MAGNETISM
Part 3 of
Vector Fields, Electricity and Magnetism (VFEM)
Year 1 module

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(originally based on notes from T. Horbury)

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Figure 1 Aurora borealis during full moon over north of Finland. (Figure credit: Jouni Jussila)

Contents

Magnetism: Course overview

Structure of the course

- *Chapter 1* introduces magnetism, including magnetic field and magnetic flux. It also discusses the absence of monopoles through Gauss's Law for magnetism.
- *Chapter 2* looks at the force experienced by moving charges in a magnetic field with application to the Hall effect. We also examine the magnetic force on wires, the magnetic torques on current-carrying wire loops and the principle of direct-current motors.
- *Chapter 3* focuses on the source of magnetic field, and more specifically at how moving charges generate magnetic fields. It introduces the Biot-Savart Law, Ampère's Law, and forces between current-carrying wires.
- *Chapter 4* examines the interplay between magnetic and electric fields. In particular, it looks at time-varying magnetic fields and how it generates an electric field through Faraday's Law. Applications, such as generators, are provided as illustration.
- *Chapter 5* introduces inductors. We learn how to calculate an inductance, we derive the energy density of the magnetic field and discuss the concept of mutual induction.
- *Chapter 6* highlights key relations from the course and provide the four Maxwell's equations, including for the non-electrostatics case, which will be studied in more depth next year.

Course format

Lecture format

There are 9 lectures covering the magnetism material.

The format of each lecture depends on the content. Usually we are working through a problem/derivation and I will use the chalkboard, and you should **make your own notes** as you follow along. When we are discussing less mathematical material, I will use PowerPoint. All the PowerPoint slides will be made available on BlackBoard. In general I will try to organise each lecture so that there will be a short break in the middle, and an opportunity to ask questions and get feedback and clarifications.

The lectures are supported by weekly office hours (see BlackBoard): one hour is in person and the other, online in the MSTeams VFEM/Magnetism channel.

Handouts

Lecture notes (present .pdf) will be made available on BlackBoard. These notes will cover the key points, including key definitions, equations, and derivations presented in the lectures. Some examples and illustrations will however only be given in the lectures.

I encourage you to make your own summary which is an excellent exercise to digest the material.

Problem sheets and seminars

There are:

- 3 problem sheets
- 2 seminars
- 1 assessed problem sheet (joined with *Electrostatics*)

Textbooks

Whilst the lectures and handout form the basis of the examinable material, textbooks are a useful resource which should not be overlooked. In learning any subject, having access to a variety of sources with different presentation styles is often useful. For example, you may find a discussion or derivation in class is not immediately clear, but reading about it in a textbook where it is presented in a different way may help to give you a clearer picture of what's going on.

Books on magnetism:

- **University Physics with Modern Physics (Hugh D. Young and Roger A. Freedman), Pearson Education, 2015 (Fourteenth edition)**
- **Introduction of electrodynamics (David J. Griffiths), Cambridge University Press, 2017 (Fourth edition)**

The first one covers most topics, is easy to read, provides nice illustrations and applications, and includes a lot of problems for practice. The second one is more challenging and detailed, and covers a lot of material related to magnetism, which is not covered in the course.

Relevant parts of the books are cited in each section of the notes:

- **Y&F 27.1** refers to Young and Freedman, Section 27.1
- **G 4.2** refers to Griffiths, Section 4.2

Key quantities, constants, and units

Table 1 Key physics quantities, constants, and units of relevance for the course

Speed of light in vacuum	c	2.9979×10^{8} m s^{-1}
Permittivity in vacuum	ϵ_0	8.8542×10^{-12} F m^{-1}
Magnetic permeability in vacuum	μ_0	$\sim 4\pi \times 10^{-7}$ H m^{-1}
Elementary electric charge	e or $\lvert q \rvert$	1.6022×10^{-19} C
Electron Volt	eV	1 eV $= 1.6022 \times 10^{-19}$ J
Mass of an electron	m_e	9.1094×10^{-31} kg
Mass of a proton	m_p	1.6726×10^{-27} kg

Following an update in the exact length of one meter, the value of μ_0 has slightly changed, though it is OK to use $4\pi \times 10^{-7}$ H m^{-1} in this course.

Coordinate systems

Note that the symbol ^ (called "hat"), as in $\hat{\mathbf{r}}$, refers to the unit vector (magnitude = 1) associated with the vector: $\hat{\mathbf{r}} = \mathbf{r}/r$.

- *Cartesian coordinate system: (x, y, z)* with unit vectors ($\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$)

  Gradient: $\nabla\Omega = \dfrac{\partial\Omega}{\partial x}\,\hat{\mathbf{x}} + \dfrac{\partial\Omega}{\partial y}\,\hat{\mathbf{y}} + \dfrac{\partial\Omega}{\partial z}\,\hat{\mathbf{z}}$

  Divergence: $\nabla\cdot\mathbf{B} = \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} + \dfrac{\partial B_z}{\partial z}$

  Curl: $\nabla\times\mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

- *Cylindrical coordinate system: (ρ,ϕ,z)* with unit vectors ($\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{k}}$)

  ρ is the distance to the z axis (not to confuse with ρ_q, the charge density) and ϕ is the azimuthal angle.

  The form in cylindrical coordinates of the following operators is given here for information, it would be given, if needed, during the exam:

  Gradient: $\nabla f = \dfrac{\partial f}{\partial\rho}\,\hat{\boldsymbol{\rho}} + \dfrac{1}{\rho}\dfrac{\partial f}{\partial\phi}\,\hat{\boldsymbol{\phi}} + \dfrac{\partial f}{\partial z}\,\hat{\mathbf{k}}$

  Divergence: $\nabla\cdot\mathbf{B} = \dfrac{1}{\rho}\dfrac{\partial(\rho\,B_\rho)}{\partial\rho} + \dfrac{1}{\rho}\dfrac{\partial B_\phi}{\partial\phi} + \dfrac{\partial B_z}{\partial z}$

  Curl: $\nabla\times\mathbf{A} = \left(\dfrac{1}{\rho}\dfrac{\partial A_z}{\partial\phi} - \dfrac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\dfrac{\partial A_\rho}{\partial z} - \dfrac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}} + \left(\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(\rho A_\phi) - \dfrac{1}{\rho}\dfrac{\partial A_\rho}{\partial\phi}\right)\hat{\mathbf{k}}$

- *Spherical coordinate system: $(r,\ \theta\ ,\phi)$* with unit vectors ($\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$)

  r is the radial distance (distance to the origin) and θ is the polar angle.

  The form in spherical coordinates of the following operators is given here for information, it would be given, if needed, during the exam:

  Gradient: $\nabla f = \dfrac{\partial f}{\partial r}\,\hat{\mathbf{r}} + \dfrac{1}{r}\dfrac{\partial f}{\partial\theta}\,\hat{\boldsymbol{\theta}} + \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial\phi}\,\hat{\boldsymbol{\phi}}$

  Divergence: $\nabla\cdot\mathbf{B} = \dfrac{1}{r^2}\dfrac{\partial(r^2 B_r)}{\partial r} + \dfrac{1}{r\sin\theta}\dfrac{\partial(\sin\theta\,B_\theta)}{\partial\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial B_\phi}{\partial\phi}$

  Curl: $\nabla\times\mathbf{A} = \left(\dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(A_\phi\sin\theta) - \dfrac{1}{r\sin\theta}\dfrac{\partial A_\theta}{\partial\phi}\right)\hat{\mathbf{r}} + \left(\dfrac{1}{r\sin\theta}\dfrac{\partial A_r}{\partial\phi} - \dfrac{1}{r}\dfrac{\partial}{\partial r}(rA_\phi)\right)\hat{\boldsymbol{\theta}} + \left(\dfrac{1}{r}\dfrac{\partial}{\partial r}(rA_\theta) - \dfrac{1}{r}\dfrac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}}$

Chapter 1

Magnetic field

$\rightarrow$ **Textbooks**: Refer to Y&F 27.1, 27.2 and 27.3

1.1 Magnetic field

- The **magnetic field** $\mathbf{B}$ is a *vector field* defined over the whole space.

- At each point $\mathbf{r}$ of space and at time t, $\mathbf{B}(\mathbf{r}, t)$ has:

  – a direction (line of direction and sense); the magnetic field of a magnet points out of its north pole and into its south pole.

  – a magnitude, B.

- The International System (SI) unit for B is the *tesla*, symbol T [1 T = 1 N A^{-1} m^{-1} = 1 kg s^{-2} A^{-1} = 1 kg C s^{-3}]. Also (not used in the course): Gauss [1 G = 10^{-4} T]

- A **magnetic field line** is a line which, at each point $\mathbf{r}$, is tangent to the magnetic field vector, $\mathbf{B}(\mathbf{r})$, at that point.

Typical values for B, the magnitude of $\mathbf{B}$, are:

- Hospital MRI machine: 1 T (which is a lot!)

- Earth's magnetic field in your street: $\sim$ 30,000 nT (where 1 nT = 10^{-9} T)

- Generated by a 1 A current at 1 m: $\sim$ 200 nT

- Interplanetary space near the Earth: $\sim$ 5 nT

1.2 Magnetic material (non-examinable)

A **magnet** is a material which generated a magnetic field. The main origin for magnetism in material is linked to electrons in the atoms which are acting as tiny magnets.

The **magnetic permeability**, μ_m, is a measure of the magnetisation of the material in response to an applied magnetic field:

$$\mu_m = B/H_{ext} \tag{1.1}$$

where B is the magnetic field magnitude within the material resulting from the presence of H_{ext}. The quantity H_{ext} represents the "strength" of the external, magnetizing magnetic field applied and is expressed in A m^{-1}, keeping in mind that it arises from an external electric current (e.g., in a wire) (see Section 3). The unit of μ_m is in H m^{-1} where H is the symbol for the *henry* [1 H m^{-1} = 1 T m A^{-1} = 1 N A^{-2}].

Usually, a material is characterised by its **relative permeability**, μ_r (see the *Electronics* course):

$$\mu_r = \mu/\mu_0 \tag{1.2}$$

where μ_0 is the magnetic permeability in vacuum (see Table 1).

A closely-related property of a material is the **magnetic (volume) susceptibility**, χ_m. It also measures the degree a material responds to the presence of an applied magnetic field but is dimensionless:

$$\chi_m = \frac{\text{Magnetisation within the material}}{H_{ext}} \tag{1.3}$$

The material's magnetisation represents the magnetic dipole moment [A m^2] (see Section 2.3.2) per unit volume [A m^{-1}].

A permanent magnet is a material which can be magnetised by an external magnetic field and remains magnetised after the external field has been removed. The property responsible for this effect is the **ferromagnetism** or the **ferrimagnetism**. Such materials have a high, positive susceptibility to magnetic fields, and are efficiently attracted by a magnet. They are used in electromagnets, transformers (see Section 4.3), and inductors (see Chp 5).

Examples of a **ferromagnetic material** include iron, nickel, and cobalt. Examples of a **ferrimagnetic material** include mixed oxides, known as ferrites, which give their name to the ferrimagnetism.

Another example of ferrimagnetic material is the mineral magnetite (Fe_3O_4). A lodestone is a rare form of a naturally-magnetised piece of magnetite. A freely suspended elongated lodestone tends to align approximatively with the north–south (pointing truly towards the magnetic north); it was historically used in magnetic compasses. Lodestones were also used to magnetise iron and steel needles (in compass) by rubbing the needle with the lodestone.

In contrast to permanent magnets, **paramagnetic material** do not remain magnetised after the external magnetic field has been removed. They do not retain any magnetisation. Such materials have a low, though positive, magnetic susceptibility. Examples include aluminium, lithium, and magnesium.

Finally, **diamagnetic materials**, such as copper, water, and glass, have a weak, negative susceptibility to magnetic field: they are slightly repelled by an external magnetic field and do not retain any magnetic properties when the external magnetic field is removed.

1.3 Magnetic flux

The **magnetic flux** Φ through a surface $\mathscr{S}$ is defined by the following surface integral:

$$\Phi = \iint_{\mathscr{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} \tag{1.4}$$

It is expressed in *weber*, symbol Wb [1 Wb = 1 T m^2].

The magnetic flux is the amount of field threading through the surface $\mathscr{S}$. The surface can be open (see Fig.1.1) or closed (see Fig.1.2).

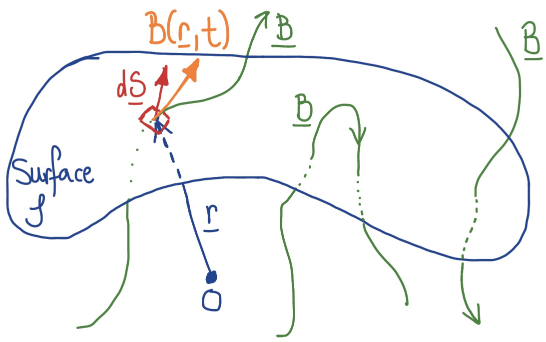

Figure 1.1 A set of magnetic field lines (in green) crossing through an open surface $\mathscr{S}$ (blue). The magnetic flux is derived by assessing $\mathbf{B} \cdot \mathrm{d}\mathbf{S}$ at each point $\mathbf{r}$ of the surface: the surface element, $\mathrm{d}\mathbf{S}$ (red), is locally perpendicular to the surface S and $\mathbf{B}(\mathbf{r}, t)$ (orange) is the magnetic field vector at position vector $\mathbf{r}$. The magnetic field lines are tangent to the local magnetic field vector. O corresponds to the origin of the coordinate system.

