3.F Duality

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definition : linear functional T, on V is
$$T: V \to \mathbb{F}, \quad T \in \mathcal{L}(V, \mathbb{F})$$

definition : the dual space V' of V is set of all linear functional
$$V' := \mathcal{L}(V, \mathbb{F})$$

lemma : $\dim(V') = \dim(V)$

definition : If $v_1, \dots, v_n$ is basis of V, then dual basis of $v_1, \dots, v_n$ is the list of $\phi_1, \dots, \phi_n \in V'$ where
$$\phi_j(v_k) = \delta_{jk}$$

proposition : dual basis of basis of V is basis of V'

proof : let $a_1, \dots, a_n \in \mathbb{F}$ s.t. $\sum_{i=1}^{n} a_i \phi_i = 0 \ \forall v \in V \ (\phi_i : V \to \mathbb{F})$
$$\sum_{i=1}^{n} a_i \phi_i(v_j) = \sum_{i=1}^{n} a_i \delta_{ij} = a_j = 0 \quad \forall j \in [\![1, n]\!]$$
$\Rightarrow \{\phi_i\}$ is linearly independent

definition : let $T \in \mathcal{L}(V, W)$. the dual map of T is $T' \in \mathcal{L}(W', V')$
$$T'(\phi) := \phi \circ T \quad \text{for } \phi \in W' = \mathcal{L}(W, \mathbb{F})$$
$$\Rightarrow T'(\phi) : V \to W \to \mathbb{F} \in \mathcal{L}(V, \mathbb{F}) = V'$$

proposition : $(S+T)' = S' + T' \quad \forall S, T \in \mathcal{L}(V, W)$

proof : $(S+T)'(\phi) = \phi \circ (S+T) = \phi \circ S + \phi \circ T = S' + T'$

proposition : $(\lambda T)' = \lambda T'$, $\lambda \in \mathbb{F}$

proof : $\lambda T'(\phi) = \phi \circ (\lambda T) = \lambda \phi \circ T = \lambda T'$

proposition : $(ST)' = T'S'$

proof : $(ST)'(\phi) = \phi \circ (ST) = \phi \circ S \circ T = (\phi \circ S) \circ T = T'(\phi \circ S) = T'(S'(\phi)) = T' \circ S'(\phi) = T'S'(\phi)$

definition : for $U \subseteq V$ annihilator U^0 of U is
$$U^0 := \{\phi \in V' : \phi(u) = 0 \ \forall u \in U\} \subseteq V$$

proposition : if $U \subseteq V$, then $U^0 < V'$

proof : (S1) $\phi(v) = 0$ functional $\in U^0$ ✓

(S2) let $\phi_1, \phi_2 \in U^0 \Rightarrow \phi_1(u) + \phi_2(u) = 0 + 0 = 0 \ \forall u \in U$
$\Rightarrow \phi_1 + \phi_2 \in U^0$ ✓

(S3) $\phi \in U^0, \lambda \in \mathbb{F} \Rightarrow \lambda \phi(u) = \lambda \cdot 0 = 0 \ \forall u \in U$
$\Rightarrow \lambda \phi \in U^0$ ✓

lemma : V is finite dimensional and $U < V$ then

$$\dim(U) + \dim(U^\circ) = \dim(V)$$

proof : let $u_1, \cdots, u_m$ be basis of U and $u_1, \cdots, u_m, \cdots, u_n$ be basis of V and $\phi_1, \cdots, \phi_m, \cdots, \phi_n$ be basis of V'

suppose $\phi \in \operatorname{span}(\phi_{m+1}, \cdots, \phi_n) \Rightarrow \phi = c_{m+1}\phi_{m+1} + \cdots + c_n\phi_n$, $c_i \in \mathbb{F}$

