7.D Polar Decomposition and Singular Value Decomposition

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definition: if T is positive operator
$\sqrt{T}$ denotes the unique positive square root

7.45 Polar Decomposition

Suppose $T \in \mathcal{L}(V)$. Then there exists an isometry $S \in \mathcal{L}(V)$ such that

$$T = S\sqrt{T^*T}.$$

analogy: $z \in \mathbb{C}$, $\underset{\text{real}}{z = \bar{z}} \Leftrightarrow \underset{\text{hermitian}}{T = T^*}$, $\underset{\text{unit circle}}{|z| = z\bar{z} = 1} \Leftrightarrow \underset{\text{isometry}}{T^*T = I}$

$$z = \frac{z}{|z|}|z| = \underbrace{\left(\frac{z}{|z|}\right)}_{\in \text{ unit circle}}\underbrace{\sqrt{z\bar{z}}}_{\sqrt{T^*T}} \quad \text{if } z \neq 0$$

definition: $T \in \mathcal{L}(V)$. singular values of T are eigenvalues of $\sqrt{T^*T}$, each with multiplicity $\dim(E(\lambda, \sqrt{T^*T}))$

7.51 Singular Value Decomposition

Suppose $T \in \mathcal{L}(V)$ has singular values $s_1, \dots, s_n$. Then there exist orthonormal bases $e_1, \dots, e_n$ and $f_1, \dots, f_n$ of V such that

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$$

for every $v \in V$.

$$\mathcal{M}(T, (\{e_i\}), (\{f_i\})) = \operatorname{diag}(s_1, \dots, s_n)$$

7.52 Singular values without taking square root of an operator

Suppose $T \in \mathcal{L}(V)$. Then the singular values of T are the nonnegative square roots of the eigenvalues of T^*T, with each eigenvalue λ repeated dim $E(\lambda, T^*T)$ times.