Vector Calculus

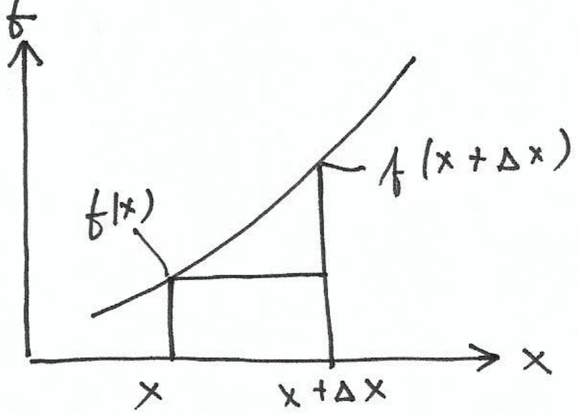
1. Differentiation (mostly review)

1.1 Ordinary differentiation

notation used here usually (Riley)

$f = f(x)$

dependent variable — function — independent variable



Derivative $\dfrac{df}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Differential $df = \dfrac{df}{dx}\, dx$ two different meanings!

df is the "tangent approximation to the change in f"
(Boas, standard)

alternatively (Leibniz) dx is "infinitesimal"
df is change in f (Riley, nonstandard, useful)

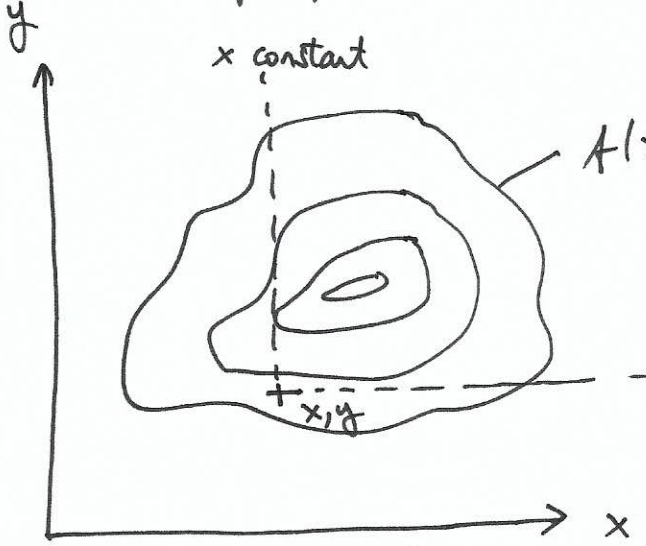
Chain rule $f = f(x(t))$

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

1.2 Partial differentiation of scalar fields

More than one independent variable

$f = f(x,y)$ a scalar field e.g. temperature



Partial derivative

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f(x,y)}{\partial x} \right)_y = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

implicit y is constant

use when needed for clarity

Total differential

$$\boxed{df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy}$$

THE KEY EQUATION

is the "tangent-plane" approximation to Δf
(or the change in f for infinitesimal changes dx, dy)

[no assumption x, y are orthogonal, only independent] example

$$f(x,y) = x^2 e^y + y^3$$

calculate the first and second partial derivatives

$$\frac{\partial f}{\partial x} = 2x e^y$$

$$\frac{\partial f}{\partial y} = x^2 e^y + 3y^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = 2e^y$$

$$\frac{\partial^2 f}{\partial y^2} = x^2 e^y + 6y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = 2x e^y$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2x e^y$$

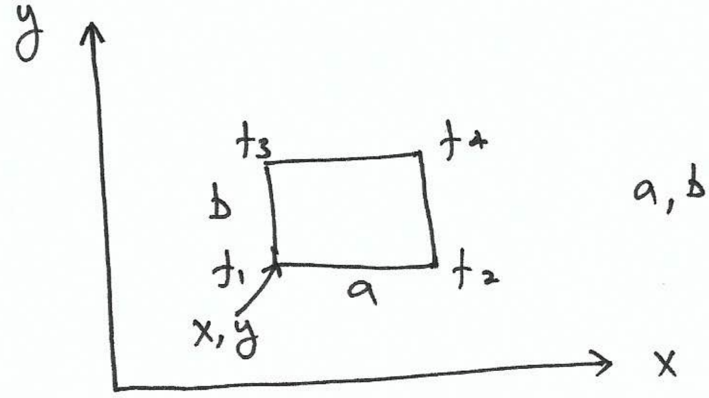
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$ Clairault's theorem

always true provided 1st + 2nd derivatives are continuous

3 A means not examinable

Clairault's theorem

very rough, not rigorous, to show the idea



a, b very small

$$\frac{\partial f}{\partial x} \simeq \frac{f_2 - f_1}{a}$$

$$\frac{\partial f}{\partial y} \simeq \frac{f_3 - f_1}{b}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \simeq \frac{\frac{(f_4 - f_3)}{a} - \frac{(f_2 - f_1)}{a}}{b} = \frac{f_1 + f_4 - (f_2 + f_3)}{a\,b}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \simeq \frac{\frac{(f_4 - f_2)}{b} - \frac{(f_3 - f_1)}{b}}{a} = \frac{f_1 + f_4 - (f_2 + f_3)}{a\,b}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

1.3 Partial differentiation of vector fields

2 D vector field defines a vector at every x, y (e.g. velocity)

$$\underline{A}(x,y) = A_x(x,y)\,\hat{\underline{i}} + A_y(x,y)\,\hat{\underline{j}}$$

e.g. $\underline{A} = xy\,\hat{\underline{i}} + x^2y^2\,\hat{\underline{j}}$

2 components defined by 2 scalar fields

By analogy define partial derivative

$$\frac{\partial \underline{A}}{\partial x} = \lim_{\Delta x \to 0} \frac{\underline{A}(x+\Delta x, y) - \underline{A}(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{A_x(x+\Delta x, y)\,\hat{\underline{i}} + A_y(x+\Delta x, y)\,\hat{\underline{j}} - A_x(x,y)\,\hat{\underline{i}} - A_y(x,y)\,\hat{\underline{j}}}{\Delta x}$$

$$\frac{\partial \underline{A}}{\partial x} = \frac{\partial A_x}{\partial x}\,\hat{\underline{i}} + \frac{\partial A_y}{\partial x}\,\hat{\underline{j}}$$

i.e. partial derivative of each component

Differential $d\underline{A} = \frac{\partial \underline{A}}{\partial x}dx + \frac{\partial \underline{A}}{\partial y}dy$

1.4 Exact differential

Really a total differential that needs to be recognised!

From before, the total differential is $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

Now a particular differential

$$A(x,y)\,dx + B(x,y)\,dy \qquad (1)$$

is a total differential of some parent function f

if $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$ $\quad$ (because $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$)

This condition is sufficient.

& so we say (1) is an exact differential.