1.4 Source: No magnetic monopole

Empirically, the net (total) magnetic flux through a *closed* surface $\mathscr{S}$ (see Fig.1.2) is always zero:

$$\oiint_{\mathscr{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0 \tag{1.5}$$

↪ This equation is sometimes called **Gauss's Law for Magnetism** (integral form), this is one of the four Maxwell's equations. The circle on the surface integral means that the surface $\mathscr{S}$ is closed.

Let's look at the consequence of Eq.(1.5). Every magnetic field line, that enters a closed volume, leaves it again. The magnetic field lines do not have a beginning or an end. Every single magnetic field line is a **loop**. This is fundamentally different from what you saw for electric field lines.

Recall from vector fields that, for a general vector field $\mathbf{D}$, the **Divergence Theorem** (see *Vector Calculus*) states:

$$\iiint_{\mathscr{V}} \nabla \cdot \mathbf{D} \, \mathrm{d}V = \oiint_{\mathscr{S}} \mathbf{D} \cdot \mathrm{d}\mathbf{S} \tag{1.6}$$

where the closed surface $\mathscr{S}$ encloses the volume $\mathscr{V}$.

Combining Eq.(1.5) and Eq.(1.6) valid for any surface $\mathscr{S}$, we can make $\mathscr{S}$ infinitesimally small, which yields that everywhere in space:

$$\nabla \cdot \mathbf{B} = 0 \tag{1.7}$$

↪ This is the differential form of **Gauss's Law for Magnetism**. It states that the magnetic field is a divergence-free field. Unlike electric fields $\mathbf{E}$, magnetic fields $\mathbf{B}$ are not generated by stationary magnetic charges or **monopoles** (isolated magnet with only one pole).

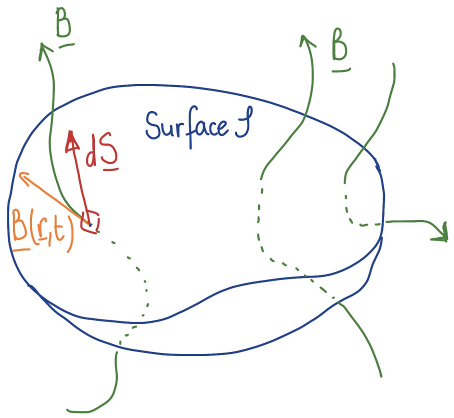

Figure 1.2 Same as Fig.1.1, but for a closed surface $\mathscr{S}$ (blue) enclosing a volume V (similarly to a balloon). In that case, the total magnetic flux through $\mathscr{S}$ is zero.

Recall **Gauss's Law** for electric fields (see the *Electrostatics* course), another Maxwell's equation. The differential and integral forms are respectively:

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0} \tag{1.8}$$

$$\oiint_{\mathscr{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q_{encl}}{\epsilon_0} \tag{1.9}$$

where ρ_q is the charge density (electric charge per unit volume [C m^{-3}]) and Q_{encl}, the total electric charge included in the volume $\mathscr{V}$ enclosed by $\mathscr{S}$. Electric fields are generated by stationary charged particles (with an electric charge).

For magnetic fields, there are no equivalent single stationary magnetic charges or monopoles. Eq.(1.5) and Eq.(1.7) are hence sometimes referred as **no magnetic monopole** equations.

↪ So, if you do not have a permanent magnet (see Section 1.2), how are magnetic fields generated? They are generated by charged particles in motion (see Chapter 3). However, we are going to look first at how charged particles are affected by magnetic fields (see Chapter 2).

Chapter 2

Motion of moving charges in a magnetic field

2.1 Lorentz force on a charged particle

→ **Textbooks**: Refer to Y&F 27.4 and G 5.1.2

↪ A charge q in motion in an electromagnetic field experiences the **Lorentz force**:

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{2.1}$$

where $\mathbf{v}$ is the velocity of the charge. It is composed of:

- the electric force: $\mathbf{F_{elec}} = q\,\mathbf{E}$ (see the *Electrostatics* course)
- the magnetic force: $\mathbf{F_{mag}} = q\,\mathbf{v} \times \mathbf{B}$

Use the Right Hand Rule (RHR) to get the direction of $\mathbf{F_{mag}}$ (see Fig.2.1): using your right hand, point your index along $\mathbf{v}$, your middle finger, along $\mathbf{B}$ and the thumb will point along $\mathbf{F_{mag}}$ for a positive charge. If you are not at ease with the vector product, please review it (see Mathematical Methods in the Physical Sciences, by M.L.Boas (3^{rd} edition), Chp3, Sect.4, p.103; Mathematical Methods for Physics and Engineering, by K.F. Riley et al., Sect. 7.6, p.222).

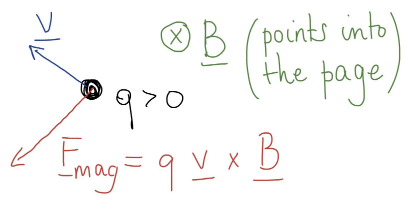

Figure 2.1 The magnetic force (red) results from the vector product between the particle's velocity (blue) and the magnetic field vector (green); its pointing depends on the sign of the charge. It would point in the opposite sense for $q < 0$.

The magnetic force $\mathbf{F_{mag}}$:

- depends on $\mathbf{v}$, hence it is not a (typical) conservative force.

- is perpendicular to $\mathbf{B}$, so the magnetic field lines, tangent to the local magnetic field vector $\mathbf{B}$, are *not* lines of force (unlike electric field lines).

Work W done: $\mathrm{d}W = \mathbf{F_{mag}} \cdot d\mathbf{r}$. But: $\mathbf{F_{mag}} = q\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{B}$, which is perpendicular to $\mathrm{d}\mathbf{r}$ Hence: $\mathbf{F_{mag}} \cdot \mathrm{d}\mathbf{r} = 0$. Magnetic force can never do work on a particle.

↪ The **kinetic energy of a particle is not changed**, but the particle can accelerate. Indeed, $\mathbf{F_{mag}}$ is perpendicular to $\mathbf{v}$, so $\mathbf{F_{mag}}$ cannot change the magnitude of the velocity, only its direction.

Gyration motion: Consider a particle moving perpendicular to $\mathbf{B}$ in a *uniform* magnetic field (see Fig.2.2). The magnetic force is always perpendicular to $\mathbf{v}$ and $\mathbf{B}$, and is constant since $|\mathbf{v}|$ and $|\mathbf{B}|$ are constant. Hence the motion is **circular**.

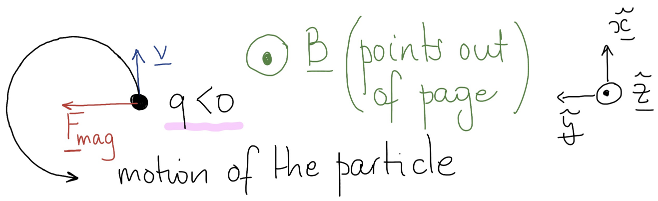

Figure 2.2 Gyration motion of a negatively-charged particle in a uniform magnetic field. Here $\mathbf{v}$ is perpendicular to $\mathbf{B}$, meaning, $v_z = 0$. The gyro-motion would be clockwise for a particle positively charged. Always be careful about the sign of the electric charge.

From Newton's Second Law applied to the charge particle of mass m moving along a direction perpendicular to a uniform magnetic field $\mathbf{B}$:

$$m\, r\, \omega^2 \;=\; q\, v\, B \tag{2.2}$$

where $v = \omega\, r$, where ω is the angular speed and r, the radius of the orbit. Hence:

$$\omega \;=\; \frac{q\, B}{m} \tag{2.3}$$

where ω is called **cyclotron (angular) frequency** or **gyro-frequency**. Note that:

- ω is independent of r & v: all particles of same mass and charge have the same gyro-motion and gyro-frequency. This is the basis of the cyclotron particle acceleration.
- though it is often called "frequency", ω is an angular speed or angular frequency: $\omega = 2\pi f$, where f is the "true" frequency. The period T is $1/f$.

The gyro-motion is characterised by the **radius of gyration**, also called the *Larmor radius*, **gyro-radius**, or **cyclotron radius, defined as**:

$$r_L \;=\; \frac{m\, v_\perp}{q\, B} \tag{2.4}$$

where $v_\perp$ refers to the component of the particle's velocity which is perpendicular to $\mathbf{B}$.

In the example considered above (see Fig.2.2), the motion of the particle is reduced to the plane of gyration (plane perpendicular to $\mathbf{B}$), that is, the ($\hat{\mathbf{x}}$–$\hat{\mathbf{y}}$) plane: $\mathbf{v}_\perp = \mathbf{v}$.

↪ What is happening regarding the motion in the $\hat{\mathbf{z}}$ direction? The magnetic force has no component along this direction, so the speed of the particle along this direction remains unchanged (assuming that no other force acts on the particle). If $v_z \neq 0$, the motion is **helical**: it has the shape of an helix (see Fig.2.3).

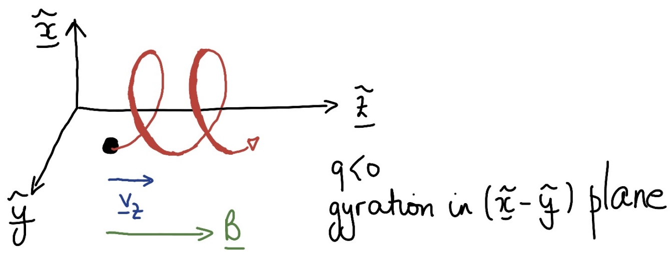

Figure 2.3 Motion (red) of a negatively-charged particle in a uniform magnetic field. The particle's velocity has a component along $\hat{\mathbf{z}}$ which is combined with the gyration motion in the ($\hat{\mathbf{x}}$–$\hat{\mathbf{y}}$) plane, hence the helical motion.

↪ Practice of the material covered in this section is offered in *Problem Sheet 1* (Q1-Q4). Q3 illustrates the principle of a mass spectrometer.

2.2 Hall effect

→ **Textbooks**: Refer to Y&F 27.9

Charges moving in an electrical conductors in a magnetic field $\mathbf{B}$ are subject to the Lorentz force (see Fig.2.4). This has the effects of inducing a potential difference across the conductor, the so-called **Hall voltage**.

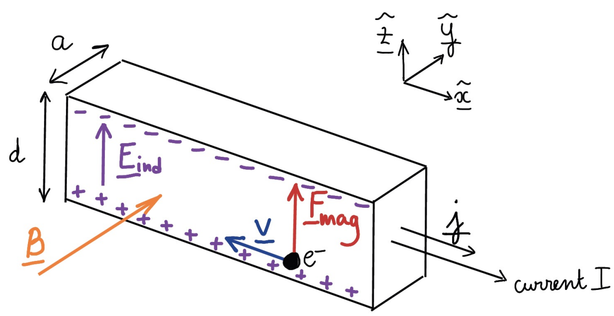

Figure 2.4 Conductor in which charge carriers are electrons (e.g., metals). In the presence of a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{y}}$, the charge carriers (here electrons) which were moving originally along ($-\hat{\mathbf{x}}$), are deflected towards the top of the conductors by the magnetic force, $\mathbf{F}_{\mathbf{mag}}$ along $\hat{\mathbf{z}}$. This yields charge separation which drives an induced electric field $\mathbf{E}_{\mathbf{ind}} = E_{ind}\hat{\mathbf{z}}$. This field exercises an electric force $\mathbf{F}_{\mathbf{elec}}$ (along ($-\hat{\mathbf{z}}$) as $q < 0$), which opposes $\mathbf{F}_{\mathbf{mag}}$.

Let's call n, the number density of charge carriers (number of charges per unit volume, unit: m^{-3}). The **current density** $\mathbf{j}$ (unit: $\mathrm{A\ m^{-2}}$ where $[1\ \mathrm{A} = 1\ \mathrm{C\ s^{-1}}]$) is given by:

$$\mathbf{j} = n\, q\, \mathbf{v} \tag{2.5}$$

The current density $\mathbf{j}$ is along $\hat{\mathbf{x}}$; it is always oriented along the direction of motion of the positive charges, independent of the sign of the charge carriers.

The cross-sectional area A of the conductor carrying a current I is given by: $A = a\, d$. Hence, the current density is given by:

$$\mathbf{j} = \frac{I}{A}\hat{\mathbf{x}} \tag{2.6}$$

Combining Eq.(2.5) and (2.6) yields the velocity of the charge carriers:

$$\mathbf{v} = \frac{I}{A\, n\, q}\hat{\mathbf{x}} \tag{2.7}$$

This expression corresponds to the drift velocity, velocity of the flow of charges. For metal, the charge carriers are electrons ($q < 0$): the particle's (drift) velocity $\mathbf{v}$ is along $(-\hat{\mathbf{x}})$.

Due to the presence of a magnetic field $\mathbf{B}$, the charge carriers experience a magnetic force $\mathbf{F_{mag}} = q\,\mathbf{v} \times \mathbf{B}$ which deflects charges. $\mathbf{F_{mag}}$ is along $\hat{\mathbf{z}}$ (see Fig.2.4). The charge carriers, here electrons, are deflected upwards, so there is charge separation: more electrons at the top and fewer at the bottom of the conductor. This induces an electric field $\mathbf{E_{ind}}$ across the conductor, which acts *against* the deflection of the electrons. The electrons experience an electric force $\mathbf{F_{elec}} = q\,\mathbf{E_{ind}}$ along $(-\hat{\mathbf{z}})$.

