8.A Generalised Eigenvectors and Nilpotent Operators

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Null Space of Powers of an Operator

theorem: let $T \in \mathcal{L}(V)$ then

$$\{0\} = \ker(T^0), \quad \ker(T^n) \subset \ker(T^{n+1}) \quad \forall\, n \in \mathbb{Z}_{\geq 0}$$

proof: let $n \in \mathbb{Z}_{\geq 0}$, $v \in \ker(T^n)$

$\Rightarrow T^n v = 0 \Rightarrow T(T^n v) = T^{n+1} v = 0 \Rightarrow v \in \ker(T^{n+1})$

theorem: let $T \in \mathcal{L}(V)$, $m \in \mathbb{Z}_{\geq 0}$ s.t.

$\ker(T^m) = \ker(T^{m+1})$

$\Rightarrow \ker(T^{m+n}) = \ker(T^{m+n+1}) \quad \forall\, n \in \mathbb{Z}_{\geq m}$

proof: we know $\subset$ so we need $\supset$

let $v \in \ker(T^{n+m+1})$

$\Rightarrow T^{m+1}(T^n v) = 0$

$\Rightarrow T^n v \in \ker(T^{m+1}) = \ker(T^m)$

$\Rightarrow T^m(T^n v) = T^{m+n} v = 0$

$\Rightarrow v \in \ker(T^{m+n})$ ∎

theorem: $T \in \mathcal{L}(V)$, $n = \dim(V)$ then

$$\ker(T^n) = \ker(T^{n+m}) \quad \forall\, m \in \mathbb{Z}_{\geq 0}$$

proof: only need $\ker(T^n) = \ker(T^{n+1})$

assume $\ker(T^n) \neq \ker(T^{n+1})$

$\Rightarrow \{0\} \subsetneq \ker(T^0) \subsetneq \ker(T) \subsetneq \ker(T^2) \subsetneq \cdots \subsetneq \ker(T^n) \subsetneq \ker(T^{n+1})$

$\Rightarrow \dim(\ker(T^k))$ increase at least by 1 as k increase

$\Rightarrow \dim(\ker(T^{n+1})) \geq n+1 \Rightarrow$ contradiction ∎

theorem: $T \in \mathcal{L}(V)$, $n = \dim(V)$ then

$$V = \ker(T^n) \oplus \operatorname{im}(T^n)$$

proof: let $v \in \ker(T^n) \cap \operatorname{im}(T^n)$

$\Rightarrow T^n v = 0$, $\exists\, u \in V$ s.t. $T^n u = v$

$\Rightarrow T^n v = T^{2n} u = 0 \Rightarrow u \in \ker(T^{2n}) = \ker(T^n)$

$\Rightarrow T^n u = v = 0$

$\Rightarrow \ker(T^n) \cap \operatorname{im}(T^n) = \{0\}$

$\Rightarrow \ker(T^n) + \operatorname{im}(T^n)$ is $\oplus$

$\Rightarrow \dim(\ker(T^n) \oplus \operatorname{im}(T^n)) = \dim(\ker(T^n)) + \dim(\operatorname{im}(T^n)) = \dim(V)$ ☒

Generalised Eigenvectors

definition: let $T \in \mathcal{L}(V)$, $v \in V \setminus \{0\}$, $\lambda \in \mathbb{F}$ is e-value of T.

v is generalised eigenvector if

$\exists j \in \mathbb{Z}_{>0}$ s.t. $(T - \lambda I)^j v = 0$

definition: let $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$ is e-value of T.

generalized eigenspace of T with λ is

$G(T, \lambda) = \{ v \in V \mid \exists j \in \mathbb{Z}_{>0} \text{ s.t. } (T - \lambda I)^j v = 0 \}$

alternatively: $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$ then

$G(T, \lambda) = \ker((T - \lambda I)^{\dim(V)})$

proof : clearly $\ker((T - \lambda I)^{\dim(V)}) \subset G(T, \lambda)$

$\{0\} \subset \ker((T - \lambda I)) \subset \cdots \subset \ker((T - \lambda I)^{\dim(V)}) = \ker(\cdots$

$\Rightarrow$ if $v \in \ker((T - \lambda I)^j)$ for some j

then $v \in \ker((T - \lambda I)^{\dim(V)})$ ☒

theorem: let $T \in \mathcal{L}(V)$, $\lambda_i \in \mathbb{F}$, $v_i \in V$, $i \in [\![1, m]\!]$ where

v_i is generalized e-vector with distinct e-value λ_i

then $\{v_i\}$ is linearly independent set.

proof : suppose $0 = a_i v_i$, $a_i \in \mathbb{F}$

and k = largest integer s.t. $(T - \lambda_1 I)^k v_1 \neq 0$

let $w = (T-\lambda_1 I)^k v_1 \Rightarrow (T-\lambda_1 I)w = (T-\lambda_1 I)^{k+1} v_1 = 0$

$\Rightarrow Tw = \lambda_1 w \Rightarrow (T-\lambda I)w = (\lambda_1 - \lambda)w \quad \forall \lambda \in \mathbb{F}$

$\Rightarrow (T-\lambda I)^n w = (\lambda_1 - \lambda)^n w \quad \forall \lambda \in \mathbb{F}$, $n = \dim(V)$,

$v_j \in \ker((T-\lambda_j I)^n)$,

$[T-\lambda_i I, T-\lambda_j I] = [T, T-\lambda_j I] - \lambda_i [I, T-\lambda_j I]$ $\to 0$

$= [T,T] - \lambda_j [T,I] = 0$

$\Rightarrow (T-\lambda I)^k \prod_{i=2}^{m} (T-\lambda_i I)^n a_j v_j = 0$

$= (T-\lambda I)^k \prod_{i=2}^{m} (T-\lambda_i I)^n a_1 v_1$

$= \prod_{i=2}^{m} (T-\lambda_i I)^n a_1 w$

$= \prod_{i=2}^{m} (\lambda_1 - \lambda_i)^n a_1 w$

$\Rightarrow a_1 = 0$, repeat for $j \in [\![2, m]\!]$

$\Rightarrow a_i = 0 \ \forall j$ ∎

Nilpotent Operators

definition: $T \in \mathcal{L}(V)$ is nilpotent if
$\exists j \in \mathbb{Z}_{>0}$ s.t. $T^j = 0$

theorem: if $N \in \mathcal{L}(V)$ is nilpotent,
then $N^{\dim(V)} = 0$

proof: $\Rightarrow G(0, N) = V = \ker((N - 0I)^{\dim(V)})$
$= \ker(N^{\dim(V)}) \Rightarrow N^{\dim(V)} = 0$

theorem: let $N \in \mathcal{L}(V)$ is nilpotent then $\exists$ basis of V s.t.

$\mathcal{M}(N) = \begin{pmatrix} 0 & & * \\ & \ddots & \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \end{pmatrix}$

proof : $\ker(N^{\dim(V)}) = V$

choose basis $\ker(N)$

extend basis to $\ker(N^2)$

⋮

" " to V

then at least first coloumn is all 0's

as it's basis of $\ker(N) := \{e_{i,1}\}$

next coloumns that's not all 0's from $\ker(N^2)$ basis vectors $:= \{e_{i,2}\}$

applying $Ne_{i,2} \in \ker(N)$

which is linear combination of $\{e_{i,1}\}$

$\Rightarrow$ coloumns have 0 below diagonal

can be continued $\blacksquare$