Unitary and Orthosymplectic 3d $\mathcal{N} = 4$ Quiver Gauge Theories

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Coulomb branch
Higgs branch

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   Higgs branch

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Supersymmetry

- Symmetries are central object of study in theoretical physics
- Can introduce *supersymmetries* to link bosonic and fermionic fields (allows SQCD, superstring theory and supergravity)

$$Q\,|\text{fermion}\rangle = |\text{boson}\rangle\,,\;\; Q\,|\text{boson}\rangle = |\text{fermion}\rangle$$

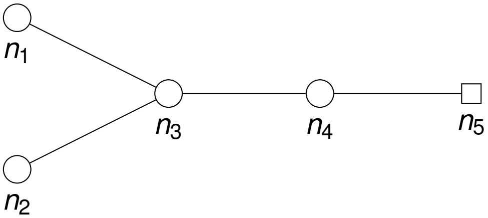
- Fields now packaged into supermultiplets [1] (in our case vector multiplets and hypermultiplets)

$$V(\phi) = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2$$

Moduli spaces

Branches

- Due to scalars vacuum is non-trivial [4]
- A moduli spaces is a space of the parameters of a geometric problem
- Vacuum expectation values parameterise moduli space [5]
- Moduli space of quiver gauge theories naturally decomposes into two branches [6]
- *Coulomb branch* from vector multiplet contribution
- *Higgs branch* from hypermultiplet contribution
- Quaternionic dimension easy to read off from quivers [7]



Moduli spaces

Hilbert series

- Moduli space is naturally a variety
- Useful tool to characterise geometric spaces
- Can extract information about moduli spaces from Hilbert series [8]
- Let $R = \oplus_{i \in \mathbb{N}} R_i$ be a graded polynomial ring over the algebraic variety $\mathcal{V}$.

$$\mathrm{HS}(t) = \sum_i \dim_{\mathbb{C}}(R_i) t^i$$

- Complex dimension can be read off from Hilbert series [9]

$$\mathrm{HS}(t) = \frac{Q(t)}{(1-t)^d}$$

- 9+1 dimensional string theory

Direction	0	1	2	3	4	5	6	7	8	9
NS5	x	x	x	x	x	x				
D3	x	x	x				-			
D5	x	x	x					x	x	x

- Coulomb branch is the space of D3 branes between NS branes
- Higgs branch is the space of the D3 branes between D5 branes.

Quivers

3d $\mathcal{N} = 4$

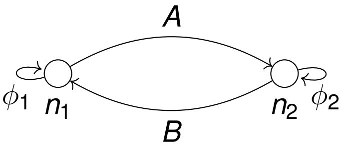
- Supersymmetric actions can get very messy and complicated
- Difficult to determine important details such as symmetries and dynamics
- Use quivers to encode symmetry information of theory [2]

- circle node : vector multiplet transforming in the adjoint representation of the gauge group
- square node : flavour group
- edge : hypermultiplet transforming in the bifundamental representation of the neighbouring nodes

Quivers

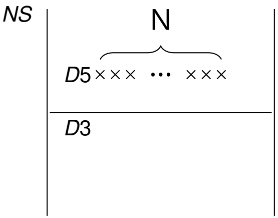
4d $\mathcal{N} = 2$

- Vectormultiplets decompose into a vectormultiplet and a chiral multiplet, ϕ_i in the adjoint representation
- hypermultiplets become biderectional and decompose into a chiral and anti-chiral multiplet A and B in fundamental and antifundamental representation
- for orthosymplectic quivers, a hypermultiplet becomes one chiral multiplet in the bifundamnetal representation as an edge is a half-hypermultiplet
- Can easily read off *superpotential* [3]



$$W = \mathrm{tr}(AB\phi_1 - BA\phi_2)$$

- Given a brane system, draw a projection of it in the 3,4,5 direction and the 6 direction.
- An interval between NS5 branes = circle node
- circle node number = number of D3 branes in the interval
- if there are D5 branes then a square node is connected
- squre node number = number of D5 branes
- the quiver is called an *electric quiver*



Definition

a *magnetic quiver* is obtained by:

- taking the brane system for an electric quiver
- rotating it to 7,8,9 direction and 6 direction projection
- exchanging NS5 with D5 branes

Theorem

3d Mirror Symmetry

- *Coulomb branch of electric quiver = Higgs branch of magnetic quiver*
- *Higgs branch of electric quiver = Coulomb branch of magnetic quiver*