Electrons continue to accumulate at the top of the conductors until an equilibrium is reached. This is reached when the Lorentz force (Eq.(2.1)) on electrons becomes zero:

$$\mathbf{E_{ind}} = -\,\mathbf{v} \times \mathbf{B} \tag{2.8}$$

As $\mathbf{v}$ is perpendicular to $\mathbf{B}$ and from Eq.(2.7), this implies:

$$\mathbf{E_{ind}} = -\,\frac{I\, B}{A\, n\, q}\hat{\mathbf{z}} \tag{2.9}$$

The sign of $\mathbf{E_{ind}}$ depends on the charge of the charge carriers. As the carriers are here electrons, $\mathbf{E_{ind}}$ is along $(+\hat{\mathbf{z}})$. If the sign of the charge carriers is not known, it can be derived from the measure of $\mathbf{E_{ind}}$.

The **potential difference**, V_H, between opposite edges of the electrical conductor (see *Electrostatics*) due to the applied magnetic field is called **Hall voltage** or **Hall electromotive force**: $\mathbf{E_{ind}} = -\nabla V_H$. It can be measured by a voltmeter connected on each side of the conductor along the z-axis. Its magnitude is:

$$V_H = E_{ind}\, d = v\, B\, d = B\, d\, I/(A\, n\, |q|) \tag{2.10}$$

Depending on which physical quantities are known or measured, Eq.(2.10) can be used to derive information on the magnetic field magnitude or the number density: **Hall probe**. There are industrial applications of it.

$\hookrightarrow$ An application of the Hall effect to a biker is proposed in *Problem Sheet 1* (Q5).

2.3 Current-carrying loops

2.3.1 Magnetic force on wires

→ **Textbooks**: Refer to Y&F 27.6 and 27.7, and G 5.1.3

Charges moving in a conductor, such as a wire, are usually representing a current. Current-carrying wires in the presence of a magnetic field are subject to a force. Let's assess it.

Consider charges which are moving at a velocity $\mathbf{v}$ in a wire.

Thin current-carrying wire: For a thin wire (infinitesimally thin wire, so do not have to worry about the Hall effect), there are n_ℓ charges *per metre*. Moving at speed v, the associated current, I (which is a scalar quantity), is:

$$I = n_\ell \, q \, v \tag{2.11}$$

Consider a vector length element, $\mathrm{d}\boldsymbol{\ell}$, along a wire (see Fig.2.5). There are $n_\ell \, \mathrm{d}\ell$ charges moving at a velocity $v \, \mathrm{d}\hat{\boldsymbol{\ell}}$.

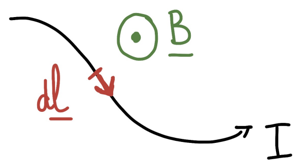

Figure 2.5 Vector length element, $\mathrm{d}\boldsymbol{\ell}$, along a wire (black) carrying a current I and placed in a magnetic field, $\mathbf{B}$. The element is oriented in the same direction as the current.

The magnetic force experienced by one charge due to $\mathbf{B}$ is: $\mathbf{F_{mag}} = q\mathbf{v} \times \mathbf{B}$.
Hence, the total force on all charges present in length $\mathrm{d}\boldsymbol{\ell}$ is:

$$\mathrm{d}\mathbf{F_{mag}} = (n_\ell \, \mathrm{d}\ell) \, q \, v \, \mathrm{d}\hat{\boldsymbol{\ell}} \times \mathbf{B} = n_\ell \, q \, v \, \mathrm{d}\boldsymbol{\ell} \times \mathbf{B} = I \mathrm{d}\boldsymbol{\ell} \times \mathbf{B} \tag{2.12}$$

↪ The **total magnetic force on a wire** of length $\mathscr{L}$, carrying a current I, in the presence of a magnetic field, $\mathbf{B}$, is:

$$\mathbf{F_{mag}} = \int_0^{\mathscr{L}} I \, \mathrm{d}\boldsymbol{\ell} \times \mathbf{B} \tag{2.13}$$

Though $\mathbf{B}$ can change along the wire, we usually have constant $\mathbf{B}$.

For a straight wire, though $\mathbf{B}$ does not have to be perpendicular to the wire, if it is, Eq.(2.13) is reduced to:

$$F_{mag} = B \, I \, \mathscr{L} \tag{2.14}$$

where B is assumed to be constant. I is usually constant along the wire.

Current-carrying loops:

In general, currents flow in wire loops. What is the net force in such a case?

Consider a rectangular, current-carrying loop $\mathscr{C}$ in a uniform magnetic field (see Fig.2.6).

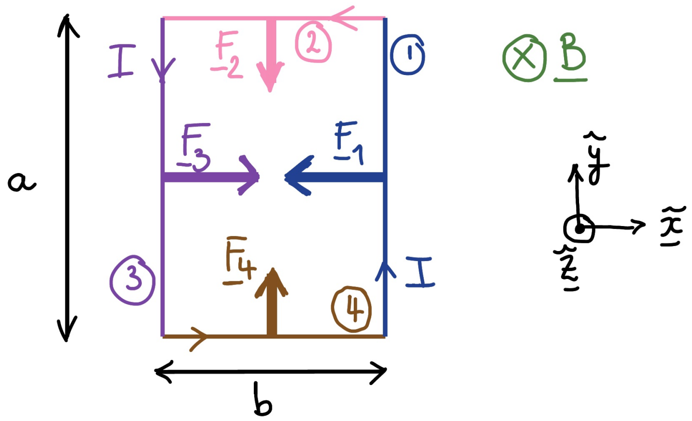

Figure 2.6 Rectangular, current loop placed in a magnetic field, $\mathbf{B}$. A magnetic force, $\mathbf{F_X}$ $(X = 1, 2, 3, 4)$, acts on each of the segments of the current loop.

The magnetic force acting on each of four segments of the current loop is (cf. Eq.(2.13)):

$$\begin{aligned} \mathbf{F_1} &= -\,B\,I\,a\,\hat{\mathbf{x}} \\ \mathbf{F_2} &= -\,B\,I\,b\,\hat{\mathbf{y}} \\ \mathbf{F_3} &= +\,B\,I\,a\,\hat{\mathbf{x}} \\ \mathbf{F_4} &= +\,B\,I\,b\,\hat{\mathbf{y}} \end{aligned} \tag{2.15}$$

This yields a total force on the closed loop:

$$\mathbf{F_{mag}} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \mathbf{F_4} = \mathbf{0} \tag{2.16}$$

One can always decompose complex-shape planar loop into many rectangular loops (with two opposite edges being part of the irregular loop). By summing their contributions and as the magnetic forces from the two other edges cancel out, one can show that the net force on the irregular, planar, current loop is zero (e.g., see Fig.27.34 in Young and Freedman, 2015).

↪ There is **no net force on any current-carrying closed loop** in a *uniform* magnetic field:

$$\mathbf{F_{mag}} = \oint_{\mathscr{C}} I\,\mathrm{d}\boldsymbol{\ell} \times \mathbf{B} = \mathbf{0} \tag{2.17}$$

2.3.2 Magnetic torque and moment

→ **Textbooks**: Refer to Y&F 27.7

Consider a rectangular current-carrying loop placed in a uniform field, $\mathbf{B} = B\hat{\mathbf{x}}$:

- The angle between the normal $\hat{\mathbf{n}}$ to the loop plane and the magnetic field direction is θ; it also corresponds to the angle between the loop plane and the vertical (see Fig.2.7).
- The segments (1) and (3), when projected in the (x–y) plane, are parallel to the x-axis, while the segments (2) and (4) are parallel to the y-axis.
- The current loop is carrying a current, I.

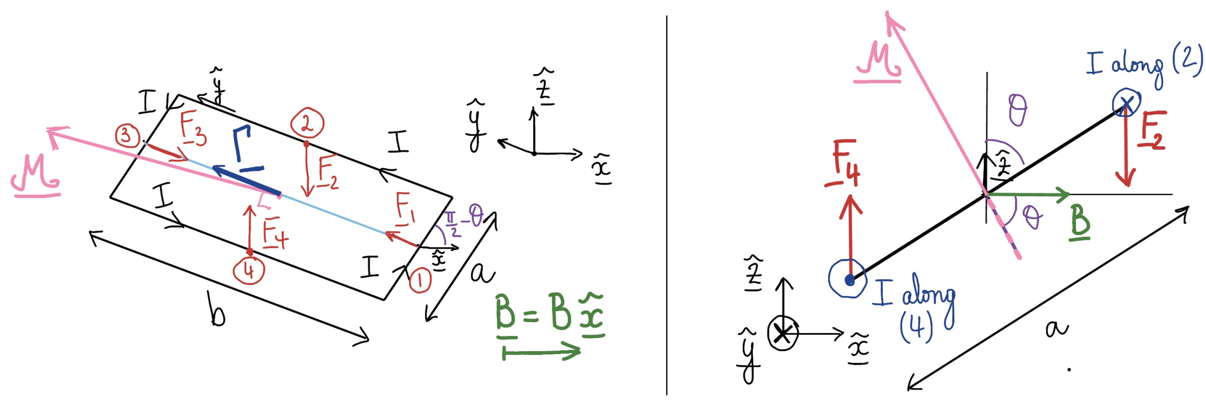

Figure 2.7 (Left) Rectangular current-carrying loop placed in a magnetic field, $\mathbf{B}$. A magnetic force, $\mathbf{F_X}$ ($X = 1, 2, 3, 4$), acts on each of the segments of the current loop. The central (rotation) axis corresponds here to the y axis (thin, light blue line). The magnetic (dipole) moment $\boldsymbol{\mathcal{M}}$ (pink) is perpendicular to the plane of the loop, while the magnetic torque $\boldsymbol{\Gamma}$ is along $\hat{\mathbf{y}}$. (Right) Same as on the left, but side view in the (x–z) plane. The central (rotation) axis as well as the torque are perpendicular to the page.

Although there is no net force on the current loop in a uniform magnetic field (see Section 2.3.1 and Eq.(2.17)), there is usually a non-zero torque around the central axis (corresponding here to the y-axis):

- You have seen the concept of a torque in the *Mechanics* course: **Torque $\boldsymbol{\Gamma}$** = Distance $\mathbf{r}$ to the axis of rotation × Force $\mathbf{F}$
- The magnetic force, $\mathbf{F_2}$, acting on the segment (2), located at a distance $\mathbf{r_2} = ((a/2)\sin\theta, 0, (a/2)\cos\theta)$ from the origin (centre of the loop, on the y-axis), is: $\mathbf{F_2} = -I\,B\,b\,\hat{\mathbf{z}}$.
- The torque $\boldsymbol{\Gamma_2}$, acting on segment (2), is: $\boldsymbol{\Gamma_2} = \mathbf{r_2} \times \mathbf{F_2} = B\,I\,b(a/2)\sin\theta\,\hat{\mathbf{y}}$

- The torque, $\mathbf{\Gamma_4}$, on segment (4), is the same as $\mathbf{\Gamma_2}$.
- Segments (1) and (3) are associated with magnetic forces which are aligned (with the y-axis) and of opposite directions, so their net torque with respect to any point is zero.

The total magnetic torque, $\mathbf{\Gamma}$, on a current loop is:

$$\mathbf{\Gamma} = A\,B\,I\,\sin\theta\,\hat{\mathbf{y}} \tag{2.18}$$

where $A = a\,b$ corresponds to the area of the loop.

We define a *vector area*, $\mathbf{A}$, such that $|\mathbf{A}|$ is the area of the loop and $\hat{\mathbf{A}}$ is perpendicular to the plane containing the loop and oriented with respect to the current in a *Right-Hand Sense* (RHS) (see Fig.2.8). Then:

$$\mathbf{\Gamma} = I\,\mathbf{A} \times \mathbf{B} \tag{2.19}$$

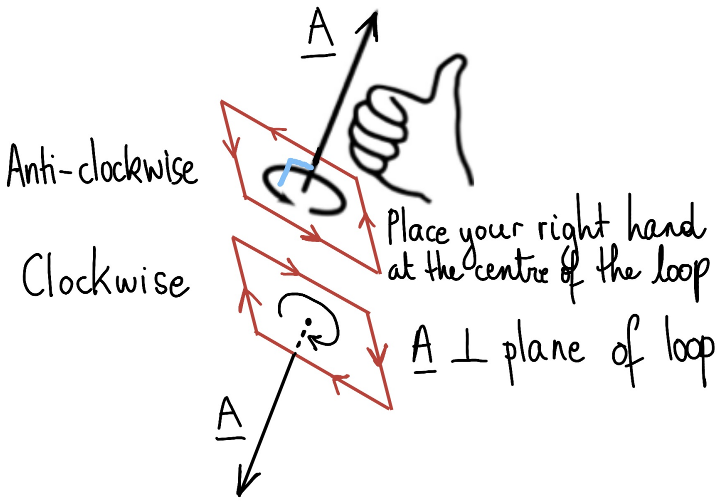

Figure 2.8 $\mathbf{A}$ is oriented with respect to the current in a right-hand sense. Put your right hand inside the loop, which can be of any shape. Apply the right-hand rule closing your fingers (except the thump); your fingers are curved in the same direction as the current flowing through the loop (red); your thump points in the same direction as $\mathbf{A}$. "Clockwise" and "anti-clockwise" on the schematic refer to the orientation as seen from above.

