Quiver Gauge Theory Calculations

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1 Hilbert Series of the Coulomb branch 3d $\mathcal{N} = 4$ $U(1)$ gauge theory with n flavours

For the 3d N=4 U(1) gauge theory with n flavours we consider the following quiver with one gauge and one flavour node.

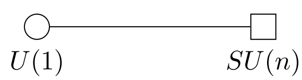


Figure 1: The quiver diagram of U(1) gauge theory with n flavours.

The general unrefined Hilbert series is then given by the monopole formula [1]

$$HS_{unref}(t) = \sum_{m \in \Gamma^*/\mathcal{W}} P(t,m) t^{2\Delta}$$

where P is the classical dressing factor and Δ is R-charge or magnetic charge. As Γ is the weight lattice and $\mathcal{W}$ gives the symmetry of the roots their quotient is the set of integers $\mathbb{Z}$ that is being summed over.
The classical dressing factor can be determined through the product

$$P(t,m) = \prod_{i=1}^{r} \frac{1}{1 - t^{2d_i(m)}}$$

where $r = 1$ is the rank of $\mathfrak{u}(1)$ and $d_i = 1$ is the degree of Casimir invariant of $\mathfrak{u}(1)$. As for each m, $m \in Z(U(\mathfrak{u}(1)))$ and the degree of the polynomial m is 1. Hence the classical dressing factor is

$$P(t,m) = \frac{1}{1-t^2} \ .$$

The R-charge is given by

$$\Delta = \Delta_V + \Delta_H = -\sum_{a \in \Phi_+} |\alpha(m)| + \frac{1}{2} \sum_{i=1}^{n} \sum_{\rho_i \in \mathcal{R}_i} |\rho_i(m)|$$

where n is the number of flavours of the flavour node and $\Phi_+ = \{\alpha(H_i)\}$ is the set of positive roots of $\mathfrak{u}(1)$ s.t. the commutator $[H_i, E_\alpha] = \alpha(H_i)E_\alpha$ with the basis H_i of the Cartan subalgebra of $\mathfrak{u}(1)$ and the extended basis E_α of $\mathfrak{u}(1)$. However for U(1) - as an Abelian group - the whole algebra is commutative which implies that $\Phi_+ = \emptyset$ and the first term of the R-charge vanishes.
As U(1) is one-dimensional, the hypermultiplets $\mathcal{R}_i$ all have weight factor 1 which reduces the R-charge to

$$\Delta = \frac{1}{2}\sum_{i=1}^{n}|m| = \frac{1}{2}n|m| \ .$$

The unrefined Hilbert series can now finally be computed as

$$HS_{unref} = \sum_{m\in\Gamma^*/\mathcal{W}} P(t,m)t^{2\Delta} = \sum_{m\in\mathbb{Z}} \frac{t^{n|m|}}{1-t^2} = 2\sum_{m=1}^{\infty}\frac{(t^n)^m}{1-t^2} + \frac{1}{1-t^2} = \frac{1+t^n}{(1-t^2)(1-t^n)} \ .$$

The Hilbert series can be refined by inserting $z^{J(m)}$ into the sum where $J(m) = \sum_{i=1}^{\mathrm{rank}(U(1))} m_i = m$ as $\mathrm{rank}(U(1)) = 1$ and hence $z^{J(m)} = z^m$ [2]. Therefore the refined Hilbert series is

$$\begin{aligned} HS_{ref} = \sum_{m\in\Gamma^*/\mathcal{W}} P(t,m)z^m t^{2\Delta} = \sum_{m\in\mathbb{Z}} P(t,m)z^m t^{n|m|} &= \frac{1}{1-t^2}\left(1+\frac{zt^n}{1-zt^n}+\frac{z^{-1}t^n}{1-z^{-1}t^n}\right) \\ &= \frac{1-t^{2n}}{(1-t^2)(1-zt^n)(1-z^{-1}t^n)};. \end{aligned}$$

1.1 Moduli Space

For n = 1,2,3 flavours, the $HS_{unref,n}$ is

$$\begin{aligned} &HS_{unref,n=1}(t) = \frac{1}{(1-t)^2} = \sum_{i=0}^{\infty}(i+1)t^i \\ &\implies \mathcal{M}_{\mathcal{C}} = \mathbb{C}^2 \\ &HS_{unref,n=2}(t) = \frac{1+t^2}{(1-t^2)^2} = \sum_{i=0}^{\infty}(2i+1)t^{2i} \\ &\implies \mathcal{M}_{\mathcal{C}} = \mathbb{C}^2/Z_2 \\ &HS_{unref,n=3}(t) = \frac{1+t^3}{(1-t^2)(1-t^3)} = 1+t^2+2t^3+t^4+2t^5+2t^6+2t^7+... \\ &\implies \mathcal{M}_{\mathcal{C}} = \mathbb{C}^2/Z_3 \end{aligned}$$

The inductive ansatz for the moduli space is $\mathbb{C}^2/Z_n$. Its HS matches what we found earlier.

1.2 Global Symmetry

The global symmetry of the Coulomb branch can be read off of the quiver of the theory if the nodes are balanced [3]. In our case it is not balanced unless $n = 2$. When $n = 2$, the quiver nodes without the flavour groups becomes the Dynkin diagram for the Lie algebra of the global symmetry which is $A_1 = \mathfrak{su}(2)$ whose group is $SU(2)$. Otherwise, the global symmetry is hidden as $U(1)^z$ where z is the number of times the $U(1)$ factor appears in the theory. In our case it is $U(1)$.

