6.A Inner Products and Norms

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definition: inner product $\langle \cdot,\cdot \rangle : V^2 \to \mathbb{F}$ has following properties

- postivity
  $\langle v,v\rangle \geq 0 \ \forall v \in V$
- definiteness
  $\langle v,v\rangle = 0 \iff v=0$
- additivity in first slot
  $\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle \ \forall (u,v,w) \in V^3$
- homogeneity in first slot
  $\langle \lambda u, v\rangle = \lambda \langle u,v\rangle \ \forall (\lambda,u,v) \in \mathbb{F}\times V\times V$
- conjugate symmetry
  $\langle u,v\rangle = \overline{\langle v,u\rangle} \ \forall (u,v) \in V^2$

proposition: for fixed $u \in V$, the fuction $v \mapsto \langle v,u\rangle$ is a linear functional ($\in \mathcal{L}(V,\mathbb{F})$)

$\Rightarrow \langle 0,u\rangle = 0 \ \forall u \in V$

$\Rightarrow \langle u,0\rangle = \overline{\langle 0,u\rangle} = 0 \ \forall u \in V$

proposition: $\langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle \ \forall (u,v,w) \in V^3$

proof : $\langle u, v+w\rangle = \overline{\langle v+w,u\rangle} = \overline{\langle v,u\rangle + \langle w,u\rangle} = \langle u,v\rangle + \langle u,w\rangle$

proposition: $\langle u,\lambda v\rangle = \bar{\lambda}\langle u,v\rangle \ \forall (\lambda,u,v) \in \mathbb{F}\times V\times V$

proof : $\langle u,\lambda v\rangle = \overline{\langle \lambda v,u\rangle} = \bar{\lambda}\overline{\langle v,u\rangle} = \bar{\lambda}\langle u,v\rangle$

definition : norm

$\|v\| := \sqrt{\langle v,v\rangle}$ for $v \in V$

proposition : $\|\lambda v\| = |\lambda|\,\|v\| \ \forall (\lambda,v) \in \mathbb{F}\times V \Rightarrow \|v\|=0 \iff v=0$

proof : $\|\lambda v\|^2 = \langle \lambda v,\lambda v\rangle = \lambda\langle v,\lambda v\rangle = \lambda\bar{\lambda}\langle v,v\rangle = |\lambda|^2\|v\|^2$

definition : $u,v \in V$ are orthogonal if $\langle u,v\rangle = 0$

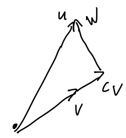
theorem : Pythagorean theorem

if u,v are orthogonal, then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

theorem : orthogonal decomposition

suppose $(u,v) \in V^2$ with $v \neq 0$. let $c = \frac{\langle u,v\rangle}{\|v\|^2}$ and $w = u - cv$ then

$\langle w,v\rangle = 0$ and $u = cv + w$



theorem : Cauchy-Schwarz Inequality

suppose $(u,v) \in V^2$ then $|\langle u,v\rangle| \leq \|u\|\cdot\|v\|$

$|\langle u,v\rangle| = \|u\|\cdot\|v\| \iff \exists c \in \mathbb{F}$ s.t. $u = cv$

proof : if $v=0$, trivial. assume $v=0$

$\Rightarrow u = \langle u,v\rangle$ ...

$u = \frac{\quad}{\|v\|^2} v + w$ where $\langle v, w \rangle = 0$

pythagorian theorem $\Rightarrow$ $\|u\|^2 = \left| \frac{\langle u, v \rangle}{\|v\|^2} \right|^2 \|v\|^2 + \|w\|^2$

$= \frac{|\langle u, v \rangle|^2}{\|v\|^2} + \|w\|^2 \geq \frac{|\langle u, v \rangle|^2}{\|v\|^2}$ ■

$\therefore$ equality $\Leftrightarrow$ $\|w\|^2 = 0$ $\Leftrightarrow$ $u = \frac{\langle u, v \rangle}{\|v\|^2} v$ ■

theorem: triangular inequality

suppose $(u, v) \in V^2$ then

$\|u + v\| \leq \|u\| + \|v\|$

equality $\Leftrightarrow$ $\exists c \in \mathbb{R}_{\geq 0}$ s.t. $u = cv$

proof: $\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\langle u, v \rangle$

$\leq \|u\|^2 + \|v\|^2 + 2|\langle u, v \rangle|$

$\leq \|u\|^2 + \|v\|^2 + 2\|u\| \cdot \|v\|$

$= (\|u\| + \|v\|)^2$ ■

2nd inequality becomes equality $\Leftrightarrow$ $|\langle u, v \rangle| = \|u\| \cdot \|v\|$ $\Leftrightarrow$ $u = cv, c \in F$

1st inequality becomes equality $\Leftrightarrow$ $\mathrm{Re}\langle u, v \rangle = |\langle u, v \rangle|$

$\Rightarrow \langle u, v \rangle = \bar{c}\|v\| \Rightarrow \mathrm{Re}\langle u, v \rangle = \|v\| \mathrm{Re}(c) = |c|\|v\| \Rightarrow c \in \mathbb{R}, c \geq 0$

theorem: parallelogram Equality

suppose $(u, v) \in V^2$ then

$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$

proof: cross terms cancel