$\hookrightarrow$ Let's define the **magnetic dipole moment $\mathcal{M}$** of a current-carrying planar wire loop by:

$$\mathcal{M} = I\,\mathbf{A} \tag{2.20}$$

This is the vector associated with a current I flowing in a loop $\mathbf{A}$. For the example given in Fig.2.7, $\mathbf{A}$ has components along $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$. The relation (2.20) is valid for any arbitrary shapes for the loop, as long as the loop is planar.

↪ The total **magnetic torque** on a loop of magnetic dipole moment $\boldsymbol{\mathcal{M}}$ due to the presence of a magnetic field $\mathbf{B}$ is given by:

$$\boldsymbol{\Gamma} = \boldsymbol{\mathcal{M}} \times \mathbf{B} \tag{2.21}$$

This is a key result. **Magnetic field lines can exert torques on current loops!**

Note that for the example illustrated in Figure 2.7, θ is the angle between $\boldsymbol{\mathcal{M}}$ and $\mathbf{B}$.

The torque vector $\boldsymbol{\Gamma}$ is in the plane of the loop and is perpendicular to $\mathbf{B}$. If $\mathbf{B}$ is perpendicular to the plane of the loop, that is, $\mathbf{B} \parallel \mathbf{A}$ (both vectors parallel or anti-parallel), the torque is zero.

Note: If we have N loops, as in a coil (see Fig.2.9), we can say:

- either $\boldsymbol{\mathcal{M}} = N\, I\, \mathbf{A}$ and then $\boldsymbol{\Gamma} = \boldsymbol{\mathcal{M}} \times \mathbf{B}$
- or define $\boldsymbol{\mathcal{M}}$ for one loop: $\boldsymbol{\mathcal{M}} = I\, \mathbf{A}$ and then write: $\boldsymbol{\Gamma} = N\, \boldsymbol{\mathcal{M}} \times \mathbf{B}$.

Figure 2.9 Coil with N loops.

2.4 Direct-current motors

→ **Textbooks**: Refer to Y&F 27.8

Electric motors use electrical energy to do mechanical work.

Consider the current-carrying wire loop in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{x}}$ illustrated in Fig.2.7. The loop is free to rotate about an axis, here the y axis.

Let's consider the stable equilibrium, position with no torque ($\Gamma = 0$) and when $\boldsymbol{\mathcal{M}}$ and $\mathbf{B}$ are parallel (pointing in the same direction, $\theta = 0°$).

To rotate the loop originally in this equilibrium position, we have to do *work* on it.

↪ The work W done against the torque to rotate the loop from 0° to an angle ψ is:

$$W = \int_{\theta=0}^{\psi} |\Gamma| \, \mathrm{d}\theta = \int_{\theta=0}^{\psi} \mathcal{M}B \sin\theta \, \mathrm{d}\theta = [-\mathcal{M}B \cos\theta]_0^{\psi} = \mathcal{M}B\,(1 - \cos\psi) \qquad (2.22)$$

where $\Gamma(\theta)$ is the torque experienced by the loop.

- Conservative force, as the work only depends on loop orientation angle (start/end, not of what happened in between, for instance independent of the angular speed).

- Maximum work for $\psi = 180°$ when $\boldsymbol{\mathcal{M}}$ and $\mathbf{B}$ are anti-parallel (pointing in opposite directions). This is an unstable equilibrium position. If moving slightly away from this position, the loop will rotate to go back to the stable equilibrium position ($\theta = 0°$).

- Key to check signs and senses of the magnetic torque, of the magnetic dipole moment, and of the current

- Identify who is doing work on whom

If we start from the unstable equilibrium, the loop will work on us towards stable equilibrium. How to have it keep going beyond the stable equilibrium? If we reverse the sense of the current in the loop, the stable equilibrium position will become an unstable equilibrium and the loop will continue rotating: we have a motor!

An illustration of a simple **motor** is given in Fig.2.10. To make a motor, we need to switch the sense of the current every half rotation: this is done through a **commutator**. This keeps the sense of the torque the same (for a given orientation of the loop) and hence allows the loop to keep rotating in the same direction.

The open-ended current-carrying loop (one turn loop here) represents the moving part of the motor, the so-called **rotor**. The ends of the rotor wires are attached to circular conducting segments (isolated from each other) that form a commutator (and which rotates with the rotor). Each segment makes contact with a metal brush, which is connected to an external circuit that includes a source of an **ElectroMotive Force** (EMF) which causes a current to flow into/out of the rotor. For reminder, examples of a device which provides an EMF are batteries (converting chemical energy into electrical energy).

- When the rotor is in an horizontal position (see Fig.2.10 (left)), it is experiencing a torque, which causes the rotor to rotate counter-clockwise such that $\boldsymbol{\mathcal{M}}$ will align with $\mathbf{B}$.

- When this is reached (stable equilibrium, rotor in a vertical position), the brush is in contact with both segments of the commutator and no current flows through the rotor, as shown in Fig.2.10 (centre). Because of inertia, the rotor continues its counter-clockwise rotation.

- When the rotor goes beyond the vertical position, current flows again through the rotor but in the opposite direction: each segment of the commutator makes now contact with the other brush.

- Having switched the sense of current means that the vertical position illustrated in Fig.2.10 (centre) has switched from stable to unstable equilibrium. The rotor keeps rotating counter-clockwise.

- When the rotor reaches the horizontal position (Fig.2.10 (right)), the magnetic moment, and thus the magnetic torque, are in the same direction as at the start (Fig.2.10 (left)). The rotor continues rotating.

This is an idealised view of a motor, but we are focusing here on the physics involved, not on the engineering details.

We do a fixed amount of work per rotation, so faster rotation produces more power (corresponding of the amount of work per unit time). Note that the torque is not changed by increasing the rotation rate. To increase the power, one can:

- increase the magnetic field (strong magnets)

- increase the number of turns in the loop (magnetic moment increased)

- have multiple loops at different angles (smoother and more efficient rotation)

Tesla cars have peak power over 700 kW (power dissipated).

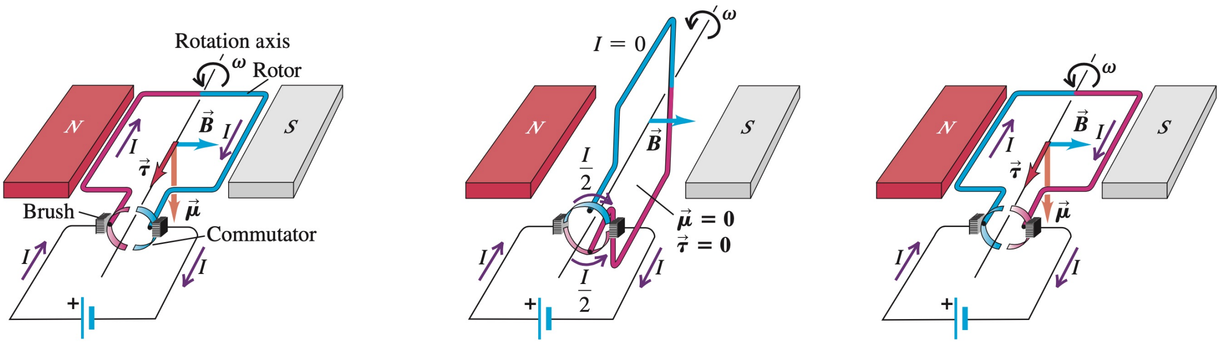

Figure 2.10 Schematic diagram of a simple direct-current motor. Note that on the schematic, the torque Γ is called τ. Note that on this figure, the magnetic moment, $\boldsymbol{\mathcal{M}}$, is identified as $\boldsymbol{\mu}$ (Figure from Young and Freedman, 2015, Section 27.8)

Chapter 3

Generation of magnetic fields by moving charges

3.1 Biot-Savart Law

→ **Textbooks**: Refer to Y&F 28.2 and G 5.2.1.

The infinitesimal magnetic field, $\mathrm{d}\mathbf{B}$, generated at point $\mathbf{r}$ by a current element $I\mathrm{d}\boldsymbol{\ell}$ located at $\mathbf{r}_\ell$ is given by:

$$\mathrm{d}\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\boldsymbol{\ell}\times\hat{\mathbf{r}}_0}{r_0^2} \tag{3.1}$$

where r_0 is the distance from the element to where the field is measured, at point P in Fig.3.1. The unit vector $\hat{\mathbf{r}}_0$ (and hence vector $\mathbf{r}_0 = r\hat{\mathbf{r}}_0 = \mathbf{r}-\mathbf{r}_\ell$) is oriented from the current element towards where the field is measured. I has to be steady (time-independent).

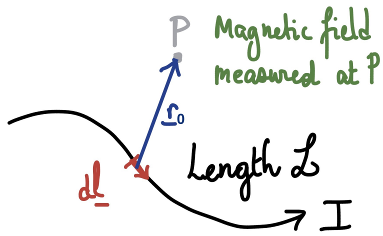

Figure 3.1 Vector length element, $\mathrm{d}\boldsymbol{\ell}$, part of a wire (black) carrying a current I. The current element, $I\mathrm{d}\boldsymbol{\ell}$, generates an infinitesimal magnetic field $\mathrm{d}\mathbf{B}$ at point P located at $\mathbf{r}$ from the element.

Biot-Savart Law: The total magnetic field vector generated at point P by the part of the wire of length $\mathscr{L}$ is given by:

$$\mathbf{B} = \int_{\text{wire}} \mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi}\int_0^{\mathscr{L}} \frac{I\,\mathrm{d}\boldsymbol{\ell}\times\hat{\mathbf{r}}_0}{r_0^2} \tag{3.2}$$

↪ See *Problem Sheet 2* (Q3) and *VFEM Seminar 4* for practice on how to apply the Biot-Savart Law and calculate magnetic fields from current configuration.

3.2 Ampère's Law

→ **Textbooks**: Refer to Y&F 28.4-6 and G 5.3.3.

3.2.1 The Law

Ampère's Law (integral form): The line integral of the magnetic field around a closed path $\mathscr{C}$ is proportional to the *net* current I through an open surface $\mathscr{S}$ enclosed by the path $\mathscr{C}$ and producing the magnetic field $\mathbf{B}$:

$$\oint_{\mathscr{C}} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 I_{\text{encl}} \tag{3.3}$$

where $\mathrm{d}\boldsymbol{\ell}$ is a vector segment (or vector length element) along the loop $\mathscr{C}$. The current passing through the surface $\mathscr{S}$ is: $I_{encl} = \iint_{\mathscr{S}} \mathbf{j} \cdot \mathrm{d}\mathbf{S}$ where $\mathbf{j}$ is the current density. The loop $\mathscr{C}$ is oriented consistently with the (positive) direction of the current (and surface $\mathscr{S}$), following the right-hand rule (see Fig.3.2). The integral form is given by:

$$\oint_{\mathscr{C}} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \iint_{\mathscr{S}} \mathbf{j} \cdot \mathrm{d}\mathbf{S} \tag{3.4}$$

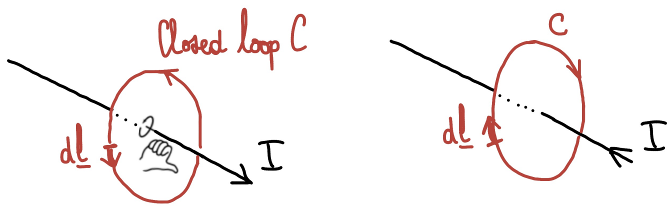

Figure 3.2 Orientation of path $\mathscr{C}$ with respect to the positive direction in which the current I is flowing (I positive here). Put your right hand inside the loop and your thumb pointing towards the direction of the current. The orientation of the loop $\mathscr{C}$ is given by the other fingers when you curl them, as illustrated.

↪ Note that, a total enclosed net current reduced to zero (because there is no current flowing through or because the currents cancel out) does not necessarily imply that the magnetic field $\mathbf{B}$ is zero everywhere along the path $\mathscr{C}$. **Ampère's Law** only says that its line integral along the closed loop is.

↪ **Ampère's Law** is only valid in the absence of time-varying electric fields and of external fields (e.g., magnetic material) and in the presence of steady current (magnetostatics). In the presence of time-varying electric fields, Maxwell added another term; with this new term, the equation becomes **Maxwell-Ampère's Law**.

Ampère's Law (differential form):

Applying **Stokes' Theorem** (see *Vector Calculus* course) to the vector field $\mathbf{B}$,

$$\oint_{\mathscr{C}} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \iint_{\mathscr{S}} (\nabla \times \mathbf{B}) \cdot \mathrm{d}\mathbf{S} \tag{3.5}$$

Eq.(3.4) can be re-written as:

$$\iint_{\mathscr{S}} (\nabla \times \mathbf{B}) \cdot \mathrm{d}\mathbf{S} = \mu_0 \iint_{\mathscr{S}} \mathbf{j} \cdot \mathrm{d}\mathbf{S} \tag{3.6}$$

Since this is true for any surface $\mathscr{S}$ (including an infinitesimally small one), this implies that everywhere in space:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{3.7}$$

↪ Practice **Ampère Law**: see *PS2* (Q1-Q2), *PS3* (Q1), and Sections 3.2.2–3.2.3.