$\Rightarrow \phi(u_1, \cdots, u_m) = 0 \Rightarrow \phi \in U^\circ \Rightarrow \operatorname{span}(\phi_{m+1}, \cdots, \phi_n) \subseteq U^\circ$

suppose $\phi \in U^\circ \Rightarrow \phi \in V' \Rightarrow \phi = c_1\phi_1 + \cdots + c_m\phi_m + \cdots c_n\phi$, $c_i \in \mathbb{F}$

for $i \in [\![1, m]\!]$ $\phi(u_i) = c_i = 0$ as $\phi \in U^\circ$ and $u_i \in U$

$\Rightarrow \phi = c_{m+1}\phi_{m+1} + \cdots + c_n\phi_n \Rightarrow U^\circ \subseteq \operatorname{span}(\phi_{m+1}, \cdots, \phi_n)$

theorem : suppose V, W are finite dimensional and $T \in \mathcal{L}(V, W)$, then

(a) $\ker(T') = (\operatorname{Im}(T))^\circ$

(b) $\dim(\ker(T')) = \dim(\ker(T)) + \dim(W) - \dim(V)$

proof : (a) let $\phi \in \ker(T') \Leftrightarrow T'(\phi) = \phi \circ T = 0 \Leftrightarrow (\phi \circ T)(v) = \phi(Tv) = 0$ $\forall v \in V$

$\Leftrightarrow \phi(T_v) = 0$ $\forall T_v \in \operatorname{Im}(T) \Leftrightarrow \phi \in (\operatorname{Im}(T))^\circ$

$\Rightarrow \ker(T') \subseteq (\operatorname{Im}(T))^\circ$ and $\ker(T') \supseteq (\operatorname{Im}(T))^\circ$ ∎

(b) $\dim(\operatorname{Im}(T)) + \dim(\ker(T)) = \dim(V)$

$\dim((\operatorname{Im}(T))^\circ) + \dim(\operatorname{Im}(T)) = \dim(W) \Leftarrow \operatorname{Im}(T) < W$

$\Rightarrow \dim(\ker(T')) + \dim(\operatorname{Im}(T)) = \dim(W)$ ∎

theorem : T surjective $\Leftrightarrow$ T' injective

proof : $T \in \mathcal{L}(V, W)$ surjective $\Leftrightarrow \operatorname{Im}(T) = W \Leftrightarrow (\operatorname{Im}(T))^\circ = \{0\}$ as $\dim(\operatorname{Im}(T)) = \dim(W)$

$\Leftrightarrow (\ker(T')) = \{0\} \Leftrightarrow T'$ injective

theorem : V and W finite dimensional and $T \in \mathcal{L}(V, W)$, then

(a) $\dim(\operatorname{Im}(T')) = \dim(\operatorname{Im}(T))$

(b) $\operatorname{Im}(T') = (\ker(T))^\circ$

proof : (a) $T' \in \mathcal{L}(W', V') \Rightarrow \dim(\operatorname{Im}(T')) + \dim(\ker(T')) = \dim(W')$

$\Rightarrow \dim(\operatorname{Im}(T')) + \dim((\operatorname{Im}(T))^\circ) = \dim(W)$

$\Rightarrow \dim(\operatorname{Im}(T')) = \dim(\operatorname{Im}(T))$ ∎

(b) let $\phi \in \operatorname{Im}(T') \Rightarrow \exists \psi$ s.t. $T'(\psi) = \phi$

if $v \in \ker(T) \Rightarrow \phi(v) = (T'(\psi))(v) = (\psi \circ T)(v) = \psi(Tv) = \psi(0) = 0$

$\Rightarrow \phi \in (\ker(T))^\circ \Rightarrow \operatorname{Im}(T') \subseteq (\ker(T))^\circ$

$\dim(\operatorname{Im}(T')) = \dim(\operatorname{Im}(T)) = \dim(V) - \dim(\ker(T))$