Example 1

is $2xy\,dx + x^2\,dy$ exact?

$$\underset{A}{2xy}\,dx + \underset{B}{x^2}\,dy$$

$\frac{\partial A}{\partial y} = 2x = \frac{\partial B}{\partial x}$ $\quad$ yes, exact

What is the parent function?

$$\frac{\partial f}{\partial x}(x,y) = 2xy$$

$\therefore\ f = x^2y + g(y) \quad \Rightarrow \frac{\partial f}{\partial y} = x^2 + \cancel{\frac{\partial g}{\partial y}}\ \frac{dg}{dy}$

$\therefore\ \frac{dg}{dy} = 0 \qquad \therefore\ g = \text{const}$

$\therefore$ parent function $f = x^2y + \text{const}$

Example 2

$$2xy\,dx - x^2\,dy \qquad (2)$$

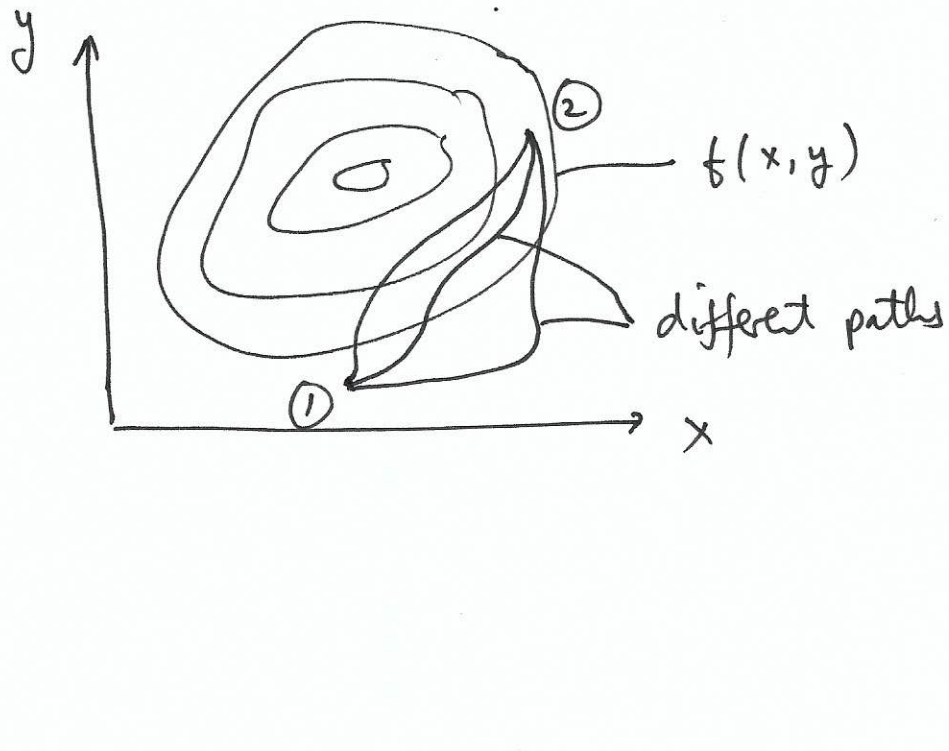
$A = 2xy$, $B = -x^2$

$$\frac{\partial A}{\partial y} = 2x \neq \frac{\partial B}{\partial x} \qquad \underline{\text{not exact}}$$

(I bet) you can't find a parent function of which (2) is the total differential

* The integral of an exact differential between two end states is independent of the path

$$\int_{①}^{②} df = \int_{①}^{②} A\,dx + \int_{①}^{②} B\,dy = f_{②} - f_{①}$$



1.5 Chain rule with partial differentiation

$f = f(x,y)$

Look at 3 cases

① $y = y(x)$ only one independent variable (x or y)

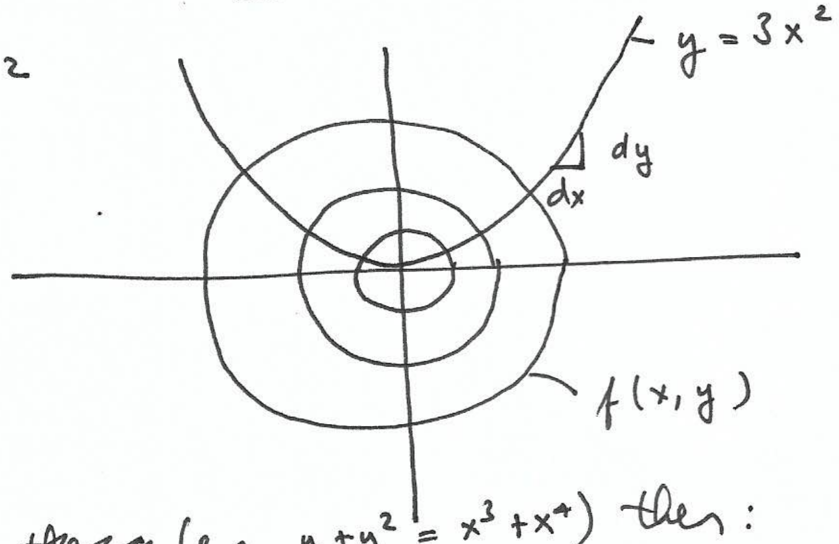
② $x = x(t)$, $y = y(t)$
only one independent variable (t)

③ $x = x(u,v)$, $y = y(u,v)$

Case ① $y = y(x)$, what is $\frac{df}{dx}$?

e.g. $f(x,y) = x^2 + y^2$

$y = 3x^2$



if you can substitute:

$f(x) = x^2 + 9x^4$

$\frac{df}{dx} = 2x + 36x^3$

maybe can't substitute, ~~thesa~~ (e.g. $y + y^2 = x^3 + x^4$) then:

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ but $dy = \frac{dy}{dx}dx$

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\frac{dy}{dx}dx$

$$\boxed{\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}}$$

check $\frac{df}{dx} = 2x + 2y \cdot 6x = 2x + 6x^2 \cdot 6x = 2x + 36x^3$ ✓

Case 2 $\quad x = x(t) \qquad y = y(t)$

$$dx = \frac{dx}{dt}\,dt \qquad dy = \frac{dy}{dt}\,dt$$

$$df = \frac{\partial f}{\partial x}\frac{dx}{dt}\,dt + \frac{\partial f}{\partial y}\frac{dy}{dt}\,dt$$

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

using same example as before

$$f(x,y) = x^2 + y^2$$

$x(t) = 2t \qquad y(t) = 12t^2 \qquad$ (this is same curve $y = 3x^2$)

$$\frac{df}{dt} = 2x.2 + 2y.24t$$

$$= 8t + 576t^3$$

same path, so check

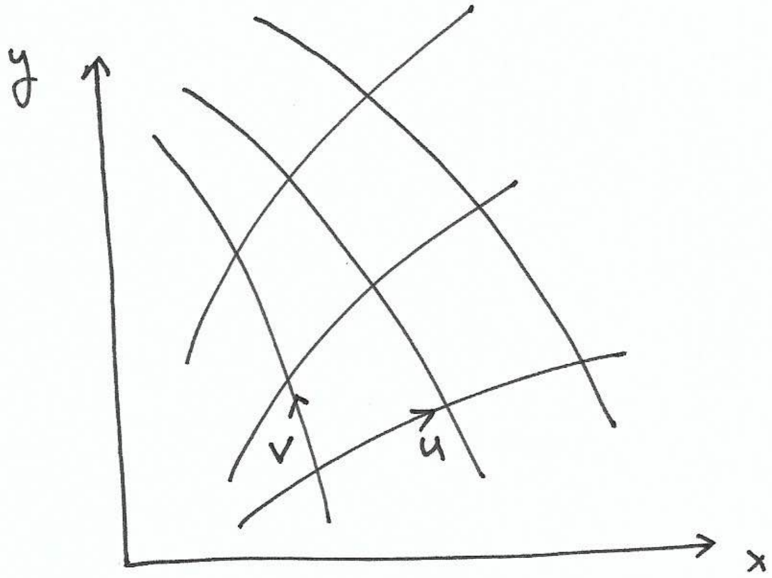
$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} = (2x + 36x^3)\,2$$

$$= 4x + 72x^3$$

$$= 8t + 576t^3 \qquad ✓$$

Case 3

$$x = x(u,v) \qquad y = y(u,v)$$



$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

now:

