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Unitary and Orthosymplectic 3d $\mathcal{N} = 4$ Quiver Gauge Theories

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Abstract

The use of quivers as a tool for investigating supersymmetric gauge theories allows us to determine properties such as the moduli space and global symmetry that would otherwise have been very computationally intensive to find. Specifically, we apply quivers to 3d $\mathcal{N} = 4$ gauge theories and also employ other techniques such as Hilbert Series and brane constructions. We first analyse the unitary $U(1)$ gauge theory with N flavours - also known as SQED - as well as the related $U(2)$ gauge theory with N flavours. Then, we turn to the more exotic orthosymplectic $Sp(1) \times Sp(2)$ gauge theory with 4 flavours and attempt to construct a magnetic quiver. Even though methods from literature do not let us find the correct magnetic quiver, we still obtain useful analysis of the theory by considering brane constructions.

Contents

Chapter 1

Introduction

Since Emmy Noether introduced Noether's theorem in 1918 [1], symmetries have been the central subject of the study of theories in physics. These are especially important in Quantum Field Theory where some dynamics and interactions can be inferred from the gauge and global symmetries acting on the bosonic and fermionic fields. Although bosonic and fermionic fields can mix among themselves by symmetry transformations, it was believed for a long time that these two types of fields are firmly separate, partly due to the fundamental differences in their properties and ways of quantising them.
This changed with the introduction of *supersymmetry* which proposes additional symmetries that mix bosonic and fermionic fields into each other. The motivation for doing this stems from many mathematical benefits gained from the extra symmetry such as confinement in supersymmetric quantum chromodynamics, supergravity and superstring theory that have the potential to enhance our understanding of the universe at the most fundamental level.
The generators of the supersymmetry are called *supercharges* and can act on states of the theory

$$\hat{Q}\,|\text{fermion}\rangle = |\text{boson}\rangle\ ,\ \hat{Q}\,|\text{boson}\rangle = |\text{fermion}\rangle\ ,$$

changing their type. This suggests that, rather than considering states as separate entities, they should be packaged into *supermultiplets*; this is similar to the set of eigenstates of a harmonic oscillator with raising and lowering operators corresponding to the supercharges, but with a finite number of states per supermultiplet [2].
The simplest supersymmetric action we can write down is known as the *Wess-Zumino action* for one supersymmetry ($\mathcal{N} = 1$) and one supermultiplet

$$S = \int d^4x \left[\partial^\mu \chi^\dagger \partial_\mu \chi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - \left|\frac{\partial W}{\partial \chi}\right|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial \chi^2}\psi\psi - \frac{1}{2}\frac{\partial^2 W^\dagger}{\partial \chi^{\dagger 2}}\bar{\psi}\bar{\psi}\right]$$

where W is known as the *superpotential* [2]. As we can see, we have a kinetic term that depends on the fields and a scalar potential term that is constructed from the superpotential. More generally we can have many supermultiplets, resulting in a scalar potential given by

$$V(\phi) = \sum_i \left|\frac{\partial W}{\partial \chi_i}\right|^2 \tag{1.1}$$

where χ_i are the scalar fields of our theory. This tells us that a stationary superpotential corresponds to a free theory and will become important later.
With more supersymmetries, we will have more supermultiplets. For the 3-dimensional case with $\mathcal{N} = 4$ that we are considering, we have 4 pairs of supercharges which give us vector multiplets containing a gauge field, real scalar fields and spinors along with hypermultiplets containing a complex scalar field, its conjugate and spinors [3].

Chapter 2

Quivers

For our purposes, we want to specifically look at gauge theories that form the backbone of many aspects of modern physics such as the Standard Model. A gauge theory is a theory with a Lagrangian that is invariant under a transformation given by a Lie group[1]. These can either be Abelian, such as electromagnetism, or non-Abelian, as given by Yang-Mills theory [4]. As mentioned in the previous section, finding the symmetries of a theory is essential for understanding solutions and their dynamics.
For supersymmetric gauge theories, however, this is often difficult as the Lagrangians can be very long with many terms hiding the important details. To illustrate this, we have included the action for supersymmetric quantum electrodynamics (SQED)

$$\begin{aligned} S_{SQED} = \int d^4x \Bigg[& -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{\vartheta}{32\pi^2} F^{\star}_{\mu\nu}F^{\mu\nu} - \frac{i}{e^2}\lambda\sigma^{\mu}\partial_{\mu}\bar{\lambda} + \sum_{i=1}^{N}(|\mathcal{D}_{\mu}\phi_i|^2 - i\bar{\psi}\bar{\sigma}^{\mu}\mathcal{D}_{\mu}\psi_i) \\ & + \frac{D^2}{2e^2} + \sum_{i=1}^{N}(|F_i^2| - \sqrt{2}q_i(\phi_i\bar{\lambda}\bar{\psi}_i - \phi_i^{\dagger}\lambda\psi_i) + q_i D|\phi_i|^2) \Bigg] \end{aligned}$$

also known as the $U(1)$ Abelian gauge theory with N flavours, or N electrons [2, 5]. We recognise the Maxwell term, a vector multiplet and N hypermultiplets in the first line, and interaction terms in the second line.
To make this easier to analyse, we now introduce *quiver* diagrams in which we can encode all the important information from the action. In this case we write

Figure 2.1: Quiver diagram for SQED in $\mathcal{N} = 4$ notation.

where we have one circle *gauge node* to denote the $U(1)$ gauge symmetry and one square *flavour node* to denote the $SU(N)$ flavour symmetry. In the quiver diagram in Fig. 2.1 the gauge node represents the vector multiplet transforming in the adjoint representation[2] of the gauge group and the edge represents the hypermultiplets transforming in the bifundamental representation of the neighbouring nodes. We will turn back to analyse this particular quiver later.
For a general quiver, we can also read off the superpotential easily. To do this, we transform

[1]Lie groups are groups that are also differentiable manifolds. In practice this corresponds to continuous transformations such as rotations.

[2]Representations map abstract group elements to matrices that can act on sets of states.

the diagram into an $\mathcal{N}=2$ quiver by considering a piece of a quiver with two gauge nodes of ranks n_1 and n_2 as shown in 2.2. As $\mathcal{N}=4$ supermultiplets can be considered to be constructed from $\mathcal{N}=2$ supermultiplets, we have to look at the decomposition of vector multiplets and hypermultiplets. In this notation the adjoint $\mathcal{N}=4$ vector multiplet becomes an $\mathcal{N}=2$ adjoint vector multiplet and an $\mathcal{N}=2$ adjoint chiral multiplet (a third type of supermultiplet denoted by ϕ_1 and ϕ_2) and the bifundamental $\mathcal{N}=4$ hypermultiplet decomposes into $\mathcal{N}=2$ fundamental and antifundamental chiral multiplets (denoted by A and B).

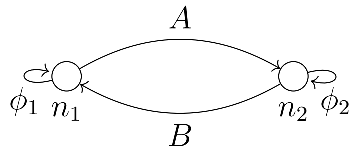

Figure 2.2: Quiver piece in $\mathcal{N}=2$ notation.

We now assign the chiral multiplets to vector spaces according to their representations. Here, ϕ_1 and ϕ_2 correspond to $n_1 \times n_1$ and $n_2 \times n_2$ matrices respectively [6]. A and B can be considered to map between the nodes as $n_1 \times n_2$ and $n_2 \times n_1$ matrices respectively. The superpotential can be read off as the trace of the difference of the closed paths in this directed graph as given by (2.1). For larger and more complex quivers, other contributions will simply be added up.

$$W = \mathrm{tr}(AB\phi_1 - BA\phi_2) \tag{2.1}$$

In the case where one of the nodes is a flavour node, there will be only one term as there is no adjoint chiral multiplet from a flavour node and therefore only one closed path. Hence we can find different conditions to make the superpotential stationary as mentioned in the previous chapter. These are known as the F-flat conditions and can be written as

$$\frac{\partial W}{\partial \phi_1} = AB = 0$$

where in the case of Fig. 2.2 we will have four different conditions.

