7.A Self-Adjoint and Normal Operators

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definition: adjoint, T^*

let $T \in \mathcal{L}(V,W)$. $T^*: W \to V$ s.t.

$$\langle Tv, w\rangle = \langle v, T^*w\rangle \quad \forall (v,w) \in V \times W$$

comment: Reisz representation theorem $\Rightarrow$ $\exists$ unique u s.t. $\phi(v) = \langle v, u\rangle$

given w, let $\phi(v) = \langle Tv, w\rangle$ then T^*_w is such unique u $\Rightarrow$

$\phi(v) = \langle v, T^*w\rangle \quad \forall v$

proposition: T^* is a linear map.

$T \in \mathcal{L}(V,W) \Rightarrow T^* \in \mathcal{L}(W,V)$

proof: let $v \in V$, $(w_1, w_2) \in W^2$, $\lambda \in F$

$$\Rightarrow \langle v, T^*(w_1+w_2)\rangle = \langle Tv, w_1+w_2\rangle$$
$$= \langle Tv, w_1\rangle + \langle Tv, w_2\rangle$$
$$= \langle v, T^*w_1\rangle + \langle v, T^*w_2\rangle$$
$$= \langle v, T^*w_1 + T^*w_2\rangle$$

$\Rightarrow T^*(w_1+w_2) = T^*w_1 + T^*w_2$

and

$$\langle v, T^*(\lambda w)\rangle = \langle Tv, \lambda w\rangle = \bar{\lambda}\langle Tv, w\rangle$$
$$= \bar{\lambda}\langle v, T^*w\rangle = \langle v, \bar{\bar{\lambda}} T^*w\rangle$$

$\Rightarrow T^*(\lambda w) = \lambda T^*w$ ∎

properties of adjoint

(a) $(S+T)^* = S^* + T^*$

(b) $(\lambda T)^* = \bar{\lambda} T^*$ $\quad (\lambda \in F, T \in \mathcal{L}(V))$

(c) $(T^*)^* = T$

(d) $I^* = I$ $\quad$ (identity: $Iv := v$)

(e) $(ST)^* = T^* S^*$ where $S \in \mathcal{L}(V,W)$, $T \in \mathcal{L}(W,U)$

proof: (a) let $S, T \in \mathcal{L}(V,W)$

$$\langle v, (S+T)^*w\rangle = \langle (S+T)v, w\rangle$$
$$= \langle Sv, w\rangle + \langle Tv, w\rangle$$
$$= \langle v, S^*w\rangle + \langle v, T^*w\rangle$$
$$= \langle v, (S^*+T^*)w\rangle \;∎$$

(b) $\langle v, (\lambda T)^*w\rangle = \langle \lambda Tv, w\rangle = \lambda\langle Tv, w\rangle$

$= \lambda\langle v, T^*w\rangle = \langle v, \bar{\lambda} T^*w\rangle$ ∎

(c) $\langle v, (T^*)^*w\rangle = \langle T^*v, w\rangle = \overline{\langle w, T^*v\rangle}$

$= \overline{\langle Tw, v\rangle} = \langle v, Tw\rangle$

(d) $\langle v, I^*w\rangle = \langle Iv, w\rangle = \langle v, w\rangle = \langle v, Iw\rangle$ ∎

(e) $\langle v, (ST)^*w\rangle = \langle STv, w\rangle = \langle Tv, S^*w\rangle = \langle v, T^*S^*w\rangle$ ∎

Rank, Nullity of adjoint

(a) $\ker(T^*) = (\operatorname{im}(T))^\perp$

(b) $\operatorname{im}(T^*) = (\ker(T))^\perp$

(c) $\ker(T) = (\operatorname{im}(T^*))^\perp$

(d) $\operatorname{im}(T) = (\ker(T^*))^\perp$

proof: (a) let $T \in \mathcal{L}(V,W)$, $w \in W$

$w \in \ker(T^*) \iff T^*w = 0 \iff \langle v, T^*w\rangle = \langle Tv, w\rangle = 0 \quad \forall v \in V$

$\iff w \in (\operatorname{im}(T))^\perp$

(a) $\iff$ (d)

$T \to T^* \Rightarrow$ (a) $\iff$ (c))

definition: conjugate transpose of M_{ij} is $\overline{M_{ji}}$

theorem: let $T \in \mathcal{L}(V,W)$, $\{e_i \mid i \in [\![1,n]\!]\}$ and $\{f_j \mid j \in [\![1,m]\!]\}$ are orthonormal basis sets of V and W

$$\mathcal{M}(T^*, f_j, e_i) = \left(\mathcal{M}(T, e_i, f_j)\right)^\dagger$$

proof: kth column of $\mathcal{M}(T)$ is $\mathcal{M}(T)_{\cdot k} = \mathcal{M}(Te_k) = A_{jk} f_j$

orthonormality $\Rightarrow \langle f_i, f_j\rangle = \delta_{ij} \Rightarrow \langle Te_k, f_j\rangle = \langle A_{ik} f_i, f_j\rangle = A_{ik}\langle f_i, f_j\rangle$
$= A_{ik}\delta_{ij} = A_{jk}$

$\Rightarrow \mathcal{M}(T)_{\cdot k} = \langle Te_k, f_j\rangle f_j \Rightarrow \mathcal{M}(T)_{jk} = \langle Te_k, f_j\rangle$

$T \to T^*$, $e_i \leftrightarrow f_j$:

$\mathcal{M}(T^*)_{ik} = \langle T^* f_k, e_i\rangle = \langle f_k, Te_i\rangle = \overline{\langle Te_i, f_k\rangle} = \overline{\mathcal{M}(T)_{ki}}$ ▨

lemma: given $V(\mathbb{C})$, $T \in \mathcal{L}(V)$

if $\langle Tv, v\rangle = 0 \ \forall v \in V$, then $T = 0$

proof: $4\langle Tu, w\rangle = \langle T(u+w), u+w\rangle - \langle T(u-w), u-w\rangle$
$+ i\left(\langle T(u+iw), u+iw\rangle - \langle T(u-iw), u-iw\rangle\right)$

$\forall (u,w) \in V^2$.