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

<u>Keeping v constant</u>

$$df = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}du + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}du.$$

so

$$\left(\frac{\partial f}{\partial u}\right)_v = \left(\frac{\partial f}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x\left(\frac{\partial y}{\partial u}\right)_v$$

similarly

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

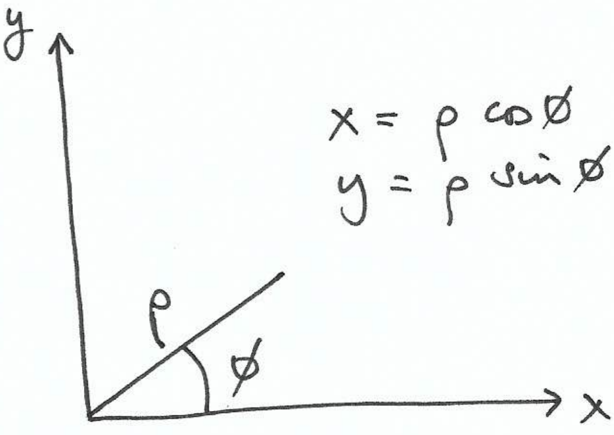
(subscripts implicit)

Sometimes refered to as "change of variables"

Example plane polar coordinates

$x(\rho,\phi)$, $y(\rho,\phi)$

find $\left(\frac{\partial f}{\partial \rho}\right)_\phi$ $\left(\frac{\partial f}{\partial \phi}\right)_\rho$



$x = \rho\cos\phi$
$y = \rho\sin\phi$

e.g. $f = x^2 + y^2$

~~$x = \rho\cos\phi$~~

Now of course $f = x^2 + y^2 = \rho^2$

so directly $\frac{\partial f}{\partial \rho} = 2\rho$ $\frac{\partial f}{\partial \phi} = 0$

Formally:

$$\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \rho}$$

$$= 2x\cos\phi + 2y\sin\phi$$

$$= 2\rho\cos^2\phi + 2\rho\sin^2\phi$$

$$\frac{\partial f}{\partial \rho} = 2\rho$$

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi}$$

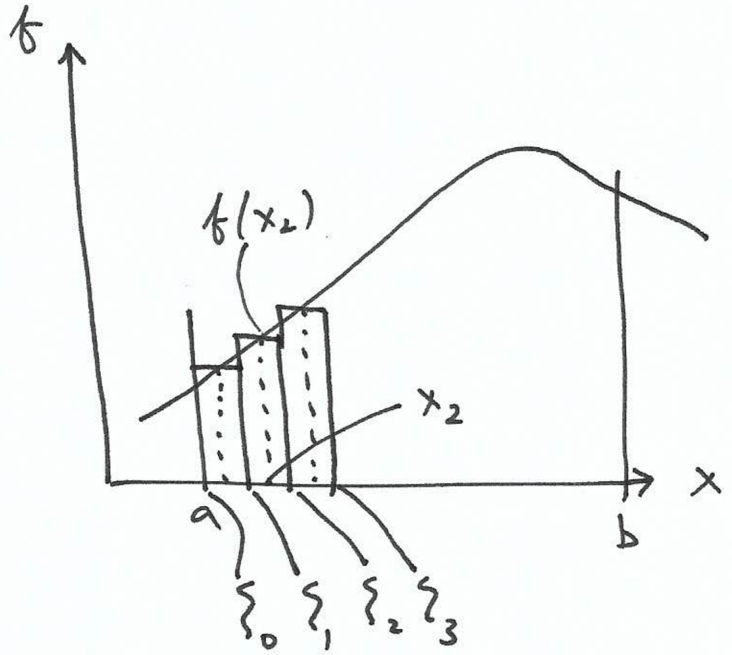
$$= 2x\rho(-\sin\phi) + 2y\rho\cos\phi$$

$$= 2\rho\cos\phi\,\rho(-\sin\phi) + 2\rho\sin\phi\,\rho\cos\phi$$

$$\frac{\partial f}{\partial \phi} = 0$$ as expected

2. 2D integration

2.1 1D integration (review)



Integral over interval a to b of $f(x)$ is limit of Riemann sum

$$S = \sum_{i=1}^{n} f(x_i)(\xi_i - \xi_{i-1})$$

x_i is any point between ξ_{i-1} and ξ_i

$\int_a^b f(x)dx$ is limit of S as $n \to \infty$

and width of each strip $\xi_i - \xi_{i-1} \to 0$ (if sum exists)

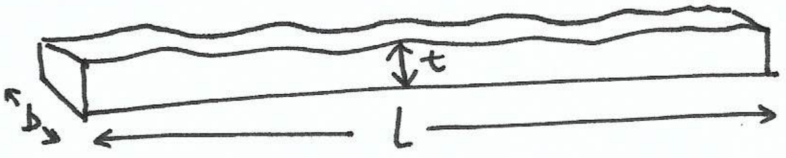
Note that widths do not have to be uniform.

1D integral is 'area under curve'.

Better to consider as "weighted sum" because of many different uses of integration.

example : a rod of uniform density ρ, width b, varying thickness $t(x)$

mass per unit length $\lambda(x) = \rho\, b\, t(x)$



length $\quad L = \int_0^L dx$

mass $\quad M = \int_0^L \lambda\, dx \qquad$ 0th moment of λ

c of grav $\quad \bar{x} = \frac{1}{M} \int_0^L \lambda\, x\, dx \qquad$ 1st moment of λ

radius of gyration $\quad x_r^2 = \frac{1}{M} \int_0^L \lambda\, x^2\, dx \qquad$ 2nd moment of λ

all weighted sums

<u>Fundamental theorem of calculus</u> (integration from first principles is hard)

provides connection between integration and differentiation

$$F(b) - F(a) = \int_a^b f(x)\, dx \quad \text{where } f = \frac{dF}{dx}$$

(simple proof e.g. Riley)

F is the anti-derivative of f.

Many possible functions F that differ by a constant.

Often we don't need to find limit of a sum, we just need to find a function that differentiates to $f(x)$.