2.1 Moduli Spaces

When analysing a theory, we are interested in characterising their vacuum states. Theories such as the Higgs boson or instantons [7] give rise to non-vanishing vacuum expectation values (VEVs) of the scalar fields. This means there are inequivalent vacuum states given by the different VEVs, or *moduli*, which correspond to the continuous vacuum field configurations. Due to the many scalar fields in quiver gauge theories, the vacuum structure is especially rich.
We can define a moduli space of vacua, parametrised by these VEVs, that encapsulates our degrees of freedom of transformations we can perform on the fields in a vacuum state without adding potential energy (i.e. remaining in a vacuum state). For supersymmetric theories, we can find the moduli space by considering the F-flat conditions that minimise the scalar potential on the dynamic superfields as seen in (1.1).
In the quivers we will analyse, the moduli space decomposes into a Coulomb branch $\mathcal{C}$, corresponding to VEVs of scalar fields in vector multiplets, and a Higgs branch $\mathcal{H}$, corresponding to VEVs of scalar fields in hypermultiplets, of which the union is the full moduli space [8]. In the literature, moduli space branches are often referred to as algebraic varieties[3] but can also be

[3] An algebraic variety is the space of solutions of a system of polynomials.

seen as manifolds (or orbifolds) that encode geometric information about the space [9]. We will aim to compute the two branches separately.
Further study of the moduli space can reveal rich structures such as the moduli space metric which governs the natural evolution of the state of the system within the moduli space according to a geodesic equation [7].

2.2 Hilbert Series

To analyse the moduli spaces of quivers, we need to develop the mathematical machinery to do so. The Hilbert Series is a useful tool for characterising geometric spaces by counting the number of linearly independent polynomials, or operators, on it at each degree.
We start by taking some algebraic variety $\mathcal{V}$ on a graded polynomial ring R and define the Hilbert Series of $\mathcal{V}$ as

$$\mathrm{HS}_{\mathcal{V}}(t) = \sum_i \dim(R_i) t^i \tag{2.2}$$

where R_i is the set of operators of degree i and $0 \leq t < 1$ is a dummy variable, referred to as the fugacity.
To illustrate how this works in practise let us consider an example that will gain importance later. Consider the variety $\mathbb{C}^2$ where the ring now corresponds to polynomials constructed from the generators z_1 and z_2, which are complex numbers, and is graded by the degree at which they appear. At zeroth order we have one operator $R_0 = \{1\}$, at first order we have two operators, $R_1 = \{z_1, z_2\}$, and at second order we have three $R_2 = \{z_1^2, z_1 z_2, z_2^2\}$. We can see that in general $\dim(R_i) = i + 1$ and therefore

$$\mathrm{HS}_{\mathbb{C}^2}(t) = \sum_i (i+1) t^i = \frac{1}{(1-t)^2} \ .$$

As can be seen from this example, the Hilbert Series is generally the quotient of two polynomials in t and will have a pole at $t = 1$. The order of this pole encodes the complex dimension of the moduli space [10] which can be seen to be 2 in this case, as expected.
The Hilbert Series also allows the extraction of the full moduli space by matching the number of operators at each degree to a known reference [11], although in practise this is sometimes not practical when these spaces are very complicated.

2.2.1 Coulomb Branch

We want to consider the Coulomb branch first. Before doing anything else, we can compute the quaternionic[4] dimension of the Coulomb branch $\mathcal{C}$ of a quiver, given by

$$\dim_{\mathbb{H}} \mathcal{C} = \sum_i \mathrm{rank}(G_i)$$

where G_i are the gauge groups of the theory. Conceptually, this corresponds to the number of vector multiplets which each carry one quaternionic degree of freedom. This means the dimension can be easily read off the quiver as the sum of the gauge node numbers.
The general Hilbert Series for the Coulomb branch is given by the monopole formula which

[4] Quaternions are a four-dimensional extension to the complex numbers with the generators $1, i, j, k$.

counts the dressed monopole operators graded by their R-charge, which is a charge associated with an internal symmetry called the R-symmetry [12]. More information about this can be found at [7]. There are multiple ways of writing this but we use the expression from [13] as

$$\mathrm{HS}_{\mathcal{C}}(t) = \sum_{\vec{m}_1} \sum_{\vec{m}_2} \cdots \sum_{\vec{m}_x} t^{2\Delta(\vec{m}_1, \vec{m}_2, \cdots, \vec{m}_x)} \prod_{i=1}^{x} P_{G_i}(t^2, \vec{m}_i)$$

where the overall gauge group is $G_1 \times G_2 \times \cdots \times G_x$. Respectively, P_{G_i} and $\vec{m}_i$ are the classical dressing factor and magnetic fluxes associated with the gauge group on the i$^{\text{th}}$ node. Each magnetic flux from $\vec{m} = {}^{\Gamma(\hat{G})}/_{W(\hat{G})}$ has a bare monopole operator associated with it which is shown in [14]. This is why we are summing over them to take into account all operators. For each gauge group G we have the weight lattice Γ of its GNO dual group $\hat{G}$ and its Weyl group W. Details about all of the above named objects and concepts can be found in [2, 3]. Having obtained the bare monopole operators, we now proceed to dress them. This dressing accounts for additional multiplicities arising from residual gauge symmetries and is captured by the classical dressing factors P_{G_i} [12]. In the following, we will lay out some computational tools used to apply the monopole formula.
In particular, we can summarise the range of magnetic fluxes $\vec{m} = (m_1, m_2, \cdots, m_r)$ for some useful Lie groups in the table below where r is the rank of the group [13].

G	$\vec{m}$
$U(r)$	$m_1 \geq m_2 \geq \cdots \geq m_r \geq -\infty$
$Sp(r)$	$m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$

The classical dressing factor is given by

$$P_G(t^2) = \prod_{i=1}^{r} \frac{1}{1 - t^{2d_i}}$$

where d_i is the degree of the i$^{\text{th}}$ Casimir invariant[5] associated with the unbroken residual gauge group [3, 12, 15]. As this is often difficult to compute, we can also write the results for the most useful gauge groups for our purposes. To do this, we define a sequence of non-increasing fluxes $\vec{n}$ in $\vec{m}$ and collect all repeating fluxes in $\vec{n}_{res} = (a_1^{r_1}, a_2^{r_2}, \cdots, a_k^{r_k})$ where the integer a_i^j is repeated j times in $\vec{n}$ such that $r_1 + r_2 + \cdots + r_k = r$. We can then write the classical dressing factors as shown in the table below [13].

G	$\vec{n}$	$\vec{n}_{res}$	P_G
$U(r)$	$(m_1, m_2, \cdots, m_r)$	$(a_1^{r_1}, a_2^{r_2}, \cdots, a_k^{r_k})$	$\prod_{i=1}^{k} \prod_{j=1}^{r_i} \frac{1}{1-t^{2j}}$
$Sp(r)$	$(m_1, m_2, \cdots, m_r)$	$(a_1^{r_1}, a_2^{r_2}, \cdots, a_{k-1}^{r_{k-1}}, 0^{r_k})$	$\prod_{l=1}^{r_k} \frac{1}{1-t^{4l}} \prod_{i=1}^{k-1} \prod_{j=1}^{r_i} \frac{1}{1-t^{2j}}$

The final ingredient to the monopole formula is the R-charge Δ, which is put together from contributions of vector multiplets Δ_{vec} and hypermultiplets Δ_{hyp}. For each gauge node we can compute Δ_{vec} as

$$\Delta_{vec}(\vec{m}) = - \sum_{\alpha \in \Phi_+} |\alpha(\vec{m})|$$

[5] The Casimir invariant is an operator that commutes with all members of a Lie algebra. A well-known example is $\hat{J}^2$ for angular momentum.

where Φ_+ is the set of positive roots[6] of G.
The other contribution Δ_{hyp}, as expected, comes from the edges between the nodes given by

$$\Delta_{hyp}(\vec{m}) = \frac{1}{2}\sum_{i=1}^{N}\sum_{\rho_i \in R_i} |\rho_i(\vec{m})|$$

where we are summing over N flavours and the weights ρ_i associated with each flavour representation R_i. Summing these two contributions gives us the total R-charge [7]. As this is often difficult to compute, we have again included a table for the Δ of useful quivers below [5].