3.2.2 Application to a long, straight wire

→ **Textbooks**: Refer to Y&F 28.7 (Example 28.7)

Let's calculate the magnetic field $\mathbf{B}$ generated by a current flowing through a long, straight wire. Consider the cylindrical coordinate system $(\rho,\ \phi,\ z)$ where the z-axis is along the wire, pointing in the same direction as the current.

The Biot-Savart Law, Eq.(3.1), implies that:

- the magnetic field $\mathbf{B}$ produced by the wire does not have any component along the wire, that is, along $\hat{\mathbf{z}}$: $B_z = 0$
- $\mathrm{d}\boldsymbol{\ell} \times \hat{\mathbf{r}}_0$, hence $\mathbf{B}$, are along $\hat{\boldsymbol{\phi}}$, as $\mathbf{r_0}$ has only components along $\hat{\boldsymbol{\rho}}$ and $\hat{\mathbf{z}}$ and $I\mathrm{d}\boldsymbol{\ell}$, the current element, is along $\hat{\mathbf{z}}$.

Regarding the zero radial component of $\mathbf{B}$, we can alternatively invoke **Gauss's Law for Magnetism**: no start/end to a magnetic field lines or apply the integral form to a cylinder placed along the wire. Additionally, the cylindrical symmetry of the problem implies that B is independent of ϕ and z. As a result, the magnetic field is reduced to its azimuthal component, which is only a function of ρ:

$$\mathbf{B}(\mathbf{r}) = B_\phi(\rho)\hat{\boldsymbol{\phi}} \tag{3.8}$$

Because of this symmetry, a good choice for the integration path $\mathscr{C}$ is a circle symmetric about the wire (see Fig.3.3).

Let's apply **Ampère's Law** to the loop $\mathscr{C}$:

- The current flowing through a surface enclosed by $\mathscr{C}$ is I carried by the wire.
- As the current I is out of the page, the loop $\mathscr{C}$ is oriented anti-clockwise (Right Hand Rule).
- As $\mathbf{B}$ is aligned with $\mathrm{d}\boldsymbol{\ell}$ at any point of the circle $\mathscr{C}$ (see Eq.(3.8)),

$$\oint_{\mathscr{C}} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = 2\pi\rho B_\phi \tag{3.9}$$

- Hence, **Ampère's Law** is reduced to: $2\pi\rho B_\phi = \mu_0 I$

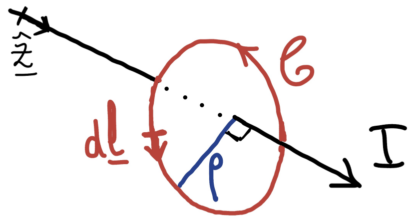

Figure 3.3 Application of Ampère's Law to a long, straight wire (black) carrying a current I to assess the magnetic field over a circle, $\mathscr{C}$, of radius ϕ and symmetric about the wire (i.e., centred on the wire, in a plane perpendicular to the wire). The vector length element, $d\boldsymbol{\ell}$, is oriented along $\hat{\boldsymbol{\phi}}$ for a loop $\mathscr{C}$ oriented anti-clockwise. The wire is assumed to be very thin, its cross section is neglected. For the impact of the wire cross section, see Q2 in *Problem Sheet 2*.

$\hookrightarrow$ The magnitude B $(= |B_\phi|)$ of the magnetic field vector $\mathbf{B}$ at a distance ρ of the long, straight wire carrying a current I is:

$$B = \frac{\mu_0 I}{2\pi\rho} \tag{3.10}$$

$\hookrightarrow$ The magnetic field vector, $\mathbf{B}$, is oriented along $d\boldsymbol{\ell}$ (here $\hat{\boldsymbol{\phi}}$) in a right hand sense, as illustrated in Fig.3.4.

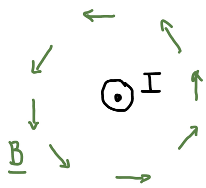

Figure 3.4 Orientation of the magnetic field vector generated by a long, straight wire carrying a current I (I positive here).

There is a magnetic field *everywhere* in space due to the presence of the current I. The radius ρ assumed for the loop $\mathscr{C}$ was arbitrary. Eq.(3.10) is valid throughout space.

3.2.3 Application to a long, straight solenoid

$\rightarrow$ **Textbooks**: Refer to Y&F 28.7 (Example 28.9)

A solenoid (or coil) is a wire with multiple parallel loops, which amplifies the magnetic field compared to just having one loop. Here there is only vacuum within the volume of the coil; if it is not the case, one needs to consider the relative permeability μ_r of whatever material we place within this volume (see Section 1.2 and the *Basic Electronics* course).

Let's calculate the magnetic field generated by an infinitely long, straight solenoid with $\mathscr{N}$ turns per metre and carrying a current I (see Fig.3.5).

Consider a rectangular loop $CDEF$ of length $\mathscr{L}$ (blue in the figure). The current I_{encl} crossing through the rectangular loop is oriented into the page (shown in orange in the figure); hence, from the Right Hand Rule, the loop $CDEF$ is oriented clockwise. Its magnitude is: $I \mathscr{N} \mathscr{L}$.

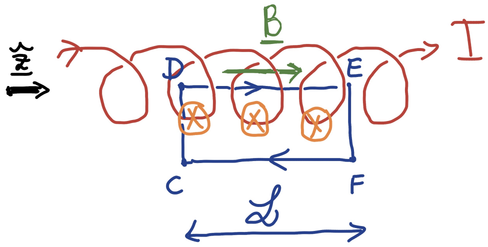

Figure 3.5 Current I (red) is flowing through the long, straight solenoid with $\mathscr{N}$ turns per metre. The vertical rectangle $CDEF$ (blue) of length $\mathscr{L}$ is going through the centre of the solenoid and used as integration path for applying **Ampère's Law**.

↪ The magnetic field $\mathbf{B}$, perpendicular to the current, points along the solenoid and is uniform inside: $\mathbf{B} = B_z\hat{\mathbf{z}}$. The magnetic field is zero outside the solenoid. As same amount of current independently of the width DC or EF, the magnetic field does not have a radial component.

From **Ampère's Law**:

$$\oint_{CDEF} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \int_{DE} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = B_z \mathscr{L} = \mu_0 I \mathscr{N} \mathscr{L} \tag{3.11}$$

↪ The magnetic field generated inside a long, straight solenoid is:

$$B = \mu_0 \mathscr{N} I \tag{3.12}$$

↪ The magnetic field vector, $\mathbf{B}$, points along $\mathrm{d}\boldsymbol{\ell}$, hence, along $+\hat{\mathbf{z}}$.

For a solenoid with a finite length, the magnetic field configuration is more complex, as illustrated in Fig.3.6.

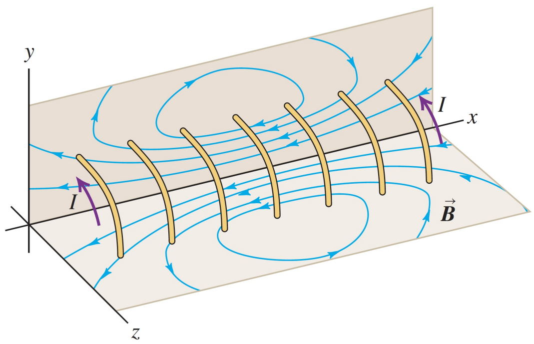

Figure 3.6 Magnetic field lines (blue) generated by a solenoid of finite length. Only a few loops (yellow) of current (violet) are shown. (After Young & Freedman (2015), p. 962).

3.3 Force between current-carrying, parallel wires

Consider two long, parallel wires, distant a apart, both carrying a current I (see Fig.3.7). Here, the currents are parallel (same direction and sense).

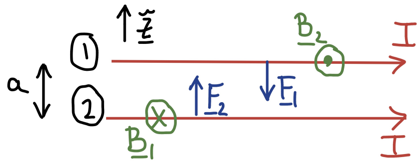

Figure 3.7 Long wire (1) carrying current I generates a magnetic field $\mathbf{B_1}$ at wire (2), which experiences force $\mathbf{F_2}$. Similarly, the long wire (2) carrying current I generates a magnetic field $\mathbf{B_2}$ at wire (1), which experiences force $\mathbf{F_1}$.

Based on Ampère's Law, current-carrying wire (1) generates, at the location of wire (2), a magnetic field $\mathbf{B_1}$ oriented into the page (see Eq.(3.10)):

$$B_1 = \frac{\mu_0 I}{2\pi a} \tag{3.13}$$

Wire (2) carries the current I. Over length $\mathscr{L}$ it experiences a force oriented towards wire (1) (see Eq.(2.13)):

$$\mathbf{F}_{2,\mathscr{L}} = I \int_0^{\mathscr{L}} \mathrm{d}\boldsymbol{\ell} \times \mathbf{B} = B_1 I \mathscr{L}\, \hat{\mathbf{z}} \tag{3.14}$$

↪ The force per unit length experienced by wire (2) and induced by wire (1) is:

$$\mathbf{F_2} = B_1 I\, \hat{\mathbf{z}} = \frac{\mu_0 I^2}{2\pi a}\, \hat{\mathbf{z}} \tag{3.15}$$

↪ Wire (1) feels an equal but opposite force, $\mathbf{F_1} = -\,\mathbf{F_2}$.

Force between the two long, current-carrying wires:

- If the currents are **parallel**, the force is **attractive**.
- If the currents are **anti-parallel**, the force is **repulsive**.

Chapter 4

Time-varying magnetic fields

→ **Textbooks**: Refer to Y&F 29.1.

Michael Faraday made key experiments at the Royal Institution of Great Britain in London in the 19^{th} century. All the following actions induce an **ElectroMotive Force** (EMF) [which allows the conversion of a form of energy into an electric current flowing in a circuit] and hence induce a current in the coil of wire:

- Moving a magnet towards or away from the coil
- Moving a second coil towards or away from the original coil
- Changing the current in the second coil

All these experiments change the magnetic flux through the coil. The process at play is **electromagnetic induction**: how a varying magnetic flux through a circuit (e.g., loop of wire or a coil) acts as a source of current in the circuit. Though there are two phenomena involved here as we will see, the observable phenomenon depends only on the relative motion of the coil (conductor) and the magnetic field (magnet or secondary coil).

↪ You have done some of these experiments this year during the First Year Lab.

4.1 Faraday's Law

→ **Textbooks**: Refer to Y&F 29.2+29.5 and G 7.2.1.

Faraday's Law of Induction (general): The EMF induced in a closed loop $\mathscr{C}$ is equal to the negative of the *rate of change* of the magnetic flux through the loop:

$$\epsilon = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \tag{4.1}$$

where the magnetic flux $\Phi = \iint_{\mathscr{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S}$ and $\mathscr{S}$ is an open surface bounded by the loop $\mathscr{C}$ (see Section 1.3). Do notice the minus sign in Eq.(4.1). This form of Faraday's Law is always true, even when the path $\mathscr{C}$ is not stationary.

There are two different ways to have a change in the magnetic flux.

$$\mathrm{d}\Phi = \iint_{\mathscr{S}(t+\mathrm{d}t)} \mathbf{B}(t+\mathrm{d}t)\cdot \mathrm{d}\mathbf{S} \;-\; \iint_{\mathscr{S}(t)} \mathbf{B}(t)\cdot \mathrm{d}\mathbf{S} \tag{4.2}$$

The magnetic flux Φ can change in time due to:

- a time-varying loop $\mathscr{C}$ where here at the position of $\mathscr{C}$ a conductor has to be placed (e.g., moving a loop of wire through a steady magnetic field or changing the shape of the loop). In that case, the **EMF is magnetic**, meaning, related to moving charges: **motional EMF** (see Section 4.4).
- a time-varying magnetic field. In that case, the **EMF is electric**. An electric field is induced everywhere in space by the change in magnetic field.

In the rest of the chapter (except in Section 4.4), we focus on the latter, that is: ↪ the magnetic field $\mathbf{B}$ is varying in time, while $\mathscr{C}$ **is stationary.**

In that case, Eq.(4.2) is reduced to:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \iint_{\mathscr{S}} \frac{\mathbf{B}(t+\mathrm{d}t)-\mathbf{B}(t)}{\mathrm{d}t}\cdot \mathrm{d}\mathbf{S} = \iint_{\mathscr{S}} \frac{\partial \mathbf{B}}{\partial t}\cdot \mathrm{d}\mathbf{S} \tag{4.3}$$

Experimentally, if a wire is placed at the location of the loop $\mathscr{C}$ enclosing $\mathscr{S}$:

- a current flows through this wire if $\mathbf{B}$ is changing with time
- there is no current if $\mathbf{B}$ is steady.

So how can a magnetic field act on charges at rest?

↪ A **time-varying magnetic field induces an electric field**, which in turn induces a current.

↪ The electric field is induced independently of the presence of charges at $\mathscr{C}$.

The EMF in the loop $\mathscr{C}$ results from the induced electric field $\mathbf{E}$:

$$\epsilon = \oint_{\mathscr{C}} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} \tag{4.4}$$

where the RHS is the line integral of the induced electric field $\mathbf{E}$ around the closed loop.

Combining Eq.(4.1), (4.3) and (4.4) yields the **integral form of Faraday's Law**:

$$\oint_{\mathscr{C}} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\iint_{\mathscr{S}} \frac{\partial \mathbf{B}}{\partial t}\cdot \mathrm{d}\mathbf{S} \tag{4.5}$$

where $\mathscr{C}$ is the stationary loop that encloses the open surface $\mathscr{S}$. $\mathscr{C}$ and $\mathrm{d}\mathbf{S}$ are oriented self-consistently in a right-hand sense (see Fig.4.1).

