10.B Determinant

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of an Operator

definition; $T \in \mathcal{L}(V)$, $\{\lambda_i\}$ repeated e-value of T for $V(\mathbb{C})$
or of $T_{\mathbb{C}}$ for $V(\mathbb{R})$

determinant of T is

$$\det(T) = \prod_i \lambda_i$$

corollary; $T \in \mathcal{L}(V)$, $n = \dim(V)$, $p \in \mathcal{P}(\mathbb{F})$ characteristic of T, then
$\det(T) = (-1)^n a_0$ where $p(z) = \sum_{i=0}^{n} a_i z^i$

corollary; $p(z) = z^n - \operatorname{tr}(T) z^{n-1} + \cdots + (-1)^n \det(T)$

theorem; $T \in \mathcal{L}(V)$ invertible $\Leftrightarrow \det(T) \neq 0$

proof; if $V(\mathbb{C})$ then T invertible $\Leftrightarrow$ T injective $\Leftrightarrow$ ($T - \lambda I$ not injective $\Rightarrow \lambda \neq 0$)
$\Leftrightarrow$ e-values of $T \neq 0 \Leftrightarrow \det(T) \neq 0$
if $V(\mathbb{R})$ e-values of $T \neq 0 \Leftrightarrow$ e-values of $T_{\mathbb{C}} \neq 0 \Leftrightarrow \det(T) \neq 0$ ☒

theorem; $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbb{F})$ characteristic of T then

$$p(z) = \det(zI - T)$$

proof; let $(\lambda, z) \in \mathbb{C}$ then

$T - \lambda I = T - zI + zI - \lambda I = 0$
$\Leftrightarrow zI - T = (z - \lambda) I$
$\therefore \lambda$ e-value of $T \Leftrightarrow (z-\lambda)$ e-value of $zI - T$ $\qquad \ker(T) = \ker(-T)$
$\Rightarrow \dim(\ker((T - \lambda I)^{\dim(V)})) = \dim(\ker((zI - T - (z-\lambda)I)^{\dim(V)}))$
$\Rightarrow$ same multiplicity

☒

of a Matrix

definition; a **permutation** of $(1, \cdots, n)$ is a list $(m_1, \cdots, m_n)$ containing each of $(1, \cdots, n)$ exactly once
$\operatorname{perm}(n)$ = set of all permutations of $(1, \cdots, n)$

definition; the **sign** of a permutation $(m_1, \cdots, m_n)$ is given by
let c = # of $(j,k) \in \mathbb{Z}^2$ s.t. $1 \leq j < k \leq n$
& j apears after k in the permutation
then, $\operatorname{sign}(m_1, \cdots, m_n) := (-1)^c$

lemma; interchanging 2 entries in a permutation results in
sign $\to$ $-1 \times$ sign

proof; let P_1, P_2 be permutations s.t. (interchanging 2 entries of P_1) $= P_2$
$\Rightarrow$ Let $P_1 = (\cdots, a, \overset{\cdots}{j}, b, \cdots)$, $P_2 = (\cdots, b, \overset{\cdots}{j}, a, \cdots)$
$P_1 \to P_2 \Rightarrow c \mathrel{+}= 1$ if $a < b$
$c \mathrel{+}= -1$ if $a > b$

$c \mathrel{+}= 2$ if $a < j < b$
$c \mathrel{+}= -2$ if $a > j > b$

$$c \mathrel{+}= 0 \text{ if } \left.\begin{array}{l}(a<b<j \text{ or} \\ b<a<j \text{ or} \\ j<a<b \text{ or} \\ j<b<a)\end{array}\right\} \text{sign invariant}$$

$\square$

definition: determinant of a $n \times n$ matrix A

$$\det(A) := \sum_{\mathrm{perm}(n)} \mathrm{sign}(m_1, \cdots, m_n) \prod_{i=1}^{n} A_{m_i, i}$$

lemma: let A $n \times n$ matrix, B is A with 2 columns interchanged
then $\det(A) = -\det(B)$

proof: if 2 columns are exchanged in A then
in each term in the sum of $\det(A)$,
the product doesn't change but
the sign flips

