3.B Null spaces and range

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$\mathcal{L}(V, W)$ is a set of all linear maps from V to W

$\mathcal{L}(V, W)$ is a vectorspace

definition: let $T \in \mathcal{L}(V,W)$. Null space, null $T = \{v \in V : Tv = 0\} = \ker(T)$, kernel of T

theorem: T is injective $\iff \ker(T) = \{0\}$

proof: $(\Rightarrow)$ $Tu = Tv \Rightarrow u = v$

let $u, v \in \ker(T)$

$\Rightarrow Tu = Tv = 0 \Rightarrow u = v$

$\therefore$ only one element in $\ker(T)$

$0 \in \ker(T)$ always ✓

$(\Leftarrow)$ let $u, v \in V$ s.t. $Tu = Tv$

$\Rightarrow Tu - Tv = 0 = T(u-v)$

$\Rightarrow u - v \in \ker(T) = \{0\}$

$\Rightarrow u - v = 0 \Rightarrow u = v$ ✓

definition: Range of T = Image of $T = \mathrm{Im}(T) := \{Tv : v \in V\}$

Fundamental Theorem of Linear Maps

let V is finite dimentional and $T \in \mathcal{L}(V,W)$

then $\mathrm{Im}(T)$ is finite dimentional and

$$\dim(V) = \dim(\ker(T)) + \dim(\mathrm{Im}(T))$$

proof: we know $\ker(T), \mathrm{Im}(T) < V$

$\Rightarrow \dim(\ker(T)) \leqslant \dim(V)$

let $\dim(\ker(T)) = m$

$\Rightarrow \exists$ basis set $\{u_1, \cdots, u_m\}$ of $\ker(T)$

$\Rightarrow \exists$ basis set of $V \subset \{u_1, \cdots, u_m\}$

let basis set of $V = \{u_1, \cdots, u_m, v_1, \cdots v_n\}$

$\Rightarrow \dim(V) = n+m$, $v = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n$, $a_i, b_i \in \mathbb{F}$

$\Rightarrow Tv = \underbrace{a_1Tu_1 + \cdots + a_mTu_m}_{=0} + b_1Tv_1 + \cdots + b_nTv_n$

$\Rightarrow \mathrm{span}(\{Tv_1, \cdots, Tv_n\}) = \{Tv : v \in V\} = \mathrm{Im}(T)$

$\Rightarrow$ if $\{Tv_1, \cdots, Tv_n\}$ is linearly independant, $\dim(\mathrm{Im}(T)) = n$

consider $c_1Tv_1 + \cdots + c_nTv_n = 0$

$\Rightarrow T(c_1v_1 + \cdots + c_nv_n) = 0$

$\Rightarrow c_1v_1 + \cdots + c_nv_n \in \mathrm{Im}(T)$

$\Rightarrow c_1v_1 + \cdots + c_nv_n = d_1u_1 + \cdots + d_mu_m$

$\Rightarrow c_i = d_i = 0 \ \forall i$ as $\{u_1, \cdots u_m, v_1, \cdots v_n\}$ is linearly independent ∎