7.C Positive Operators and Isometries

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definition: $T \in \mathcal{L}(V)$ is <u>positive</u> if

$T = T^*$ and $\langle Tv, v \rangle \geq 0 \ \forall v \in V$

definition: $R \in \mathcal{L}(V)$ is <u>square root</u> of $T \in \mathcal{L}(V)$ if

$R^2 = T$

properties: (a) $T \in \mathcal{L}(V)$ is positive

$\Longleftrightarrow$

(b) $T = T^*$ and e-value $\lambda_i \geq 0 \ \forall i$

$\Longleftrightarrow$

(c) $\exists$ positive $R \in \mathcal{L}(V)$ s.t. $R^2 = T$

$\Longleftrightarrow$

(d) $\exists \ R \in \mathcal{L}(V)$ s.t. $R = R^*$ and $R^2 = T$

$\Longleftrightarrow$

(e) $\exists \ R \in \mathcal{L}(V)$ s.t. $R^* R = T$

proof: (a) $\Rightarrow$ let $Tv = \lambda v$ then

$0 \leq \langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle \Rightarrow$ (b)

spectral theorem $\Rightarrow$ orthonormal $\{e_i\}$ = e-vector of T

$\Rightarrow$ let $Te_i = \lambda_i e_i$ (Not Einstein convention)

(b) $\Rightarrow \lambda_i \geq 0 \ \forall i \Rightarrow$ let $R \in \mathcal{L}(V)$ s.t. $Re_j = \sqrt{\lambda_j}\, e_j$

$\Rightarrow R$ is positive, $R^2 e_j = \lambda_j e_j = Te_j \Rightarrow$ (c) $\Rightarrow$ (d)

$\Rightarrow R = R^*$, $T = R^2 = RR^* \Rightarrow$ (e)

$\Rightarrow \ T^* = (R^*R)^* = R^*(R^*)^* = R^*R = T$

$\Rightarrow \langle Tv, v \rangle = \langle R^*Rv, v \rangle = \langle Rv, Rv \rangle \geq 0 \ \forall v \in V$

$\Rightarrow$ (a) $\boxtimes$

theorem: positive operator has unique positive square root proof : let $T \in \mathcal{L}(V)$ positive

$\Rightarrow \exists\ \lambda \geq 0$ s.t. $Tv = \lambda v$

let R be positive, $R^2 = T$

$\Rightarrow$ let $\{e_i\}$ eigenbasis of $R \in \mathcal{L}(V)$

$\Rightarrow \exists \{\lambda_i \geq 0\}$ s.t. $Re_j = \sqrt{\lambda_j}\, e_j$,

$v \in V \Rightarrow v = a_i e_i$

$\Rightarrow Rv = a_i \sqrt{\lambda_i}\, e_i$

$\Rightarrow R^2 v = a_i \lambda_i e_i = Tv = \lambda v = \lambda a_i e_i$

$\Rightarrow a_i(\lambda - \lambda_i) = 0 \quad \forall i$

$$\Rightarrow v = \sum_{\{j \mid \lambda_j = \lambda\}} a_j e_j$$

$\Rightarrow Rv = a_j \sqrt{\lambda}\, e_j = \sqrt{\lambda}\, v$

R is uniquely determined for each e-vector v of T

definition: $S \in \mathcal{L}(V)$ is isometry if

$$\|Sv\| = \|v\|$$

corollary: isometry preserves norms

7.42 Characterization of isometries

Suppose $S \in \mathcal{L}(V)$. Then the following are equivalent:

(a) S is an isometry;

(b) $\langle Su, Sv\rangle = \langle u, v\rangle$ for all $u, v \in V$;

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(c) $Se_1, \ldots, Se_n$ is orthonormal for every orthonormal list of vectors $e_1, \ldots, e_n$ in V;

(d) there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $Se_1, \ldots, Se_n$ is orthonormal;

(e) $S^*S = I$;

(f) $SS^* = I$;

(g) S^* is an isometry;

(h) S is invertible and $S^{-1} = S^*$.

proof: 229 p of LADR

7.43 **Description of isometries when $\mathbf{F} = \mathbf{C}$**

Suppose V is a complex inner product space and $S \in \mathcal{L}(V)$. Then the following are equivalent:

(a) S is an isometry.

(b) There is an orthonormal basis of V consisting of eigenvectors of S whose corresponding eigenvalues all have absolute value 1.

proof: 231 p of LADR