$\square$

lemma: let A $n \times n$ matrix with 2 columns equal
then $\det(A) = 0$

proof: exchange the equal columns
$\Rightarrow \det(A) = -\det(A) \Rightarrow \det(A) = 0$ $\square$

notation: $n \times n$ matrix $A = \left(A_{i,j} \right) \downarrow i \; \overset{j}{\rightarrow}$

kth column $A_{\cdot,k} = \left(A_{i,k} \right) \downarrow i$

$\Rightarrow A = (\quad A_{\cdot,j} \quad) \rightarrow j$

corollery: let $n \times n$ matrix $A = (A_{\cdot,1}, \cdots, A_{\cdot,n})$ and
$(m_1, \cdots, m_n) \in \mathrm{perm}(n)$ then

$$\det((A_{\cdot,m_1}, \cdots, A_{\cdot,m_n})) = \mathrm{sign}(m_1, \cdots, m_n) \det(A)$$

theorem: let $(k, n) \in \mathbb{Z}^2_{>0}$ s.t. $1 \leq k \leq n$ and A be $n \times n$ matrix.
fixing all columns of A except $A_{\cdot,k} \in \mathbb{F}^n$,
consider the function $\phi : \mathbb{F}^n \rightarrow \mathbb{F}$ s.t.
$\phi(A_{\cdot,k}) = \det(A)$

$\phi(A_{\cdot,k}) = \det(A)$

then ϕ is a linear map

proof: let $(a,b) \in \mathbb{F}^n \times \mathbb{F}^n$, $a = (a_i)\downarrow i$, $b = (b_i)\downarrow i$

$$\Rightarrow \phi(a+b) = \sum_{\mathrm{perm}(n)} \mathrm{sign}(m_1, \cdots, m_n) \prod_{i=1}^{n} A_{m_i,i}$$

where $i=k$ in $\prod$, $A_{m_k,k} = a_{m_k} + b_{m_k}$

so the product can be expanded

$$\phi(a+b) = \sum \text{ with } a + \sum \text{ with } b$$

$$= \phi(a) + \phi(b)$$

let $\lambda \in \mathbb{F}$ then

$$\phi(\lambda a) = \sum_{\mathrm{perm}(n)} \mathrm{sign}(m_1, \cdots, m_n) \prod_{i=1}^{n} A_{m_i,i}, \quad A_{m_k,k} = \lambda a_{m_k}$$

$$= \lambda \sum (\cdots), \quad A_{m_k,k} = a_{m_k}$$

$$= \lambda \phi(a)$$

theorem: let A and B be $n \times n$ matrices then

$$\det(AB) = \det(A)\det(B) = \det(BA)$$

proof: let e_k be a column vector $\begin{array}{cl} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow k\text{th row} \\ \\ \\ \\ \end{array} \end{array}$

$$\Rightarrow A e_k = A_{\cdot,k} = A_{i,k} e_i$$

$$\Rightarrow \det(AB) = \det((ABe_1, \cdots, ABe_n))$$

$$= \det((AB_{m_1,1}e_{m_1}, \cdots, AB_{m_n,n}e_{m_n}))$$

$$= \left(\prod_{i=1}^{n} B_{m_i,i}\right) \det(Ae_{m_1}, \cdots, Ae_{m_n}) \quad \leftarrow \text{Einstein convention}$$

$$= \left(\sum_{m_1=1}^{n} \cdots \sum_{m_n=1}^{n}\right) \left(\prod_{i=1}^{n} B_{m_i,i}\right) \det(Ae_{m_1}, \cdots, Ae_{m_n})$$

$$= \sum_{\mathrm{perm}(n)} \left(\prod_{i=1}^{n} B_{m_i,i}\right) \det(Ae_{m_1}, \cdots, Ae_{m_n})$$

$$= \sum \left(\prod_{i=1}^{n} B_{m_i,i}\right) \mathrm{sign}(m_1, \cdots, m_n) \det(A)$$

