1.C Subspaces

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* V is vector space over $\mathbb{F}$.
  - $S1: 0 \in U$
  - $S2: u_1, u_2 \in U \Rightarrow u_1+u_2 \in U$
  - $S3: u \in U$ and $\lambda \in \mathbb{F} \Rightarrow \lambda u \in U$

Sum of Subspaces

union of subspaese are not subspace in general

definition Sum of subsets:

let $U_1, \cdots, U_m \subset V$ sum of $U_1, \cdots, U_m$ is set of all possible sums of elements

$$U_1 + \cdots + U_m := \{u_1 + \cdots u_m : u_1 \in U_1, \cdots, u_m \in U_m\}$$

elements of $U_1 + \cdots + U_m$ is denoted $u_1 + \cdots + u_m$

Direct Sums

definition Diret sum:

if each element of $U_1 + \cdots + U_m$ can be written in olny one way as sum of $u_1 + \cdots + u_m$ where $u_j \in U_j$

example: $U < \mathbb{F}^3$ where $u_3 = 0$, $W < \mathbb{F}^3$ where $w_1 = w_2 = 0$

condition for vector space: let U_i is subspace of V

$U_1 \cap \cdots \cap U_m = \{0\}$ where 0 is identity $\Leftrightarrow U_1 + \cdots + U_m = U_1 \oplus \cdots \oplus U_m$

proof: $(\Rightarrow)$ let $U+W$ is direct sum $\Rightarrow 0 \in U+W$

$v \in U \cap W \Rightarrow 0 = v + (-v)$ where $v \in U$, $-v \in W$

unique representation of $0 \Rightarrow v = 0$

$(\Leftarrow)$ $U \cap W = \{0\}$ let $u \in U$, $w \in W$ and $u + w = 0$

$u + w = 0 \Rightarrow u = -w \in W \Rightarrow u \in U \cap W \Rightarrow u = 0 = w$ ☒

Exercises 1.C in Linear Algebra Done Right

1. is following set subspace of $\mathbb{F}^3$?

a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

algebraic way

$S1: (0,0,0) = 0$

$S2:$ $a_1 + 2a_2 + 3a_3 = 0$ and $b_1 + 2b_2 + 3b_3 = 0$

$\Rightarrow (a_1 + b_1) + 2(a_2 + b_2) + 3(a_3 + b_3) = 0 + 0 = 0$ kind of like homogeneous condition

$\Rightarrow a + b \in U$

$S3:$ $\lambda x_1 + 2\lambda x_2 + 3\lambda x_3 = \lambda 0 = 0$

$\Rightarrow \lambda x \in U$

geometric way

equation of the plane is $\underline{r}\cdot\underline{\hat{n}} = d$

$\Rightarrow$ the set is a plane with normal $(1,2,3)\cdot\frac{1}{\sqrt{1^2+2^2+3^2}}$ whose shortest distance from $(0,0,0)$ is 0 i.e. $(0,0,0) \in U$

plane $\Rightarrow$ S2, S3 origin $\Rightarrow$ S1

3. $\{ f: (-4,4) \to \mathbb{R}$ s.t. $f'(-1) = 3f(2) \} < \mathbb{R}^{(-4,4)}$?

S1: $f(x) = 0 \in U$

S2: if $f, g \in U \Rightarrow f'(-1) = 3f(2)$ and $g'(-1) = 3g(2)$

$f'(-1) + g'(-1) = 3(f(2) + g(2)) = (f(x)+g(x))'|_{x=-1}$

S3: if $f \in U$ and $\lambda \in \mathbb{F} \Rightarrow f'(-1) = 3f(2) \Rightarrow \lambda f'(-1) = 3\lambda f(2) = (\lambda f)'(-1)$

$\Rightarrow \lambda f \in U$

4. $U = \{ f: [0,1] \to \mathbb{R}$ s.t. $\int_0^1 f = b \in \mathbb{R} \}$

$U < \mathbb{R}^{[0,1]} \Leftrightarrow b = 0$

proof: $(\Rightarrow)$ S1 $\Rightarrow f(x) = 0 \in U \Rightarrow \int_0^1 0 = 0 \Rightarrow b = 0$

$(\Leftarrow)$ S2: $f, g \in U \Rightarrow \int (g+f) = \int g + \int f = 0 + 0 = 0$

$\Rightarrow f + g \in U$

S3: $f \in U, \lambda \in \mathbb{F} \Rightarrow \int \lambda f = \lambda \int f = \lambda \cdot 0 = 0$

$\Rightarrow \lambda f \in U$

5. $\mathbb{R}^2 < \mathbb{C}^2$?

$\mathbb{R}^2 = \{ (x_1, x_2) : x_1, x_2 \in \mathbb{C}, \mathrm{Im}(x_1) = \mathrm{Im}(x_2) = 0 \}$

$\mathbb{C}^2 = \{ (x_1, x_2) : x_1, x_2 \in \mathbb{C} \}$

note that $\mathbb{R}^2$ is over $\mathbb{C}$ in this case

$\Rightarrow$ S3 means if $x \in \mathbb{R}^2, \lambda \in \mathbb{C} \Rightarrow \lambda x \in \mathbb{R}^2 \leftarrow$ contradiction if $\mathrm{Im}(\lambda) \neq 0$

6. (a) $\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^3 = x_2^3 \} < \mathbb{R}^3$?