Quiver type	$\Delta(\vec{m})$
$U(r)$ with N flavours	$\frac{N}{2}\sum_{i=1}^{r}\lvert m_i\rvert - \sum_{i<j}\lvert m_i - m_j\rvert$
$Sp(r)$ with N flavours	$(N-2)\sum_{i=1}^{r}\lvert m_i\rvert - \sum_{i<j}\lvert m_i - m_j\rvert - \sum_{i<j}\lvert m_i + m_j\rvert$

2.2.2 Higgs Branch

Again, we start by computing the quaternionic dimension of the Higgs branch $\mathcal{H}$ of a quiver which is given by

$$\dim_{\mathbb{H}}\mathcal{H} = \#\text{hypermultiplets} - \sum_i \dim(G_i)$$

where G_i are the gauge groups of the theory [18]. This corresponds to the number of hypermultiplets which, similarly to the vector multiplets, carry one quaternionic degree of freedom with the number of constraints imposed by the requirement of invariance under the gauge group action subtracted.
Due to the very different nature of the Higgs branch, we cannot use the monopole formula to make statements about it. Instead we use the Molien-Weyl projection that gives us the Hilbert Series as

$$\mathrm{HS}_{\mathcal{H}}(t) = \int_G d\mu_G \mathrm{PE}\left[\sum_{i=1}^{N_{hyp}} \chi_{R_i}(\omega)\chi'_{R'_i}(\omega')t - \sum_{i=1}^{N_r}\chi_{R''_i}(\omega)t^{d_i}\right] \tag{2.3}$$

where N_{hyp} is the number of $\mathcal{N}=2$ chiral multiplets and N_r is the number of F-flat relations from the hypermultiplet contributions. The plethystic exponential defined as

$$\mathrm{PE}[f(x_1, x_2, ...)] = \exp\left(\sum_{k=1}^{\infty}\frac{f(x_1^k, x_2^k, ...)}{k}\right)$$

helps us count the number of operators at each degree.
To visualise how this works, we can use the example of $\mathbb{C}^2$ for which we computed the Hilbert Series before. For each basis operator z_1 and z_2 we have one factor of PE[t] so our Hilbert Series

[6]A root α is a linear form defined by the commutator $[H_i, E_\alpha] = \alpha(H_i)E_\alpha$ where H_i is in the Cartan subalgebra (maximum commutative subalgebra of $\mathfrak{g}$, the Lie algebra of G) and E_α is not. More about this can be found at [16].

[7]As $0 \leq t < 1$, we want the R-charge to be positive so that the monopole formula converges. If we do not have enough hypermultiplets and $\Delta \leq 0$, this theory is called *bad* which requires much more rigorous treatment [17]

will be $\mathrm{HS}_{\mathbb{C}^2}(t) = \mathrm{PE}[2t] = \frac{1}{(1-t)^2}$.
We can take into account how the hypermultiplets and vector multiplets transform by inserting the characters[8] of the respective representations and summing over them. The continuous fugacities ω_i that specify a group element will be integrated over later so that we are only left with t. The power d_i is the degree of our F-flat relation which is typically 2. As we expect our moduli space to be invariant under the action of the gauge groups, we have to quotient out the gauge group action which is done by integrating over the gauge group using the Haar measure

$$\int_G d\mu_G = \frac{1}{(2\pi i)^r} \oint_{|\omega_1|=1} \oint_{|\omega_2|=1} \cdots \oint_{|\omega_r|=1} \frac{d\omega_1}{\omega_1}\frac{d\omega_2}{\omega_2}\cdots\frac{d\omega_r}{\omega_r} \prod_{\alpha\in\Phi_+}\left(1-\prod_{i=1}^{r}\omega_i^{\alpha_i}\right)$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_r)$ are the dummy fugacities mentioned above with $r = \mathrm{rank}(G)$. The term Φ_+ again refers to the set of positive roots of G [13]. If we have multiple gauge nodes, we can simply multiply their Haar measures and integrate.

2.3 Brane Constructions

Quiver gauge theories are often low-energy limits of brane constructions. Branes are extended string-like objects in superstring theory that can act as boundary conditions for strings but are treated as dynamical objects themselves that can interact and give rise to gauge theories [19]. For unitary quivers we will consider three types of branes in 9+1 dimensional spacetime. Their extensions are as listed in the table below.

Brane	0	1	2	3	4	5	6	7	8	9
NS5	×	×	×	×	×	×				
D5	×	×	×					×	×	×
D3	×	×	×				×			

Here the NS5, D5 and D3 branes span the marked dimensions respectively, 0 being time, and are pointlike in all others. The only exception is the D3 brane which does not span dimension 6 but is only finite or semi-infinite. It has to start and/or end on an NS5 or D5 brane [20]. As an example consider a setup of branes in Fig. 2.3 projected along the 6 dimension on the horizontal axis and along the 3, 4 or 5 dimension on the vertical axis.

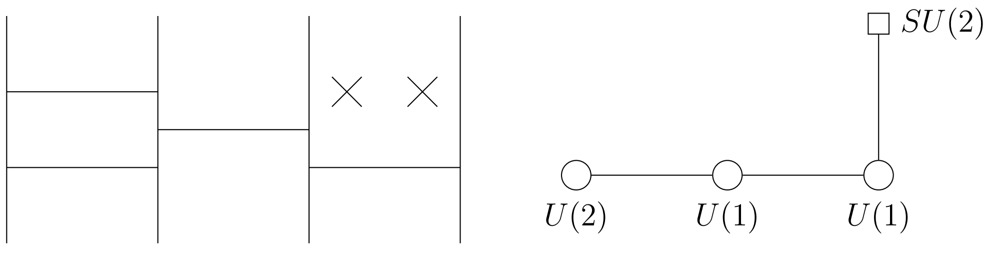

Figure 2.3: Example brane construction with associated unitary quiver.

We can now translate the brane construction into a quiver. Each finite interval between NS5 branes becomes a gauge node and the number of D3 branes between them corresponds to the number label on it. If we have D5 branes, we add a flavour node with the number of D5 branes [6]. We can also apply this method to generate the brane construction for any unitary quiver.

[8] A character is the trace of the matrix associated with a group element in a particular representation.

Chapter 3

$U(1)$ Gauge Theory with N Flavours

We will first look at the SQED, also known as the $U(1)$ gauge theory with N flavours, described by the quiver shown in Fig. (3.1). Our goal is to understand some properties about this theory and then look at related theories.

Figure 3.1: The quiver diagram of $U(1)$ gauge theory with N flavours.

