6.C Orthogonal Complements and Minimisation Problems

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definition: $U \subset V$, then orthogonal complement of U is

$$U^\perp := \{ v \in V \mid \langle v, u \rangle = 0 \ \forall u \in U \}$$

the set of all vectors in V that are orthogonal to $\forall u \in U$

proposition: (a) if $U \subset V$ then $U^\perp < V$

(b) $\{0\}^\perp = V$, $V^\perp = \{0\}$

(c) if $U \subset V$ then $U \cap U^\perp \subset \{0\}$

(d) if $U, W \subset V$ and $U \subset W$ then $W^\perp \subset U^\perp$

proof: (a) S1: $0 \in U^\perp$ ✓

S2: let $v, w \in U^\perp \Rightarrow \langle v, u \rangle = \langle w, u \rangle = 0 \ \forall u \in U$

$\Rightarrow \langle v + w, u \rangle = \langle v, u \rangle + \langle w, u \rangle = 0 \ \forall u \in U$

$\Rightarrow v + w \in U^\perp$ ✓

S3: let $v \in U^\perp$ and $\lambda \in \mathbb{F} \Rightarrow \langle v, u \rangle = 0 \ \forall u \in U$

$\Rightarrow \langle \lambda v, u \rangle = \lambda \langle v, u \rangle = \lambda \cdot 0 = 0 \ \forall u \in U$ ✓

(c) if $v \in U \cap U^\perp$ then $\langle v, v \rangle = 0 \Rightarrow v = 0$ ☒

(d) $U \subset W \subset V$, let $v \in W^\perp \Rightarrow \langle v, u \rangle = 0 \ \forall u \in W \Rightarrow \langle v, u \rangle = 0 \ \forall u \in U$

$\Rightarrow v \in U^\perp \Rightarrow W^\perp \subset U^\perp$

theorem: let $U < V$ finite then

$$V = U \oplus U^\perp$$

proof: let $v \in V$ and $\{ e_i \mid i \in [\![1, m]\!] \}$ orthonormal basis of U

$\Rightarrow v = \underbrace{\langle v, e_i \rangle e_i}_{u} + \underbrace{v - \langle v, e_i \rangle e_i}_{w}$

$\Rightarrow \langle w, e_j \rangle = \langle v, e_j \rangle - \langle u, e_j \rangle = \langle v, e_j \rangle - \langle v, e_i \rangle \underbrace{\langle e_i, e_j \rangle}_{\delta_{ij}} = 0 \quad \forall j$

$\Rightarrow w \in U^\perp \Rightarrow V = U + U^\perp$

we know $U \cap U^\perp = \{0\} \Rightarrow V = U \oplus U^\perp$

corollary: $\dim(U^\perp) = \dim(V) - \dim(U)$

proposition: $U < V$ finite then $U = (U^\perp)^\perp$

proof: let $u \in U$ then $\langle u, v \rangle = 0 \ \forall v \in U^\perp \Rightarrow u \in (U^\perp)^\perp \Rightarrow U \subset (U^\perp)^\perp$

let $v \in (U^\perp)^\perp$ then $v = u + w$ where $u \in U$ and $w \in U^\perp$ as $v \in V$

$\Rightarrow v-u=w \in U^\perp$, $U \subset (U^\perp)^\perp \Rightarrow u \in (U^\perp)^\perp \Rightarrow v-u=w \in (U^\perp)^\perp$

$\Rightarrow v-u=w \in U^\perp \cap (U^\perp)^\perp = \{0\} \Rightarrow v=u \in U \Rightarrow (U^\perp)^\perp \subset U$ ∎

definition: $U < V$ finite. the <u>orthogonal projection</u> of V onto U is $P_U \in \mathcal{L}(V)$ defined as for $v \in V$, $v = u + w$ where $u \in U$, $w \in U^\perp$ then $P_U v = u$ (direct sum $\Rightarrow$ P_U is well-defined)

proposition: (a) $P_U u = u \ \forall u \in U$, $P_U w = 0 \ \forall w \in U^\perp$

(b) $\mathrm{im}(P_U) = U$, $\ker(P_U) = U^\perp$

(c) $v - P_U v \in U^\perp$

(d) $P_U^2 = P_U$

(e) $\|P_U v\| \leq \|v\|$

(f) for all orthonormal basis $\{e_i\}$ of U

$P_U v = \langle v, e_i \rangle e_i$

proof : (d) let $v = u + w$ with $u \in U$, $w \in U^\perp$.

$\Rightarrow P_U^2 v = P_U(P_U v) = P_U u = u = P_U v$, $\forall v \in V$

(e) let $v = u + w$ with $u \in U$, $w \in U^\perp$

$\Rightarrow \|P_U v\|^2 = \|u\|^2 \leq \|u\|^2 + \|w\|^2 = \|v\|^2$

(f) let $v = u + w$ with $u \in U$, $w \in U^\perp$,

$\{e_i\}$ be orthonormal basis of U

$\Rightarrow \langle e_i, e_j \rangle = \delta_{ij}$

$\langle w, e_i \rangle = 0 \ \forall i$

$P_U v = u = \langle u, e_i \rangle e_i$

$= \langle u, e_i \rangle e_i + \langle w, e_i \rangle e_i$

$= (\langle u, e_i \rangle + \langle w, e_i \rangle) e_i$

$= \langle u+w, e_i \rangle e_i$

$= \langle v, e_i \rangle e_i$ ∎

theorem: <u>Minimisation problem</u>

let $U < V$ and $\dim(U) < \infty$, $u \in U$, $v \in V$

then $\|v - P_U v\| \leq \|v - u\|$

also, the inequality becomes equality $\Leftrightarrow u = P_U v$

proof : $\|v - P_U v\|^2 \leq \|v - P_U v\|^2 + \|P_U v - u\|^2$, $v - P_U v \in U^\perp$ and $P_U v - u \in U \Rightarrow \langle v - P_U v, P_U v - u \rangle = 0$

$= \|(v - P_U v + P_U v - u)\|^2 \Leftarrow$ Pythagoras

$= \|v - u\|^2$ ∎

- $\|v - P_U v\|^2 = \|v-u\|^2 \Leftrightarrow \|P_U v - u\|^2 = 0 \Leftrightarrow P_U v = u$

examples, excersises : later