5.A Invariant Subspaces

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definition: invariant subspace

let $T \in \mathcal{L}(V)$. $U < V$ is invariant under T

if $Tu \in U \;\forall u \in U$

$T|_U$ is T with a domain U

$\Rightarrow$ if U is invariant under T then $T|_U \in \mathcal{L}(U)$

example: a) $\{0\}$

b) V

c) $\ker(T)$

d) $\operatorname{Im}(T)$

proposition: V is finite dimensional, $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$ then

λ is eigenvalue of T

$\Leftrightarrow$ $T - \lambda I$ is not injective

$\Leftrightarrow$ $T - \lambda I$ is not surjective

$\Leftrightarrow$ $T - \lambda I$ is not bijective

proof: $Tv = \lambda v \Leftrightarrow (T - \lambda I)v = 0 \Rightarrow T - \lambda I$ not injective

operator injective $\Leftrightarrow$ surjective $\blacksquare$

theorem: if $\{\lambda_i\}$ are distinct eigenvalues and $\{v_i\}$ are corresponding eigenvectors then $\{v_i\}$ are linearly independent

proof: suppose $\{v_i\}$ are linearly dependent and k is smallest integer s.t.

$v_k \in \operatorname{span}(v_1, \cdots, v_{k-1}) \Rightarrow \{v_1, \cdots, v_{k-1}\}$ are linearly independent

$\Rightarrow v_k = a_j v_j \;,\; j \in [\![1, k-1]\!] \Rightarrow \lambda_k v_k = \lambda_k a_j v_j$ ①

$\Rightarrow Tv_k = T a_j v_j \Rightarrow \lambda_k v_k = \lambda_j a_j v_j \;,\; j \in [\![1, k-1]\!]$ ②

$\Rightarrow$ ①-②: $0 = (\lambda_k - \lambda_j) a_j v_j \;,\; j \in [\![1, k-1]\!] \;,\; \lambda_k - \lambda_j \neq 0$

$\Rightarrow a_j = 0 \;\forall j$ contradiction

corollary: $T \in \mathcal{L}(V)$ has at most $\dim(V)$ distinct eigenvalues

definition: suppose $T \in \mathcal{L}(V)$ and $U < V$ is invariant under T

the restriction operator $T|_U \in \mathcal{L}(U)$ is

$T|_U(u) := Tu$ for $u \in U$

the quotient operator $T/U \in \mathcal{L}(V/U)$ is

$(T/U)(v + U) := Tv + U \quad v \in V$