Applying **Stokes' Theorem** to $\mathbf{E}$ over the loop $\mathscr{C}$ (see *Vector Calculus* course):

$$\oint_{\mathscr{C}} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = \iint_{\mathscr{S}} (\nabla\times\mathbf{E})\cdot \mathrm{d}\mathbf{S} \tag{4.6}$$

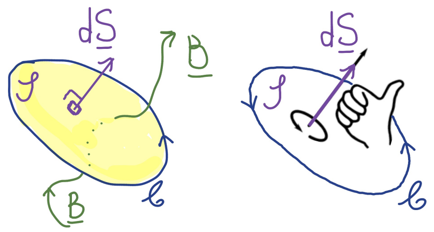

Figure 4.1 (Left) Magnetic flux Φ calculated through an open surface $\mathscr{S}$ (highlighted in yellow on the left) bounded by the closed loop $\mathscr{C}$. (Right) $\mathscr{C}$ and $\mathrm{d}\mathbf{S}$ are oriented in a right-hand sense.

Hence:

$$\iint_{\mathscr{S}} (\nabla \times \mathbf{E}) \cdot \mathrm{d}\mathbf{S} = -\iint_{\mathscr{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} \tag{4.7}$$

As Eq.(4.7) is valid for any surface $\mathscr{S}$, including infinitesimally small, if $\mathscr{S}$ is not changing in time, this yields the **differential form of Faraday's Law**:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.8}$$

$\hookrightarrow$ **Time-varying magnetic field acts as a source of electric field.**

Remember:

- Changes in a magnetic field in time induce electric fields.
- This only drives a current if mobile charges (e.g., in a wire) are present.
- Electric field is not just where we put the loop, it is everywhere.

$\hookrightarrow$ For practice of *Faraday's Law*, see *Problem Sheet 3* (Q2) along with the rest of this chapter.

4.2 Lenz's Law

$\rightarrow$ **Textbooks**: Refer to Y&F 29.3.

What is the consequence of the minus sign in **Faraday's Law**?

Faraday's Law states that:

$$\epsilon = \oint_{\mathscr{C}} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\iint_{\mathscr{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} \tag{4.9}$$

where $\mathscr{C}$ enclosing $\mathscr{S}$ is stationary here. The sense of $\mathrm{d}\mathbf{S}$ and that of $\mathrm{d}\boldsymbol{\ell}$ are defined by the Right-Hand Rule (see Fig.4.1). Let's consider $\mathrm{d}\mathbf{S}$ oriented up. In that case, $\mathscr{C}$ is oriented anti-clockwise. This also means that the magnetic flux Φ through $\mathscr{S}$ is positive for $\mathbf{B}$ along $\mathrm{d}\mathbf{S}$, here up and (see Fig.4.2).

The time-varying magnetic field induces an electric field $\mathbf{E}$. If the magnetic flux Φ is increasing over time, the induced EMF is negative: $\mathbf{E}$ (or more specifically, the component of $\mathbf{E}$ along $\mathrm{d}\boldsymbol{\ell}$) and $\mathrm{d}\boldsymbol{\ell}$ have different senses (see Fig.4.2).

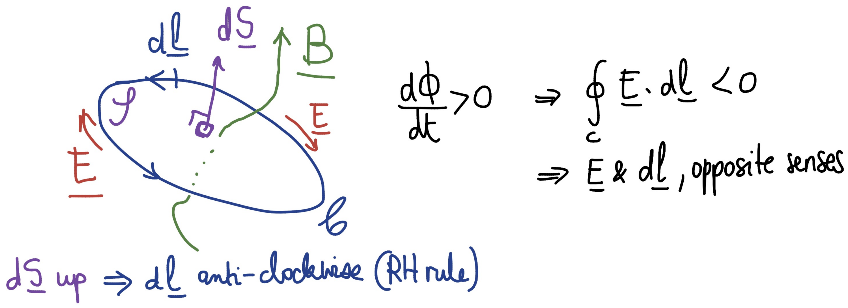

Figure 4.2 Orientation of $\mathbf{E}$ derived from **Faraday's Law**. $\mathrm{d}\mathbf{S}$ and $\mathrm{d}\boldsymbol{\ell}$ need to be self-consistent (in the right-handed sense). Here the magnetic flux Φ (calculated through an open surface $\mathscr{S}$ bounded by the closed loop $\mathscr{C}$ (blue)) is positive and is assumed to increase over time; hence, from **Faraday's Law**, the EMF ϵ is negative: $\mathbf{E}$ has an opposite sense to $\mathrm{d}\boldsymbol{\ell}$.

If mobile charges are present at $\mathscr{C}$ (e.g., wire placed there), this drives a current (see Fig.4.3). This current then produces a magnetic field (apply **Ampère's Law** to a loop perpendicular to $\mathscr{C}$ and circling I) (light green). Inside the loop, the induced magnetic field $\mathbf{B_{ind}}$ is oriented down, so negative magnetic flux: this acts against the increase in Φ originally assumed.

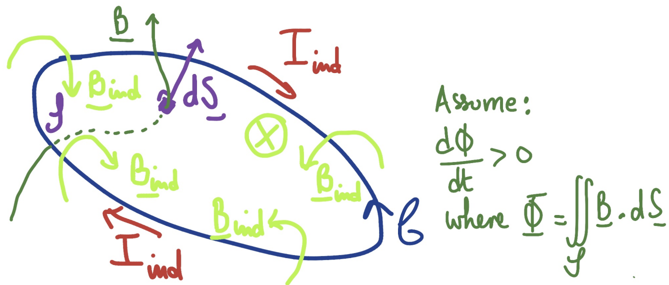

Figure 4.3 Induced magnetic field $\mathbf{B_{ind}}$ (light green) from **Ampère's Law**, resulting from the induced current I_{ind} from **Faraday's Law**, for the case of increasing Φ over time. Φ is the magnetic flux through the surface $\mathscr{S}$ enclosed by $\mathscr{C}$, a stationary loop oriented anti-clockwise as seen from above. The induced field $\mathbf{B_{ind}}$ (light green) acts such as to reduce the original magnetic flux through $\mathscr{C}$.

Lenz's Law: When a change in the magnetic field produces an induced current, the direction of the current flow is such as to produce a magnetic field which opposes the original change in the field.

↪ This is in fact the meaning of the minus sign in Faraday's Law: negative feedback.

↪ Lenz's Law is a corollary of Faraday's Law. It helps to check the direction of the induced current.

4.3 Faraday's Law and the electric field

→ **Textbooks**: Refer to Y&F 29.5.

Let's calculate the electric field induced by a conductor, here a solenoid, in the time-varying magnetic field.

Consider an infinite solenoid with $\mathscr{N}$ turns per meter, a radius a and carrying a current I (see Fig.3.5). Recall, the magnetic field generated is (see Section 3.2.3):

- inside the solenoid: $B = \mu_0 I \mathscr{N}$
- outside the solenoid: $B = 0$

To calculate the electric field at a distance ρ from the axis of the solenoid, we consider a circular contour $\mathscr{C}$ of radius ρ, symmetric around the solenoid and the surface $\mathscr{S}$ of the associated disk (see Fig.4.4). The magnetic flux through $\mathscr{S}$ is:

$$\Phi = \iint_{\mathscr{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = B\left(\pi a^2\right) = \mu_0 \mathscr{N} I \pi a^2 \tag{4.10}$$

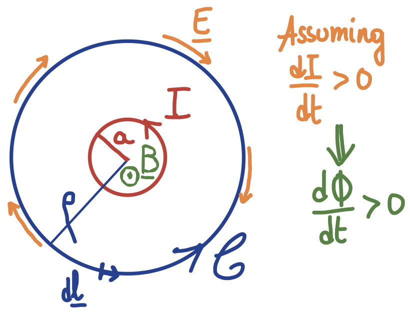

Figure 4.4 Cross section of the solenoid (red) of radius a carrying a current I. Circular contour $\mathscr{C}$ (blue) of radius ρ, centred with respect to the solenoid and in a plane perpendicular to the symmetry axis of the solenoid. The surface $\mathscr{S}$ corresponds to the associated disk. I (red) is oriented anti-clockwise, which means that $\mathbf{B}$ (green) is pointing out of the page. For $\mathrm{d}\mathbf{S}$ parallel to $\mathbf{B}$, $\mathscr{C}$ (hence $\mathrm{d}\boldsymbol{\ell}$) has to be oriented anti-clockwise. The electric field $\mathbf{E}$ induced by the varying I is shown in orange. As $\mathrm{d}I/\mathrm{d}t > 0$, $\mathrm{d}\Phi/\mathrm{d}t > 0$ and $\mathbf{E}$ is in the opposite sense to $\mathrm{d}\boldsymbol{\ell}$.

If we vary the current I with time, the magnetic flux Φ varies as well:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \mu_0 \mathscr{N} \pi a^2 \frac{\mathrm{d}I}{\mathrm{d}t} \tag{4.11}$$

Furthermore, due to the cylindrical symmetry, $\mathbf{E}$ is only a function of ρ and $E_z = 0$. Gauss's Law implies that $E_\rho = 0$ (no enclosed charge). Hence, $\mathbf{E} = E_\phi\, \hat{\boldsymbol{\phi}}$ and:

$$\oint_{\mathscr{C}} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = 2\pi\rho E_\phi \tag{4.12}$$

where E_ϕ is positive if $\mathbf{E}$ and $\mathrm{d}\boldsymbol{\ell}$ are parallel, and is negative if they are anti-parallel.

Combining **Faraday's Law**:

$$\oint_{\mathscr{C}} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \tag{4.13}$$

with Eq.(4.11) and (4.12), yields:

$$E_\phi(\rho) = -\frac{\mu_0 \mathcal{N}}{2} \frac{a^2}{\rho} \frac{\mathrm{d}I}{dt} \tag{4.14}$$

This is a circular electric field (meaning, the field lines are circles):

- The electric field lines do not have a start or end point; this is very different from the electric field generated by charges.
- This is a non-zero electric field caused by a change in the magnetic flux at a different place in the universe.

Transformer : We could put a loop of wire at $\mathscr{C}$ in the presence of a time-varying current in the solenoid: this would induce a current in the wire at $\mathscr{C}$. This leads to the concept of the **transformer** (non-examinable).

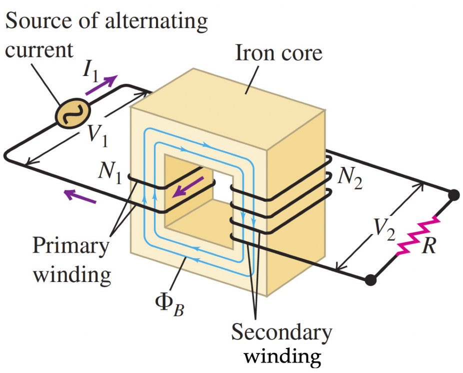

Figure 4.5 Schematic diagram of an idealized step-up transformer. The primary is connected to an AC source; the secondary is connected to a device with resistance R. (After Young & Freedman (2015), p. 1062)

We drive an alternating current in the primary winding, while we get an alternative current in the secondary winding (see Fig.4.5). By changing the number of windings in the primary (N_1) and the secondary (N_2), we can change the secondary voltage V_2 with respect to the primary, V_1: $\dfrac{V_2}{V_1} = \dfrac{N_2}{N_1}$

→ **Textbook**: Refer to Y&F 31.6.

4.4 Generators

→ **Textbooks**: Refer to Y&F 29.2 (Examples 29.3, 29.5, 29.6) and Y&F 29.4 (Example 29.9) and G 7.1.3.

A **generator** uses mechanical energy to produce electrical energy.

Side note in relation to Faraday's Law: Recall from Section 4.1 that the change in time of the magnetic flux Φ is due to a change in time of either the magnetic field $\mathbf{B}$ (e.g., Section 4.3) or the loop $\mathscr{C}$. In this section 4.4, the magnetic field is assumed steady, while the loop $\mathscr{C}$, where a conductor has to be placed, is moving at velocity $\mathbf{v}(\mathbf{r})$ over time. Hence, Eq.(4.2) is reduced to:

$$\mathrm{d}\Phi = \iint_{\mathscr{S}(t+\mathrm{d}t)} \mathbf{B} \cdot \mathrm{d}\mathbf{S} - \iint_{\mathscr{S}(t)} \mathbf{B} \cdot \mathrm{d}\mathbf{S} \tag{4.15}$$

It implies that (e.g., Textbook G 7.1.3, p. 307-308):

$$\mathrm{d}\Phi = \oint_{\mathscr{C}} \mathbf{B} \cdot (\mathbf{v} \times \mathrm{d}\boldsymbol{\ell})\, \mathrm{d}t = -\oint_{\mathscr{C}} (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\boldsymbol{\ell}\, \mathrm{d}t \tag{4.16}$$

Hence, combining with the general form of **Faraday's Law** (Eq.(4.1)), this yields:

$$\epsilon = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \oint_{\mathscr{C}} (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\boldsymbol{\ell} \tag{4.17}$$

This means is that, though the magnetic field is not changing in time here, an EMF is produced as a result of the motion of charges in the presence of a magnetic field: it is the **motional EMF**. The line integral in Eq.(4.17) represents the magnetic force per unit charge acting on the charges present in $\mathscr{C}$. On a side note, the velocity of the charges is the sum of the velocity of $\mathscr{C}$ and the drift velocity of the current, but the latter is along $\mathrm{d}\boldsymbol{\ell}$, so the integral would remain unchanged, hence the velocity in Eq.(4.17) is reduced to the velocity of the loop $\mathscr{C}$.