RHS is of the form $\langle Tv, v\rangle$

$\Rightarrow$ if $\langle Tv, v\rangle \ \forall v \in V$, then

$\langle Tu, w\rangle = 0 \quad \forall (u,w) \in V^2$

$\Rightarrow$ for $w = Tu$; $\langle Tu, w\rangle = \langle Tu, Tu\rangle \ \forall u \in V$

$\Rightarrow \|Tu\| = 0 \ \forall u \in V \Rightarrow T = 0$ ▨

lemma: let $V(\mathbb{C})$, $T \in \mathcal{L}(V)$

$\langle Tv, v\rangle \in \mathbb{R} \ \forall v \in V \iff T = T$

proof: let $v \in V$ then

$\langle Tv, v\rangle - \overline{\langle Tv, v\rangle} = \langle Tv, v\rangle - \langle v, Tv\rangle = \langle (T-T^*)v, v\rangle$

if $\langle Tv, v\rangle \in \mathbb{R} \Rightarrow LHS = 0 \Rightarrow \langle (T-T^*)v, v\rangle = 0 \ \forall v \in V$

above lemma $\Rightarrow T - T^* = 0$

if $T = T^*$, $\Rightarrow RHS = 0 \Rightarrow \langle Tv, v\rangle = \overline{\langle Tv, v\rangle} \ \forall v \in V$ ☒

lemma: given $V(\mathbb{F})$, $T \in \mathcal{L}(V)$.

If $T = T^*$ and $\langle Tv, v\rangle = 0 \ \forall v \in V$

then $T = 0$

proof: $\mathbb{F} = \mathbb{C}$ is proven. $\Rightarrow$ assume $\mathbb{F} = \mathbb{R}$

$\langle T(u+w), u+w\rangle - \langle T(u-w), u-w\rangle$

$= \langle Tu, u+w\rangle + \langle Tw, u+w\rangle - \langle Tu, u-w\rangle + \langle Tw, u-w\rangle$

$= 2\langle Tu, w\rangle + 2\langle Tw, u\rangle$, $(\langle Tu, w\rangle = \langle u, Tw\rangle = \langle Tw, u\rangle)$

$= 4\langle Tu, w\rangle$

$\Rightarrow \langle Tu, w\rangle = 0 \ \forall (u,w) \in V^2$. take $w = Tu \Rightarrow T = 0$

definition: $T \in \mathcal{L}(V)$ is <u>normal</u> if $[T, T^*] = 0$

theorem: $T \in \mathcal{L}(V)$ is normal $\iff \|Tv\| = \|T^*v\| \ \forall v \in V$

proof: $[T, T^*] = 0 \iff \langle [T^*, T]v, v\rangle = 0 \ \forall v \in V$

$\iff \langle T^*Tv, v\rangle - \langle TT^*v, v\rangle = 0 \ \forall v \in V$

$\iff \langle Tv, Tv\rangle - \langle T^*v, T^*v\rangle = 0 \ \forall v \in V$ ☒

corollary: given normal $T \in \mathcal{L}(V)$

if $Tv = \lambda v$ then $T^*v = \bar{\lambda} v$

proof: $[T, T^*] = 0 \Rightarrow [T - \lambda I, T^* - \bar{\lambda} I^*] = [T, T^* - \bar{\lambda} I^*] - \cancelto{0}{[\lambda I, T^* - \bar{\lambda} I^*]}$

$= [T, T^*] = 0 \Rightarrow T - \lambda I$ is normal

$\iff \|(T - \lambda I)v\| = \|(T^* - \bar{\lambda} I)v\| \ \forall v \in V$

$\Leftrightarrow \|(T-\lambda I)v\| = \|(T^*-\bar{\lambda} I)v\| \quad \forall v \in V$

if v is eigenvector with λ : LHS $=0$

$\Rightarrow$ v is also eigenvector of T^* with e-value $\bar{\lambda}$

theorem: let $T \in \mathcal{L}(V)$ is normal then

e-vectors of T with distint e-values are orthogonal

proof: $(u, v) \in V^2$, $(\alpha, \beta) \in \mathbb{F}^2$ s.t.

$$Tu = \alpha u, \quad Tv = \beta v$$

$$[T, T^*] = 0 \Rightarrow T^* v = \bar{\beta} v$$

$$\Rightarrow (\alpha - \beta)\langle u, v\rangle = \langle \alpha u, v\rangle - \langle u, \bar{\beta} v\rangle$$

$$= \langle Tu, v\rangle - \langle u, T^* v\rangle = 0$$

$\Rightarrow$ if $\alpha - \beta \neq 0$ then $\langle u, v\rangle = 0$