Polynomials

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coefficients for polynomials are unique

theorem: division algorithm

Suppose $p, s \in \mathcal{P}(\mathbb{F})$ with $s \neq 0$ then

$\exists$ unique $q, r \in \mathcal{P}(\mathbb{F})$ s.t. $p = sq + r$ where $\deg(r) < \deg(s)$

proof: if $\deg(p) < \deg(s)$, then take $q = 0$, $r = p$ ✓

else if $\deg(p) \geq \deg(s)$

define $T: \mathcal{P}_{\deg(p)-\deg(s)}(\mathbb{F}) \times \mathcal{P}_{\deg(s)-1}(\mathbb{F}) \to \mathcal{P}_{\deg(p)}(\mathbb{F})$ by

$$T(q,r) = sq + r$$

T is linear map and if $(q,r) \in \ker(T) \Rightarrow sq + r = 0 \Rightarrow q = r = 0$

otherwise if $sq = -r \neq 0 \Rightarrow \deg(s) + (\deg(p) - \deg(s)) = \deg(s) - 1 \Rightarrow \deg(p) < \deg(s)$ contradiction

$\Rightarrow \ker(T) = \{0\} \Rightarrow T$ is injective (unique)

$\dim(\mathcal{P}_m(\mathbb{F})) = m+1 \Rightarrow \dim(\text{domain}) = (\deg(p) - \deg(s) + 1) + (\deg(s) - 1 + 1)$

$= \deg(p) + 1 = \dim(\mathcal{P}_{\deg(p)}(\mathbb{F})) = \dim(\text{target})$

$= \overbrace{\dim(\cancel{\ker(T)})}^{0} + \dim(\mathrm{Im}(T))$

$\Rightarrow \mathrm{Im}(T) = \mathcal{P}_{\deg(p)}(\mathbb{F}) \Rightarrow T$ surjective (exist)