2 Hilbert Series of the Higgs branch 3d $\mathcal{N} = 4$ $U(1)$ gauge theory with N_f flavours

We now compute the Hilbert Series of the Higgs Branch for the $U(1)$ gauge and $SU(N)$ flavour group with N_f flavours. According to the Molien-Weyl formula the

$$HS_{higgs} = \int d\mu_G g^{\mathcal{F}_b}(t, z)$$

where

$$g^{\mathcal{F}_b} = (1 - t^2)PE([1, 0, ..., 0]zt + [0, ..., 0, 1]\frac{t}{z}) = \frac{1 - t^2}{(1 - tz)^{N_f}(1 - t/z)^{N_f}}$$

and $d\mu_G$ is the Haar Measure of the gauge group

$$\int\limits_G d\mu_G = \frac{1}{(2\pi i)^r} \oint\limits_{|z_1|=1} \frac{dz_1}{z_1} \cdots \oint\limits_{|z_r|=1} \frac{dz_r}{z_r} \prod_{\alpha^+} \left(1 - \prod_{i=1}^{r} z_l^{\alpha_l}\right)$$

where the gauge group $G = U(1)$ and $r = 1$ is the rank of $U(1)$ which is equal to the dimension of the Cartan subalgebra (in this case the whole algebra) [3]. As mentioned above, the set of roots of $\mathfrak{u}(1)$ is the empty set and therefore the product is empty which retrieves the Haar measure to be

$$\frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{dz}{z}$$

Thus, the integral can be written as:

$$HS_{higgs} = \frac{(1 - t^2)}{2\pi i} \oint dz \frac{z^{n-1}}{(1 - tz)^n (t - z)^n}$$

where we have recast N_f to n. To solve this integral, we employ the Residue theorem. Noting the result,

$$\text{res}(p, z_0) = \lim_{z \to z_0} \left[\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} ((z - z_0)^n p(z)) \right]$$

we see that for the pole at $z = t$, if $p(z)$ is our integrand, the $(t - z)^n$ term cancels and we are left with:

$$\text{res}(p, z_0) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}((1 - tz)^{-n} z^{n-1})\Big|_{z=z_0}$$

The $(n - 1)$th derivative is given by the Leibniz product rule as follows:

$$\frac{d^{n-1}}{dz^{n-1}}(f(z)g(z)) = \sum_{k=0}^{n-1} \binom{n-1}{k} f^{(k)}(z) g^{(n-k-1)}(z)$$

with $f(z) = (1 - tz)^{-n}$ and $g(z) = z^{n-1}$. Evaluating the derivative at $z = t$ yields a final result of (with $n = N_f$) :

$$HS_{Higgs} = \frac{\sum_{k=0}^{n-1} \binom{n-1}{k}^2 t^{2k}}{(1 - t^2)^{2(n-1)}}$$

2.1 Moduli Space

To find the moduli space of the Higgs branch we define the superpotential for the quiver $W = \phi UD$ where $\phi_{1\times 1}$ is the loop associated with the gauge node and $D_{N_f \times 1}$ and $U_{1\times N_f}$ map from the gauge to the flavour node and back. Hence for the Higgs branch the superpotential has to satisfy $\frac{\partial W}{\partial \phi} = UD = 0$ We define a matrix $M = DU$ such that $M^2 = D(UD)U = 0$ which characterises the moduli space as M has to be traceless and minimally nilpotent. The moduli space is hence given by

$$\begin{aligned} \mathcal{M}_{\mathcal{H}} &= \{M_{N_f \times N_f} | \text{tr} M = 0, M^2 = 0, \text{rank } M \leq 1\} \\ &= \overline{\text{min } SL(N_f)} \; . \end{aligned}$$

2.2 Global Symmetry

The global symmetry of the Higgs branch can be read off from the square nodes of the quiver of the theory. Therefore it is $SU(N_f)$ for N_f flavours. Note that this is also consistent with the mirror symmetry. When $N_f = 2$, the theory is self-dual, meaning the electric quiver is identical to the magnetic quiver. Therefore the global symmetry is $SU(2)$ which is the same result as the Coulomb branch.

3 Quivers with moduli space rank ≤ 2

3.1 Higgs branch

As we already found the matrices of the moduli space for Higgs branch of the U(1) gauge theory with N_f flavours to have rank ≤ 1. We can simply extend

this by considering following theorem:
If $A = BC$ then rank $A = min(\text{rank } B, \text{rank } C)$.
Hence we know that the gauge group has to be assigned a 2×2 matrix. The simplest quiver to achieve this is shown in Fig(2).

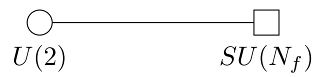


Figure 2: The quiver diagram of U(2) gauge theory with N_f flavours.

3.2 Coulomb branch

To determine the quiver that has a Coulomb branch with moduli space with matrices of rank ≤ 2 we make use of the property that the Higgs branch of the electric quiver is the Coulomb branch of the corresponding magnetic quiver. We hence move around the branes and change the plane of projection to obtain the following quiver in Fig(3).

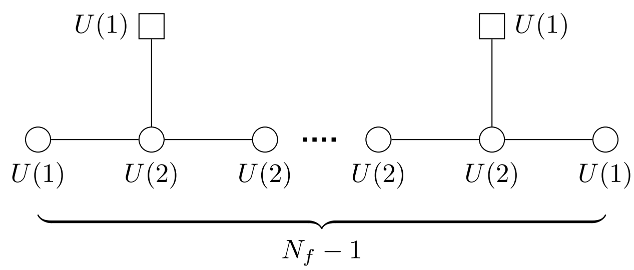


Figure 3: The magnetic quiver of U(2) gauge theory with N_f flavours. There are $N_f - 1$ circle nodes.

