8.D Jordan Form

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Basis of Nilpotent

theorem: Let $N \in \mathcal{L}(V)$ is nilpotent then

$\exists\ (v_1, \dots, v_n) \in V^n$ and $(m_1, \dots, m_n) \in \mathbb{Z}^n_{\geq 0}$ s.t.

(a) $\{ N^i v_j \mid j \in [\![1, n]\!],\ i \in [\![0, m_j]\!] \}$ is basis of V

(b) $N^{m_j+1} v_j = 0 \quad \forall j \in [\![1, n]\!]$

proof: induction on $\dim(V)$

true if $\dim(V) = 1$

assume $\dim(V) > 1$, true for U if $\dim(U) < \dim(V)$

$\ker(N) \neq \{0\} \Rightarrow$ not injective $\Rightarrow$ not surjective

and $\dim(\operatorname{im}(N)) < \dim(V)$

$\Rightarrow$ true for $N|_{\operatorname{im}(N)} \in \mathcal{L}(\operatorname{im}(N))$

$\Rightarrow \exists$ such basis $\{N^i v_j\}$ of $\operatorname{im}(N)$

$v_j \in \operatorname{im}(N) \Rightarrow \exists\ u_j \in V$ s.t. $N u_j = v_j \quad \forall j$

consider $\{N^i u_j\} := \{ N^i u_j \mid j \in [\![1, n]\!],\ i \in [\![0, m_j+1]\!] \}$

proof that $\{N^i u_j\}$ is linearly independent:

assume $\displaystyle\sum_{\{N^i u_j\}} a_{ij} N^i u_j = 0$

$\Rightarrow \displaystyle N \sum_{\{N^i u_j\}} a_{ij} N^i u_j = 0 = \underbrace{\sum_{\{N^i v_j\}}}_{\leftarrow \text{basis}} a_{ij} N^i v_j = 0$

$\Rightarrow a_{ij} = 0$ if $N(N^i u_j) \neq 0$

$\Rightarrow$ left with coefficients of $\{ N^{m_i+1} u_i \mid i \in [\![1, n]\!] \}$

$= \{ N^{m_i} v_i \mid i \in [\![1, n]\!] \} \leftarrow$ basis

$\Rightarrow a_{ij} = 0 \quad \forall i, j$ Q.E.D.

extend $\{N^i u_j\}$ to $\{N^i u_j\} \cup \{ w_i \mid i \in [\![1, p]\!] \}$ basis of V

basis of V

$\Rightarrow \quad Nw_i \in \operatorname{im}(N) = \operatorname{span}(\{N^i v_j\})$

$\Rightarrow \ \exists x_j \in \operatorname{span}(\{N^i u_j\})$ s.t. $Nw_j = Nx_j$

let $u_{n+j} = w_j - x_j \Rightarrow Nu_{n+j} = 0$

$\Rightarrow \operatorname{span}(\{N^i u_j\} \cup \{u_{n+j} \mid j \in [\![1, p]\!]\}) = V$

$\because$ it has all x_j and $u_{n+j} \Rightarrow$ it has all w_j $\blacksquare$

definition:

Suppose $T \in \mathcal{L}(V)$. A basis of V is called a ***Jordan basis*** for T if with respect to this basis T has a block diagonal matrix

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix},$$

where each A_j is an upper-triangular matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}.$$

theorem: <u>Jordan Form</u>

Let $V(\mathbb{C})$, $T \in \mathcal{L}(V)$ then

$\exists$ Jordan basis for T of V

proof: consider a nilpotent $N \in \mathcal{L}(V)$

with basis $\{N^i v_i\}$, N sends a basis vector to the next one

$\Rightarrow \mathcal{M}(N) =$ block diagonal with $A_j = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$

$\Rightarrow \exists$ Jordan basis for nilpotents

now let $\{\lambda_i\}$ distinct e-value of T

$$\Rightarrow \quad V = \bigoplus_i G(\lambda_i, T) \quad ,$$

$$(T - \lambda_j I)|_{G(\lambda_j, T)} \text{ is nilpotent}$$

$\Rightarrow \exists$ Jordan bais of $G(\lambda_j, T)$ for $(T-\lambda_j I)|_{G(\lambda_j, T)}$ $\forall j$ ∎