10.A Trace

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Change of Basis

definition: identity matrix I

$n \times n$ matrix let B = any basis

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \mathcal{M}(I, (B))$$

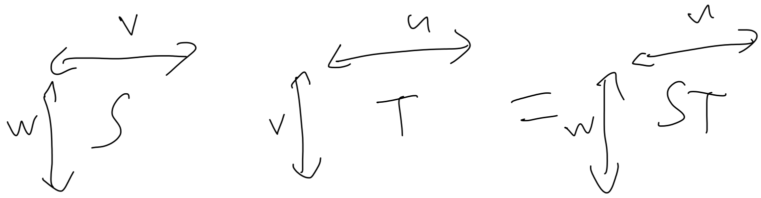
definition: square matrix A is invertible if

$\exists$ same size square matrix A^{-1} s.t.

$$AA^{-1} = A^{-1}A = I$$

lemma: let $\{u_i\}, \{v_i\}, \{w_i\}$ be bases of V, $T \in \mathcal{L}(V)$ then

$$\mathcal{M}(ST, (\{u_i\}), (\{w_i\})) = \mathcal{M}(S, (\{v_i\}), (\{w_i\}))\, \mathcal{M}(T, (\{u_i\}), (\{v_i\}))$$



lemma: let $\{u_i\}, \{v_i\}$ be bases of V then

$$\mathcal{M}(I, (\{u_i\}), (\{v_i\}))^{-1} = \mathcal{M}(I, (\{v_i\}), (\{u_i\}))$$

proof: in the above lemma, replace $w_i \to u_i$ and $S, T \to I$

$$\Rightarrow \mathcal{M}(II = I, (\{u_i\})(\{u_i\})) = \mathcal{M}(I, (\{v_i\}), (\{u_i\}))\, \mathcal{M}(I, (\{u_i\}), (\{v_i\}))$$

$= I$

and symmetry $u_i \leftrightarrow v_i$ ☒

theorem: <u>Similarity Transformation</u>

$T \in \mathcal{L}(V)$, $\{u_i\}$ and $\{v_i\}$ bases of V

let $A = \mathcal{M}(I, (\{u_i\}), (\{v_i\}))$ then

$$\mathcal{M}(T, (\{u_i\})) = A^{-1}\mathcal{M}(T, (\{v_i\}))A$$

$${}_u\big|\overline{\;T\;}^{\,u} = {}_u\big|\overline{\;A^{-1}\;}^{\,v} \;\; {}_v\big|\overline{\;T\;}^{\,v} \;\; {}_v\big|\overline{\;A\;}^{\,u}$$

proof: in first lemma $w_j \to u_j$, $S \to I$

$\Rightarrow \mathcal{M}(T, (\{u_i\})) = A^{-1}\mathcal{M}(T, (\{u_i\}), (\{v_i\}))$

in first lemma $w_j \to v_j$, $T \to I$

$\mathcal{M}(S, (\{u_i\}), (\{v_i\})) = \mathcal{M}(S, (\{v_i\}))A$ $\blacksquare$

<u>Trace: connecting Operators and Matrices</u>

definition: let $T \in \mathcal{L}(V)$, $\{\lambda_i\}$ is (repeated) evalues

of T if $V(\mathbb{C})$

of $T_{\mathbb{C}}$ if $V(\mathbb{R})$ then

<u>trace of the operator</u> T is

$$tr(T) := \sum_i \lambda_i$$

corollary: $T \in \mathcal{L}(V)$, $n = \dim(V)$, $p \in \mathcal{P}(\mathbb{F})$ characteristic of T, then

$tr(T) = -a_{n-1}$ where $p(z) = \sum_{i=0}^{n} a_i z^i$

proof: $p(z) = \prod_{i=1}^{n}(z - \lambda_i) = z^n - (\sum_{i=1}^{n}\lambda_i)z^{n-1} + \dots + (-1)^n\prod_{i=1}^{n}\lambda_i$

definition: <u>trace of</u> a $n \times n$ <u>matrix</u> A is

$$\operatorname{tr}(A) := \sum_i A_{ii}$$

lemma: let A, B $n \times n$ matrix then

$$\operatorname{tr}(AB) = \operatorname{tr}(BA)$$

proof: $(AB)_{ii} = \sum_j A_{ij} B_{ji}$

$$\Rightarrow \operatorname{tr}(AB) = \sum_{i,j} A_{ij} B_{ji} = \sum_{i,j} B_{ji} A_{ij} = \sum_{i,j} B_{ij} A_{ji} = \operatorname{tr}(BA) \quad \blacksquare$$

theorem: <u>trace independent of basis</u>

let $T \in \mathcal{L}(V)$, $\{u_i\}$, $\{v_i\}$ bases of V, then

$$\operatorname{tr}(M(T, (\{u_i\}))) = \operatorname{tr}(M(T, (\{v_i\})))$$

proof: let $A = M(I, (\{u_i\}), (\{v_i\}))$

$$\Rightarrow \operatorname{tr}(M(T, (\{u_i\}))) = \operatorname{tr}(A^{-1} M(T, (\{v_i\})) A)$$

$$= \operatorname{tr}(M(T, (\{v_i\})) \underbrace{\cancel{AA^{-1}}}_{I}) \quad \blacksquare$$

theorem: let $T \in \mathcal{L}(V)$ then

$$\operatorname{tr}(T) = \operatorname{tr}(M(T))$$

proof: $\exists$ basis of V s.t. $M(T)$ for $V(\mathbb{C})$ or $M(T_{\mathbb{C}})$ for $V(\mathbb{R})$ is uppertriangular whose diagonal are e-values

$\Rightarrow$ true in some basis $\Rightarrow$ true for all basis $\blacksquare$

Lemma: additivity of trace

let $S, T \in \mathcal{L}(V)$ then

$tr(S+T) = tr(S) + tr(T)$

proof: $tr(S+T) = tr(\mathcal{M}(S+T))$

$= tr(\mathcal{M}(S)+\mathcal{M}(T)) = tr(\mathcal{M}(S)) + tr(\mathcal{M}(T))$ ☒

theorem: $\nexists\, S, T \in \mathcal{L}(V)$ s.t. $[S,T] = I$

proof: $tr([S,T]) = tr(ST) - tr(TS) = 0$

$\therefore$ commutators are traceless

$tr(I) = \dim(V) \neq 0$ ☒