## Complex Spectral Theorem

given $V(\mathbb{C})$, $T \in \mathcal{L}(V)$ then

(a) $T$ is normal

$\iff$

(b) $\exists$ orthonormal basis of $V$ consisting of e-vectors of $T$

$\iff$

(c) $\exists$ diagonal matrix rep of $T$ w.r.t. some orthonormal basis of $V$

proof: (c) $\Rightarrow$ $\mathcal{M}(T)$ is diagonal, $\mathcal{M}(T^*) = \mathcal{M}(T)^\dagger$

$\iff$ $\mathcal{M}(T^*)$ is diagonal, 2 diag matrix commutes

$\Rightarrow [T, T^*] = 0 \iff$ (a)

Schur $\Rightarrow \exists \{e_i\}$ s.t. $\mathcal{M}(T,(e_i)) = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ 0 & & a_{n,n} \end{pmatrix}$

$\|Te_i\|^2 = \sum_{j=1}^{i} |a_{j,i}|^2$ and $\|T^* e_i\|^2 = \sum_{j=i}^{n} |a_{i,j}|^2$

(a) $\Rightarrow \|Te_i\|^2 = \|T^* e_i\|^2 \Rightarrow a_{ij} = 0$ if $i \neq j$

$\Rightarrow$ (c) $\blacksquare$

## Real Spectral Theorem

lemma: given $T \in \mathcal{L}(V)$ s.t. $T = T^*$, $(b,c) \in \mathbb{R}^2$ s.t. $b^2 < 4c$

then $T^2 + bT + cI$ is invertible

proof: $\langle (T^2 + bT + cI)v, v \rangle = \|Tv\|^2 + b\langle Tv, v\rangle + c\|v\|^2$

$\geq \|Tv\|^2 - |b| \cdot \|Tv\| \cdot \|v\| + c\|v\|^2$

$= \left(\|Tv\| - \frac{|b| \cdot \|v\|}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)\|v\|^2$

$> 0$

$T^2 + bT + cI = (T^2 + bT + cI)^*$

$\Rightarrow T^2 + bT + cI \neq 0 \Rightarrow \ker(T^2 + bT + cI) = \{0\}$

$\Rightarrow$ injective $\Rightarrow$ invertible $\blacksquare$

lemma: let $V \neq \{0\}$ and $T = T^* \in \mathcal{L}(V)$ then

$\exists$ e-value of $T$

proof: trivial for $V(\mathbb{C}) \Rightarrow$ assume $V(\mathbb{R})$

let $n = \dim(V)$, $v \in V$ s.t. $v \neq 0$

then $\{T^n v \mid n \in [\![0, n]\!]\}$ is linearly dependent

$\Rightarrow \exists \{a_i\}$ s.t. $a_i \neq 0$ for some $i$ and

$0 = a_i T^i v$

$= c\left(\prod_i^m (T^2 + b_i T + c_i I)\right)\left(\prod_i^\ell (T - \lambda_i I)\right)$

where $c \in \mathbb{R} \setminus \{0\}$, $(b_i, c_i, \lambda_i) \in \mathbb{R}^3 \ \forall i$,

$b_i^2 < 4c_i \ \forall i$, $m + \ell \geq 1$

above lemma $\Rightarrow T^2 + b_i T + c_i I \neq 0$

$\Rightarrow 0 = \prod (T - \lambda_i I) \Rightarrow \exists i$ s.t. $T - \lambda_i I = 0$

lemma: let $T \in \mathcal{L}(V)$ s.t. $T^* = T$, $U < V$ is invariant under $T$

then (a) $U^\perp$ is invariant under $T$

(b) $T|_U = (T|_U)^* \in \mathcal{L}(U)$

(c) $T|_{U^\perp} = (T|_{U^\perp})^* \in \mathcal{L}(U^\perp)$

proof (a): let $v \in U^\perp$, then

$\langle Tv, u \rangle = \langle v, Tu \rangle = 0 \quad \forall u \in U$

$\Rightarrow Tv \in U^\perp$ ▨

(b): $\langle (T|_U)u, v \rangle = \langle Tu, v \rangle = \langle u, Tv \rangle$

(if $u, v \in U \Rightarrow$) $= \langle u, (T|_U)v \rangle$ ▨

(c): (a), (b) $\Rightarrow$ (c)

theorem: Real Spectral Theorem

let $V(\mathbb{R})$, $T \in \mathcal{L}(V)$ then

(a) $T = T^*$

$\Leftrightarrow$

(b) $\exists$ orthonormal basis $\{e_i\}$ of $V$ = e-vectors of $T$

$\Leftrightarrow$

(c) $\exists \mathcal{M}(T)$ diagonal with orthonormal basis $\{e_i\}$

proof: suppose (c) $\Rightarrow \mathcal{M}(T) = (\mathcal{M}(T))^T \Rightarrow T = T^*$ (a)

(a) $\Rightarrow$ (b): if $\dim(V) = 1$ then, above lemma $\Rightarrow \exists$ e-value

$\Rightarrow$ ((a) $\Rightarrow$ (b))

assume $\dim(V) > 1$, (a) $\Rightarrow$ (b) for all smaller dim spaces

(a) $\Rightarrow \exists u \in V$ s.t. $Tu = \lambda u$, $\|u\| = 1$, $U = \operatorname{span}(u)$

$\Rightarrow U$ is eigenspace of $V \Rightarrow U$ invariant under $T$

$\Rightarrow T|_{U^\perp} = (T|_{U^\perp})^*$, $\dim(U) = 1 \Rightarrow \dim(U^\perp) = \dim(V) - 1$

$\Rightarrow \exists$ orthonormal basis $\{u_i\}$ of $U^\perp \Rightarrow \{u_i\} \cup \{u\}$ is basis of $V \Rightarrow$ (b)

(b) $\Rightarrow$ (c) ▨