9.B Operators on Real Inner Product Spaces

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Normal Operators

Lemma: let $V(\mathbb{R})$, $\dim(V)=2$, $T\in\mathcal{L}(V)$ then

(a) T is normal but not self-adjoint

$\iff$

(b) $\mathcal{M}(T)$ w.r.t. any orthonormal basis of V is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $b\neq 0$

$\iff$

(c) $\mathcal{M}(T)$ w.r.t. some orthonormal basis of V is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $b>0$

proof: let (a) be true, $\{e_1,e_2\}$ orthonormal basis,

$$\mathcal{M}(T,\{e_i\})=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\Rightarrow \|Te_1\|^2=a^2+b^2,\ \|T^*e_1\|^2=a^2+c^2$$

$$\langle Te_i,Te_i\rangle=\langle e_i,T^*Te_i\rangle$$
$$=\langle e_i,\overset{0}{\cancel{[T^*,T]}}+TT^*e_i\rangle$$
$$=\langle T^*e_i,T^*e_i\rangle$$

$$\Rightarrow \|Te_1\|=\|T^*e_1\| \Rightarrow b^2=c^2$$

but $b\neq c$ otherwise $T=T^*$

$$\Rightarrow b=-c \Rightarrow \mathcal{M}([T,T^*],\{e_i\})=0$$

$$\Rightarrow \begin{pmatrix} a & -b \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ -b & d \end{pmatrix}=\begin{pmatrix} a & b \\ -b & d \end{pmatrix}\begin{pmatrix} a & -b \\ b & d \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a^2+b^2 & ab-bd \\ ab-db & b^2+d^2 \end{pmatrix}=\begin{pmatrix} a^2+b^2 & -ab+bd \\ -ab+db & b^2+d^2 \end{pmatrix}$$

$$\Rightarrow ab-bd=-ab+bd \Rightarrow ab=bd \Rightarrow a=d\ (b\neq 0)$$

$\Rightarrow$ (b) if $b>0$ then (c)

if $b<0$ then

$$\mathcal{M}(T,(e_1,e_2))e_1=ae_1+be_2=\mathcal{M}(T,(e_1,-e_2))e_1=ae_1-b(-e_2)$$

$$\mathcal{M}(T,(e_1,e_2))e_2=-be_1+ae_2=\mathcal{M}(T,(e_1,-e_2))e_2$$

$$\Rightarrow \mathcal{M}(T,(e_1,-e_2))(-e_2)=be_1+a(-e_2)$$

$$\Rightarrow \mathcal{M}(T,(e_1,-e_2))=\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \text{ where } -b>0$$

$\Rightarrow$ (c) and computation shows $\Rightarrow$ (a) ☒

lemma: let $T \in \mathcal{L}(V)$ be normal, $U \leq V$ is invariant under T then,

(a) $U^\perp$ invariant under T

(b) U invariant under T^*

(c) $(T|_U)^* = (T^*)|_U$

(d) $T|_U \in \mathcal{L}(U)$ & $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ are normal

proof: $V = U \oplus U^\perp$ let $\{e_i\}$ be orthonormal basis of U

" $\{f_i\}$ " " of $U^\perp$

U invariant under T

$$\Rightarrow \mathcal{M}(T, \{e_i, f_j\}) = \begin{array}{c|cc} & e_i & f_i \\ \hline e_i & A & B \\ f_i & 0 & C \end{array}$$ where

$A: \{e_i\} \to \{e_i\}$

$B: \{f_i\} \to \{e_i\}$

$C: \{f_i\} \to \{f_i\}$

$\Rightarrow \|Te_j\|^2 = \sum_i |A_{ij}|^2$

$\Rightarrow \sum_j \|Te_j\|^2 = \sum_{i,j} |A_{ij}|^2$,

$\|T^* e_i\|^2 = \sum_j |A_{ij}|^2 + \sum_j |B_{ij}|^2$

$\Rightarrow \sum_i \|T^* e_i\|^2 = \sum_{i,j} |A_{ij}|^2 + \sum_{i,j} |B_{ij}|^2$

T normal $\Rightarrow \|Te_j\| = \|T^* e_j\|$

$\Rightarrow \sum_{i,j} |B_{ij}|^2 = 0 \Rightarrow B = 0$

$$\Rightarrow \mathcal{M}(T) = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$$

$\Rightarrow Tf_i \in \operatorname{span}(\{f_i\})$

$\Rightarrow U^\perp$ invariant under T (a) ✓

$$\mathcal{M}(T^*) = \mathcal{M}(T)^T = \begin{pmatrix} A^T & 0 \\ 0 & C^T \end{pmatrix}$$