$$\sum_{perm(n)}$$

$$= \det(A) \sum_{perm(n)} \left(\prod_{i=1}^{n} B_{m_i, i} \right) sign(m_1, \dots, m_n)$$

$$= \det(A) \det(B) \quad \boxtimes$$

theorem: let $T \in \mathcal{L}(V)$, $\{u_i\}$ and $\{v_i\}$ are bases of V then

$$\det(\mathcal{M}(T, (\{u_i\}))) = \det(\mathcal{M}(T, (\{v_i\})))$$

proof: let $A = \mathcal{M}(I, (\{u_i\}), (\{v_i\}))$ then

$$\det(\mathcal{M}(T, (\{u_i\}))) = \det(A^{-1} \mathcal{M}(T, (\{v_i\})) A)$$

$$= \det(\mathcal{M}(T, (\{v_i\})) \underbrace{\cancel{AA^{-1}}}_{I}) \quad \boxtimes$$

theorem: let $T \in \mathcal{L}(V)$ then $\det(T) = \det(\mathcal{M}(T))$

proof: det independent of basis $\Rightarrow$ (true for a basis $\Rightarrow$ true $\forall$ basis)

T (or $T_{\mathbb{C}}$ for $V(\mathbb{R})$) can be Jordan form

which is e-values along the diagonal, upper triangular

$\Rightarrow$ consider $\det(\mathcal{M}(T, \text{Jordan})) = \sum_{perm(n)}$

permutation $(1, 2, 3, \dots, n-1, n)$ term in the sum gives the product of the diagonal

any other permutation has at least one $m_j < j$

and for such j, $\mathcal{M}(T, \text{upper-tri})_{m_j, j} = 0$

$\Rightarrow$ product $= 0$

$\Rightarrow \det(\mathcal{M}(T, \text{Jordan})) = \prod \text{diagonal} = \prod \text{e-values} = \det(T)$

the Sign of the Determinant

theorem: let $S \in \mathcal{L}(V)$ be isometry then
$$|\det(S)| = 1$$

proof: if $V(\mathbb{R})$, then $\langle \cdot, \cdot \rangle$ in $V_{\mathbb{C}}$ is

$$\langle u+iv, x+iy \rangle = \langle u, x \rangle + \langle u, iy \rangle + \langle iv, x \rangle + \langle iv, iy \rangle$$
$$= \langle u, x \rangle + \langle v, y \rangle + i(\langle v, x \rangle - \langle u, y \rangle)$$

$\Rightarrow S_{\mathbb{C}}(u+iv) = Su + iSv$,
$$\|S_{\mathbb{C}}(u+iv)\| = \|Su + iSv\| = \|Su\| + \|Sv\| + i(\langle Sv, Su \rangle - \langle Su, Sv \rangle)$$
$$= \|u\| + \|v\| + i(\langle v, u \rangle - \langle u, v \rangle) = \|u+iv\|$$

$\Rightarrow S_{\mathbb{C}}^* S_{\mathbb{C}} = I$ in $V_{\mathbb{C}}$

$\Rightarrow$ if λ_i = e-value of S (or $S_{\mathbb{C}}$ for $V(\mathbb{R})$) then $|\lambda_i| = 1$

$\Rightarrow |\det(S)| = 1$ ∎

the sign of $\det(T)$ in $V(\mathbb{R})$:

$\sqrt{T^*T}$ is positive $\Rightarrow$ e-values $\geq 0 \Rightarrow \det(\sqrt{T^*T}) \geq 0$ and

$T = S\sqrt{T^*T} \Rightarrow$ if $\det(T) \neq 0$ then $\det(S)$ as same sign with $\det(T)$.