3.1 Coulomb Branch

To investigate the Coulomb branch, we employ the monopole formula. Since we only have one gauge node, it is of the form [12]

$$\mathrm{HS}_{\mathcal{C}}(t) = \sum_{\vec{m}} t^{2\Delta(\vec{m})} P_{U(1)}(t^2, \vec{m}) \ .$$

We can find the classical dressing factor with the formula given in 2.2.1 where we use the fact that $U(1)$ has one Casimir invariant of degree 1. Alternatively, we can read the classical dressing factor from the table where we have only one m and hence $r_1 = r = 1$. This gives us

$$P_{U(1)} = \frac{1}{1-t^2}$$

which concludes the first step of the computation. The R-charge is generally given by

$$\Delta = \Delta_{vec} + \Delta_{hyp}$$

but as U(1) is Abelian and therefore has no associated roots, the vector multiplet contribution to the R-charge will vanish. For the hypermultiplets we can determine each to contribute

$$\Delta_{hyp} = \frac{1}{2}|m|$$

and as they transform in the bifundamental representation of $U(1)$ and $SU(N)$ which have dimensions 1 and N respectively, there will be N hypermultiplets with a total contribution and total R-charge of

$$\Delta = \frac{1}{2}N|m|$$

which can be put into the equation. The Coulomb branch Hilbert Series can now finally be computed as

$$\mathrm{HS}_{\mathcal{C}} = \sum_{m\in\mathbb{Z}} P_{U(1)} t^{2\Delta} = \sum_{m=-\inf} \frac{t^{N|m|}}{1-t^2} = 2\sum_{m=1}^{\infty} \frac{(t^N)^m}{1-t^2} + \frac{1}{1-t^2} = \frac{1+t^N}{(1-t^2)(1-t^N)} \ .$$

From the order of the pole at $t = 1$ we can deduce the complex dimension of the Coulomb branch to be 2. This matches the observation that the quaternionic dimension is 1 which can be read off as the sum of the ranks of the gauge groups from the quiver.
By matching coefficients at different orders to a known reference, we can determine the Coulomb branch [11]. For $N = 1$ this gives $\mathbb{C}^2$ and for $N = 2$ we get $\mathbb{C}^2/Z_2$. Using an inductive guess that for all N, the Coulomb branch is $\mathbb{C}^2/Z_N$, we can generalise the result and find the moduli space for all theories of this type. As we are often not able to find an exact label for the moduli space, we present another way of finding this in the next section.

The global symmetry of the Coulomb branch can be read off of the quiver if the nodes are balanced. For unitary quivers, balance means that twice the gauge node number is the sum of the neighbouring node numbers. Then we consider each balanced piece to be a Dynkin diagram which gives us the Lie algebra of the symmetry group [21]. In our case, nothing is balanced unless $N = 2$. When $N = 2$, the quiver node can be treated as the Dynkin diagram $A_1 = \mathfrak{su}(2)$ of which the group is $SU(2)$. If nothing is balanced, the symmetry is $U(1)^k$. In our case, this is just $U(1)$.

3.2 Higgs Branch

We now want to take a look at the Higgs branch. In our case, the form of the Molien-Weyl formula is

$$\mathrm{HS}_{\mathcal{H}}(t) = \int_{U(1)} d\mu_{U(1)} \mathrm{PE}\left[\sum_{i=1}^{2N} \chi^{\text{source}}_{\text{fund}}(\omega)\chi^{\text{target}}_{\text{antifund}}(\omega')t - \chi^{U(1)}_{\text{adj}}(\omega)t^2\right]$$

where we take into account the $2N$ $\mathcal{N} = 2$ chiral multiplets in the fundamental and antifundamental representations of $U(1)$ and $SU(N)$ that the N $\mathcal{N} = 4$ hypermultiplets decompose into. We also take into account the adjoint representation of $U(1)$ for the $\mathcal{N} = 2$ chiral multiplet that the $\mathcal{N} = 4$ vector multiplet decomposes into. We can compute the vector multiplet contribution easily since the character of the adjoint representation of $U(1)$ is 1 and hence this factor becomes

$$\mathrm{PE}[-t^2] = 1 - t^2 \ .$$

For the hypermultiplet contribution, we consider the fundamental and antifundamental representations of $U(1)$ and $SU(N)$ which results in

$$\mathrm{PE}([1, 0, ..., 0]_y zt + [0, ..., 0, 1]_y \frac{t}{z}) = \frac{1}{(1-tz)^N(1-t/z)^N}$$

where y are the fugacities associated with $SU(N)$ and will be set to 1. The next step of the computation is the Haar measure $d\mu_G$, but as $U(1)$ has no roots, this will be

$$\frac{1}{2\pi i}\oint_{|z|=1} \frac{dz}{z} \ .$$

Thus, the integral can be written as

$$\mathrm{HS}_{\mathcal{H}} = \frac{(1-t^2)}{2\pi i} \oint_{|z|=1} dz \frac{z^{N-1}}{(1-tz)^N (t-z)^N}$$

and using the residue theorem yields

$$\mathrm{HS}_{\mathcal{H}} = \frac{\sum_{k=0}^{N-1} \binom{N-1}{k}^2 t^{2k}}{(1-t^2)^{2(N-1)}} \ .$$

From the order of the pole at $t = 1$ we can again read off the complex dimension of the Higgs branch which is $2(N-1)$. This matches our expectation of being twice the quaternionic dimension of $N-1$. As mentioned above, sometimes we cannot extract a closed form expression of the moduli space by matching coefficients. Therefore we explicitly look at the F-flat condition.

First, we read off the superpotential for the quiver $W = \phi AB$, where $\phi_{1\times1}$ is the loop associated with the gauge node and $A_{N\times1}$ and $B_{1\times N}$ map from the gauge to the flavour node and back. Hence for the Higgs branch, the superpotential has to satisfy $\frac{\partial W}{\partial \phi} = AB = 0$. We define an auxiliary matrix $M = BA$ such that $M^2 = B(AB)A = 0$. This characterises the moduli space since M has to be traceless and minimally nilpotent. The Higgs branch is hence given by the space of all matrices satisfying these conditions and can be summarised as

$$\mathcal{H} = \{M_{N\times N} | \mathrm{tr} M = 0, M^2 = 0, \mathrm{rank}\ M \leq 1\} = \bar{\mathcal{O}}^{\mathfrak{sl}(N)}_{\mathrm{min}}$$

which is also called the the minimum nilpotent orbit of $\mathfrak{sl}(N)$. This type of space often appears when considering the Higgs branch of quiver gauge theories. For more information the reader may be directed to [22].

The global symmetry of the Higgs branch can be read off from the flavour nodes of the quiver of the theory. Therefore it is $SU(N)$ for N flavours. When $N = 2$, the global symmetry of the Coulomb branch is the same as the Higgs branch. In fact, both Hilbert Series are equivalent, meaning $\mathcal{C} = \mathcal{H}$. This is related to the concept of *magnetic quivers* where the Coulomb branch of the original quiver is the Higgs branch of the magnetic quiver. In this case, the quiver is *self-dual*. More details on the magnetic quiver of this theory are given in the appendix A.1. We will discuss and apply this concept later.

3.3 Related Theories

3.3.1 Higgs Branch with rank ≤ 2

To explore properties of theories close to SQED, we consider theories with a similar Higgs branch to the one found above. Specifically, we want to find the theory with the Higgs branch

$$\mathcal{H} = \{M_{N\times N} | \mathrm{tr} M = 0, M^2 = 0, \mathrm{rank}\ M \leq 2\} \ .$$

Going from the moduli space to an associated theory is generally difficult but in this case it will be possible. If we want the auxiliary matrix to have rank $M \leq 2$, we need to have a superpotential $W = \phi AB$ where $\phi_{2\times2}$, $A_{N\times2}$ and $B_{2\times N}$. The simplest quiver to achieve this is shown below where we modify the gauge group from $U(1)$ to $U(2)$. The quaternionic dimension

Figure 3.2: The quiver diagram of U(2) gauge theory with N flavours.

of the Coulomb branch of this theory will be 2 and the quaternionic dimension of the Higgs branch will be $2N - 4$.

The result can also be generalised to the condition rank $M \leq k$ where we will find the $U(k)$ gauge theory with N flavours, where $k \leq N$.
We now aim to compute some properties of this quiver.

Coulomb Branch

We first want to look at the Coulomb branch of this theory and use the monopole formula. Using the table with $r_1 = 2$, we find the classical dressing factor as

$$P_{U(2)} = \frac{1}{1-t^2}\frac{1}{1-t^4}$$

which coincides with the two Casimir invariants of $U(2)$ with degrees 1 and 2.