$\Rightarrow U$ invariant under T^* (b) ✓

let $S = T|_U \in \mathcal{L}(U)$, $v \in U$

$\Rightarrow \langle u, S^* v\rangle = \langle Su, v\rangle = \langle Tu, v\rangle = \langle u, T^* v\rangle \ \forall u \in U$

(b) $\Rightarrow T^* v \in U \Rightarrow S^* v = T^* v \Rightarrow (T|_U)^* = (T^*)|_U$ (c) ✓

$[T, T^*] = 0 \Rightarrow [T, T^*]|_U = 0 = [T|_U, (T^*)|_U] = [T|_U, (T|_U)^*]$

$U \to U^\perp$ and $U^\perp \to U$ shows $[T|_{U^\perp}, (T|_{U^\perp})^*] = 0$ (d) ✓ ☒

theorem: let $V(\mathbb{R})$, $T \in \mathcal{L}(V)$ then

(a) T normal

$\Longleftrightarrow$

(b) $\exists$ orthonormal basis s.t. $\mathcal{M}(T)$ is block diagonal whose each block is 1×1 or 2×2 of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $b>0$

proof: (b) $\Rightarrow$ $T^*T = \begin{pmatrix} T^*T_1 & & 0 \\ & \ddots & \\ 0 & & T^*T_n \end{pmatrix}$ where $T^*T_i = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}\begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix} = TT^*_i$

$= \begin{pmatrix} TT^*_1 & & 0 \\ & \ddots & \\ 0 & & TT^*_n \end{pmatrix} = TT^*$ (a) ✓

(a), induction on $\dim(V)$

$\dim(V) = 1 \Rightarrow$ (b)

$\dim(V) = 2 \Rightarrow$ if $T = T^*$ then Real Spectral theorem $\Rightarrow$ (b)

elif $[T, T^*] = 0$ then 1st lemma $\Rightarrow$ (b)

assume $\dim(V) > 2$ and (b) for U s.t. $\dim(U) < \dim(V)$

let $U < V$ s.t. invariant under T and $\dim(U) = \min(1,2)$

if $\dim(U) = 1$,

choose $u \in U$ s.t. $\|u\| = 1$

$\Rightarrow \{u\}$ orthonormal basis of U, $T|_U \in \mathcal{L}(U)$ is 1×1

else $\dim(U) = 2$,

$\Rightarrow$ No eigenvector of $T \Rightarrow T \neq T^*$

and

$\Rightarrow T|_U \in \mathcal{L}(U)$ is normal

$\Rightarrow \exists$ desired basis of U

$\Rightarrow U^\perp$ invariant under T, $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ normal,

$V = U \oplus U^\perp \Rightarrow \dim(U^\perp) < \dim(V)$

$\Rightarrow \exists$ desired form of basis of $U^\perp$ ∎

Isometries

theorem: let $V(\mathbb{R})$, $S \in \mathcal{L}(V)$ then

(a) S is isometry

$\Longleftrightarrow$

(b) $\exists$ orthonormal basis of V s.t.

$\mathcal{M}(S) =$ block diagonal where

each block is

1×1 : (1) or (-1)

2×2 : $\begin{pmatrix} \cos\theta & -\sin\theta \\ \end{pmatrix}$ [illegible]

2×2 : $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $\theta \in (0, \pi)$

proof i (a) $\Rightarrow S^*S = I = SS^* \Rightarrow S$ is normal

$\Rightarrow 2\times 2$ is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $b>0$ and

1×1 is (λ)

w.r.t. some basis $\{e_i\}$

$\Rightarrow \exists e_j$ s.t. $Se_j = \lambda e_j \Rightarrow |\lambda| = 1$ and

$\exists e_j, e_{j+1}$ s.t. $Se_j = ae_j + be_{j+1}$

$\Rightarrow 1 = \|e_j\|^2 = \|Se_j\|^2 = a^2 + b^2$

$\Rightarrow \exists \theta \in (0, \pi)$ s.t. $a = \cos\theta$, $b = \sin\theta$ ✓

(b) $\Rightarrow V = \bigoplus_i U_i$ s.t. $U_i < V$ and $\dim(U_i) \in \{1, 2\}$ $\forall i$

$\Rightarrow S|_{U_j} \in \mathcal{L}(U_j)$ is isometry and U_j invariant under S

$v \in V \Rightarrow v = \sum_i u_i$ where $u_i \in U_i$ $\langle u_i, u_j \rangle \propto \delta_{ij}$

$$\begin{aligned}\Rightarrow \|Sv\|^2 &= \|\sum_i Su_i\|^2 \quad \Leftarrow Su_i \in U_i \\ &= \sum_i \|Su_i\|^2 \quad \Leftarrow S|_{U_j} \text{ isometry} \\ &= \sum_i \|u_i\|^2 \\ &= \|v\|^2 \Rightarrow (a)\end{aligned}$$

$\boxtimes$