* This is the first of several integral theorems we will meet *

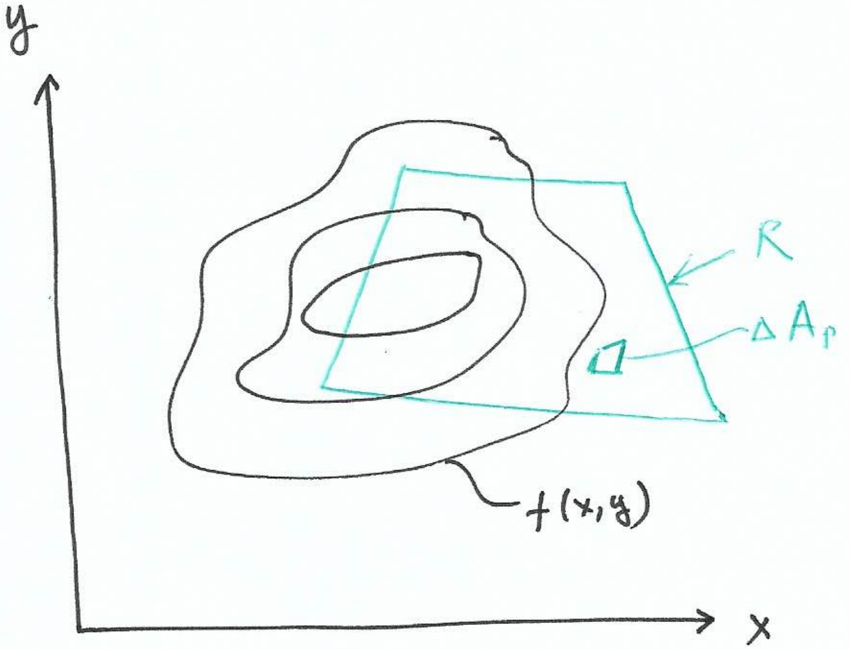
2.2 2 D integrals

The 2D integral of $f(x,y)$ over a region R

$\iint_R f(x,y)\, dA$ is the limit

of Riemann sum

$$S = \sum_{p=1}^{n} f(x_p, y_p)\, \Delta A_p$$

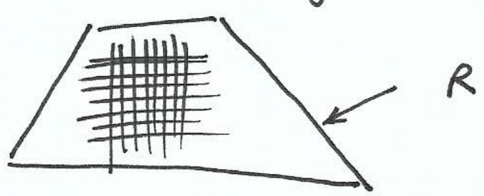


as $n \to \infty$, area $\Delta A_p \to 0$

the shape and relative sizes of ΔA_p do not matter.

2D integral is like "volume under surface", but generally is a weighted sum.

Could use uniform grid $\Delta x\, \Delta y$. Then as $n \to \infty$ area elements are infinitesimal $dA = dx\, dy$



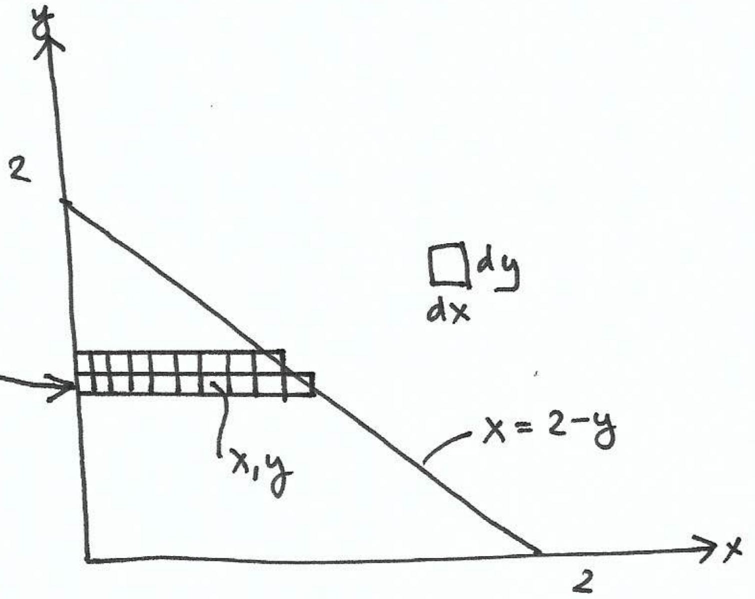
Example

$f(x,y) = x + y$

R is region bounded by $x=0$, $y=0$, $y = 2-x$

We want $\Sigma f(x,y)\,dx\,dy$

Consider sum for a single slice at fixed y

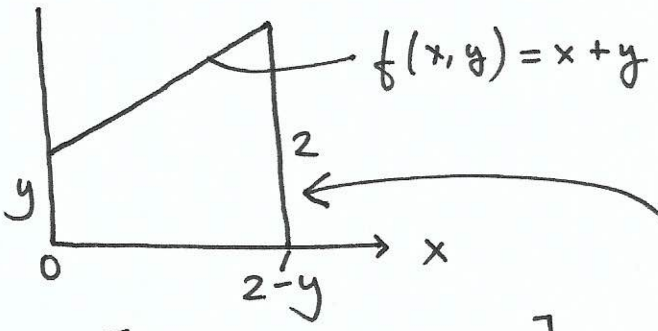


$dV = \Sigma f(x,y)\,dx\,dy$

$= dy \; \Sigma f(x,y)\,dx$

$= dy \displaystyle\int_{x=0}^{2-y} (x+y)\,dx$ (y fixed)

cross section looks like:



$dV = dy \left[\frac{x^2}{2} + yx\right]_0^{2-y} = dy \left[\frac{(4-4y+y^2)}{2} + 2y - y^2\right]$

$dV = dy\left[2 - \frac{y^2}{2}\right] = A(y)\,dy$ $\quad A(y) \equiv$ cross section of slice

Now sum up slices

$$V = \Sigma\, dV = \int_{y=0}^{2}\left(2 - \frac{y^2}{2}\right)dy = \left[2y - \frac{y^3}{6}\right]_0^2 = 4 - \frac{4}{3} = \underline{\underline{\frac{8}{3}}}$$

more compactly.

$$V = \int_{y=0}^{2} \left[\int_{x=0}^{2-y} (x+y)\,dx \right] dy$$

"iterated integral"

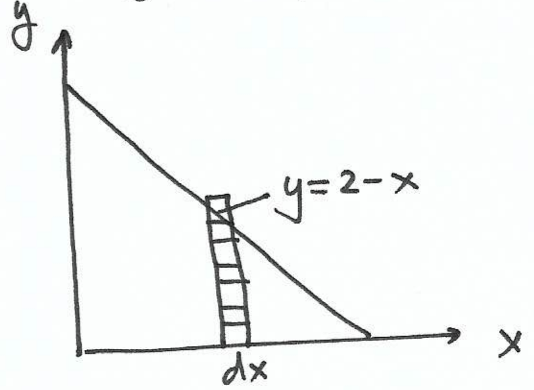
write limits explicitly

not needed. Convention is inner integral first.

N.B. Convention used here "inner to outer" is Boas convention. Others (Riley) use "right to left". Watch out!

