8.C Characteristic and Minimal Polynomials

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The Cayley-Hamilton Theorem

definition: let $V(\mathbb{C})$, $T \in \mathcal{L}(V)$, $\{\lambda_i\}$ distinct e-values, $\{d_i\}$ multiplicities of T

<u>charicteristic polynomial</u> of T is

$$\prod_i (z - \lambda_i)^{d_i}$$

corollary: $V(\mathbb{C})$, $T \in \mathcal{L}(V)$ then

(a) degree of characteristic polynomial $= \dim(V)$

(b) zeros of characteristic polynomial $=$ e-values

theorem: <u>Cayley-Hamilton theorem</u>

Let $V(\mathbb{C})$, $T \in \mathcal{L}(V)$, $q(z) =$ characteristic polynomial of T

then $q(T) = 0$

proof: let $\{\lambda_i\}$ e-values of T, $\{d_i\}$ multiplicities.

$$\Rightarrow (T - \lambda_j I)^{d_j}\big|_{G(\lambda_j, T)} = 0$$

$$q(T)\big|_{G(\lambda_j, T)} = \prod_i (T - \lambda_i I)^{d_i}\big|_{G(\lambda_j, T)} = 0 \quad \forall j$$

$$\Rightarrow q(T) = 0$$

The Minimal Polynomial

definition: <u>monic polynomial</u> is a polynomial with highest degree coefficient $= 1$

lemma : $T \in \mathcal{L}(V)$ then

$\exists !$ monic polynomial p of smallest degree s.t. $p(T)=0$

proof : let $n = \dim(V)$

$\Rightarrow \{I, T, T^2, \cdots, T^{n^2}\}$ is not linearly independent in $\mathcal{L}(V)$

$\Leftarrow \dim(\mathcal{L}(V)) = n^2$

let $m \in \mathbb{Z}_{>0}$ be smallest s.t. $\{T^i \mid i \in [0, m]\}$ is linearly dependent

$$\Rightarrow 0 = \sum_{i=0}^{m-1} a_i T^i + T^m$$

let $p \in P(\mathbb{F})$

$$p(z) = \sum_{i=0}^{m-1} a_i z^i + z^m$$

$\Rightarrow \exists$ monic p s.t. $p(T)=0$

m smallest $\Rightarrow$ if monic $q \in P(\mathbb{F})$ with degree $< m$ then $q(T) \neq 0$

if q degree $= m$ and $q(T)=0$ then $(p-q)(T)=0$

$\deg(p-q) < m \Rightarrow q = p$

definition : let $T \in \mathcal{L}(V)$

<u>minimal polynomial</u> p of T is

the unique monic p of smallest $\deg(p)$ s.t. $p(T)=0$

theorem : $T \in \mathcal{L}(V)$, $q \in P(\mathbb{F})$ then

$q(T) = 0 \iff q$ is multiple of minimal polynomial of T

proof : let $p \in P(\mathbb{F})$ be minimal polynomial of T

$(\Leftarrow)$: let $s \in P(\mathbb{F})$ s.t. $q = ps$

$\Rightarrow q(T) = p(T)s(T) = 0 \cdot s(T) = 0$ ✓

$(\Rightarrow)$: let $q(T) = 0$, $(r, s) \in P^2(\mathbb{F})$ s.t.

$q = ps + r$ and $\deg(r) < \deg(p)$

$\Rightarrow q(T) = p(T)s(T) + r(T) = r(T) = 0$

$\Rightarrow r = 0$ ∎

corollery : $V(\mathbb{C})$, $T \in \mathcal{L}(V)$ then

characteristic of T is multiple of minimal of T

theorem: $T \in \mathcal{L}(V)$ then

zeros of minimal of T = eigenvalues of T

proof: let $p(z)$ be minimal of T and

$\lambda \in \mathbb{F}$ s.t. $p(\lambda) = 0$ then

$p(z) = (z - \lambda) q(z)$ where q is monic.

$p(T) = (T - \lambda I) q(T) = 0$, $q(T) \neq 0$

$\Rightarrow T - \lambda I = 0 \Rightarrow \lambda$ is eigenvalue of T ✓

let $\lambda \in \mathbb{F}$ is e-value of T

$\Rightarrow \exists\, v \in V \setminus \{0\}$ s.t. $Tv = \lambda v$

$\Rightarrow T^j v = \lambda^j v \quad \forall j \in \mathbb{Z}_{\geq 0}$

$\Rightarrow 0 = p(T) v = p(\lambda) v \Rightarrow p(\lambda) = 0$ ■