let $U_1, \cdots, U_m < V$, $\Gamma : U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m$

$\Gamma(u_1, \cdots, u_m) := u_1 + \cdots + u_m$

then Γ is injective $\Leftrightarrow$ $U_1 + \cdots + U_m$ is direct sum

theorem: $U_1 + \cdots + U_m$ is direct sum $\Leftrightarrow \dim(U_1 + \cdots + U_m) = \dim(U_1) + \cdots + \dim(U_m)$

proof: Γ is injective $\Leftrightarrow \dim(\ker(\Gamma)) = 0 \Leftrightarrow \dim(Im(\Gamma)) = \dim(U_1 \times \cdots \times U_m)$

Γ is by definition surjective $\Leftrightarrow \dim(Im(\Gamma)) = \dim(U_1 + \cdots + U_m)$

$\Leftrightarrow \dim(U_1 \times \cdots \times U_m) = \dim(U_1 + \cdots + U_m)$ ■

let $v \in V$, $U < V$

definition: $v + U := \{v + u : u \in U\} \subset V$

let $U < V$

definition: the quotient space V/U is the set of all affine subsets of V parallel to U

$V/U := \{v + U : v \in V\}$

let $U < V$, $v, w \in V$

theorem: $v - w \in U \Leftrightarrow v + U = w + U \Leftrightarrow (v + U) \cap (w + U) \neq \emptyset$

proof: if $v - w \in U$, $u \in U$

$\Leftrightarrow v + u = w + v - w + u$, $v - w + u \in U$

$\Leftrightarrow v + u \in w + U \Leftrightarrow v + U \subseteq w + U$

symmetry $v \leftrightarrow w \Leftrightarrow w + U \subseteq v + U \Leftrightarrow v + U = w + U$

if $(v + U) \cap (w + U) \neq \emptyset$,

$\Rightarrow \exists\, u_1, u_2 \in U$ s.t. $v + u_1 = w + u_2 \Rightarrow v - w = u_2 - u_1 \in U$ ■

let $U < V$ and $v, w \in V$, $\lambda \in \mathbb{F}$

definition: addition and scalar multiplication on V/U

$(v + U) + (w + U) = (v + w) + U$

$\lambda(v + U) = \lambda v + U$

proof that addition is injective:

let $v', w' \in V$ s.t. $v' + U = v + U$, $w' + U = w + U$

$\Rightarrow v'-v \in U,\ w'-w \in U \Rightarrow (v'+w')-(v+w) \in U$

$\Rightarrow (v'+w')+U = (v+w)+U$

proof that multiplication is injective

let $v' \in V$ s.t. $v'+U = v+U$, $\lambda \in \mathbb{F}$

$\Rightarrow v'-v \in U \Rightarrow \lambda(v'-v) \in U \Rightarrow \lambda v' + U = \lambda v + U$

corollary: V/U is a vector space

additive identity: $0+U = U$, aditive inverse of $v+U$: $-v+U$

definition: quotient map. π is a linear map

let $U<V$, $v \in V$, $\pi: V \to V/U$

$\pi(v) := v+U$

theorem: if $\dim(V)$ is finite and $U<V$

$\Rightarrow \dim(V/U) = \dim(V) - \dim(U)$

proof: F.T.L.M $\dim(V) = \dim(\ker(\pi)) + \dim(\operatorname{Im}(\pi))$

$\ker(\pi) = U$ as $u+U = U = 0+U \quad \forall u \in U$

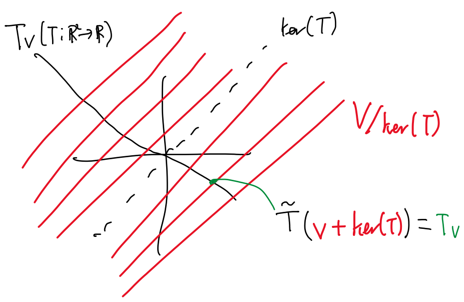
$\operatorname{Im}(\pi) = V/U$

$\Rightarrow \dim(V) = \dim(U) + \dim(V/U)$

definition: $\tilde{T}$

let $T \in \mathcal{L}(V,W)$, $\tilde{T}: V/(\ker(T)) \longrightarrow W$

$\tilde{T}(v+\ker(T)) := Tv$



lemma: $\tilde{T}$ is injective

proof: $v \in V$. if $\tilde{T}(v+\ker(T)) = 0 \Rightarrow Tv = 0 \Rightarrow v \in \ker(T)$

$\Rightarrow v+\ker(T) = \ker(T) \Rightarrow v=0 \Rightarrow \ker(\tilde{T}) = \{0\} \Rightarrow$ injective

lemma: $\operatorname{Im}(T) = \operatorname{Im}(\tilde{T})$ by definition

lemma: $V/\ker(T) \cong \operatorname{Im}(T)$

proof: modify $\tilde{T}$ to $\tilde{T}: V/\ker(T) \to \operatorname{Im}(\tilde{T})$

by definition of image, $\tilde{T}$ is surjective $\Rightarrow \tilde{T}$ is bijective

as $\operatorname{Im}(\tilde{T}) = \operatorname{Im}(T)$, $\tilde{T}$ is an isomorphism between $V/\ker(T)$ and $\operatorname{Im}(T)$

Exercises 3.E

1. definition: graph of T

   let $T: V \to W$

   $$\text{graph of } T := \{(v, Tv) \in V \times W : v \in V\}$$

   T is linear map $\iff$ graph of $T < V \times W$

3. $\mathcal{L}(V_1 \times \cdots \times V_m, W) \Rightarrow T: (v_1, \cdots, v_m) \to w$

   $T_i : v_i \to w$ $\quad T_i$

7. $v, x \in V$, $\quad U, W < V$

   $v + U = x + W$

   $(v - x) + U = W$

   $(v - x) \in V$

   $x - v + v + U = x + U = 2x - v + W$

   $v - x + u \in W$

   $v - x \in W \Rightarrow$

   $\Rightarrow x - v \in W$

   $U = x - v + W = W$ ☐

8. $A \subseteq V \iff \lambda v + (1-\lambda) w \in A$, $v, w \in A$, $\lambda \in \mathbb{F}$

   $\lambda(v - w) + w \in A$