$\ker(T) < V \Rightarrow \dim(\ker(T)) + \dim((\ker(T))^\circ) = \dim(V)$

$\Rightarrow \dim(\operatorname{Im}(T')) = \dim((\ker(T))^\circ)$ ∎

theorem : T injective $\Leftrightarrow$ T' surjective

proof : $T \in \mathcal{L}(V, W)$ injective $\Leftrightarrow \ker(T) = \{0\} \Leftrightarrow \dim((\ker(T))^\circ) = \dim(V')$, as $(\ker(T))^\circ < V'$

$\Leftrightarrow (\ker(T))^\circ = V' \Leftrightarrow \operatorname{Im}(T') = V'$, $T' \in \mathcal{L}(W', V') \Leftrightarrow T'$ surjective

definition : transpose of matrix A is

$$(A^T)_{kj} = A_{jk}$$

propositions : $(A + C)^T = A^T + C^T$, $(\lambda A)^T = \lambda A^T$ for $\lambda \in \mathbb{F}$

$(AC)^T = C^T A^T$

theorem: let $T \in \mathcal{L}(V,W)$. then $\mathcal{M}(T') = (\mathcal{M}(T))^T$

proof: let $k \in [\![1,n]\!]$, $j \in [\![1,m]\!]$, $\{\phi_i\}$ is basis of V', $\{\psi_j\}$ is basis of W', $\{v_k\}$ basis of V, $\{w_j\}$ basis of W, $\mathcal{M}(T) = A$, $\mathcal{M}(T') = C$

$$\Rightarrow T'(\psi_j) = C_{ij}\,\phi_i = \psi_j \circ T$$

$$\Rightarrow \psi_j \circ T(v_k) = C_{ij}\,\phi_i(v_k) = C_{ij}\,\delta_{ik} = C_{kj}$$

$$= \psi_j(A_{ik} w_i) = A_{ik}\,\psi_j(w_i) = A_{ik}\,\delta_{ji} = A_{jk}$$

$$\Rightarrow A_{jk} = C_{kj} = (A^T)_{kj} \Rightarrow A^T = C$$

definition: given m by n matrix A over $\mathbb{F}$

row rank is dimension of span of rows of A in $\mathbb{F}^{1,n}$

column rank " " columns " in $\mathbb{F}^{m,1}$

lemma: $\dim(\mathrm{Im}(T)) =$ column rank of $\mathcal{M}(T)$

proof: $\{v_c\}$, $c \in [\![1,n]\!]$ is basis of V and $\{w_r\}$, $r \in [\![1,m]\!]$ is basis of W

$\mathcal{M}(Tv) = \mathcal{M}(w)$ where $v \in V$, $w \in \mathrm{Im}(T)$

$\Rightarrow f: \mathrm{span}(\{Tv_c\}) \rightarrow \mathcal{M}(\mathrm{span}(\{Tv_c\}))$ is isomorphism

$\mathcal{M}(\mathrm{span}(\{Tv_c\})) = \mathrm{span}(\{\mathcal{M}(Tv_c)\}) \cong \mathrm{span}(\{Tv_c\}) = \mathrm{Im}(T)$

$\Rightarrow \dim(\mathrm{Im}(T)) = \dim(\mathrm{span}(\{\mathcal{M}(Tv_c)\})) = \dim(\mathrm{span}(\{\text{columns of } \mathcal{M}(T)\}))$

$=$ column rank

theorem: row rank $=$ column rank $\forall A \in \mathbb{F}^{m,n}$

proof: define $T: \mathbb{F}^{n,1} \rightarrow \mathbb{F}^{m,1}$ by $Tx = Ax$ where $A \in \mathbb{F}^{m,n}$

$\Rightarrow \mathcal{M}(T) = A$

$\Rightarrow$ column rank of $A = \dim(\mathrm{Im}(T)) = \dim(\mathrm{Im}(T'))$

$=$ column rank of $\mathcal{M}(T')$

$=$ column rank of $(\mathcal{M}(T))^T$

$=$ column rank of A^T

$=$ row rank of A