9.A Complexification

Tuesday 10 September 2024 19:09

Complexification of a Vector Space

definition: let $V(\mathbb{R})$. complexification of V is

vector space $V_\mathbb{C}$ over $\mathbb{C}$ s.t. $V_\mathbb{C} \cong V^2$

the isomorphism $\phi: V^2 \to V_\mathbb{C}$ is $\phi(u,v) = u + iv$

theorem: let $V(\mathbb{R})$, $\{v_j\}$ basis of V

then $\{v_j\}$ is also basis of $V_\mathbb{C}$

corollery: $\dim(V) = \dim(V_\mathbb{C})$

proof: spanning in $V_\mathbb{C}$

$v_j \in \text{span}(\{v_j\})$, $iv_j \in \text{span}(\{v_j\})$

$\Rightarrow V_\mathbb{C} = \text{span}(\{v_j\})$

let $\lambda_j v_j = 0$ where $\lambda_j \in \mathbb{C}$

$\Rightarrow \text{Re}(\lambda_j) v_j = 0$ and $\text{Im}(\lambda_j) v_j = 0$

v_j basis in $V(\mathbb{R}) \Rightarrow \text{Re}(\lambda_j) = 0 = \text{Im}(\lambda_j) \quad \forall j$ ∎

Complexification of an Operator

definition: let $V(\mathbb{R})$, $T \in \mathcal{L}(V)$. complexification of T is

$T_\mathbb{C}(u + iv) := Tu + iTv$ for $(u,v) \in V^2$

$\Rightarrow T_\mathbb{C} \in \mathcal{L}(V_\mathbb{C})$

corollery: let $V(\mathbb{R})$, $\{v_j\}$ basis of V, $T \in \mathcal{L}(V)$ then

$\mathcal{M}(T, \{v_j\}) = \mathcal{M}(T_\mathbb{C}, \{v_j\})$

theorem: all operator on nonzero finite-dim vector space has an invariant subspace of dim 1 or 2

proof: already proven for $\mathbb{F} = \mathbb{C}$.

assume $V(\mathbb{R})$, $T \in \mathcal{L}(V)$

let $a+ib$ e-value of $T_{\mathbb{C}}$ where $(a,b) \in \mathbb{R}^2$

$\Rightarrow \exists\ (u,v) \in V^2 \setminus \{0\}$

s.t. $T_{\mathbb{C}}(u+iv) = (a+ib)(u+iv)$

$= Tu + iTv = (au-bv) + i(av+bu)$

$\Rightarrow Tu = au-bv$, $Tv = av+bu$

$\Rightarrow$ span(u,v) is invariant under T ∎

The Minimal Polynomial of the Complexification

theorem: $V(\mathbb{R})$, $T \in \mathcal{L}(V)$ then

minimal of $T_{\mathbb{C}}$ = minimal of T

proof: $p \in \mathcal{P}(\mathbb{R})$ be minimal of T

$T_{\mathbb{C}}^n(u+iv) = T^n u + i T^n v$

$\Rightarrow p(T_{\mathbb{C}}) = (p(T))_{\mathbb{C}} = 0_{\mathbb{C}} = 0$

let $q \in \mathcal{P}(\mathbb{C})$ be any monic s.t. $q(T_{\mathbb{C}}) = 0$

$q(z) = z^{\deg(q)} + \sum_{i=0}^{\deg(q)-1} a_i z^i$

$r(z) := z^{\deg(q)} + \sum_{i=0}^{\deg(q)-1} \operatorname{Re}(a_i) z^i$

$\Rightarrow r(z)$ is monic and $\deg(q) = \deg(r) \geq \deg(p)$

$\Rightarrow p$ is minimal of $T_{\mathbb{C}}$ ∎

Eigenvalues of the Complexification

theorem: let $V(\mathbb{R})$, $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{R}$ then

λ is e-value of $T \iff \lambda$ is e-value of $T_\mathbb{C}$

proof1: $(\Rightarrow)$: $\exists v \in V \setminus \{0\}$ s.t. $Tv = \lambda v \Rightarrow T_\mathbb{C} v = \lambda v$ ✓

$(\Leftarrow)$: $\exists (u,v) \in V^2 \setminus \{0\}$ s.t. $T_\mathbb{C}(u+iv) = \lambda(u+iv)$