3.3 Hilbert Series of the Moduli Space

3.3.1 Hilbert series of the Coulomb Branch

To find the Hilbert series, we first note that the rank of the gauge group U(2) is 2, and the Langlands dual group is itself. The magnetic fluxes are hence specified by: $m_1 \geq m_2 \geq -\infty$. To find the positive roots of $\mathfrak{u}(2)$, we first decompose it: $\mathfrak{u}(2) = \mathfrak{u}(1) \bigoplus \mathfrak{su}(2)$. Since U(1) is Abelian, the set of the positive root is that of $\mathfrak{su}(2)$, given by $\phi^+ = \{e_1 - e_2\}$ in the Cartesian basis. The degrees of the Casimirs are 1 and 2, giving a classical dressing factor of:

$$P(\vec{m}) = \frac{1}{1-t}\frac{1}{1-t^2} \tag{1}$$

The vector and hyper multiplets are given by:

$$\Delta_{vector} = -\left|\sum_{k=1}^{r}(r+1-2k)m_k\right| = -|m_1 - m_2| \tag{2}$$

$$\Delta_{hyper} = \frac{N_f}{2}\sum_i\sum_j |w_i(\vec{a}) - w_j(\vec{a})|, \tag{3}$$

where $w_i(\vec{a})$ is the i^{th} component of the tuple of GNO fluxes, denoted by $w(\vec{a})$. The GNO fluxes for flavor nodes are all 0, so $\Delta_{hyper} = \Delta_{hyper}(\vec{a}, \vec{0})$.

Hence:

$$\Delta_{hyper} = \frac{N_f}{2}(|m_1| + |m_2|) - |m_1 - m_2| \tag{4}$$

.

From this, we get the Hilbert Series:

$$HS_{U(2),N_f} = \frac{(1-t^{N_f})(1-t^{N_f-1})}{(1-t^{N_f/2-1})^2(1-t^{N_f/2})^2(1-t)(1-t^2)} \tag{5}$$

3.3.2 Hilbert series of the Higgs Branch

We use the formula:

$$HS_{\mathcal{H}}(t) = \int_G d\mu_G \frac{PE\left[\sum_{i=1}^{2N_{hyp}} \chi_{R_i}(\omega)\chi'_{R'_i}(\omega')t\right]}{PE\left[\sum_{i=1}^{N_r} \chi_{R''_i}(\omega)t^{d_i}\right]} \tag{6}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_r)$ are dummy variables for integration with $r = \text{rank}(G)$. $\chi_R(w)$ are the characters of the representation R of the group. As each edge in a quiver represents a hypermultiplet transforming in the fundamental and antifundamental representation of the groups of the neighbouring nodes G and G' respectively, the numerator is the sum of the characters of the representation under which i^{th} hypermultiplet is charged i.e. R_i is the fundamental representation of a node G and R'_i is the antifundamental representation of the other node G'. The denominator is a prefactor encoding the relations obtained from the F-term conditions given by the superpotential where R''_i is the representation of the i^{th} relation and d_i is the degree of the relation. Typically in $\mathcal{N} = 4$ theories, R''_i will be the adjoint and the relation will be at quadratic: $d_i = 2$ [10].

The character of the fundamental representation of U(2) is given by:

$$\chi_{R_i}(z_1, z_2) = z_1 + z_2 \tag{7}$$

The antifundamental representation is calculated by taking the complex conjugate of the fugacities. Hence the prefactor is:

$$PE\left[(z1+z2)\left(\frac{1}{z1}+\frac{1}{z2}\right)t^2\right]^{-1} = \left(1-t^2\right)^2\left(-\frac{t^2 z_2}{z_1}+1\right)\left(-\frac{t^2 z_1}{z_2}+1\right) \tag{8}$$

The Haar measure is given by:

$$\int d\mu_{U(2)}(z_1, z_2) = \frac{1}{2} \int_{|z_1|=1} \frac{dz_1}{z_1} \int_{|z_2|=1} \frac{dz_2}{z_2} \left(\frac{1}{z_1} - \frac{1}{z_2} \right) (z_1 - z_2) \tag{9}$$

The rest of the integrand is given by computing the character of the fundamental representation of U(2) and the antifundamental representation of SU(N) (and vice-versa), and taking the plethystic exponential, as such [14]:

$$PE \left[(z_1 + z_2) [0, \ldots, 0, 1]_y + \left(\frac{1}{z_1} + \frac{1}{z_2} \right) [1, 0, \ldots, 0]_y \right] = \prod_{i=0}^{N} \prod_{a=1}^{2} \left(1 - t z_a \frac{y_i}{y_{i+1}} \right) \left(1 - t z_a^{-1} \frac{y_{i+1}}{y_i} \right), \tag{10}$$

where $[0, \ldots, 0, 1]_y$ is the character of the fundamental representation of SU(N).