S1: ✓

S2: $x, y \in \mathbb{R}^3$ $(x+y)_1 = x_1 + y_1$, $(x+y)_2 = x_2 + y_2$

$x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ in $\mathbb{R}$ ✓

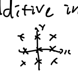
S3: ✓

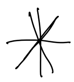
(b) $\{ (x_1, x_2, x_3) \in \mathbb{C}^3 : x_1^3 = x_2^3 \} < \mathbb{C}^3$?

$x_1^3 = x_2^3 = x_2^3 e^{2\pi i k} \Rightarrow x_1 = x_2 e^{\frac{2\pi i k}{3}}$

$\Rightarrow$ Not S2

7. find $U \neq \{0\}$, $U \subset \mathbb{R}^2$ s.t. closed under addition and additive inverse but not $U < \mathbb{R}^2$

it's $\mathbb{Z}^2$ 

8.  any 2 linearly independent line crossing origin

9. $\{ f: \mathbb{R} \to \mathbb{R}$ s.t. $\exists p > 0$ $f(x) = f(x+p)$ $\forall x \} < \mathbb{R}^{\mathbb{R}}$?

S1: $f(x) = 0$

S2: $f(x) + g(x) = f(x+p) + g(x+q) \overset{?}{=} h(x+l)$

$\Leftrightarrow ap = bq$ where $a, b \in \mathbb{Z}$

$\Leftrightarrow \frac{p}{q} = \frac{b}{a} \in \mathbb{Q}$

$\therefore$ This is condition

11. $U_1, \cdots, U_m < V$

S1: ✓

S2: $a, b \in U_1 \cap \cdots \cap U_m \Rightarrow a, b \in U_1$ and $\cdots$ and U_m

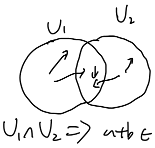
$\Rightarrow a+b \in$ $\Rightarrow a+b \in$ ✓

S3:

12. $(\Rightarrow)$: $U_1 \cup U_2 < V$ let $|U_1| < |U_2|$

S1 $\Rightarrow 0 \in U_1 \cap U_2$, S2 $\Rightarrow a, b \in U_1 \cup U_2 \Rightarrow a+b \in U_1 \cup U_2$

$\Rightarrow$ if $a, b \in U_1 / U_2 \Rightarrow a+b \in U_1 \Rightarrow a+b \notin U_2 / U_1$



$\Rightarrow$ if $a, b \in U_1 \cap U_2 \Rightarrow a+b \in U_1 \cap U_2$

if $a \in U_1/U_2, b \in U_2/U_1$, $a, b \notin U_1 \cap U_2$

$a+b \in U_1 \cup U_2 \Rightarrow a+b \in U_1/U_2$ or U_2/U_1

$a+b \in U_1/U_2$: $\Rightarrow a+b \in U_1 \Rightarrow (a+b) - a = b \in U_1$

15. $U + U = U$

16. $+$ is commutative $\Rightarrow W + U = U + W$

17.

18. $U + I = U \Rightarrow I \subset U \quad \forall U < V$

$\Rightarrow I = \{0\}$

if $U + U^{-1} = I \Rightarrow U = 0 \Rightarrow$ only I has inverse

19. $U_1 + W = U_2 + W$

$\Rightarrow U_1 \subset U_2 + W, \quad W \subset U_2 + W$

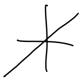
$U_2 \subset U_1 + W, \quad W \subset U_1 + W$

only if $W = 0$

20. $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$

find $W < \mathbb{F}^4$ s.t. $\mathbb{F}^4 = U \oplus W$

(x, x)



$W = \cancel{\{(a, b, a, b) \in \mathbb{F}^4 : a \neq b \in \mathbb{F}\}} + \{0\}$

$W = \{(a, 0, b, 0) \in \mathbb{F}^4\}$

$\Rightarrow U + W = \mathbb{F}^4$

$U \cap W = 0$

21. $U = \{(x, y, x+y, x-y, 2x) \in \mathbb{F}^5\}$

$W = \{(0, 0, a, b, c) \in \mathbb{F}^5\}$

$\Rightarrow W + U = \{(x, y, x+y+a, x-y+b, 2x+c)\}$

$W \cap U = 0 \Leftarrow x = y = 0$

22.

23. $U_1, U_2, W < V$

24. $f(-x) = f(x) \quad f(-x) = -f(x)$

$U_e + U_o = e(x) + o(x)$

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

$$= e(x) + o(x)$$

$\Rightarrow U_e + U_o = \mathbb{R}^{\mathbb{R}}$

$U_e \cap U_o = \{f(x) : f(x) = f(-x) = -f(-x)\}$

$\Rightarrow U_e \cap U_o = 0$