Check going the other way, doing the y integral first.

$$V = \int_{x=x_{min}}^{x_{max}} \int_{y=y_{min}}^{y_{max}} f(x,y)\,dy\,dx$$



$$= \int_{x=0}^{2} \int_{y=0}^{2-x} (x+y)\,dy\,dx$$

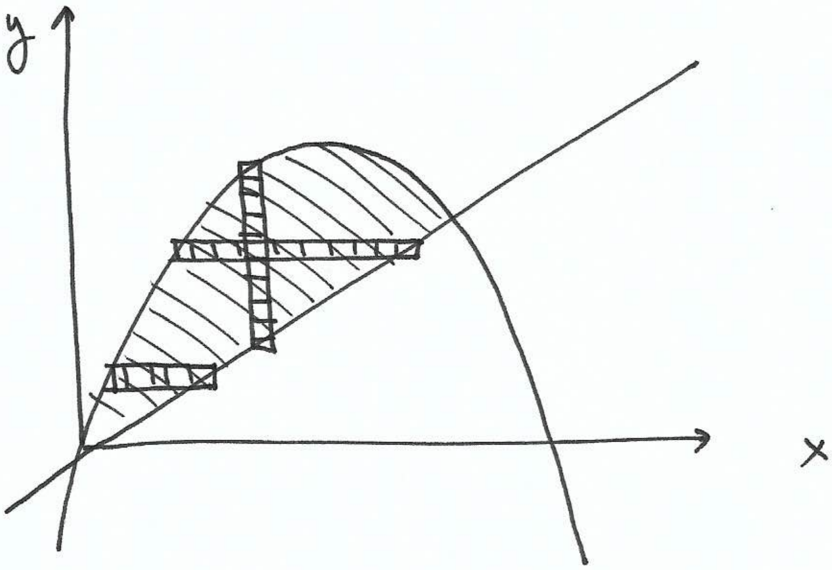
$$= \int_{x=0}^{2} \left[xy + \frac{y^2}{2} \right]_0^{2-x} dx = \int_0^2 2 - \frac{x^2}{2}\,dx$$

$$= \left[2x - \frac{x^3}{6} \right]_0^2 = \frac{8}{3}$$ as before

** the limits on the integrals for x, y depend on the order you do the integrals !!!

always choose the order of integration to make your life easy

example :



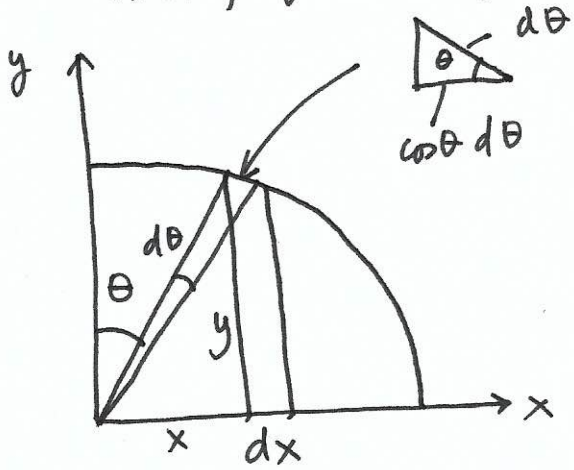
integrate $f(x,y)$ over the shaded area.

This one : do the y integral first (convince yourself)

2.3 Change of variables in 2D integrals – The Jacobian

Reminder: change of variables in 1D integration

Area of quadrant of a circle, radius 1. Expect $\frac{\pi}{4}$.



$$A = \int_0^1 y \, dx \qquad x^2 + y^2 = 1$$
$$y = (1 - x^2)^{1/2}$$

$$A = \int_0^1 (1 - x^2)^{1/2} \, dx$$

change variables $x \to \theta$

* Remember: change 3 things

① the integrand: express $f(x)$ as $f(x(\theta))$

$x = \sin\theta \quad f(x(\theta)) = (1 - \sin^2\theta)^{1/2} = \cos\theta$

② the limits $\quad 0 < x < 1 \equiv 0 < \theta < \pi/2$

③ the differential

$$dx = \frac{dx}{d\theta} \, d\theta$$

$$x = \sin\theta \qquad \frac{dx}{d\theta} = \cos\theta$$

Put it all together:

$$A = \int_0^1 (1 - x^2)^{1/2} \, dx = \int_0^{\pi/2} \cos\theta . \cos\theta \, d\theta = \int_0^{\pi/2} \frac{1}{2}\left[\cancel{\cos 2\theta} + 1\right] = \left[\frac{\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{4}$$

$$dx = \frac{dx}{d\theta} \, d\theta$$

is "how much x changes as θ change by $d\theta$"

unevenly spaced!

same num, different spacing

1D Jacobian

Now: <u>change of variables in 2D integrals</u>

$$\iint_R f(x,y)\,dx\,dy \longrightarrow \iint_{R'} f(x(u,v),y(u,v))\,|J|\,du\,dv$$

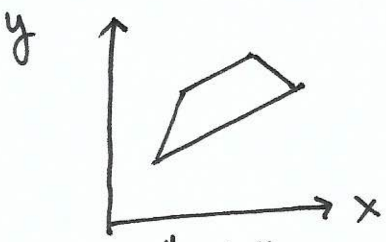
Jacobian

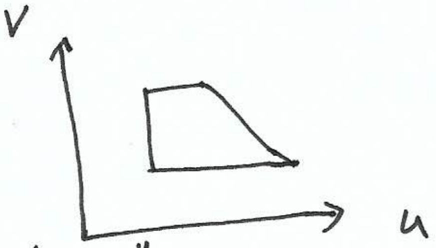
Change 3 things

① the integrand: express $f(x,y)$ as $f(x(u,v),y(u,v))$

② the limits

<u>Draw</u> (yes!) R_{xy} as R'_{uv}





$$\int_{y=y_{min}}^{y_{max}}\int_{x=x_{min}}^{x_{max}} \rightarrow \int_{v=v_{min}}^{v_{max}}\int_{u=u_{min}}^{u_{max}}$$

in xy or uv you may need to split into more than one integral

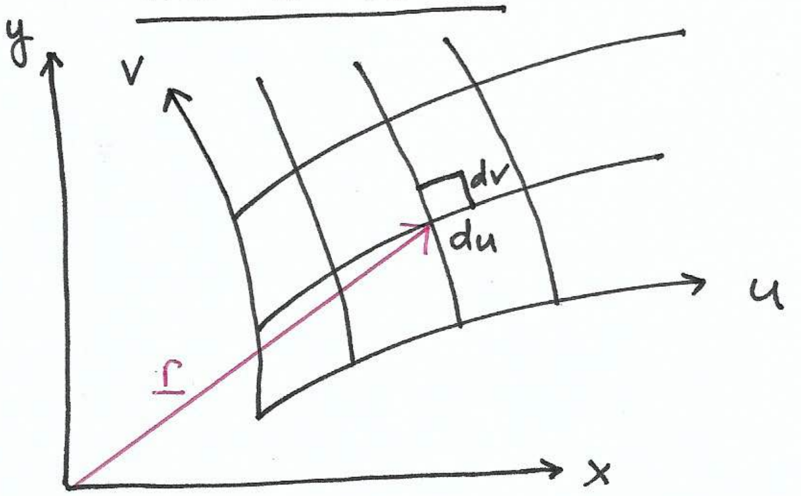
③ transform the differential

$$dx\,dy \rightarrow |J|\,du\,dv$$

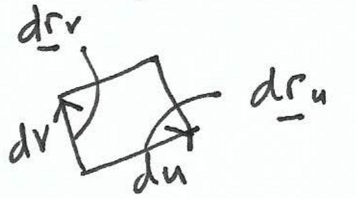
What area in the x,y plane does infinitesimal change du dV cover?