The aim here is to highlight the difference between varying magnetic field (see Eq.(4.5)) and moving charges in a wire placed at $\mathscr{C}$ (see Eq.(4.17)). You do not need to remember Eq.(4.15)-(4.16) and the line integral in (4.17). They are **only given for context**. What you need to be able to do concerning generators is **to apply Faraday's Law given by Eq.(4.1)**, as illustrated in Sections 4.4.1, 4.4.2, and 4.4.3.

4.4.1 Simple alternator

The magnetic flux Φ through a conducting loop is changing by rotating the loop in a fixed magnetic field (see Fig.4.6). The loop rotates at a rate ω from an external drive (e.g., you push it with your hand).

The magnetic flux through one loop is:

$$\Phi_1 = \iint_A \mathbf{B} \cdot d\mathbf{S} = BA\cos\theta = BA\cos(\omega t) \tag{4.18}$$

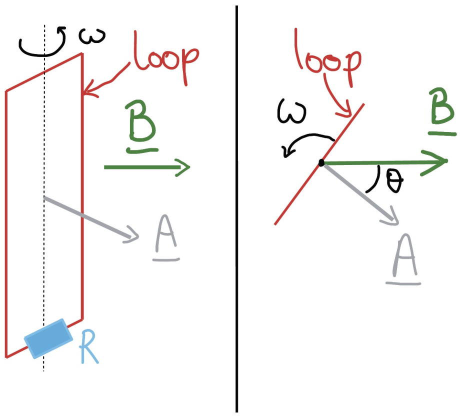

Figure 4.6 Schematic of a generator (simple alternator): (Left) The loop has an area A (perpendicular to the surface of the loop), N turns, rotates at a rate ω and is placed in a magnetic field **B**. (Right) As seen from above.

For N loops, this yields: $\Phi = BAN\cos(\omega t)$
Hence, the total change in the magnetic flux Φ is:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\,BAN\omega\sin(\omega t) \tag{4.19}$$

Faraday's Law (Eq.(4.1)) implies that the induced EMF is:

$$\epsilon = BAN\omega\sin(\omega t) \tag{4.20}$$

The current though the resistor of resistance R is:

$$I = \frac{|\epsilon|}{R} = \frac{NBA\,\omega}{R}\,\sin(\omega t) \tag{4.21}$$

The **power dissipated**, which corresponds to the electrical power generated from the mechanical motion, is:

$$P = I\,|\epsilon| = \left[\frac{NBA\,\omega}{R}\,\sin(\omega t)\right]\,[BAN\omega\sin(\omega t)] = \frac{(NBA\,\omega)^2}{R}\,\sin^2(\omega t) \tag{4.22}$$

We do not get it for free! Due to **Faraday's Law**, a current I flows in the loops. The associated magnetic dipole moment, $\boldsymbol{\mathcal{M}}$, has a magnitude:

$$\mathcal{M} = NIA = \frac{N^2BA^2\omega}{R}\,\sin(\omega t) \tag{4.23}$$

The torque on the loops is:

$$\mathbf{\Gamma} = \boldsymbol{\mathcal{M}} \times \mathbf{B} \tag{4.24}$$

Hence (see Fig.4.6):

$$\Gamma = \mathcal{M}B\,\sin\theta \tag{4.25}$$

The **power spent in rotating the loops** at the rate ω is:

$$P = \Gamma\omega = \mathcal{M}B\,\omega\sin\theta = \frac{N^2B^2A^2\omega^2}{R}\,\sin^2(\omega t) \tag{4.26}$$

This corresponds to the power dissipated (see Eq.(4.22)): **the energy is conserved!**

4.4.2 Slidewire generator

→ **Textbooks**: Refer to Y&F 29.2 (Example 29.5).

Consider a circuit made of three fixed wires and a conducting rod. The rod is pushed, moving at a speed v, in the presence of a uniform, stationary magnetic field $\mathbf{B}$ (see Fig.4.7).

The circuit has an area A which is increasing with time as the rod is moved at $\mathbf{v} = v\,\hat{\mathbf{x}}$:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathscr{L}v\mathrm{d}t}{\mathrm{d}t} = \mathscr{L}\,v \tag{4.27}$$

In addition, $\mathbf{A}$ is chosen to be pointing out of the page (upward). This means that a positive EMF would be directed anti-clockwise.

Hence, the magnetic flux through the circuit placed in the uniform, stationary magnetic field changes with time as:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\iint_A \mathbf{B}\cdot\mathrm{d}\mathbf{S} = \mathbf{B}\cdot\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = -B\,\mathscr{L}\,v \tag{4.28}$$

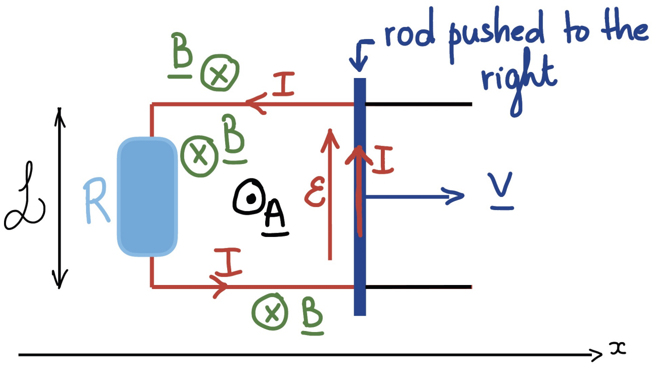

Figure 4.7 Schematic of the circuit composed of three wires, a resistor of resistance R, and a conducting rod. The circuit has an area $\mathbf{A}$ and a width $\mathscr{L}$. The rod is pushed along the horizontal wires at velocity $\mathbf{v} = v\hat{\mathbf{x}}$. The induced current in the circuit is I.

From **Faraday's Law**, this implies:

$$\epsilon = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = B\mathscr{L}v \tag{4.29}$$

This drives a current in the circuit which is anti-clockwise (positive ϵ), of magnitude:

$$I = \frac{B\mathscr{L}v}{R} \tag{4.30}$$

The **power dissipated** in the circuit is:

$$P = I^2R = \frac{B^2\mathscr{L}^2v^2}{R^2}R = \frac{B^2\mathscr{L}^2v^2}{R} \tag{4.31}$$

↪ Where does this power come from?

- We have to push the rod.
- The rod experiences a **Lorentz force** due to the current flowing through it:

$$\mathbf{F} = I \int_0^{\mathscr{L}} \mathrm{d}\boldsymbol{\ell} \times \mathbf{B} \tag{4.32}$$

  which is oriented to the left $(-\hat{\mathbf{x}})$, opposing the motion.
- The opposite force (same magnitude but to the right) has to be imposed on the rod such that it can continue moving at v.
- The magnitude of the force is:

$$F = I\mathscr{L}B = \left(\frac{B\mathscr{L}v}{R}\right)\mathscr{L}B = B^2\mathscr{L}^2\frac{v}{R} \tag{4.33}$$

- The associated **power to push the rod** is:

$$P = Fv = \frac{B^2\mathscr{L}^2v^2}{R} \tag{4.34}$$

  which is the same as the electrical power dissipated (see Eq.(4.31)).

4.4.3 Motional electric field in a slidewire generator

→ **Textbooks**: Refer to Y&F 29.4.

Let's consider the slidewire generator again. Instead of applying Faraday's Law (see Section 4.4.2), we now consider the charges in the rod as the rod moves.

In the presence of a magnetic field $\mathbf{B}$, charges undergo the **magnetic force** (see Section 2.1):

$$\mathbf{F}_{\mathbf{mag}} = q\,(\mathbf{v} \times \mathbf{B}) \tag{4.35}$$

In a metal, the mobile charges are the electrons $(q < 0)$. Hence, they experience a force $\mathbf{F}_{\mathbf{mag,e}}$ along $(-\hat{\mathbf{y}})$. As a result of the charge separation, an electric field $\mathbf{E}$, is produced to cancel out the magnetic force, $\mathbf{F}_{\mathbf{mag,e}}$. The resulting electric force, $\mathbf{F}_{\mathbf{elec,e}}$, has the same magnitude but opposite in sense to the magnetic field, $\mathbf{F}_{\mathbf{mag,e}}$. The Lorentz force, $\mathbf{F}$ (see Section 2.1), is thus reduced to $\mathbf{0}$:

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \mathbf{0} \tag{4.36}$$

Hence:

$$\mathbf{E} = -\,\mathbf{v} \times \mathbf{B} \tag{4.37}$$

↪ The electric field points along $(-\hat{\mathbf{y}})$ (see Fig. 4.8).

The total potential along the rod is given by:

$$\epsilon = E\mathscr{L} = v\mathscr{L}B \tag{4.38}$$

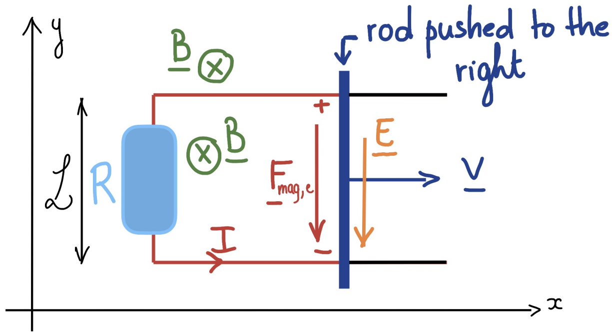

Figure 4.8 Schematic of the circuit composed of three wires, a resistor of resistance R, and a conducting rod. The rod is pushed along the horizontal wires at velocity $\mathbf{v} = v\hat{\mathbf{x}}$. The induced current in the circuit is I. Electrons feel force $\mathbf{F_e}$. This generates the motional electric field, $\mathbf{E}$.

It drives a current around the circuit given by:

$$I = \frac{\epsilon}{R} = \frac{v\mathscr{L}B}{R} \tag{4.39}$$

This is the same result as what we got when applying **Faraday's Law** (see Eq.(4.30))!

↪ These are not two separate effects; they are two ways of looking at the same phenomenon.

We have to be careful when to use the motional electric field, but it shows us that there are links between motion, electric field, and electromagnetic induction.

4.5 Frame transformations (non-examinable)

→ Non examinable, though illustrating the two ways at looking at the general form of Faraday's Law (Eq.(4.1)).

Let look at varying magnetic field in two different ways:

1. **A stationary charge with a moving magnet**
   Let's consider loop $\mathscr{C}$, a virtual, planar loop oriented counter-clockwise, of area A, and crossing the location of the positive charge q. The magnet is moving down at a constant velocity $\mathbf{v}$ through the centre of the loop.

   - As the magnet (with $\mathbf{B}$ oriented upward) moves down through the loop, the magnetic flux $\Phi = \iint_A \mathbf{B} \cdot \mathrm{d}\mathbf{S}$ is positive and increases (see Fig.4.9).
   - At the location of a positive charge q, the electric field $\mathbf{E}$ is oriented into the page, based on **Faraday's Law** (see Eq.(4.5)).
   - Hence the charge is subject to a force $\mathbf{F}$ into the page.

2. **A moving charge with a stationary magnet**

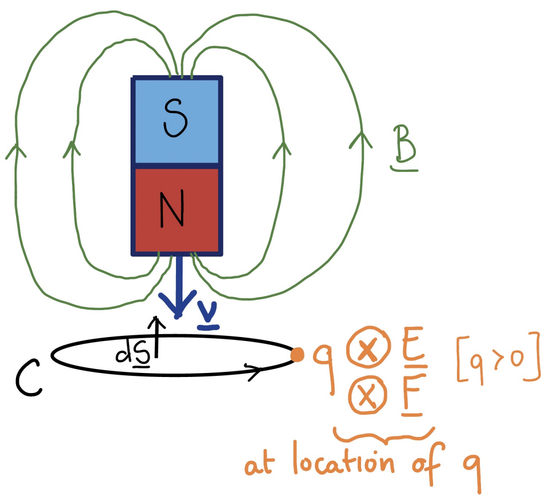

Figure 4.9 Schematic of the magnet moving at constant $\mathbf{v}$ through the virtual loop $\mathscr{C}$ centred about the magnet axis and crossing the location of the positive charge q, where an electric field $\mathbf{E}$ is produced.

- There is no electric field as the magnet, and hence the magnetic field, are stationary.
- The charge is subject to **Lorentz force** (see Section 2.1). which is oriented into the page (see Fig.4.10).

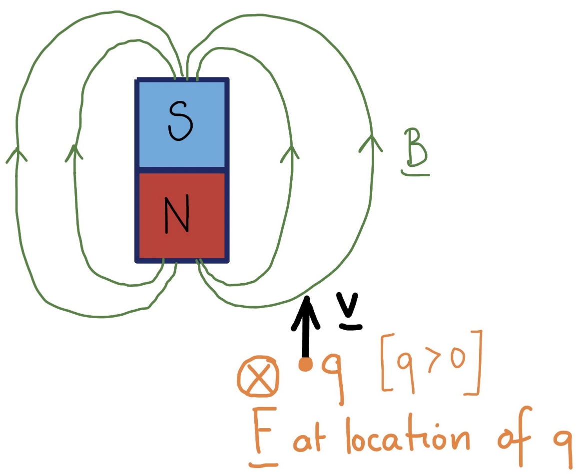

Figure 4.10 Schematic of the positive charge q moving through a magnetic field $\mathbf{B}$ generated by a stationary magnet.

