8.B Decomposition of an Operator

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On Complex Spaces

theorem: $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbb{F})$ then

$\ker(p(T))$ and $\mathrm{im}(p(T))$ are invariant under T

proof: let $v \in \ker(p(T))$ then $p(T)v = 0$

$\Rightarrow (p(T))Tv = Tp(T)v = 0 \quad \Leftarrow [T, p(T)] = 0$

$\Rightarrow Tv \in \ker(p(T))$

let $v \in \mathrm{im}(p(T))$ then $\exists\, u \in V$ s.t. $p(T)u = v$

$Tv = p(T)Tu \Rightarrow Tv \in \mathrm{im}(p(T))$ $\blacksquare$

theorem: $V(\mathbb{C})$, $T \in \mathcal{L}(V)$, $(\lambda_1, \cdots, \lambda_m) \in \mathbb{C}^m$ are distinct e-value of T

then

(a) $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$

(b) $G(\lambda_j, T)$ invariant under T $\forall j \in [\![1, m]\!]$

(c) $(T - \lambda_j I)|_{G(\lambda_j, T)}$ nilpotent $\forall j \in [\![1, m]\!]$

proof: let $n = \dim(V)$

$G(\lambda_j, T) = \ker((T - \lambda_j I)^n) = \ker(p(T))$ where $p(z) = (z - \lambda_j)^n$

$\Rightarrow$ (b) $\Rightarrow$ (c)

if $n = 1$, (a)

assume $n > 1$, (a) for U with $\dim(U) < n$

V is over $\mathbb{C} \Rightarrow \exists$ e-value of $T \Rightarrow m \geq 1$,

$V = G(\lambda_1, T) \oplus U$, where $U = \mathrm{im}((T - \lambda_1 I)^n)$

each invariant under T, $\dim(U) < n$

$\Rightarrow$ (a) for $U \Rightarrow U = G(\lambda_2, T|_U) \oplus \cdots \oplus G(\lambda_m, T|_U)$

let $k \in [\![2, m]\!] \Rightarrow G(\lambda_k, T|_U) \subset G(\lambda_k, T)$ $\forall k$

let $v \in G(\lambda_k, T) \Rightarrow v = v_1 + u$ where $v_1 \in G(\lambda_1, T)$, $u \in U$

induction $\Rightarrow v = v_1 + \sum_j v_j$ where $v_j \in G(\lambda_j, T|_U)$

e-vectors are linearly independent

$\Rightarrow v_i = 0$ if $i \neq k$, $i \in [\![1, m]\!]$

$\Rightarrow v_1 = 0 \Rightarrow v = u \in U \Rightarrow v \in G(\lambda_k, T|_U)$ $\blacksquare$

corollary: let $V(\mathbb{C})$, $T \in \mathcal{L}(V)$ then

$\exists$ basis of V with generalised e-vectors of T

Multiplicity of Eigenvalue

definition: let $T \in \mathcal{L}(V)$. multiplicity of e-value λ of T is
$$\dim(G(\lambda, T))$$
$$= \dim(\ker((T-\lambda I)^{\dim(V)}))$$

corollary: $V(\mathbb{C})$, $T \in \mathcal{L}(V)$, $\{\lambda_i\}$ = e-values
$$\sum_i \text{multiplicity of } \lambda_i = \dim(V)$$

Block Diagonal Matrices

A ***block diagonal matrix*** is a square matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix},$$

where $A_1, \dots, A_m$ are square matrices lying along the diagonal and all the other entries of the matrix equal 0.

theorem: $V(\mathbb{C})$, $T \in \mathcal{L}(V)$, $\{\lambda_i\}$ = e-values, $\{d_i\}$ = multiplicities

then $\exists$ basis of V s.t.
$$\mathcal{M}(T) = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix} \text{ where } A_j = \begin{pmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{pmatrix} \quad d_j \text{ by } b_j$$

proof: $(T-\lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent $\forall j$

choose basis of $G(\lambda_j, T)$

s.t. $\mathcal{M}((T-\lambda_j I)|_{G(\lambda_j,T)}) = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$

$$\Rightarrow \mathcal{M}(T|_{G(\lambda_j,T)}) = \mathcal{M}((T-\lambda_j I)|_{G(\lambda_j,T)}) + \mathcal{M}(\lambda_j I|_{G(\lambda_j,T)})$$

$$= \begin{pmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{pmatrix}$$

basis of $G(\lambda_j, T)$ for all j = basis of V ☒

Square Roots

lemma: let $N \in \mathcal{L}(V)$ is nilpotent. then

$I + N$ has a square root

proof: $\sqrt{1+x} = 1 + \sum_{i=1}^{\infty} a_i x^i$

$\exists\, m \in \mathbb{Z}_{>0}$ s.t. $N^m = 0$

ansatz: $\sqrt{I+N} = I + \sum_{i=1}^{m-1} a_i N^i$

$$\left(I + \sum_{i=1}^{m-1} a_i N^i\right)^2 = I + 2a_1 N + (2a_2 + a_1^2)N^2 + (2a_3 + 2a_1a_2)N^3 + \cdots + (2a_{m-1} + \cdots)N^{m-1}$$

$= I + N \Rightarrow$ $m-1$ unknowns, $m-1$ equations ☒

lemma: let $V(\mathbb{C})$, $T \in \mathcal{L}(V)$ is invertible then,

T has a square root

proof: $\{\lambda_i\}$ distinct e-values of T

$\exists$ nilpotent $N_i \in \mathcal{L}(G(\lambda_i, T))$ s.t.

$T|_{G(\lambda_i, T)} = \lambda_i I + N_i \quad \forall i$,

T invertible $\Rightarrow \lambda_i \neq 0 \quad \forall i$

$\Rightarrow T|_{G(\lambda_i, T)} = \lambda_i \left(I + \frac{1}{\lambda_i} N_i\right)$

$\Rightarrow$ let $R_i = \sqrt{T|_{G(\lambda_i, T)}}$

let $v \in V$

$\Rightarrow$ $v = \sum_i u_i$ where $u_i \in G(\lambda_i, T)$

$\Rightarrow$ let $R \in \mathcal{L}(V)$

$$Rv = \sum_i R_i u_i$$

theorem: let $k \in \mathbb{Z}_{>0}$, $V(\mathbb{C})$, $T \in \mathcal{L}(V)$ invertible then

T has k^{th} root $\forall k$

proof: $(I+N)^{\frac{1}{k}}$ exists (taylor series)

$$T|_{G(\lambda_i, T)} = \lambda_i \left(I + \frac{1}{\lambda_i} N_i\right)$$

$\Rightarrow$ let $R_i = \left(T|_{G(\lambda_i, T)}\right)^{\frac{1}{k}}$

let $v \in V$

$\Rightarrow$ $v = \sum_i u_i$ where $u_i \in G(\lambda_i, T)$

$\Rightarrow$ let $R \in \mathcal{L}(V)$

$$Rv = \sum_i R_i u_i$$