5.C Eigenspaces and Diagonal Matrices

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definition: suppose $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$ then the eigenspace $E(\lambda, T)$ of T corresponding to λ is

$$E(\lambda, T) := \ker(T - \lambda I)$$

it is the set of all eigenvectors of T corresponding to λ + 0 vector

$T - \lambda I \in \mathcal{L}(V) \Rightarrow \ker(T - \lambda I) < V$ and $T|_{E(\lambda,T)} = \lambda I$

proposition: V is finite dimensional and $T \in \mathcal{L}(V)$, $\lambda_1, \cdots, \lambda_m$ are distinct eigenvalues of T then

$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is a direct sum

also,

$\dim(E(\lambda_1, T)) + \cdots + \dim(E(\lambda_m, T)) \leq \dim(V)$

proof: suppose $\exists$ some $u_j \in E(\lambda_j, T)$ s.t. $u_1 + \cdots + u_m = 0$

as eigenvectors with distinct eigenvalues are linearly independent, $u_j = 0 \; \forall j$

$\Rightarrow u_1 + \cdots + u_m$ where $u_j \in E(\lambda_j, T)$ has unique representation. ☑

also, $\dim(E(\lambda_1, T)) + \cdots + \dim(E(\lambda_m, T)) = \dim(E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)) \leq \dim(V)$

theorem: conditions equivalent to diagonalisability

suppose $T \in \mathcal{L}(V)$ where $\dim(V) = n$, $\{\lambda_i : i \in [\![1, m]\!]\}$ is distinct eigenvalues of T

then the following are equivalent:

(a) T is diagonisable

(b) V has basis consisting of eigenvectors of T

(c) $\exists$ 1-dimensinal $U_1, \dots, U_n < V$, each invariant under T s.t. $V = U_1 \oplus \cdots \oplus U_n$

(d) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$

(e) $\dim(V) = \sum_{i=1}^{m} \dim(E(\lambda_i, T))$

proof: (a) $\Leftrightarrow$ (b):

$Tv_j = \lambda_j v_j \Leftrightarrow \mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

(b) $\Rightarrow$ (c):

let $\{v_j : j \in [\![1, n]\!]\}$ basis of V is also eigenvectors of T.

$\Rightarrow$ $\operatorname{span}(v_j)$ is 1-dimentional subspace invariant under T

$v = \sum_{j=1}^{n} c_j v_j$, $c_j \in \mathbb{F} \; \forall v \in V$ is unique representation and $c_j v_j \in \operatorname{span}(v_j)$

$\Rightarrow V = \operatorname{span}(v_1) \oplus \cdots \oplus \operatorname{span}(v_n)$

(c) $\Rightarrow$ (b):

let $v_j \in U_j$, $v_j \neq 0$. as U_j is invariant under T, v_j is eigenvector of T

$\Rightarrow c_j v_j \in U_j$, $c_j \in \mathbb{F}$ so direct sum $\Rightarrow v = \sum_{j=1}^{n} c_j v_j$ is unique $\Rightarrow \{v_j : j \in [\![1, n]\!]\}$ is basis of V

$\therefore$ (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c)

(b) $\Rightarrow$ (d):

let $\{v_j : j \in [\![1, n]\!]\}$ basis of V is also eigenvectors of T.

$\exists i \in [\![1, m]\!]$ s.t. $v_j \in E(\lambda_i, T) \; \forall j \in [\![1, n]\!] \Rightarrow v \in E(\lambda_1, T) + \cdots + E(\lambda_m, T) \; \forall v \in V$

(d) $\Rightarrow$ (e):

already shown by dimensionnality of direct sum

(e) $\Rightarrow$ (b):

let $\{v_i \dots\}_i$ be a basis of $E(\lambda_i, T)$. $\bigcup_{i=1}^{m} \{v_i \dots\}$ has length $n \Leftarrow$ (e)

re-lable the union of all basis of $E(\lambda_i, T)$ as $\{v_j : j \in [\![1, n]\!]\}$

suppose $a_j v_j = 0$ where $a_j \in \mathbb{F}$, $j \in [\![1, n]\!]$

let $u_i = a_k v_k$ s.t. $v_k \in E(\lambda_i, T)$

$\Rightarrow a_j v_j = \sum_{i=1}^{m} u_i$ eigenvectors with distinct eigenvalue are linearly independent

$\Rightarrow u_i = 0 \; \forall i \in [\![1, m]\!] \Rightarrow a_k v_k = 0$ s.t. $v_k \in E(\lambda_i, T)$ and v_k is basis of $E(\lambda_i, T)$

$\Rightarrow a_k = 0 \; \forall k \Rightarrow a_j = 0 \; \forall j \in [\![1, n]\!] \Rightarrow \{v_j : j \in [\![1, n]\!]\}$ is linearly independent

$$\therefore \begin{tikzcd} (a) \arrow[r, Leftrightarrow] & (b) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & (c) \\ (e) \arrow[ur, Rightarrow] & (d) \arrow[l, Rightarrow] & \end{tikzcd}$$

corollery: if $T \in \mathcal{L}(V)$ has $\dim(V)$ distinct eigenvalues, T is diagonalisable