$\det(S) \in \{-1, 1\}$ and consider $E(-1, S) = \{v \in V \mid Sv = -v\}$

isometry $\Rightarrow \exists$ basis s.t. $\mathcal{M}(S)$ = block diagonal with (1) or (-1) or $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\Rightarrow$ if $\dim(E(-1,S)) = 2$, $\det(S|_{E(-1,S)}) = \cos^2\theta + \sin^2\theta = 1 \quad \forall \theta \in (0, \pi)$

if $\dim(E(-1,S)) = 1$, $\det(S|_{E(-1,S)}) = \det((-1)) = -1$

one dimensional $E(-1,S)$ is reversing the direction of the line $E(-1,S)$

$\therefore$ $\det(T) > 0 \Rightarrow$ direction reversed even times

$\det(T) < 0 \Rightarrow$ " " odd "

Volume

lemma: $T \in \mathcal{L}(V)$ then

$|\det(T)| = \det(\sqrt{T^*T})$

proof: $T = S\sqrt{T^*T}$

$\det(T) = \det(S)\det(\sqrt{T^*T})$, $|\det(S)| = 1$ ☒

definition: box in $\mathbb{R}^n$ is a set of the form

$\{(y_1, \cdots, y_n) \in \mathbb{R}^n \mid x_j < y_j < x_j + r_j \text{ for } j \in [\![1, n]\!]\}$

where $r_i \in \mathbb{R}_{>0}$ are side lengths & $(x_1, \cdots, x_n) \in \mathbb{R}^n$

definition: the volume of a box B in $\mathbb{R}^n$ with side lengths $\{r_i\}$ is

$\text{volume}(B) = \prod_i r_i$

definition: let $\Omega \in \mathbb{R}^n$ the volume of Ω is

$\text{Volume}(\Omega) = \inf\left(\sum_i \text{volume}(B_i)\right)$

infimum is over all sequences B_i's s.t. $\left(\bigcup_i B_i\right) \supset \Omega$

notation: let T be a function defined on Ω then

$T(\Omega) = \{Tx \mid x \in \Omega\}$

lemma: Let $T \in \mathcal{L}(\mathbb{R}^n)$ positive, $\Omega \in \mathbb{R}^n$ then

$\text{volume}(T(\Omega)) = \det(T)\,\text{volume}(\Omega)$

proof: positive $\Rightarrow$ hermitian $\Rightarrow$ RST $\Rightarrow \exists$ orthonormal basis

with e-values $\{\lambda_i\}$

positive $\Rightarrow \lambda_i \geq 0 \Rightarrow$ each $r_i \overset{T}{\mapsto} \lambda_i r_i$

$\Rightarrow \text{volume}(T(\Omega)) = \prod_i (r_i \lambda_i) = \left(\prod_i r_i\right)\left(\prod_j \lambda_j\right)$ ☒

lemma: let $S \in \mathcal{L}(\mathbb{R}^n)$ isometry, $\Omega \in \mathbb{R}^n$ then

$$\text{Volume}(S(\Omega)) = \text{volume}(\Omega)$$

proof: $\|Sx - Sy\| = \|S(x-y)\| = \|x-y\|$

metric invariant ∎

theorem: let $T \in \mathcal{L}(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ then

$$\text{volume}(T(\Omega)) = |\det(T)| \, \text{volume}(\Omega)$$

proof: $T = S\sqrt{T^*T}$ where S is isometry

$$\Rightarrow \text{volume}(T(\Omega)) = \text{volume}(S(\sqrt{T^*T}(\Omega)))$$
$$= \text{volume}(\sqrt{T^*T}(\Omega))$$
$$= \det(\sqrt{T^*T}) \, \text{volume}(\Omega)$$
$$= |\det(T)| \, \text{volume}(\Omega)$$ ∎

10.56 Definition ***differentiable, derivative,*** $\sigma'(x)$

Suppose Ω is an open subset of $\mathbf{R}^n$ and σ is a function from Ω to $\mathbf{R}^n$. For $x \in \Omega$, the function σ is called ***differentiable*** at x if there exists an operator $T \in \mathcal{L}(\mathbf{R}^n)$ such that

$$\lim_{y \to 0} \frac{\|\sigma(x+y) - \sigma(x) - Ty\|}{\|y\|} = 0.$$

If σ is differentiable at x, then the unique operator $T \in \mathcal{L}(\mathbf{R}^n)$ satisfying the equation above is called the ***derivative*** of σ at x and is denoted by $\sigma'(x)$.