- Coulomb branch, $\mathcal{M}_{\mathcal{C}}$:

$$\dim_{\mathbb{H}}(\mathcal{M}_{\mathcal{C}}) = \sum_i \mathrm{rank}(G_i)$$

where G_i are the gauge groups

- Higgs branch, $\mathcal{M}_{\mathcal{H}}$:

$$\dim_{\mathbb{H}}(\mathcal{M}_{\mathcal{H}}) = (\#\text{ of hypermultiplet}) - \sum \dim(\text{vector multiplet})$$

where the vectormultiplets are
adjoint representation of gauge groups

- Each edge connecting nodes G and G' has $\dim(R^{G}_{\text{fund}}) \times \dim(R^{G'}_{\text{fund}})$ hypermultiplets

Definition

a node is *balanced* if

$$\text{node number} \times 2 = \sum \text{neighbouring node numbers}$$

- Coulomb branch, $\mathcal{M}_{\mathcal{C}}$:

$$\text{Dynkin diagram formed by balanced nodes}$$
$$\times U(1)^n \text{ where } n \text{ is the number of ovebalanced nodes}$$

- Higgs branch, $\mathcal{M}_{\mathcal{H}}$:

$$\text{the flavour group}$$

- Use monopole formula to count dressed monopole operators at each conformal dimension [7, 10]

$$\mathrm{HS}(t) = \sum_{m \in \Gamma/\mathcal{W}} t^{2\Delta(m)} P(t, m)$$

$$= \sum_{\vec{m}_1} \sum_{\vec{m}_2} \cdots \sum_{\vec{m}_x} t^{2\Delta(\vec{m}_1, \vec{m}_2, \cdots, \vec{m}_x)} \prod_{i=1}^{x} P_{G_i}(t, \vec{m}_i)$$

1. Determine classical dressing factors
2. Compute conformal dimension $\Delta = \Delta_{\mathrm{vec}} + \Delta_{\mathrm{hyp}}$ [11]
3. Perform sums

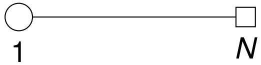
- Use Molien-Weyl formula to determine the Hilbert series the Higgs branch and the quotient by the gauge group action [4, 7]

$$\mathrm{HS}_{\mathcal{H}}(t) = \int_G d\mu_G \frac{\mathrm{PE}\left[\sum_{i=1}^{2N_{hyp}} \chi_{R_i}^{G_i}(\omega)\chi_{R_i'}^{G_i'}(\omega')t\right]}{\mathrm{PE}\left[\sum_{j=1}^{N_{vec}} \chi_{R_j''}^{G_j}(\omega'')t^2\right]}$$

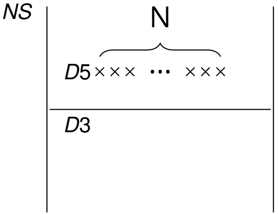
1. Compute characters for the representations of vector multiplets and hypermultiplets
2. Determine Haar measures
3. Perform integrals

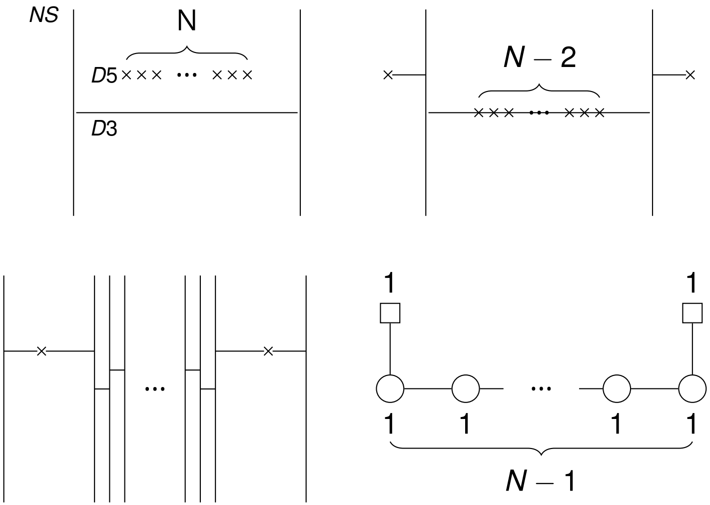
$U(1)$ gauge theory with N flavours

- Consider SQED



- Can see $\dim_{\mathbb{H}} \mathcal{C} = 1$ and $\dim_{\mathbb{H}} \mathcal{H} = N - 1$
- Can write down associated brane construction [3]