Hence the Hilbert Series is given by:

$$g_{U(2),N} = \int d\mu_{U(2)}(z_1, z_2) = \frac{1}{2} \int_{|z_1|=1} \frac{dz_1}{z_1} \int_{|z_2|=1} \frac{dz_2}{z_2} \left(\frac{1}{z_1} - \frac{1}{z_2} \right) (z_1 - z_2) \frac{\left(1-t^2\right)^2 \left(-\frac{t^2 z_2}{z_1} + 1\right) \left(-\frac{t^2 z_1}{z_2} + 1 \right.}{\prod_{i=0}^{N} \prod_{a=1}^{2} \left(1 - t z_a \frac{y_i}{y_{i+1}} \right) \left(1 - t z_a^{-1} \right.} \tag{11}$$

Unrefining by setting $y_i = 1, 0 \leq i \leq N$, we obtain:

$$g_{U(2),N} = \int d\mu_{U(2)}(z_1, z_2) = \frac{1}{2} \int_{|z_1|=1} \frac{dz_1}{z_1} \int_{|z_2|=1} \frac{dz_2}{z_2} \left(\frac{1}{z_1} - \frac{1}{z_2} \right) (z_1 - z_2) \frac{\left(1-t^2\right)^2 \left(-\frac{t^2 z_2}{z_1} + 1\right) \left(-\frac{t^2 z_1}{z_2} + \right.}{\left[(1 - t z_1) \left(1 - t z_1^{-1}\right) (1 - t z_2) \left(1 - t \right. \right.} \tag{12}$$

One can compute the integrals using the residue theorem. There is no clear generalisation of the Hilbert Series for an arbitrary flavour SU(N). Instead, we compute for 3, 4 and 5 flavours, and make an inductive ansatz on the general form that the Hilbert Series should adopt:

$$g_{U(2),SU(3)} = \frac{1 + 4t^2 + t^4}{(1 - t^2)^4}$$

$$g_{U(2),SU(4)} = \frac{1 + 7t^2 + 12t^4 + 7t^6 + t^8}{(1 - t^2)^8}$$

$$g_{U(2),SU(5)} = \frac{1 + 12t^2 + 53t^4 + 88t^6 + 53t^8 + 12t^{10} + t^{12}}{(1 - t^2)^{12}}$$

We note from the quiver that the complex dimension of the Higgs branch of our theory is 4N - 8, which is why we computed the Hilbert Series for $N \geq 3$. Firstly, the numerator is a palindromic polynomial of the form $P_{2N-4}(t^2)$ of degree equal to the quaternionic dimension 2N-4 of the Higgs Branch. As expected, the order of the pole gives the complex dimension of the Higgs Branch. In conclusion, the Hilbert Series can generally be written as:

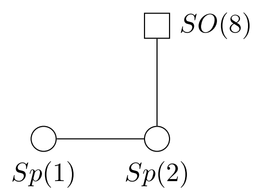
$$g_{U(2),SU(N)} = \frac{P_{2N-4}(t^2)}{(1-t^2)^{4N-8}} \tag{14}$$

3.4 Dimension of the Moduli Space

For the quiver depicted in Fig(2), the dimension of the Coulomb branch is $\dim_{\mathbb{H}}(\mathcal{M}_{\mathcal{C}}) = 2$ which is the sum of the gauge node numbers. This can be checked again by the Hilbert series in Sec(3.3.1) where the order of the pole at $t = 1$ is $\dim_{\mathbb{C}}(\mathcal{M}_{\mathcal{C}}) = 4$. Furthermore, the Higgs branch of the same quiver is the Coulomb branch of the magnetic quiver shown in Fig(3). Summing over the gauge node numbers of the magnetic quiver, we obtain $\dim_{\mathbb{H}}(\mathcal{M}_{\mathcal{H}}) = 2 + 2(N_f - 3) = 2N_f - 4$.

4 The Magnetic Quiver for Sp(1) — Sp(2) with Framing SO(8) on Sp(2)

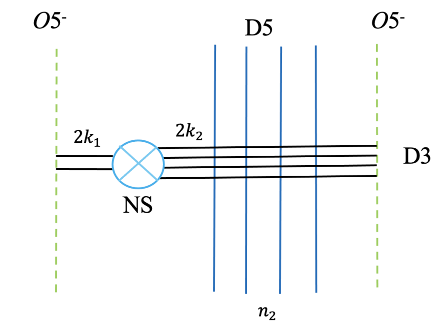
Given following quiver,

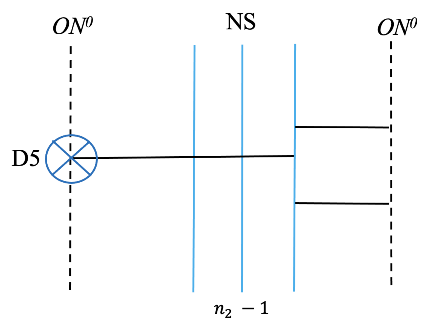


we want to find the magnetic phase using the methods for orthosymplectic quivers outlined in [4], [5]. To do so, we first draw the corresponding unitary brane system and add $O3$ planes to obtain the orthosymplectic brane system. As described in [5], $O3$ planes change type as they cross NS5 or D5 branes. However, the symplectic gauge node corresponds to an $O3^+$ plane. Therefore this method cannot be used as it does not allow two nodes of the same gauge group type next to each other. We instead constructed the brane system with the method outlined in [6]. This method has two gauge nodes with $Sp(k_1)$ and $Sp(k_2)$ gauge groups and n_1 and n_2 flavours respectively. In our case therefore $k_1 = 1$, $k_2 = 2$ and $n_1 = 0$, $n_2 = 4$. This gives us the following brane system

Each $Sp(k_i)$ node corresponds to $2k_i$ D5 branes attached to an $O5^-$ plane on one side an ending on the NS brane on the other. As we have $k_1 = 1$ and $k_2 = 2$, we have 2 and 4 D3 branes spanned between the $O5^-$ plane and the NS brane on each side. The $n_2 = 4$ flavours on the $Sp(2)$ gauge node are encoded by 4 D5 branes, between the NS5 brane and the $O5^-$ plane on the right and crossing the D5 branes. On the left side there are no D5 branes as $n_1 = 0$. After transitions that take into account the conservation of the linking number and S-duality, using the rules in [7], we get the following mirror theory:

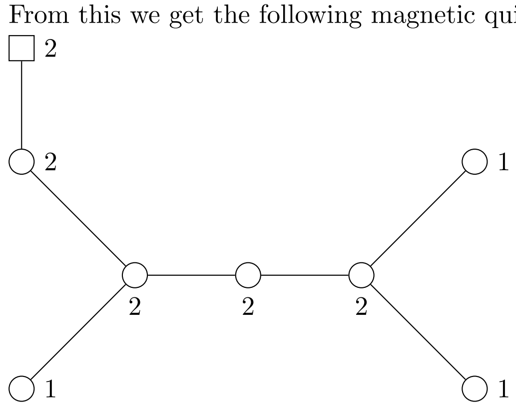
However, our theory differs from the ones given by [6]. In our theory, Sp(1) is flavourless and $k_1 < k2$, which creates complications as it pertains to translating





the brane into the quiver. However, from [6], the quiver must have unitary gauge groups and an SO(8) global symmetry with potential enhancements. Furthermore, we have to make sure the dimensions of the mirror theory are consistent with those of the original theory.

From this we get the following magnetic quiver:



Other configurations for the magnetic quiver either do not match the dimensions, or do not have an SO(8) global symmetry linked to the a D4 Dynkin diagram.

4.1 Dimension of the Moduli Space

Using the same method we used in Sec(3.4), for Coulomb branch, we have $Sp(1) \times Sp(2)$ as the gauge groups. Therefore $\dim_{\mathbb{H}}(\mathcal{M}_{\mathcal{C}}) = 1 + 2 = 3$. and for Higgs banch, summing over the range of gauge groups of the magnetic quiver, we obtain $\dim_{\mathbb{H}}(\mathcal{M}_{\mathcal{H}}) = 17$.

4.2 Hilbert Series of Coulomb Branch

We could use the monopole formula to compute the Hilbert Series. However, in the theory, conventionally labelled as bad since the number of flavours is $N_f = 2k$, there exists at least one monopole operator that violates the unitary bound [11], and the Hilbert series is ill-defined through the monopole formula. As such, we instead find the Hilbert Series of the Higgs Branch of the magnetic phase.

We use the formula [10]:

$$HS(t) = \int_G d\mu_G \,\mathrm{Pfc}(w,t)\mathrm{PE}\left[\sum_{i=1}^{7} \chi^{G_i}_{\text{fund}}(w)\,\chi^{G'_i}_{\text{anti}}(w')\,t + \chi^{G_i}_{\text{anti}}(w)\,\chi^{G'_i}_{\text{fund}}(w')\,t\right], \tag{15}$$

where χ represents the character of the fundamental / antifundamental representation of the ith hypermultiplet connecting the G_i and $G'i$ groups.

The antifundamental representations can be found by taking the complex conjugate of the fugacities in the fundamental representation : $z_i \mapsto \frac{1}{z_i}$.

The character of the fundamental representation of U(N) is given by:

$$\chi_{U(N)} = \sum_{i=1}^{N} z_i \tag{16}$$

The character of the fundamental representation of SU(2) is given by:

$$\chi_{SU(2)}(z) = z + \frac{1}{z} \tag{17}$$

The prefactor Pfc is given by:

$$\mathrm{Pfc}(w,t) = \mathrm{PE}\left[\sum_{i=1}^{N_r} \chi_{R''_i}(w)t^2\right]^{-1}, \tag{18}$$

corresponding to the character of the adjoint representation of the gauge group, and we sum over the number of nodes.

Assigning fugacities as,

G	fugacity
$SU(2)$	x
$U(1)$	w_1
$U(1)$	w_2
$U(3)$	$w_3 \equiv (z_1, z_2, z_3)$
$U(4)$	$w_4 \equiv (z_4, z_5, z_6, z_7)$
$U(4)$	$w_5 \equiv (z_8, z_9, z_{10}, z_{11})$
$U(2)$	$w_6 \equiv (z_{12}, z_{13})$
$U(2)$	$w_7 \equiv (z_{14}, z_{15})$

For our quiver, the prefactor given by:

$$\begin{aligned} Pfc = PE[(&\chi_{adj,U(1)}(w_1) + \chi_{adj,U(1)}(w_2) + \chi_{adj,U(3)}(w_3) + \chi_{adj,U(4)}(w_4) \\ &+\chi_{adj,U(4)}(w_5) + \chi_{adj,U(2)}(w_6) + \chi_{adj,U(2)}(w_7))t^2], \end{aligned} \tag{19}$$

where the character of the adjoint representation of U(N) is given by:

$$\chi_{\text{adj}}(z_1, \dots, z_N) = \sum_{i,j=1}^{N} z_i z_j^{-1} \tag{20}$$

For our quiver, the remaining term is :

$$\begin{aligned} PE\left[\sum_{i=1}^{2*7} \chi_{R_i}(w)\, \chi_{R_i'}(w')\, t\right] = PE\,\big[(&\chi_{f,U(1)}(w_1)\chi_{af,SU(2)}(x) + \chi_{f,U(1)}(w_1)\chi_{af,U(3)}(w_3) \\ &+\chi_{f,U(1)}(w_2)\chi_{af,U(3)}(w_3) + \chi_{f,U(3)}(w_3)\chi_{af,U(4)}(w_4) + \chi_{f,U(4)}(w_5)\chi_{af,U(4)}(w_4)+ \\ &\chi_{f,U(4)}(w_5)\chi_{af,U(2)}(w_6) + \chi_{f,U(4)}(w_5)\chi_{af,U(2)}(w_7) + (f \leftrightarrow af))t \end{aligned} \tag{21}$$

where f, a denote fundamental and antifundamental respectively. The explicit computations were omitted from the paper.

