3.D Invertibility and Isomorphic Vector Spaces

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definition: if $T \in \mathcal{L}(V,W)$ is invertible, its inverse $T^{-1} \in \mathcal{L}(W,V)$ is unique s.t. $TT^{-1} = T^{-1}T = I$

theorem: T is invertible $\iff$ T injective and surjective

proof: $(\Rightarrow)$ suppose $T \in \mathcal{L}(V,W)$ is invertible and $u, v \in V$ s.t. $Tu = Tv$

$\Rightarrow u = T^{-1}Tu = T^{-1}Tv = v \Rightarrow u = v \Rightarrow T$ injective

suppose $w \in W \Rightarrow w = T(T^{-1}w) \Rightarrow T^{-1}w \in V$ and $w \in Im(T)$

$\Rightarrow T$ surjective

$(\Leftarrow)$ T injective and surjective $\Rightarrow$ we can define $w \in W$, $S \in \mathcal{L}(W,V)$ s.t. $T(Sw) = w$

$\Rightarrow Sw$ is unique for each w and $T \circ S = I$ on W

and $T((S \circ T)v) = T \circ S(Tv) = Tv \Rightarrow (S \circ T)v = v \Rightarrow S \circ T = I$ on V

S is linear: $T(S(w_1 + w_2)) = T \circ S(w_1 + w_2) = w_1 + w_2 = T(Sw_1) + T(Sw_2)$

$T(\lambda Sw) = \lambda T \circ S(w) = \lambda w = T(S(\lambda w))$ ∎

theorem: 2 finite dimensional V, W over $\mathbb{F}$

$V \cong W \iff \dim(V) = \dim(W)$

proof: $(\Rightarrow)$ suppose $V \cong W \Rightarrow \exists$ invertible $T \in \mathcal{L}(V,W)$

$\Rightarrow \ker(T) = \{0\}$, $Im(T) = W \Rightarrow \dim(V) = \underbrace{\dim(\ker(T))}_{0} + \dim(Im(T))$

$= \dim(W)$

$(\Leftarrow)$ suppose $\dim(V) = \dim(W)$, $\{v_i\}$ basis of V, $\{w_j\}$ basis of W

let $T: V \to W$, $T(c_i v_i) := c_j w_j \Rightarrow T \in \mathcal{L}(V,W)$

$\Rightarrow \ker(T) = \{0\}$ as $\{w_j\}$ is linearly independent $\Rightarrow T$ injective

and $\{w_j\}$ spans $W \Rightarrow Im(T) = W \Rightarrow T$ surjective ∎

theorem: suppose $\{v_1, \cdots, v_n\}$ basis of V and $\{w_1, \cdots, w_m\}$ basis of W

$\Rightarrow \mathcal{M}$ is isomorphism between $\mathcal{L}(V,W)$ and $\mathbb{F}^{m,n}$

$\Rightarrow \mathcal{L}(V,W) \cong \mathbb{F}^{m,n}$

proof: $\mathcal{M}$ is linear

if $\mathcal{M}(T) = 0 \Rightarrow Tv_k = 0 \ \forall k \in [\![1, n]\!] \Rightarrow T = 0$ as $\{v_k\}$ is linearly independent

$\Rightarrow \ker(\mathcal{M}) = \{0\} \Rightarrow \mathcal{M}$ injective

suppose $A \in \mathbb{F}^{m,n}$ let $T \in \mathcal{L}(V,W)$ st. $Tv_k := A_{jk} w_j \Rightarrow \mathcal{M}(T) = A$

$\Rightarrow A \in Im(\mathcal{M})$, $Im(\mathcal{M}) < \mathbb{F}^{m,n} \Rightarrow Im(\mathcal{M}) = \mathbb{F}^{m,n} \Rightarrow \mathcal{M}$ surjective

theorem: $\dim(\mathcal{L}(V,W)) = \dim(V) \cdot \dim(W)$

theorem: V finite dimensional and $T \in \mathcal{L}(V)$ then,

T injective $\iff$ T surjective $\iff$ T invertible

T injective $\Longleftrightarrow$ T surjective $\Longleftrightarrow$ T invertible

proof: only need to prove 1st equivalence

if T injective $\Rightarrow \ker(T) = \{0\} \Rightarrow \dim(V) = \dim(\mathrm{Im}(T)) + 0$

$\Rightarrow$ T surjective

if T surjective $\Rightarrow \dim(\mathrm{Im}(T)) = \dim(V)$

$\Rightarrow \dim(V) = \dim(V) + \dim(\ker(T)) \Rightarrow \dim(\ker(T)) = 0$

$\Rightarrow$ T injective $\blacksquare$