As $U(2)$ does have associated roots, we will have a vector multiplet contribution to the R-charge

$$\Delta_{vec} = -|m_1 - m_2|$$

and also contribution from the hypermultiplets

$$\Delta_{hyp} = \frac{N}{2}(|m_1| + |m_2|) - |m_1 - m_2| \ .$$

Finally, we can compute the Hilbert Series

$$HS_{\mathcal{C}} = \frac{(1-t^{2N})(1-t^{2N-2})}{(1-t^{N-2})^2(1-t^N)^2(1-t^2)(1-t^4)}$$

where we can confirm the complex dimension for the Coulomb branch is 4 [21]. As is the case for the previously considered SQED quiver, the global symmetry here is $U(1)$ except if $N = 4$ in which case, it is enhanced to $SU(2)$. As the flavour symmetry is usually the Higgs branch symmetry, in this case there is no N for which the global symmetries of the Coulomb branch and the Higgs branch match. Hence it is expected that this theory has no self-dual case.

Higgs Branch

For the Higgs branch, we again consider the Molien-Weyl formula which is of the form

$$HS_{\mathcal{H}}(t) = \int\limits_{U(2)} d\mu_{U(2)} \mathrm{PE}\left[\sum_{i=1}^{2N} \chi_{\text{fund}}^{\text{source}}(\omega)\chi_{\text{antifund}}^{\text{target}}(\omega')t - \chi_{\text{adj}}^{U(2)}(\omega)t^2\right] \ ;$$

very similarly to the form of the $U(1)$ case. The overall F-flat Hilbert Series can be summarised by

$$PE\left[(z_1 + z_2)\,[0,\ldots,0,1] + \left(\frac{1}{z_1} + \frac{1}{z_2}\right)[1,0,\ldots,0]\right] \mathrm{PE}\left[-(z_1+z_2)\left(\frac{1}{z_1}+\frac{1}{z_2}\right)t^2\right]$$

which depends on z_1 and z_2 as we now have to perform two line integrals due to the rank of $U(2)$. The Haar measure is given by

$$\int d\mu_{U(2)}(z_1, z_2) = \frac{1}{2} \int\limits_{|z_1|=1} \frac{dz_1}{z_1} \int\limits_{|z_2|=1} \frac{dz_2}{z_2} \left(\frac{1}{z_1} - \frac{1}{z_2} \right) (z_1 - z_2)$$

where we can see that z_1 and z_2 are being integrated out. As this integral is quite tricky by hand, we are using Mathematica [23] to perform the computation. Some results for different N are given below

$$\mathrm{HS}_{\mathcal{H},N=3} = \frac{1 + 4t^2 + t^4}{(1 - t^2)^4}$$

$$\mathrm{HS}_{\mathcal{H},N=4} = \frac{1 + 7t^2 + 12t^4 + 7t^6 + t^8}{(1 - t^2)^8}$$

$$\mathrm{HS}_{\mathcal{H},N=5} = \frac{1 + 12t^2 + 53t^4 + 88t^6 + 53t^8 + 12t^{10} + t^{12}}{(1 - t^2)^{12}} .$$

We first note that the complex dimensions of confirms the pattern $4N - 8$ as expected. The general form of the Hilbert Series will be

$$\mathrm{HS}_{\mathcal{H}} = \frac{P_{2N-4}(t^2)}{(1 - t^2)^{4N-8}}$$

where the numerator is a palindromic polynomial of order $2N - 4$ in t^2.

3.3.2 Coulomb Branch with rank ≤ 2

We can now also consider a theory that has a Coulomb branch similar to the Higgs branch of the SQED theory. As discussed briefly in the above, this should be the magnetic quiver of the $U(2)$ theory shown in Fig. 3.3. Finding the magnetic quiver is often difficult and can be done by looking at the brane construction of the theory. For unitary quivers, we can find the magnetic quiver by performing some specified manipulations, bringing the brane construction into the *magnetic phase* and then reading off the magnetic quiver [24]. For brevity we have not included these manipulations here but the reader may be directed to the appendix A.2 for more details.

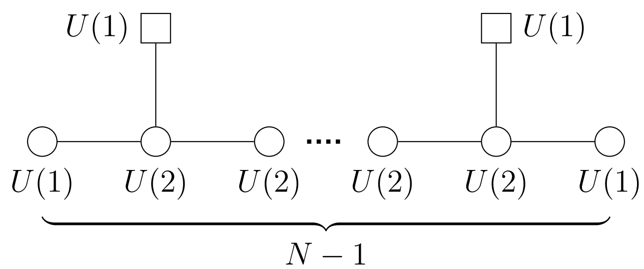

Figure 3.3: The magnetic quiver of U(2) gauge theory with N flavours for $N \geq 4$.

It can be seen that there is no N for which the magnetic quiver matches the original theory so our earlier suspicion that there exists no self-dual case can be confirmed.

Chapter 4

$Sp(1) \times Sp(2)$ Gauge Theory with 4 Flavours

In addition to the very simple $U(1)$ gauge theory, we now want to also analyse a more exotic orthosymplectic gauge theory and see how these can arise.

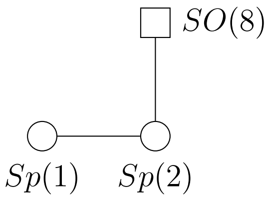

Figure 4.1: The quiver diagram for the $Sp(1) \times Sp(2)$ gauge theory with 4 flavours.

The theory we will choose for this is the $Sp(1) \times Sp(2)$ gauge theory with 4 flavours as shown in Fig. 4.1. Here, we have to recognise that the rank of $SO(2N)$ is N and therefore we have $SO(8)$ for 4 flavours. Additionally, the edge between the symplectic and orthogonal nodes now represents half-hypermultiplets and not full hypermultiplets as the fundamental representation of $SO(2N)$ is pseudo-real instead of complex as would be the case for an $SU(N)$ node. This will slightly change the way we would extract the F-flat condition and hypermultiplet contribution to the R-charge from the quiver given in a table in 2.2.1.
First, we want to compute the dimensions of the moduli space as before. For the Coulomb branch, we will have $\dim_{\mathbb{H}}\mathcal{C} = 3$ and for the Higgs branch $\dim_{\mathbb{H}}\mathcal{H} = 11$.

4.1 Coulomb Branch

We want to now go on to determine some properties of the Coulomb branch of our orthosymplectic theory. We start with the classical dressing factors which will be

$$P_{\mathrm{Sp}(1)}(t^2) = \frac{1}{1-t^4}$$
$$P_{\mathrm{Sp}(2)}(t^2) = \frac{1}{(1-t^4)(1-t^8)}$$

and so we can write the overall factor as

$$P = \prod_{i=1}^{2} P_{G_i}(t^2) = \frac{1}{(1-t^4)^2(1-t^8)}$$

which moves us on to the R-charge. For the vector multiplet contribution, we compute

$$\Delta_{vec,Sp(1)} = -2m$$
$$\Delta_{vec,Sp(2)} = -(4m_1 + 2m_2)$$

where we have labelled the fluxes from the gauge groups m for $Sp(1)$ and m_1, m_2 for $Sp(2)$. For the hypermultiplets, we have the contributions $\Delta_{hyp,1}$ from the $Sp(2) - SO(8)$ edge and $\Delta_{hyp,2}$ from the $Sp(1) - Sp(2)$ edge. These can be summarised as

$$\Delta_{hyp,1} = 8m_1 + 8m_2$$
$$\Delta_{hyp,2} = 8(m_1 + m_2 + 2m + |m_1 - m| + |m_2 - m|)$$

which gives us the overall R-charge

$$\Delta = 14m + 12m_1 + 14m_2 + 8|m_1 - m| + 8|m_2 - m| \ .$$

We can now substitute all of the above mentioned quantities into the monopole formula which takes the form

$$\mathrm{HS}_{\mathcal{C}} = P \sum_{m_1=0} \sum_{m_2 \geq m_1} \sum_{m=0} t^{2\Delta}$$

which we can compute to find the closed form expression

$$\mathrm{HS}_{\mathcal{C}} = \frac{1}{(1-t^4)^2(1-t^8)} \left[\frac{1}{(1-t^4)(1-t^{44})} \left(\frac{1}{1-t^{84}} - \frac{1}{1-t^{80}} \right) + \frac{1}{(1-t^{28})(1-t^{52})} \left(\frac{1}{1-t^{44}} - \frac{1}{1-t^{72}} \right) + \frac{1}{(1-t^{60})(1-t^{20})(1-t^{12})} \right] .$$

Although this might seem very long and complicated, we can still extract a lot of information from it. Each factor of

$$\frac{1}{1-t^{4k}} = \frac{f(t)}{1-t^2}$$

can be decomposed into a pole of order 1 and a rational function that does not have a pole at $t = 1$. The overall order of the pole at $t = 1$ can now be found by rewriting the function to

$$\mathrm{HS}_{\mathcal{C}} = \frac{Q(t^2)}{(1-t^2)^6}$$

and hence the order of the pole at $t = 1$ is 6 which gives us $\dim_{\mathbb{C}}\mathcal{C} = 6$. This matches our earlier observation that $\dim_{\mathbb{H}}\mathcal{C} = 3$.