To find the Haar Measure, we first find the positive roots:

$$\Delta^+ = \{\lambda_j - \lambda_k : 1 \leq j < k \leq N\}. \tag{22}$$

After representing them in the Cartesian basis for each gauge group in our theory, we then use the Haar measure formula and substitute it in the Molien-Weyl integral.

4.3 Higgs Branch of Magnetic quiver

We analyze a quiver gauge theory consisting of four U(2) gauge nodes and three U(1) gauge nodes. The integrand of the Hilbert series captures contributions from vector multiplets in the adjoint representations and hypermultiplets in bifundamental representations.

The character expressions and their contributions to the integrand are as follows:

- **Adjoint Representation of** U(2)**:**
  The character of the adjoint representation is

$$\chi_{\text{adj}}^{\text{U}(2)}(z_1, z_2) = \frac{z_1}{z_2} + 1 + \frac{z_2}{z_1},$$

  and contributes the prefactor

$$\frac{1}{(1-t^2)^2 \left(1 - \frac{t^2 z_2}{z_1}\right)\left(1 - \frac{t^2 z_1}{z_2}\right)}.$$

- **Adjoint Representation of** U(1)**:**
  The adjoint of U(1) is trivial. The only contribution arises from the F-term relation, yielding

$$(1-t^2)^3.$$

- **Fundamental Representation of** U(2)**:**
  The character is given by

$$\chi_{\text{fund}}^{\text{U}(2)}(z_1, z_2) = z_1 + z_2,$$

  and the antifundamental representation corresponds to $(1/z_1 + 1/z_2)$.

- **Bifundamental Hypermultiplets:**
  - Between U(2) nodes labeled by i and j, the bifundamental representation contributes

$$\frac{1}{(1-t^2)^2 \left(1 - \frac{t^2 z_2^{(i)}}{z_1^{(i)}}\right)\left(1 - \frac{t^2 z_1^{(i)}}{z_2^{(i)}}\right)}.$$

  - Between a U(2) node (fugacities z_1, z_2) and a U(1) node (fugacity u), the bifundamental interaction contributes

$$\frac{1}{(1 - \frac{t^2 z_1}{u})(1 - \frac{t^2 z_2}{u})(1 - \frac{t^2 u}{z_1})(1 - \frac{t^2 u}{z_2})}.$$

- **Haar Measures:**
  - For each U(2) node:

$$\int \frac{(z_1 - z_2)(1/z_1 - 1/z_2)}{z_1 z_2} \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2}.$$

  - For each U(1) node:

$$\int \frac{du}{2\pi i u}.$$

The full integrand is assembled by taking the product over all such contributions for the nodes and edges of the quiver. The resulting multi-variable contour integral is then evaluated over the unit circles of the complex fugacities using the method of residues. This yields the Hilbert series:

$$\mathrm{HS}(t) = \frac{16\left(2t^4 - t^2 + 1\right)\left(t^8 - t^6 + 3t^4 - 2t^2 + 1\right)\left(t^8 + t^6 + 2t^4 - t^2 + 1\right)\left(t^{12} + 3t^{10} + 11t^8 + 14t^6 + 11t^4 + 3t\right.}{\left(t^2 - 1\right)^6\left(t^2 + 1\right)^9\left(t^4 + 1\right)\left(t^4 + t^2 + 1\right)} \tag{23}$$

4.4 Higgs Branch

4.4.1 Moduli Space

We can write $3d$ $\mathcal{N} = 4$ quiver in $4d$ $\mathcal{N} = 2$ formalism as both have 8 supercharges under the following rules. Vectormultiplets decompose into a vectormultiplet and a chiral multiplet, ϕ in the **adjoint** representation, whereas hypermultiplets decompose into a chiral and anti-chiral multiplet Q_1 and $\tilde{Q_1}$ but for orthosymplectic quivers, an edge connecting Sp gourp and SO group denotes a half-hypermultiplet. Consequently, in going to the $\mathcal{N} = 2$ quiver, the half-hypermultiplet edge does not become a bidirectional, but a chiral multiplet in the **bifundamental** representation, which we call Q. [**?**] The quiver in Fig(4) is obtained accordingly from which we can read off the superpotential, W as the sum of all the closed loops given by

$$W = \mathrm{tr}(Q_1\phi_1\tilde{Q_1} - Q_1\phi_2\tilde{Q_1} + Q_2\phi_2 Q_2). \tag{24}$$

Hence the relavent F-term conditions are

$$\frac{\partial W}{\partial \phi_1} = Q_1\tilde{Q_1} = 0, \tag{25}$$

$$\frac{\partial W}{\partial \phi_2} = Q_2Q_2 - Q_1\tilde{Q_1} = Q_2Q_2 = 0. \tag{26}$$

As Higgs branches occur when the scalars coming from the $\mathcal{N} = 4$ vector multiplet vanish whilst the ones from the hypermultiplet take non-zero values, the F-flat space $\mathcal{F}^\flat$ is the space of solutions

$$\mathcal{F}^\flat = \{\phi_1 = \phi_2 = 0, Q_1 \neq 0, Q_2 \neq 0 \quad | \quad Q_1\tilde{Q_1} = 0, \quad Q_2Q_2 = 0\}. \tag{27}$$

Note that there are 2 relations, both in quadratic order, hence $d_i = 2$ for the t^{d_i} in the prefactor of the Higgs branch formula and they transform as the adjoint of each gauge node.

Before computing the Hilbert series, we assign the fugacities to each group as follows.