* There isn't an automatic ~~way~~ (analytic) way to do step ② - you really do need to draw it! (see example later in this lecture)

The Jacobian



what is area in x, y plane for element du dv?



position vector is a vector field $\underline{r} = x\,\hat{\underline{i}} + y\,\hat{\underline{j}}$

then $d\underline{r} = \frac{\partial \underline{r}}{\partial u} du + \frac{\partial \underline{r}}{\partial v} dv = d\underline{r}_u + d\underline{r}_v$

area of parallelogram is $dA = |d\underline{A}|$

$$d\underline{A} = d\underline{r}_u \times d\underline{r}_v = \boxed{\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\, du\, dv}$$

$$d\underline{r}_u = \frac{\partial}{\partial u}\left(x\,\hat{\underline{i}} + y\,\hat{\underline{j}}\right) du = \left(\frac{\partial x}{\partial u}\hat{\underline{i}} + \frac{\partial y}{\partial u}\hat{\underline{j}}\right) du$$

$$d\underline{r}_v = \left(\frac{\partial x}{\partial v}\hat{\underline{i}} + \frac{\partial y}{\partial v}\hat{\underline{j}}\right) dv\,.$$

$$d\underline{A} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} du\,dv = \hat{\underline{k}} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv$$

determinant
J = Jacobian

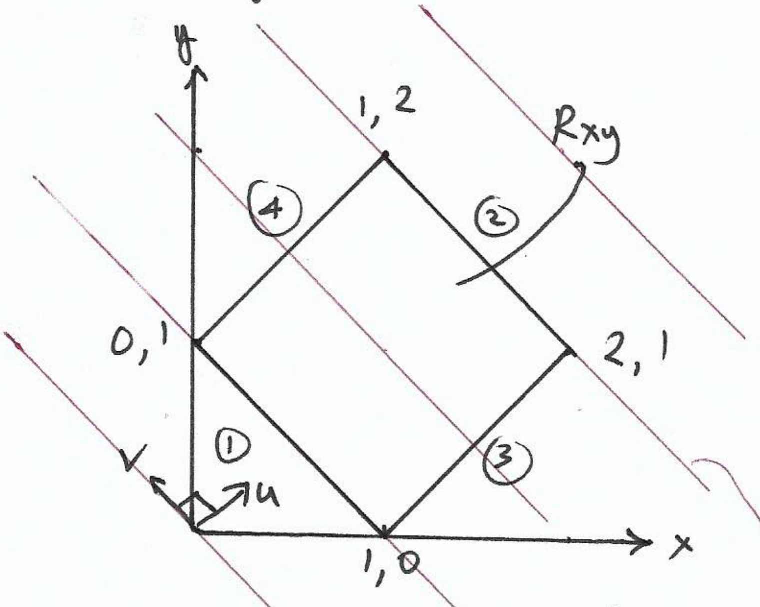
$$dA = |J|\, du\, dv$$

absolute value

sometimes written $J = \frac{\partial(x, y)}{\partial(u, v)}$

Example 2D integral and Jacobian

Integrate $f(x,y) = x+y$ over diamond region



boundaries

1. $x+y = 1$ — $u = 1$
2. $x+y = 3$ — $u = 3$
3. $y-x = -1$ — $v = -1$
4. $y-x = 1$ — $v = 1$

we will use new variables

$u = x+y \qquad x = (u-v)/2$

$v = y-x \qquad y = (u+v)/2$

check the following 2D integral using x, y

$$I = \iint_{R_{xy}} f(x,y)\, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=1-x}^{1+x} (x+y)\, dy\, dx + \int_{x=1}^{2} \int_{y=x-1}^{3-x} (x+y)\, dy\, dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_{1-x}^{1+x} dx + \int_1^2 \left[xy + \frac{y^2}{2} \right]_{x-1}^{3-x} dx$$

2 integrals, long!

$$= \int_0^1 2x^2 + 2x\, dx + \int_1^2 4 + 2x - 2x^2\, dx$$

$$= \frac{5}{3} + \frac{7}{3} = \boxed{4 = I}$$

Integral using u, v.

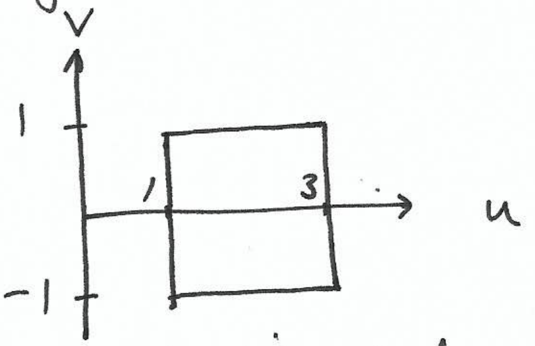
Remember 3 steps

1. Transform integrand

$$f(x,y) \to f(x(u,v), y(u,v))$$

$$\cancel{f(x+y) \to f(u)} \quad x+y \to u$$

2. Change limits. Draw R_{xy} in the u,v plane



3. Transform the differential

$$dx\,dy = |J|\,du\,dv$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2} \qquad \text{(analytic)}$$

* I wrote this wrong in the lecture

Put it all together:

$$I = \iint_{R_{uv}} f(x(u,v), y(u,v))\,|J|\,du\,dv$$

$$= \int_{v=-1}^{1} \int_{u=1}^{3} u \cdot \frac{1}{2} \cdot du\,dv = \int_{-1}^{1} \left[\frac{u^2}{4}\right]_1^3 dv$$

$$= \int_{-1}^{1} 2\,dv = \left[2v\right]_{-1}^{1} = 4 \quad \text{(as before)}$$

easy

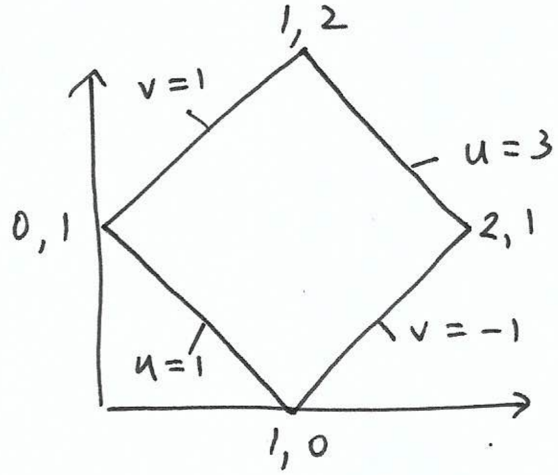
Note: u, v the natural coordinates for this problem

We derived the Jacobian analytically.