As the quiver is not balanced, we only expect a $U(1)$ global symmetry for the Coulomb branch of this theory.

4.2 Magnetic Quiver

We now want to find the magnetic quiver - the quiver with the same Higgs branch as the Coulomb branch of the original quiver and vice versa - using the methods for orthosymplectic

quivers outlined in [25, 26]. To do so, we can draw the corresponding unitary brane construction where we add horizontal $O3$ planes to generate the orthosymplectic gauge symmetry. $O3^-$ planes generate $SO(2n)$ gauge symmetry and $O3^+$ planes corresponds to $Sp(n)$ [21]. This would however require $O3$ planes to cross NS5 branes where they change type as described in [26]. Using this method hence assumes alternating orthogonal and symplectic gauge groups which is the case in most literature on orthosymplectic quivers.
In our case, however, we have two consecutive symplectic gauge nodes; therefore this method cannot be used and we instead use the approach outlined in [27]. Here we have two gauge nodes with $Sp(k_1)$ and $Sp(k_2)$ gauge groups and N_1 and N_2 flavours respectively. In our case therefore $k_1 = 1$, $k_2 = 2$ and $N_1 = 0$, $N_2 = 4$. This gives us the brane system in Fig. 4.2.

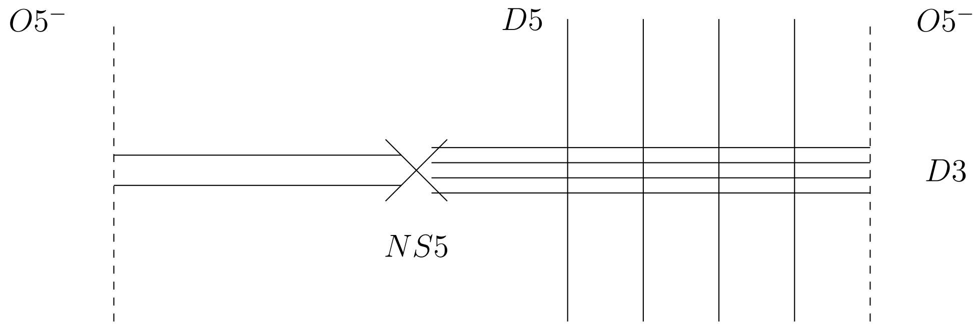

Figure 4.2: Brane construction for $Sp(1) \times Sp(2)$ theory with 4 flavours.

Each $Sp(k_i)$ node corresponds to $2k_i$ D3 branes attached to an $O5^-$ plane on one side an ending on the NS5 brane on the other. As we have $k_1 = 1$ and $k_2 = 2$, we have 2 and 4 D3 branes spanned between the $O5^-$ plane and the NS brane on each side. The $N_2 = 4$ flavours on the $Sp(2)$ gauge node are encoded by 4 D5 branes, between the NS5 brane and the $O5^-$ plane on the right and crossing the D5 branes. On the left side, there are no D5 branes as $N_1 = 0$. After using the rules outlined in [27, 28] and applying S-duality [29], we obtain the general magnetic phase given in Fig. 4.3.

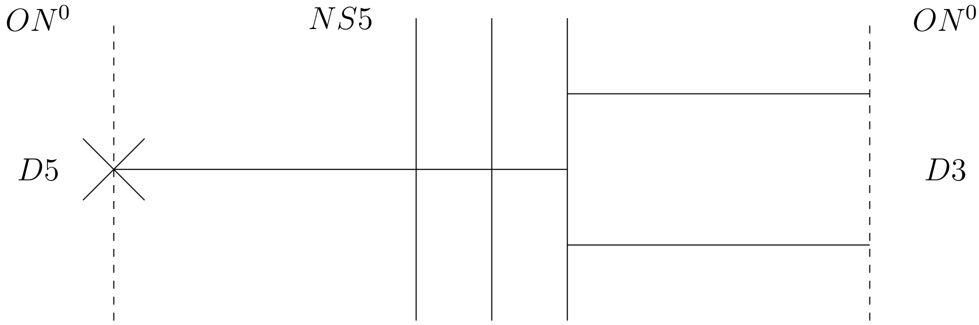

Figure 4.3: Magnetic phase for $Sp(1) \times Sp(2)$ theory with 4 flavours.

When it comes to translating this to a quiver, however, our theory differs from the ones treated by [27], as it only considers quivers of this type satisfying $k_1 < k_2$ and $N_1 + 2k_2 > 2k_1 + 1$ or $N_1 - 2k_2 + 2k_1 = 0, 1$.
To find the magnetic quiver we hence have to try a different approach. The treatment in [27] suggests that the magnetic quiver for this type of theory should be unitary. Additionally, the dimensions of Coulomb and Higgs branches have to match their swapped counterparts and the global symmetry has to be consistent.

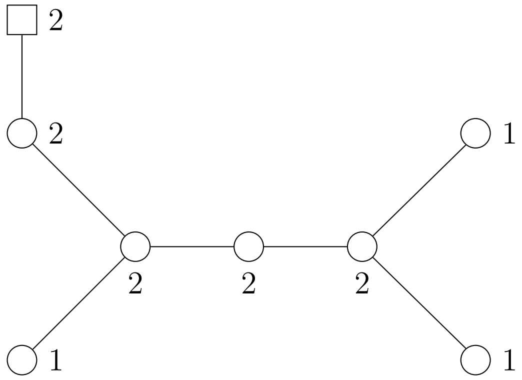

Figure 4.4: Candidate for a magnetic quiver for the $Sp(1) \times Sp(2)$ theory with 4 flavours.

The global symmetry of our original quiver would normally be the flavour symmetry $SO(8)$ but this is enhanced by a factor of $Sp(1)$ here to $Sp(1) \times SO(8)$ due to effects from the brane system [27]. We can now attempt explicit construction of the magnetic quiver by looking at variations of the approach given and trying to match the dimensions and global symmetry while preserving the overall shape of the quiver. The best possible candidate is given in Fig. 4.4.