↪ These two cases illustrate the same scenario, but in two different frames: one in the charge's frame (case 1), the other in the magnet's frame (case 2). Both frames are

inertial. Hence, the same force is experienced, but for different, apparent reasons.
$\hookrightarrow$ The electric field is different between two inertial frames. Electric fields are not frame invariant! (see **Relativity** course for more about this interesting property).

Chapter 5

Inductance

5.1 Self-inductance

→ **Textbooks**: Refer to Y&F 30.2.

5.1.1 Definition

Wires carrying currents generate magnetic fields (see Section 3).

Let's consider a current-carrying loop. The resulting magnetic field leads to a magnetic flux, Φ, threading through the loop (see Fig.5.1).

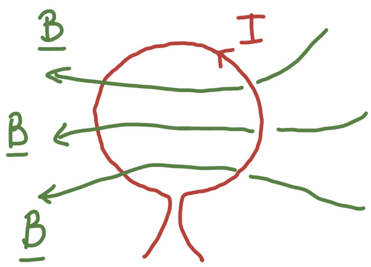

Figure 5.1 A wire loop carrying a current I generates a magnetic field $\mathbf{B}$.

Inductance: The flux Φ is proportional to I. We define the **(self-)inductance** L as this proportionality constant:

$$\Phi = L\,I \tag{5.1}$$

↪ The unit of L is the *henry*, symbol H [1 H = 1 Wb/A = 1 T m^2/A = 1 N m/A^2].

Inductor: A circuit device that is designed to have a particular inductance is called an inductor. Its symbol in a circuit diagram is illustrated in Fig.5.2.

L depends on different parameters, including the shape and size of the loop, and the number of loops. The bigger the size of the loop is, the larger is L.

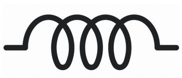

Figure 5.2 Symbol of an inductor in a circuit diagram

What happens if the current I is varied? If I varies:

- $\dfrac{\mathrm{d}\Phi}{\mathrm{d}t} \neq 0$
- **Faraday's Law** implies that an EMF is generated.
- $\epsilon = -\dfrac{\mathrm{d}\Phi}{\mathrm{d}t} = -L\dfrac{\mathrm{d}I}{\mathrm{d}t}$
- This opposes the change in the current (see Lenz's Law, Section 4.2). It would generate a current in an opposite sign.
- This leads to inertia in electrical circuits: inductors resist changes in the current in a circuit by generating a back EMF.

↪ For practice to calculate inductance, see *Problem Sheet 3* (Q2, Q3) and Section 5.1.2. See also the *Basic Electronics* course for more information.

5.1.2 How to calculate inductance

Let's consider a long, straight, thin solenoid of length $\mathscr{L}$, N turns and cross-sectional area A (see Fig.5.3). The number of turns per meter is: $\mathscr{N} = N/\mathscr{L}$.

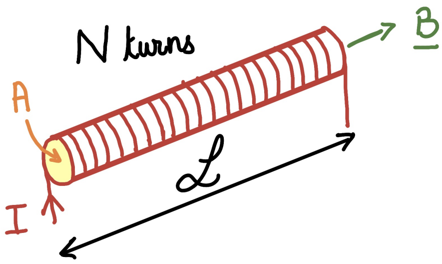

Figure 5.3 Straight solenoid of length $\mathscr{L}$, N turns, and cross-sectional area A.

When a current I is flowing through the long solenoid, the magnetic field generated is given by (see Section 3.2.3):

$$\begin{aligned} &\text{Inside: } B = \frac{\mu_0 N I}{\mathscr{L}} \text{ along the axis of the solenoid (see Fig.3.5 and 5.3)} \\ &\text{Outside: } B = 0 \end{aligned} \tag{5.2}$$

The magnetic flux is: $\Phi = BA$ for each turn.

Hence, the total flux is: $\Phi = NBA$ over the entire solenoid.
By definition (see Eq.(5.1)), the inductance for a long solenoid is:

$$L = \frac{\Phi}{I} = N\left(\frac{\mu_0 NI}{\mathscr{L}}\right)\frac{A}{I} = \frac{\mu_0 N^2 A}{\mathscr{L}} \tag{5.3}$$

Typical values: For a solenoid with length $\mathscr{L} = 10$ cm, number of loops $N = 1000$, and loop's radius $r = 1$ cm, the inductance L is (see Table 1 for μ_0):

$$L = \frac{4 \times \pi \times 10^{-7} \times 1000^2 \times \pi \times (10^{-2})^2}{0.1} = 4 \text{ mH}$$

5.1.3 Energy density of the magnetic field

→ **Textbooks**: Refer to Y&F 30.3.

An inductor carrying a current I stores energy, but where is it?
The **total energy stored** in the inductor is (see the *Basic Electronics* course):

$$U = \frac{1}{2}LI^2 \tag{5.4}$$

For the case of a solenoid (see Eq.(5.3)), the total energy is:

$$U = \frac{1}{2}\frac{\mu_0 N^2 A}{\mathscr{L}} I^2 \tag{5.5}$$

Hence, combining it with Eq.(5.2):

$$U = \frac{1}{2\mu_0}B^2 A\mathscr{L} \tag{5.6}$$

Note that: $A\mathscr{L}$ represents the volume of the solenoid.

Energy Density: Since B is constant inside the solenoid and zero outside, we conclude that the **energy density** of the magnetic field B is:

$$\frac{B^2}{2\mu_0} \tag{5.7}$$

This is a simplified proof, but the outcome still stands for a more detailed proof.
↪ An inductor carrying a current stores energy in the space around it!

For the solenoid whose parameters are given in Section 5.1.2 and for $I = 10$ A:

$$U = \frac{4 \times 10^{-3} \times 10^2}{2} = 0.2 \text{ J}$$

5.2 Mutual inductance

→ **Textbooks**: Refer to Y&F 30.1.

Let's consider two coils near each other. Each coil is part of two different circuits. If a current is flowing in one of them, it produces a magnetic field and it results in a magnetic flux threading through the other coil – and vice versa (see Fig.5.4).

For a given current I_1 in coil 1, we get a total amount Φ_2 of magnetic flux in coil 2.

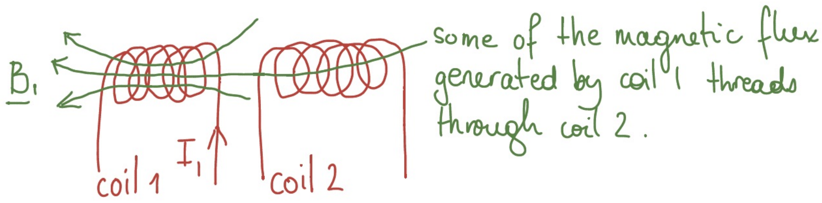

Figure 5.4 Coil 1 is carrying a current I_1 which generates a magnetic field $\mathbf{B_1}$, which extends through coil 2.

We define M_{12} as the constant of proportionality:

$$\Phi_2 = M_{12} I_1 \tag{5.8}$$

Similarly, the magnetic flux Φ_1 through coil 1 due to I_2 flowing through coil 2 is:

$$\Phi_1 = M_{21} I_2 \tag{5.9}$$

It turns out that we have always: $M_{12} = M_{21}$

Mutual Inductance: We define the **mutual inductance** as:

$$M = M_{12} = M_{21} \tag{5.10}$$

Its unit is the *henry*, symbol H.

Mutual Induction: If we vary the current I_1 in coil 1, the magnetic flux Φ_2 varies and induces an EMF ϵ_2 in coil 2 via **Faraday's Law**:

$$\epsilon_2 = -\frac{\mathrm{d}\Phi_2}{\mathrm{d}t} = -M\frac{\mathrm{d}I_1}{\mathrm{d}t} \tag{5.11}$$

Note: This only works when the current is varying. If $\frac{\mathrm{d}I_1}{\mathrm{d}t} = 0$, then $\epsilon_2 = 0$.

Mutual induction can be a good thing: it can transfer power from one place to another. This is basically the transformer that we encountered earlier (see Section 4.3). It is also the basis of wireless charging of mobile phones and electric toothbrushes.

Mutual induction can also be a bad thing: coil 1 might be an entire electrical circuit, while coil 2 might be a different circuit. If we have an AC signal in coil 1, or, for example, some digital signals like in a computer or mobile phone, they can induce voltages in coil 2. This is interference and it can be very annoying.

↪ For practice on mutual inductance, see *Problem Sheet 3* (Q4).

Chapter 6

Overview

6.1 Summing up the course

Despite its name, this section does not include everything of what needs to be known, but some important things which need to be known.

- Lorentz force: $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
- Magnetic force on a wire: $\mathbf{F} = \int_0^{\mathscr{L}} nqv \, \mathrm{d}\boldsymbol{\ell} \times \mathbf{B} = \int_0^{\mathscr{L}} I \, \mathrm{d}\boldsymbol{\ell} \times \mathbf{B}$
- Magnetic dipole moment of a closed loop: $\boldsymbol{\mathcal{M}} = I\mathbf{A}$
- Torque exerted by a magnetic field: $\boldsymbol{\Gamma} = \boldsymbol{\mathcal{M}} \times \mathbf{B}$
- Hall effect: Charges deflected until imbalance makes $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \mathbf{0}$
- Biot-Savart Law: $\mathrm{d}\mathbf{B} = \dfrac{\mu_0 I}{4\pi} \dfrac{\mathrm{d}\boldsymbol{\ell} \times \hat{\mathbf{r}}_{\mathbf{0}}}{r_0^2}$
- Ampère's Law:
  - Integral form (v1): $\oint_{\mathscr{C}} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \iint_{\mathscr{S}} \mathbf{j} \cdot \mathrm{d}\mathbf{S} \;\Leftrightarrow\; \oint_{\mathscr{C}} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 I_{encl}$
  - Differential form: $\nabla \times \mathbf{B} = \mu_0 \, \mathbf{j}$
  - μ_0 is the permeability of free space ($\mu_0 \sim 4\pi \times 10^{-7}$ H/m, see Table 1).
- Faraday's Law:
  - Integral form: $\oint_{\mathscr{C}} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\iint_{\mathscr{S}} \dfrac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S}$
  - Differential form: $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$
  - EMF: $\epsilon = -\dfrac{\mathrm{d}\Phi}{\mathrm{d}t}$
- Inductance: $\Phi = L\, I$

6.2 Maxwell's Equations

→ **Textbooks**: Refer to Y&F 29.7 and G 7.3.

Maxwell's equations describe the evolution of the electric and magnetic fields. Though their interaction, they allow electromagnetic radiation, the propagation of electromagnetic energy through space. That is light! (see *VFEM Seminar 5*).

Integral form of Maxwell's equations:

- Gauss's Law:

$$\oint\!\!\!\oint_{\mathscr{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\epsilon_0} \iiint_{\mathscr{V}} \rho_q \, \mathrm{d}V \tag{6.1}$$

- Gauss's Law for magnetism (no magnetic monopoles):

$$\oint\!\!\!\oint_{\mathscr{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0 \tag{6.2}$$

- Faraday's Law of induction:

$$\oint_{\mathscr{C}} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = - \iint_{\mathscr{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} \tag{6.3}$$

  and, in the general case of $\mathscr{S}$ varying with time:

$$\oint_{\mathscr{C}} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} + \oint_{\mathscr{C}} (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\boldsymbol{\ell} = - \frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathscr{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} \tag{6.4}$$

- Ampère-Maxwell's Law:

$$\oint_{\mathscr{C}} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \iint_{\mathscr{S}} \mathbf{j} \cdot \mathrm{d}\mathbf{S} + \mu_0 \epsilon_0 \iint_{\mathscr{S}} \frac{\partial \mathbf{E}}{\partial t} \cdot \mathrm{d}\mathbf{S} \tag{6.5}$$

where the surface $\mathscr{S}$ and volume $\mathscr{V}$ are time-independent.

Differential form of Maxwell's equations:

- Gauss's Law:

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0} \tag{6.6}$$

- Gauss's Law for Magnetism:

$$\nabla \cdot \mathbf{B} = 0 \tag{6.7}$$

- Faraday's Law of Induction:

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \tag{6.8}$$

- Ampère-Maxwell's Law:

$$\nabla \times \mathbf{B} = \mu_0 \, \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{6.9}$$

where ρ_q and $\mathbf{j}$ are the charge density $[\rho_q = e(n_i - n_e)]$ and the electric current density $[\mathbf{j} = e(n_i \mathbf{u_i} - n_e \mathbf{u_e})]$, respectively, function of position $\mathbf{r}$ and time t. n_e and n_i correspond to the electron and ion densities, respectively. Note that in the expressions of ρ_q and $\mathbf{j}$, we have assumed singly-charged ions (which is usually a good approximation).

Interestingly these equations are paired. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. Applying this relation to the electric field $\mathbf{E}$ (combining Gauss's Law and Faraday's Law) yields: $\nabla \cdot (\partial \mathbf{B} / \partial t) = 0$. Hence: $\partial / \partial t (\nabla \cdot \mathbf{B}) = 0$, therefore $\nabla \cdot \mathbf{B}$ is constant. In fact, it is zero (Gauss's Law for Magnetism), not withstanding the possible discovery one day of magnetic monopoles. Thus the 'no monopoles' law is the initial condition for the evolution of Faraday's Law.