G	fugacity	χ(fundamental)	χ(adjoint)
$Sp(1)$	w_1	$[1]_{Sp(1)}$	$[2]_{Sp(1)}$
$Sp(2)$	w_2, w_3	$[1,0]_{Sp(2)}$	$[1,2]_{Sp(2)}$
$SO(8)$	$x \equiv (x_1, x_2, x_3, x_4)$	$[1,0,0,0]_{SO(8)}$	$n.a.$

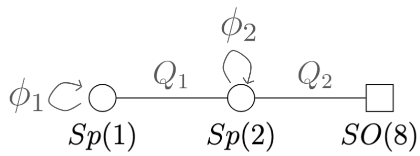


Figure 4: The orthosympletic quiver written in $\mathcal{N}=2$ formalism.

The characters can be computed using the fact that $\mathfrak{sp}(1) \cong \mathfrak{su}(2) \cong \mathfrak{so}(3)$, $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$ and the formula,

$$\chi^{SO(2N)}_{fund}(x_1, \cdots, x_N) = \sum_i^N (x_i + x_i^{-1}) = \chi^{SO(2N+1)}_{fund} - 1 \tag{28}$$

$$R_{\text{fund}} \otimes R_{\text{anti-fund}} = R_{\text{adjoint}} \oplus R_{\text{trivial}} \tag{29}$$

and lastly, the character of the anti-fundamental representation is obtained by making the substitution $x_i \leftrightarrow x_i^{-1}$ from the fundamental representation [13].

The generating function $g^{\mathcal{F}^\flat}$ is then given by the expression,

$$\begin{aligned} g^{\mathcal{F}^\flat} &= \frac{PE\left[\chi^{SO(8)}_{\text{fund}}(x)\chi^{Sp(2)}_{\text{fund}}(w_2,w_3)t + \chi^{Sp(1)}_{\text{fund}}(w_1)\chi^{Sp(2)}_{\text{anti-fund}}(w_2,w_3)t + \chi^{Sp(1)}_{\text{anti-fund}}(w_1)\chi^{Sp(2)}_{\text{fund}}(w_2,w_3)t\right]}{PE\left[\chi^{Sp(1)}_{\text{adj}}(w_1)t^2 + \chi^{Sp(2)}_{\text{adj}}(w_2,w_3)t^2\right]} \\ &= \frac{PE\left[\{2(w_1+w_1^{-1}) + (x_1+x_1^{-1}+x_2+x_2^{-1}+x_3+x_3^{-1}+x_4+x_4^{-1})\}(w_2+w_2^{-1}+w_3+w_3^{-1}+1)t\right]}{PE\left[(w_1^2+1+w_1^{-2})t^2 + (w_2+w_2^{-1}+w_3+w_3^{-1}+w_2w_3+w_2^{-1}w_3^{-1}+w_2w_3^{-1}w_2^{-1}w_3)t^2\right]} \end{aligned} \tag{30}$$

and the Haar measure of the gauge group of the theory $G = Sp(1) \times Sp(2)$ is [12]:

$$\int_G d\mu_G = \frac{1}{(2\pi i)^3} \oint_{|w_1|=1} \frac{dw_1}{w_1}(1-w_1^2) \oint_{|w_2|=1} \frac{dw_2}{w_2} \oint_{|w_3|=1} \frac{dw_3}{w_3}(1-w_2^2)(1-w_3)(1-w_2^2w_3^{-1})(1-w_3^2w_2^{-1}). \tag{31}$$

Projecting the $g^{\mathcal{F}^\flat}$ onto the space of gauge invariant operators i.e. contour integrating by the Haar measure in (31) gives the unrefined Hilbert series of the Higgs branch of the quiver if we substitute $x_i = 1\ \forall i$ which can be evaluated using the residue theorem.

The integrand is

$$\frac{}{8\pi^3 w_1 w_2 w_3 \left(-\frac{t}{w_1}+1\right)^2 (-tw_1+1)^2 \left(-\frac{t}{x_1}+1\right)(-tx_1+1)\left(-\frac{t}{x_2}+1\right)(-tx_2+1)\left(-\frac{t}{x_3}+1\right)(-tx_3+1)\left(-\frac{t}{x_4}\right.} \tag{32}$$

The refined Hilbert series is

$$\mathrm{HS}(t,x) = \frac{2t^{22}\left(t^4+t^2+1\right)x_1{}^5x_2{}^5x_3{}^5x}{\left(t^2-1\right)\left(t-x_1\right)^3\left(t^3-x_1\right)^2\left(tx_1-1\right)^3\left(t^3x_1-1\right)^2\left(t-x_2\right)^3\left(t^3-x_2\right)^2\left(tx_2-1\right)^3\left(t^3x_2-1\right)^2\left(t-x_3\right)^3} \tag{33}$$

and the unrefined Hilbert series is

$$\mathrm{HS}(t) = \frac{Q(t)}{512(t-1)^{40}(t+1)^{34}\left(t^2+1\right)^{17}\left(t^2-t+1\right)^7\left(t^2+t+1\right)^{15}}$$

(34)

where $Q(t) = (t+1)^{19}\left(t^2-t+1\right)^8\left(t^2+t+1\right)^{15}\left(-4t^{30}-7t^{29}-28t^{28}-15t^{27}-104t^{26}-42t^{25}-328t^{24}+67t^{2}\right.$
$1)^{20}+1024t^8(t+1)^{19}\left(t^2+1\right)^{17}\left(t^2-t+1\right)^7\left(t^{33}+22t^{32}+207t^{31}+1325t^{30}+6417t^{29}+25077t^{28}+82027t^{27}+\right.$
$512t^{14}\left(t^2+t+1\right)^8\left(25740t^{72}+91080t^{71}+823086t^{70}+2480500t^{69}+12613667t^{68}+33432866t^{67}+124695711t\right.$
$\left(t^2-t+1\right)^7\left(4t^{101}+571t^{100}+5019t^{99}+104630t^{98}+972084t^{97}+8904800t^{96}+63806894t^{95}+419232315t^{94}+\right.$