This quiver has a Coulomb branch with $\dim_{\mathbb{H}}\mathcal{C} = 11$ and a Higgs branch with $\dim_{\mathbb{H}}\mathcal{C} = 3$. Additionally when we consider the Coulomb branch symmetry we get a balanced piece on the right that corresponds to $SO(8)$ and a balanced piece on the bottom left corner that corresponds to $Sp(1)$. The symmetry, however, does not fully match the expected $Sp(1) \times SO(8)$ as there is an extra balanced piece on the top left that gives us an additional $Sp(1)$ making the total Coulomb branch symmetry of this quiver $Sp(1)^2 \times SO(8)$. There could be an argument that our theory has some additional enhancement which explains the $Sp(1)$ but we were not able to find any evidence of this.
Historically, the only way to confirm a magnetic quiver was to match the dimension of the moduli spaces, global symmetries, and other parameters of the theory but thanks to the discovery of the monopole formula, we can directly match the moduli spaces [3]. Therefore, to check whether the magnetic quiver is indeed correct, we compute the Hilbert Series of the Higgs branch and compare it to the Hilbert Series of the Coulomb branch of the original orthosymplectic theory. This will give a definitive answer as the Hilbert Series determines the variety [11]. As we have already demonstrated multiple examples of computing the Molien-Weyl formula, we have not included this for brevity. The Hilbert Series we obtain is

$$\mathrm{HS}_{\mathcal{H}} = \frac{1}{(1-t^2)^6}\left[\frac{16(2t^4-t^2+1)(t^8-t^6+3t^4-2t^2+1)}{(1+t^2)^9}\right.$$
$$\left.\times\frac{(t^8+t^6+2t^4-t^2+1)(t^{12}+3t^{10}+11t^8+14t^6+11t^4+3t^2+1)}{(1+t^4)(1+t^2+t^4)}\right]$$

which has a pole of order 6 as expected. Unfortunately, this Hilbert Series does not match the Coulomb branch of the original orthosymplectic quiver in Fig. 4.1. This confirms that the magnetic quiver candidate given in Fig. 4.4 cannot be correct. Due to the strict requirements imposed by the global symmetry and dimensions of the moduli space, we do not believe that there exists a unitary magnetic quiver for the theory given in Fig. 4.1. Further analysis of the brane construction will be necessary to make progress on the potentially orthosymplectic magnetic quiver.

Chapter 5

Conclusion

Over the past few months, we have worked on understanding and computing properties of supersymmetric gauge theories and their implementations as brane systems in superstring theory. Moduli spaces and their symmetries have been at the centre of this investigation, giving a good indication of the nature and dynamics of the theory.

We started by considering one of the simplest supersymmetric gauge theories - the $U(1)$ gauge theory with N flavours - and revealed a surprisingly rich vacuum structure, generated by the scalar fields of the theory. This is also where we learnt most of the basic techniques of Hilbert Series, Lie theory, supersymmetry and gauge theories that are applied to more advanced theories later.

Looking at the equivalent theory with a $U(2)$ gauge group, there was an even more complex structure. It was quite remarkable that techniques from algebraic geometry and Lie theory can be used in such a manner to analyse a physical theory.

Moving on to the orthosymplectic $Sp(1) \times Sp(2)$ theory with 4 flavours, we became more familiar with brane constructions. Although we did not find a correct magnetic quiver for our theory using the techniques applied in literature, there was some progress made on identifying and testing the restrictions imposed on it.

Further work could make progress on the Higgs branch of the orthosymplectic theory that we have not included here. It may be interesting to see a different approach to finding the magnetic quiver of this theory by understanding the mathematics of S-duality and brane dynamics in greater detail.

More broadly, the focus on supersymmetric theories, while not confirmed by particle physics discoveries as of yet, allows for the development of new mathematical ideas as well as the improvement of the understanding of symmetries and dualities of gauge theories.

Bibliography

[1] Emmy Noether. "Invariant variation problems". In: *Transport theory and statistical physics* 1.3 (1971), pp. 186–207.

[2] D. Tong (ed). "Supersymmetric Field Theory". In: *University of Cambridge, Department of Applied Mathematics and Theoretical Physics* (2021).

[3] Zhenghao Zhong. "Magnetic quivers: a new perspective on supersymmetric gauge theories". PhD thesis. Imperial College London, 2023.

[4] Arthur Jaffe and Edward Witten. "Quantum yang-mills theory". In: *The millennium prize problems* 1 (2006), p. 129.

[5] Stefano Cremonesi et al. "Monopole operators and Hilbert series of Coulomb branches of 3d $\mathcal{N} = 4$ gauge theories". In: *Journal of High Energy Physics* 2014.1 (2014), pp. 1–34.

[6] A. Hanany. *Branes and quivers in string theory (Lecture 1)*. 2019.

[7] David Tong. "TASI lectures on solitons". In: *arXiv preprint hep-th/0509216* (2005).

[8] William Ronayne and Amihay Hanany. "Brane webs and 5d $\mathcal{N} = 1$ supersymmetric field theories". PhD thesis. Imperial College London, 2023.

[9] Anton Zajac. "Iter geometria: vacuum moduli spaces of supersymmetric quiver gauge theories with 8 supercharges". PhD thesis. Imperial College London, 2020.

[10] Amihay Hanany and Rudolph Kalveks. "Quiver theories for moduli spaces of classical group nilpotent orbits". In: *Journal of High Energy Physics* 2016.6 (2016), pp. 1–61.

[11] Edward Hirst. "Machine learning for Hilbert series". In: *arXiv preprint arXiv:2203.06073* (2022).

[12] A. Hanany and M. Sperling. "Algebraic properties of the monopole formula". In: *Journal of High Energy Physics* 2017(2), pp.1-43 (2017).

[13] Mohammad Akhond et al. "Five-brane webs, Higgs branches and unitary/orthosymplectic magnetic quivers". In: *Journal of High Energy Physics* 2020.12 (2020), pp. 1–69.

[14] Vadim Borokhov, Anton Kapustin, and Xinkai Wu. "Monopole operators and mirror symmetry in three dimensions". In: *Journal of High Energy Physics* 2002.12 (2003), p. 044.

[15] Stefano Cremonesi. "3d supersymmetric gauge theories and Hilbert series". In: *Eur. Phys. J. C* (2017).

[16] Joshua Ruiter. *Root systems*. 2019.

[17] Stefano Cremonesi et al. "$T^{\rho}_{\sigma}(G)$ theories and their Hilbert series". In: *Journal of High Energy Physics* 2015.1 (2015), pp. 1–56.

[18] Zhenghao Zhong. "An exceptionally simple family of Orthosymplectic 3d $\mathcal{N} = 4$ rank-0 SCFTs". In: *arXiv preprint arXiv:2411.12802* (2024).

[19] Joseph Polchinski. “TASI lectures on D-branes”. In: *arXiv preprint hep-th/9611050* (1996).

[20] Jamie Luke Edward Rogers. *Quiver vacua geometry and fermions on branes*. The University of Liverpool (United Kingdom), 2021.

[21] E.M. Amat. “$\mathcal{N} = 4$ Gauge Theories in 3D, Hilbert Series, Mirror Symmetry and Enhanced Global Symmetries”. MA thesis. Imperial College London, 2013.

[22] Yuzhou Gu. “Nilpotent Orbits: Geometry and Combinatorics”. In: (2017).

[23] Wolfram Research Inc. *Mathematica, Version 14.2*. Champaign, IL, 2024. URL: `https://www.wolfram.com/mathematica`.

[24] Antoine Bourget, Marcus Sperling, and Zhenghao Zhong. “Decay and fission of magnetic quivers”. In: *Physical Review Letters* 132.22 (2024), p. 221603.

[25] S. Nawata et al. “Magnetic quivers and line defects—On a duality between 3d $\mathcal{N} = 4$ unitary and orthosymplectic quivers”. In: *Journal of High Energy Physics* 2022(2), pp.1-72 (2022).

[26] B. Feng and A. Hanany. “Mirror symmetry by O3-planes”. In: *Journal of High Energy Physics* JHEP11(2000)033 (2000).

[27] A. Hanany and A. Zaffaroni. “Issues on orientifolds: on the brane construction of gauge theories with SO(2n) global symmetry”. In: *Journal of High Energy Physics* JHEP07(1999)009 (1999).

[28] A. Hanany and E. Witten. “Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics”. In: *Nuclear Physics B* 492 (1–2): 152–190, hep-th/9611230 (1997).

[29] J-E Bourgine. “Fiber-base duality from the algebraic perspective”. In: *Journal of High Energy Physics* 2019.3 (2019), pp. 1–60.