NS
N
D5
D3
N − 2
1
1
1
1
1
1
N − 1

$U(1)$ gauge theory with N flavours

Coulomb branch

- Determine Hilbert series using monopole formula

$$\mathrm{HS}_{\mathcal{C}} = \frac{1 + t^N}{(1 - t^2)(1 - t^N)}$$

- $\dim_{\mathbb{C}} \mathcal{C} = 2 = 2\dim_{\mathbb{H}} \mathcal{C}$ as expected
- Matching number of operators at each order reveals Coulomb branch is orbifold $\mathbb{C}^2/Z_N$
- Global symmetry is $U(1)$ except when quiver is balanced then $SU(2)$

- Determine Hilbert series using Molien-Weyl formula

$$\mathrm{HS}_{\mathcal{H}} = \frac{\sum_{k=0}^{N-1} \binom{N-1}{k}^2 t^{2k}}{(1-t^2)^{2(N-1)}}$$

- $\dim_{\mathbb{C}} \mathcal{H} = 2(N-1) = 2\dim_{\mathbb{H}} \mathcal{H}$ as expected
- Hilbert series too complicated to extract Higgs branch directly
- Use superpotential $W = \mathrm{tr}(BA\phi)$ with condition $\frac{\partial W}{\partial \phi} = BA = 0$
- Now defining $M = AB$ gives condition $M^2 = 0$ and

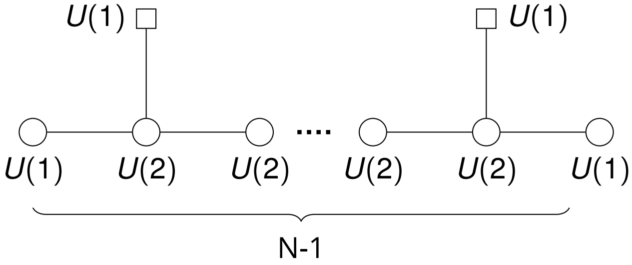
$$\mathcal{H} = \{M_{N\times N} | \mathrm{tr}M = 0, M^2 = 0, \mathrm{rank}\ M \leq 1\} = \bar{\mathcal{O}}_{\mathrm{min}}^{\mathfrak{sl}(N)}$$

- Global symmetry is $SU(N)$ (consistent with self-dual case)

$U(2)$ gauge theory with N flavours

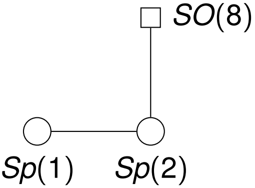
- Consider similar theory where Higgs branch rank $M \leq 2$

- Also have S-dual theory where Coulomb branch has rank $M \leq 2$



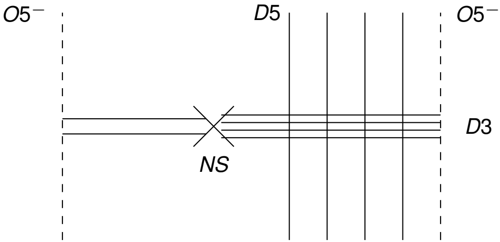
$Sp(1) \times Sp(2)$ gauge theory with 4 flavours

- Now consider more exotic orthosymplectic theory

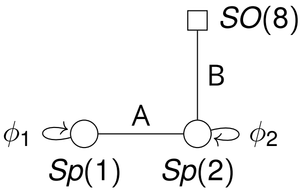


- Can see $\dim_{\mathbb{H}} \mathcal{C} = 3$ and $\dim_{\mathbb{H}} \mathcal{H} = 11$

- Try to compute Hilbert series using monopole formula but diverges
- Compute Higgs branch of magnetic quiver using Molien-Weyl formula
- Realise Sp gauge symmetry using orientifold planes [12]



- Global symmetry is $Sp(1) \times SO(8)$
- Use Molien-Weyl formula again to determine Hilbert series
- For orthosymplectic quivers edges only represent half-hypermultiplets [13]



- Use Mathematica due to difficulty of integration

Conclusion

- Computed properties of SQED $U(1)$ quiver
- Analysed related $U(2)$ gauge theory
- Extracted information from $Sp(1) \times Sp(2)$ gauge theory
- Planning to find magnetic quiver for $Sp(1) \times Sp(2)$ gauge theory to compute Coulomb branch

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