The complex dimension needs to be 22. So it does not match. For the relation $Q_1\tilde{Q}_1 = 0$ to have trivial representation $[0]$ of $Sp(1)$ whose character is just 1, then

$$g^{\mathcal{F}^\flat} = \frac{PE\left[\chi^{SO(8)}_{\text{fund}}(x)\chi^{Sp(2)}_{\text{fund}}(w_2,w_3)t+\chi^{Sp(1)}_{\text{fund}}(w_1)\chi^{Sp(2)}_{\text{anti-fund}}(w_2,w_3)t+\chi^{Sp(1)}_{\text{anti-fund}}(w_1)\chi^{Sp(2)}_{\text{fund}}(w_2,w_3)t\right]}{PE\left[\chi^{Sp(1)}_{\text{trivial}}(w_1)t^2+\chi^{Sp(2)}_{\text{adj}}(w_2,w_3)t^2\right]}$$
$$= \frac{PE\left[\{2(w_1+w_1^{-1})+(x_1+x_1^{-1}+x_2+x_2^{-1}+x_3+x_3^{-1}+x_4+x_4^{-1})\}(w_2+w_2^{-1}+w_3+w_3^{-1}+1)t\right]}{PE\left[t^2+(w_2+w_2^{-1}+w_3+w_3^{-1}+w_2w_3+w_2^{-1}w_3^{-1}+w_2w_3^{-1}w_2^{-1}w_3)t^2\right]} \tag{35}$$

Unrefined Hilbert series is

$$HS(t) = \frac{Q(t)}{512(t-1)^{40}(t+1)^{34}\left(t^2+1\right)^{17}\left(t^2-t+1\right)^7\left(t^2+t+1\right)^{15}} \tag{36}$$

where $Q(t) = (t+1)^{19}\left(t^2-t+1\right)^8\left(t^2+t+1\right)^{15}\left(-4t^{30}-7t^{29}-28t^{28}-15t^{27}-104t^{26}-42t^{25}-328t^{24}+67t^{2}\right.$
$1)^{20}+1024t^8(t+1)^{19}\left(t^2+1\right)^{17}\left(t^2-t+1\right)^7\left(t^{33}+22t^{32}+207t^{31}+1325t^{30}+6417t^{29}+25077t^{28}+82027t^{27}+\right.$
$512t^{14}\left(t^2+t+1\right)^8\left(25740t^{72}+91080t^{71}+823086t^{70}+2480500t^{69}+12613667t^{68}+33432866t^{67}+124695711t\right.$
$\left(t^2-t+1\right)^7\left(4t^{101}+571t^{100}+5019t^{99}+104630t^{98}+972084t^{97}+8904800t^{96}+63806894t^{95}+419232315t^{94}+\right.$

But still the complex dimension is not 22 but 40. So it does not match. Another guess: the relations of both $Sp(1)$ and $Sp(2)$ gauge groups transform as trivial representation i.e. $[0]_{Sp(1)}$ and $[0,0]_{Sp(2)}$ whose characters are just 1, then

$$g^{\mathcal{F}^\flat} = \frac{PE\left[\chi^{SO(8)}_{\text{fund}}(x)\chi^{Sp(2)}_{\text{fund}}(w_2,w_3)t + \chi^{Sp(1)}_{\text{fund}}(w_1)\chi^{Sp(2)}_{\text{anti-fund}}(w_2,w_3)t + \chi^{Sp(1)}_{\text{anti-fund}}(w_1)\chi^{Sp(2)}_{\text{fund}}(w_2,w_3)t\right]}{PE\left[\chi^{Sp(1)}_{\text{trivial}}(w_1)t^2 + \chi^{Sp(2)}_{\text{trivial}}(w_2,w_3)t^2\right]}$$
$$= \frac{PE\left[\{2(w_1+w_1^{-1}) + (x_1+x_1^{-1}+x_2+x_2^{-1}+x_3+x_3^{-1}+x_4+x_4^{-1})\}(w_2+w_2^{-1}+w_3+w_3^{-1}+1)t\right]}{PE\left[t^2+t^2\right]} \tag{37}$$

Unrefined Hilbert series is

$$HS(t) = \frac{Q(t)}{512(t-1)^{40}(t+1)^{34}\left(t^2+1\right)^{17}\left(t^2-t+1\right)^{7}\left(t^2+t+1\right)^{15}} \tag{38}$$

where $Q(t) = (t{+}1)^{19}\left(t^2-t+1\right)^8\left(t^2+t+1\right)^{15}\left(-4t^{30}-7t^{29}-28t^{28}-15t^{27}-104t^{26}-42t^{25}-328t^{24}+67t^{2}\right.$
$1)^{20}{+}1024t^8(t{+}1)^{19}\left(t^2+1\right)^{17}\left(t^2-t+1\right)^7\left(t^{33}+22t^{32}+207t^{31}+1325t^{30}+6417t^{29}+25077t^{28}+82027t^{27}+\right.$
$512t^{14}\left(t^2+t+1\right)^8\left(25740t^{72}+91080t^{71}+823086t^{70}+2480500t^{69}+12613667t^{68}+33432866t^{67}+124695711t\right.$
$\left(t^2-t+1\right)^7\left(4t^{101}+571t^{100}+5019t^{99}+104630t^{98}+972084t^{97}+8904800t^{96}+63806894t^{95}+419232315t^{94}+\right.$

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