But we could do it geometrically.

The question is what is the area in the x, y plane for changes du, dv

Consider the diamond



area of diamond in x, y plane is $\sqrt{2} \times \sqrt{2} = 2$

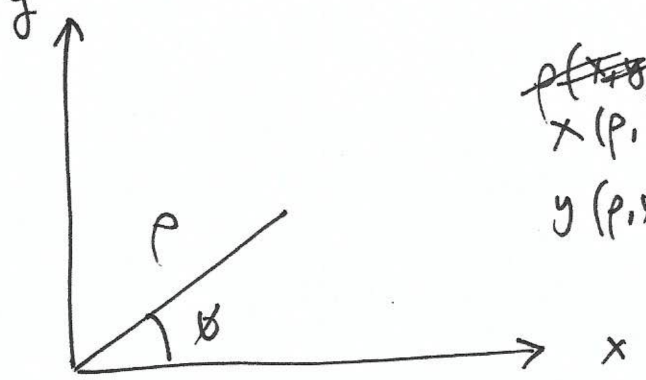
area in u, v plane is $2 \times 2 = 4$

So an element $du\,dv$ covers an area $\frac{1}{2}\,du\,dv$ in the x, y plane

$|J|$ (as we found)

2.4 Plane-polar coordinates

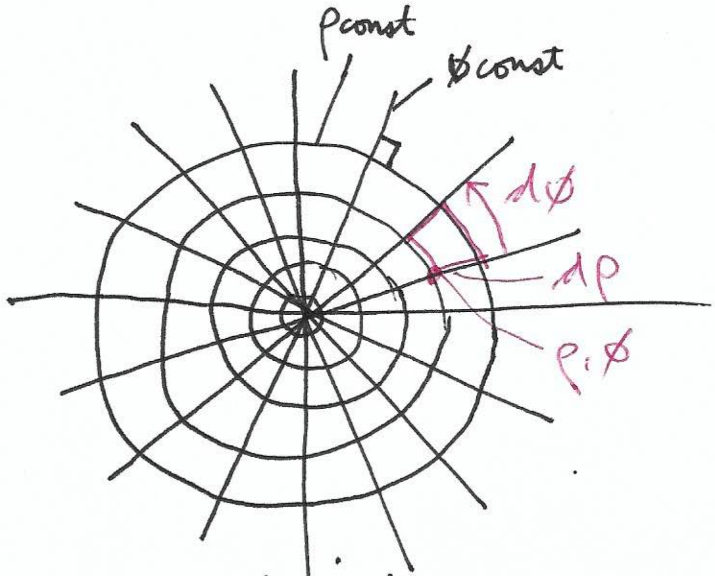
Useful for problems with circular symmetry



$$x(\rho,\phi) = \rho\cos\phi \qquad \rho(x,y) = (x^2+y^2)^{1/2}$$

$$y(\rho,\phi) = \rho\sin\phi \qquad \phi(x,y) = \tan^{-1}(y/x)$$

The coordinates are orthogonal (!)



what is dA for changes $d\rho$, $d\phi$?

geometric derivation

$$dA = \rho\, d\rho\, d\phi = |J|\, d\rho\, d\phi$$

i.e. $\quad J = \rho$

analytic derivation

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial y}{\partial \rho} \\[2ex] \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{vmatrix} = \rho\left(\cos^2\phi + \sin^2\phi\right) = \rho$$

$J = \rho$ agrees

Example

A disk of radius R has surface mass density $\sigma(x,y) = \dfrac{B}{\sqrt{x^2+y^2}}$.

What is the mass of the disk?

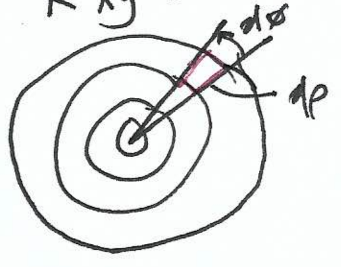
Difficult in Cartesian coords.

Problem has circular symmetry – use plane-polar coordinates

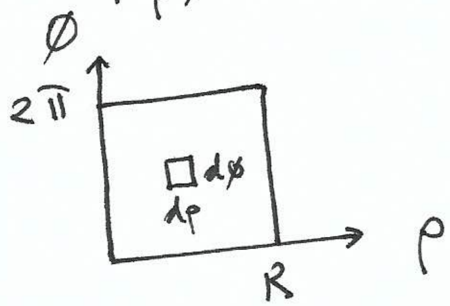
① $\sigma(x,y) \to \sigma(x(\rho,\phi), y(\rho,\phi)) = \dfrac{B}{\rho}$

② Limits

R_{xy} is a circle



$R'_{\rho\phi}$ is a rectangle



③ $|J| = \rho$

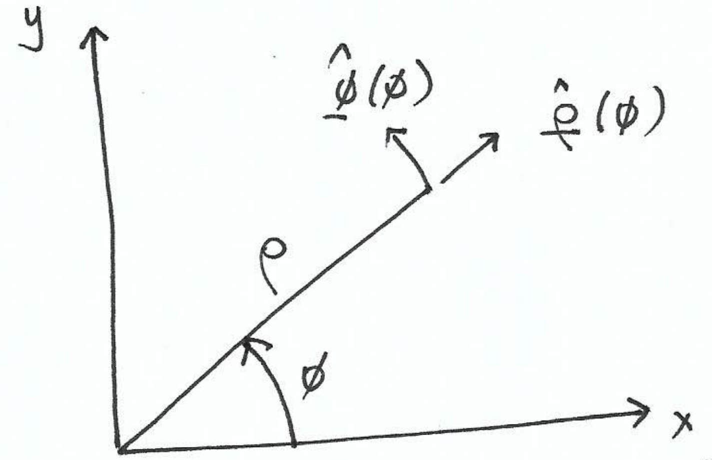
$$I = \int_0^{2\pi}\int_0^{R} \frac{B}{\rho}\, \rho \, d\rho \, d\phi$$

$$= \int_0^{2\pi} \left[B\rho \right]_0^R d\phi = BR\left[\phi\right]_0^{2\pi} = 2\pi B R$$

easy

2.5 Unit vectors in plane-polar coordinates

Differentiating vector fields in plane-polar coords is more difficult!
Useful to define unit vectors in radial $\hat{\underline{\rho}}$ and tangential $\hat{\underline{\phi}}$ directions. They both depend on ϕ!



