5.B Eigenvectors and Upper-Triangular Matrices

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definition: suppose $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbb{F})$ where $p(z) = \sum_{i=0}^{m} a_i z^i$ for $z \in \mathbb{F}$
then $p(T) := a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m$

corollary: given T, function $f: \mathcal{P}(\mathbb{F}) \to \mathcal{L}(V)$ defined by $p \to p(T)$ is linear
proof: $f(p+q) = p(T) + q(T)$, $f(\lambda p) = \lambda p(T)$

definition: if $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is
$(pq)(z) := p(z)q(z)$

corollary: any 2 polynomials of an operator commutes
$p(T)q(T) = q(T)p(T)$
proof: $p(z)q(z) = q(z)p(z) \Rightarrow pq = qp$

theorem: every operator on finite dimensional, nonzero, complex V has an eigenvalue
proof: suppose $\dim(V) = n > 0$ and $T \in \mathcal{L}(V) \Rightarrow \{v, Tv, T^2 v, \cdots, T^n v\}$ is not linearly independent as the length of the set is $n+1$
$\Rightarrow \exists \{a_i\}, i \in [0, n]$ s.t. $0 = a_0 v + a_1 T v + a_2 T^2 v + \cdots + a_n T^n v$ and not all $\{a_i\}$ are 0.
let $p \in \mathcal{P}(\mathbb{C})$ s.t. $p(T) = \sum_{i=0}^{n} a_i T^i v$, $T^0 = I \Rightarrow 0 = c \prod_{i=1}^{m} (T - \lambda_i I) v \Rightarrow T - \lambda_i I = 0$ for at least one i (Fundamental Theorem of Algebra)

definition: $T \in \mathcal{L}(V)$, $\{v_i : i \in [\![1, n]\!]\}$ basis of V, $\mathcal{M}(T) = A$ s.t.
$T v_k = A_{jk} v_j$

theorem: $T \in \mathcal{L}(V)$, $\{v_i : i \in [\![1, n]\!]\}$ then
$\mathcal{M}(T)$ with $\{v_i\}$ basis uppertriangular
$\iff T v_j \in \operatorname{span}(\{v_i \in [\![1, j]\!]\})$ for each $j \in [\![1, n]\!]$
$\iff \operatorname{span}(\{v_i \in [\![1, j]\!]\})$ for each $j \in [\![1, n]\!]$ is invariant under T

proof: second ($\iff$) is trivial
first ($\iff$) $\mathcal{M}(T) = \begin{pmatrix} * & \cdots & * \\ 0 & * & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots\ 0 & * \end{pmatrix}$
jth column $= \mathcal{M}(T v_j)$ only represented by $\{v_i : i \in [\![1, j]\!]\}$

theorem: $V \cong \mathbb{C}^n$, then $T \in \mathcal{L}(V)$ has an upper triangular matrix with some basis of V
proof: induction if $\dim(V) = 1 \Rightarrow$ any 1 by 1 matrices are triangular
suppose $\dim(V) = n > 1$ and assume the results holds for all vector spaces $\cong \mathbb{C}^{n-1}$
let v_1 be an eigenvector of T, let $U = \operatorname{span}(v_1) \Rightarrow U$ is invariant under T and $\dim(U) = 1$
$\Rightarrow \dim(V/U) = \dim(V) - \dim(U) = n - 1 \Rightarrow T/U \in \mathcal{L}(V/U)$ has triangular matrix with some basis $\{v_2 + U, \cdots, v_n + U\}$
$\Rightarrow (T/U)(v_j + U) \in \operatorname{span}(v_2 + U, \cdots, v_j + U)$ for each $j \in [\![2, n]\!]$
$\Rightarrow (T/U)(v_j + U) = T v_j + U \in \operatorname{span}(v_2 + U, \cdots, v_j + U) \Rightarrow T v_j \in \operatorname{span}(v_2, \cdots, v_j) \subset \operatorname{span}(v_1, \cdots, v_j)$ for each $j \in [\![1, n]\!]$ ∎

theorem: if $T \in \mathcal{L}(V)$ has upper-triangular matrix with some basis
then (T invertible $\iff$ (all diagonal entries $\neq 0$))
proof: suppose $\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ with basis $\{v_i : i \in [\![1, n]\!]\}$

($\Leftarrow$) if $\lambda_i \neq 0\ \forall i \in [\![1, n]\!] \Rightarrow T v_1 = \lambda_1 v_1 \Rightarrow T(\frac{1}{\lambda_1} v_1) = v_1 \Rightarrow v_1 \in \operatorname{Im}(T)$
and $T(\frac{1}{\lambda_2} v_2) = a v_1 + v_2 \in \operatorname{Im}(T)$ and $a v_1 \in \operatorname{Im}(T) \Rightarrow v_2 \in \operatorname{Im}(T)$
$\Rightarrow v_i \in \operatorname{Im}(T)\ \forall i \in [\![1, n]\!] \Rightarrow \{v_i\}$ basis of $\operatorname{Im}(T) \Rightarrow V = \operatorname{Im}(T) \Rightarrow T$ surjective $\Rightarrow T$ invertible
($\Rightarrow$) if T invertible $\Rightarrow$ (if $\lambda_1 = 0 \Rightarrow T v_1 = 0 \Rightarrow T$ not injective) $\Rightarrow \lambda_1 \neq 0$
assume $\lambda_j = 0$ for some $j \in [\![2, n]\!] \Rightarrow T v_j \in \operatorname{span}(v_1, \cdots, v_{j-1}) \Rightarrow T$ maps $\operatorname{span}(v_1, \cdots, v_j) \to \operatorname{span}(v_1, \cdots, v_{j-1})$
$\Rightarrow T$ restricted to $\operatorname{span}(v_1, \cdots, v_j)$ is not injective $\Rightarrow \exists$ [illegible]

$\Rightarrow T$ not injective $\blacksquare$

theorem: if $T \in \mathcal{L}(V)$ has upper-triangular matrix with some basis then eigenvalues of T are the diagonal entries of the upper-triangular matrix

proof: suppose $\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, $\lambda \in \mathbb{F} \Rightarrow \mathcal{M}(T-\lambda I) = \begin{pmatrix} \lambda_1-\lambda & & * \\ & \ddots & \\ 0 & & \lambda_n-\lambda \end{pmatrix}$

$\therefore$ $T-\lambda I$ not invertible $\Leftrightarrow$ $\lambda_i - \lambda = 0$ for some $i \in [\![1, n]\!]$ $\Leftrightarrow \{\lambda_i : i \in [\![1, n]\!]\}$ are eigenvalues