6.B Orthonormal Bases

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definition: $\{e_i\}$ is orthoromal if $\langle e_i, e_j \rangle = \delta_{ij}$

lemma : if $\{e_i\}$ is orthonormal vectors then

$$\|a_i e_i\|^2 = \sum_i |a_i|^2 \quad \forall a_i \in \mathbb{F}$$

proof : pythagorian theorem

corollery : orthonormal list of vectors is linearly independent

$\Rightarrow$ if length of the list $= \dim(V)$ then it's basis of V

$\Rightarrow v = a_i e_i \Rightarrow \langle v, e_i \rangle = \langle a_j e_j, e_i \rangle = a_j \langle e_j, e_i \rangle = a_j \delta_{ji} = a_i$

theorem: Gram – Schmidt Procedure

given $\{v_j \in V : j \in [\![1, m]\!]\}$ is linearly independent

let $e_1 = \frac{v_1}{\|v_1\|}$ for $j \in [\![2, m]\!]$:

$$e_j = \frac{v_j - \langle v_j, e_i \rangle e_i}{\|v_j - \langle v_j, e_i \rangle e_i\|}, \quad i \in [\![1, j-1]\!]$$

then $\{e_j\}$ is orthonormal s.t. $\mathrm{span}(\{v_j\}) = \mathrm{span}(\{e_j\})$ for $j \in [\![1, m]\!]$

proof : Induction. $\mathrm{span}(e_1) = \mathrm{span}(v_1)$ ✓

assume $1 < j < m$ and $\mathrm{span}(v_1, \cdots, v_{j-1}) = \mathrm{span}(e_1, \cdots, e_{j-1})$

$\Rightarrow v_j \notin \mathrm{span}(e_1, \cdots, e_{j-1}) \Rightarrow v_j - \langle v_j, e_i \rangle e_i \neq 0$

$\Rightarrow$ not dividing by 0 and $\|e_j\| = 1$

let $k \in [\![1, j-1]\!]$ then

$$\langle e_j, e_k \rangle = \frac{\langle v_j, e_k \rangle - \langle v_j, e_i \rangle \langle e_i, \overbrace{e_k}^{\delta_{ik}} \rangle}{\|v_j - \langle v_j, e_i \rangle e_i\|}$$

$$= \frac{\langle v_j, e_k \rangle - \langle v_j, e_k \rangle}{\|v_j - \langle v_j, e_i \rangle e_i\|} = 0$$

$\Rightarrow \{e_1, \cdots, e_j\}$ is orthonormal

from definition of $e_j \Rightarrow v_j \in \mathrm{span}(e_1, \cdots, e_j)$

$\Rightarrow \mathrm{span}(v_1, \cdots, v_j) \subseteq \mathrm{span}(e_1, \cdots, e_j)$

as $\{v_1, \cdots, v_j\}$ linearly independent,

$\Rightarrow \mathrm{span}(v_1, \cdots, v_j) \subseteq \mathrm{span}(e_1, \cdots, e_j)$

definition: Riesz Representation Theorem

Suppose V is finite, $\phi \in \mathcal{L}(V, \mathbb{F})$

Then $\exists$ unique $u \in V$ s.t.

$$\phi(v) = \langle v, u \rangle \quad \forall v \in V \quad \text{where } u = \overline{\phi(e_i)} e_i$$

proof : let $\{e_i\}$ basis of $V \Rightarrow v = \langle v, e_i \rangle e_i$

$\Rightarrow \phi(v) = \langle v, e_i \rangle \phi(e_i)$

$= \langle v, \overline{\phi(e_i)} e_i \rangle$

$\Rightarrow u = \overline{\phi(e_i)} e_i \in V \Rightarrow$ exists

assume $\phi(v) = \langle v, u_1 \rangle = \langle v, u_2 \rangle$

$\Rightarrow 0 = \langle v, u_1 \rangle - \langle v, u_2 \rangle = \langle v, u_1 - u_2 \rangle \ \forall v \in V$

$\Rightarrow u_1 - u_2 = 0 \Rightarrow$ unique