$\Rightarrow Tu + iTv = \lambda u + i\lambda v \Rightarrow Tu = \lambda u,\ Tv = \lambda v$ ∎

proof2: e-values of T = zeros of minimal of T

" of $T_\mathbb{C}$ = " of $T_\mathbb{C}$

and minimal of T = minimal of $T_\mathbb{C}$

∎

theorem: let $V(\mathbb{R})$, $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{C}$, $(u,v) \in V^2$, then

$$(T_\mathbb{C} - \lambda I)^j (u+iv) = 0 \iff (T_\mathbb{C} - \bar{\lambda} I)^j (u-iv) = 0 \quad \forall j \in \mathbb{Z}_{\geq 0}$$

proof: induction on j

true if $j=0$

assume $j>1$, true for $j-1$

$(\Rightarrow)$ $(T_\mathbb{C} - \lambda I)^j (u+iv) = 0$

$\Rightarrow (T_\mathbb{C} - \lambda I)^{j-1} \left((T_\mathbb{C} - \lambda I)(u+iv)\right) = 0$

$\Rightarrow (T_\mathbb{C} - \bar{\lambda} I)^{j-1} \left((T_\mathbb{C} - \lambda I)(u+iv)\right)^* = 0$

$\Rightarrow (T_\mathbb{C} - \bar{\lambda} I)^{j-1} \left((T_\mathbb{C} - \bar{\lambda} I)(u-iv)\right)$ ✓

$(\Leftarrow)$ replace $\lambda \to \bar{\lambda}$, $v \to -v$

∎

corollery1: λ is e-value of $T_\mathbb{C} \iff \bar{\lambda}$ is e-value of $T_\mathbb{C}$ $(j=1)$

corollery 2: multiplicity of λ = multiplicity of $\bar{\lambda}$

proof: let $\{u_j + iv_j\}$ be basis of $G(\lambda, T_\mathbb{C})$

$\Rightarrow \{u_j - i v_j\}$ is basis of $G(\bar{\lambda}, T_{\mathbb{C}})$ ∎

theorem: let $V(\mathbb{R})$, $\dim(V) \bmod 2 \equiv 1$ then

$\exists$ an eigenvalue $\forall T \in \mathcal{L}(V)$

proof: $\dim(V) = \dim(V_{\mathbb{C}}) = \sum$ multiplicity of e-value of $T_{\mathbb{C}} \equiv 1 \bmod 2$

if $\lambda \notin \mathbb{R}$ is e-value of $T_{\mathbb{C}}$ then

multiplicity of λ + multiplicity of $\bar{\lambda} = 2m \equiv 0 \bmod 2$

$\Rightarrow \sum$ multiplicity of real e-value of $T_{\mathbb{C}} \equiv 1 \bmod 2$ ∎

Characteristic Polynominl of the Complexification

lemma: let $V(\mathbb{R})$, $T \in \mathcal{L}(V)$ then

characteristic polynominl of $T_{\mathbb{C}} \in \mathcal{P}(\mathbb{R})$

proof: let $\lambda \notin \mathbb{R}$ be e-value of $T_{\mathbb{C}}$ with multiplicity m

$\Rightarrow (z-\lambda)^m (z-\bar{\lambda})^m = (z^2 - 2\,\mathrm{Re}(\lambda) z + |\lambda|^2)^m$ divides

characteristic polynomial of $T_{\mathbb{C}}$

$\Rightarrow$ all coefficients $\in \mathbb{R}$

definition: let $V(\mathbb{R})$, $T \in \mathcal{L}(V)$ then

the characteristic polynominl of T

:= " " of $T_{\mathbb{C}}$

corollories: let p be characteristic polynominl of T, then

(a) $p \in \mathcal{P}(\mathbb{R})$

(b) $\deg(p) = \dim(V)$

(c) $\{$e-values of $T\} = \{$real zeros of $p(z)\}$

theorem: <u>Cayley-Hamilton theorem</u>

let $T \in \mathcal{L}(V)$, $q \in \mathcal{P}(\mathbb{F})$ is charateristic of T

then $q(T) = 0$

proof: proven for $V(\mathbb{C}) \Rightarrow$ assume $V(\mathbb{R})$

q is chrateristic of $T_{\mathbb{C}}$

$(C\text{-}H) \Rightarrow q(T_{\mathbb{C}}) = 0 \Rightarrow q(T) = 0$ ∎

corollery: let p be minimal of T

(a) $\deg(p) \leq \dim(V)$

(b) $\exists s \in \mathcal{P}(\mathbb{F})$ s.t. $q = ps$