Appendix A

Unitary Magnetic Quivers

A.1 $U(1)$ Theory with N Flavours

To find the magnetic quiver for the $U(1)$ theory with N flavours, we first draw the brane construction associated with this theory according to 2.3 using NS5, D5 and D3 branes as shown in Fig. A.1.

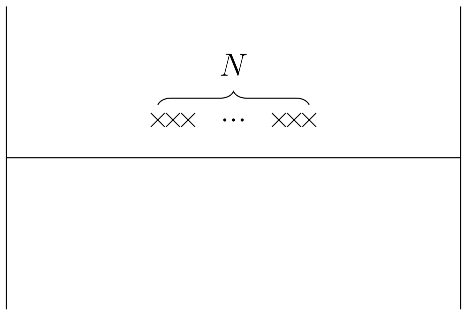

Figure A.1: Brane construction associated with $U(1)$ theory with N flavours

Now we take the two outermost D5 branes and move them past the NS5 branes, performing a Hanany-Witten transition [28]. This means that anytime a D5 brane moves past an NS5 brane, a D3 brane is created between them. This can be seen in Fig. A.2.

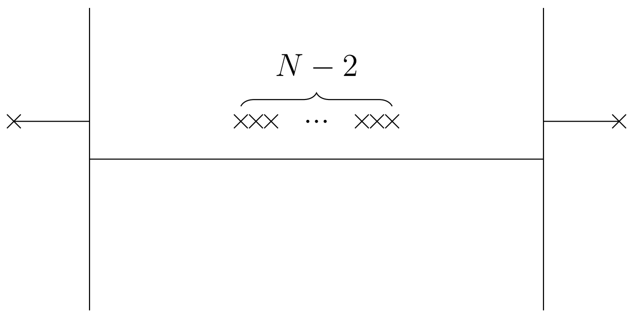

Figure A.2: Brane construction associated with $U(1)$ theory with N flavours after Hanany-Witten transition.

Now we move the D5 branes down to coincide with the D3 branes in the 3, 4 and 5 direction. We then change the projection of the brane construction from the dimensions 3, 4 or 5 on the vertical axis to the dimensions 7, 8 or 9. We will keep the dimension 6 on the horizontal axis. In this coordinate system, the D5 branes span the vertical dimension and the NS5 branes are pointlike. We also apply S-duality which exchanges D5 and NS5 brane types [6, 29]. This gives us the final magnetic phase of the brane construction Fig. A.3.

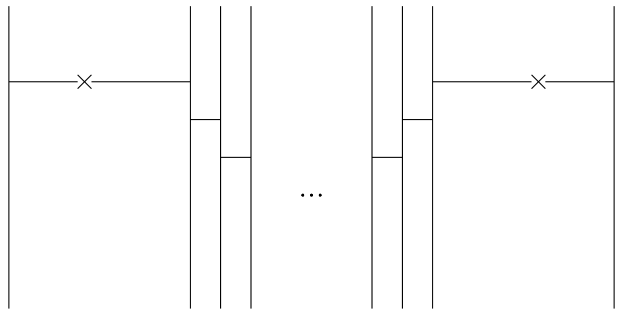

Figure A.3: Magnetic phase of brane construction associated with $U(1)$ theory with N flavours.

We can now easily translate this magnetic phase into our magnetic quiver in Fig. A.4.

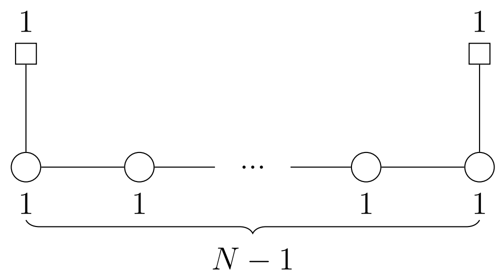

Figure A.4: Magnetic quiver for $U(1)$ theory with N flavours.

Considering the case $N = 2$ discussed in 3, we can see that all vertical NS5 branes between the pointlike D5 branes vanish and we end up with the same brane in Fig. A.3 system we started with in Fig. A.1. This confirms we are working with a self-dual theory.

A.2 $U(2)$ Theory with N Flavours

We again start by drawing the brane construction for the $U(2)$ theory with N flavours as shown in Fig. A.5.

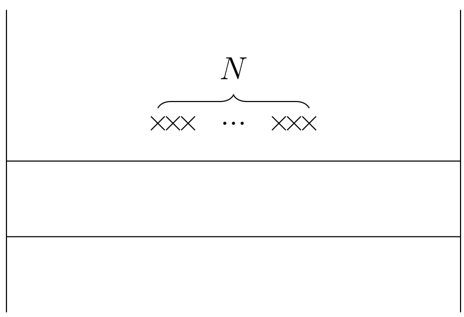

Figure A.5: Brane construction associated with $U(2)$ theory with N flavours

We then perform a Hanany-Witten transition with the four outermost D5 branes and move the D3 branes to coincide in 3, 4 and 5 direction. In Fig. A.6, the D3 brane do not quite coincide but this is only as a reminder that we indeed have two D3 branes.

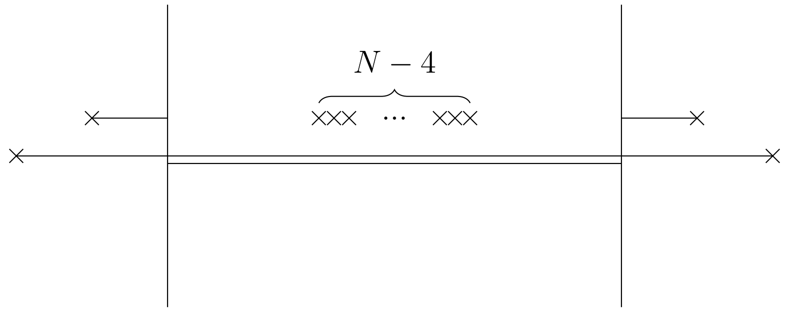

Figure A.6: Brane construction associated with $U(2)$ theory with N flavours after Hanany-Witten transition.

As in the $U(1)$ case, we move the D5 branes to coincide with both D3 branes in the 3, 4 and 5 direction and change the projection to direction 7, 8 or 9 on the vertical axis. Applying S-duality we obtain the magnetic phase in Fig. A.7.

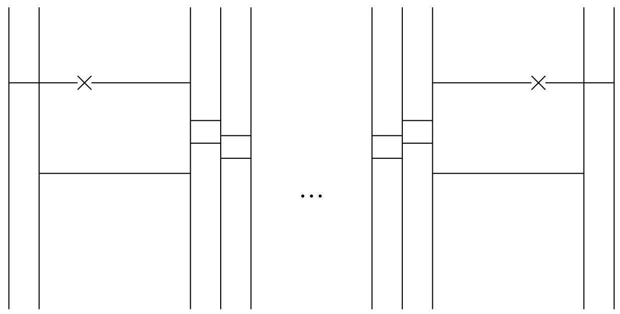

Figure A.7: Magnetic phase of brane construction associated with $U(2)$ theory with N flavours.

From this we can read off the magnetic quiver given in Fig. A.8.

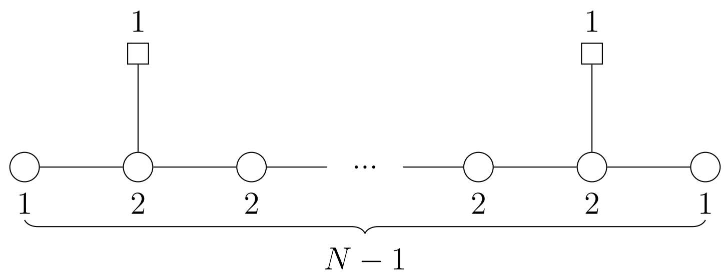

Figure A.8: Magnetic quiver for $U(2)$ theory with N flavours.

Here we can clearly confirm that there is no N for which the brane construction in Fig. A.5 is equal to the magnetic phase in Fig. A.7 and hence there is no self-dual case as mentioned in 3.