(mutually orthogonal and orthogonal to $\hat{\underline{k}}$)

To differentiate unit vectors, write in Cartesian coords

$$\hat{\underline{\rho}} = \cos\phi\,\hat{\underline{i}} + \sin\phi\,\hat{\underline{j}}$$
$$\hat{\underline{\phi}} = -\sin\phi\,\hat{\underline{i}} + \cos\phi\,\hat{\underline{j}}$$

Then
$$\frac{\partial\hat{\underline{\rho}}}{\partial\phi} = -\sin\phi\,\hat{\underline{i}} + \cos\phi\,\hat{\underline{j}} = \hat{\underline{\phi}}$$
$$\frac{\partial\hat{\underline{\phi}}}{\partial\phi} = -\cos\phi\,\hat{\underline{i}} - \sin\phi\,\hat{\underline{j}} = -\hat{\underline{\rho}}$$

** REMEMBER!

$$\frac{\partial\hat{\underline{\rho}}}{\partial\rho} = 0 \qquad \frac{\partial\hat{\underline{\phi}}}{\partial\rho} = 0$$

Differentiating vector fields using plane-polar coords

We will differentiate the position vector $\underline{r} = \rho\,\hat{\underline{\rho}}$ to derive the Jacobian for plane-polar

Generally $\quad d\underline{r} = \dfrac{\partial \underline{r}}{\partial \rho}\,d\rho + \dfrac{\partial \underline{r}}{\partial \phi}\,d\phi \qquad (\rho \equiv u,\ \phi \equiv v)$

and the Jacobian is

$$J = \left| \frac{\partial \underline{r}}{\partial \rho} \times \frac{\partial \underline{r}}{\partial \phi} \right| \text{ - modulus}$$

$$\frac{\partial \underline{r}}{\partial \rho}(\rho,\phi) = \frac{\partial}{\partial \rho}(\rho\hat{\underline{\rho}}) = \frac{\partial \rho}{\partial \rho}\hat{\underline{\rho}} + \rho\cancel{\frac{\partial \hat{\underline{\rho}}}{\partial \rho}} = \hat{\underline{\rho}}$$

$$\frac{\partial \underline{r}}{\partial \phi}(\rho,\phi) = \frac{\partial(\rho\hat{\underline{\rho}})}{\partial \phi} = \cancel{\frac{\partial \rho}{\partial \phi}}\hat{\underline{\rho}} + \rho\frac{\partial \hat{\underline{\rho}}}{\partial \phi} = \rho\,\hat{\underline{\phi}}$$

orthogonal $\therefore J = \rho$

or write

$$\frac{\partial \underline{r}}{\partial \rho} \times \frac{\partial \underline{r}}{\partial \phi} = \begin{vmatrix} \hat{\underline{\rho}} & \hat{\underline{\phi}} & \hat{\underline{k}} \\ 1 & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix} = \rho\,\hat{\underline{k}} \qquad |J| = \rho$$

~~generally: write vector field~~ $\cancel{\underline{A} = A_\rho \hat{\underline{\rho}} + A_\phi \hat{\underline{\phi}}}$ $\cancel{A_\rho(\rho,\phi)\hat{\underline{\rho}} + A_\phi(\rho,\phi)\hat{\underline{\phi}}}$

~~then e.g.~~ $\cancel{\dfrac{\partial \underline{A}}{\partial \rho} = \dfrac{\partial (}{\partial \rho}}$

Generally: write vector field

$$\underline{A} = A_\rho(\rho,\phi)\,\hat{\underline{\rho}}(\phi) + A_\phi(\rho,\phi)\,\hat{\underline{\phi}}(\phi)$$

then e.g. $\dfrac{\partial \underline{A}}{\partial \phi} = \dfrac{\partial}{\partial \phi}(A_\rho \hat{\underline{\rho}}) + \dfrac{\partial (A_\phi \hat{\underline{\phi}})}{\partial \phi}$ and use product rule

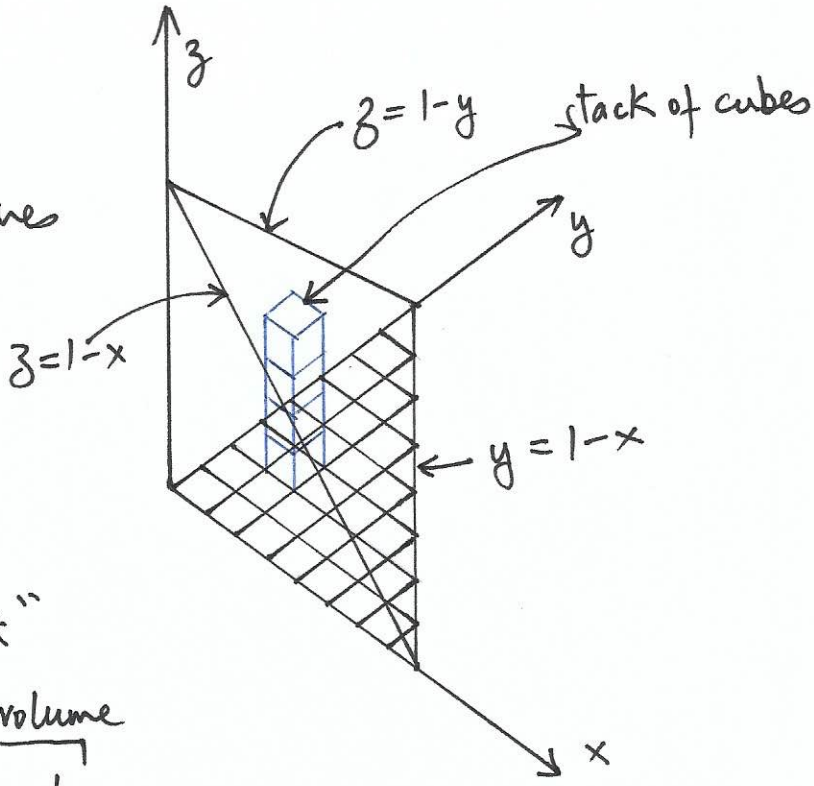
3. 3D integrals 3.1 Cartesian coords

Example

Integrate $f(x,y,z) = \alpha x$

over tetrahedron bounded by planes

$x = 0$
$y = 0$
$z = 0$
$x+y+z = 1$



Let us integrate in order

z, y, x

$$I = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} \underbrace{\alpha x}_{\text{"weight"}} \; \underbrace{dz\, dy\, dx}_{\text{cube volume}}$$

weight for 2D x, y. integral

1st integral x, y constant

$$I = \int_{x=0}^{1}\int_{y=0}^{1-x} \left[\alpha x z\right]_0^{1-x-y} dy\, dx = \alpha \int_{x=0}^{1}\int_{y=0}^{1-x} x - x^2 - xy \; dy\, dx$$

2nd integral x constant

$$I = \alpha\int_{x=0}^{1} \left[x(1-x)y - \frac{x y^2}{2}\right]_0^{1-x} dx = \alpha \int_{x=0}^{1} x\frac{(1-x)^2}{2}\, dx$$

area ~~vol.~~ of slice at constant x

$$I = \frac{\alpha}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{\alpha}{24}$$

Now: you really must try this integral in order y, x, z or x, z, y to get same answer

3.2 Jacobian in 3D

$$I = \iiint_{R_{xyz}} f(xyz)\,dx\,dy\,dz \Rightarrow \iiint_{R'_{uvw}} f(x(uvw), y(uvw), z(uvw))\,|J|\,du\,dv\,dw$$

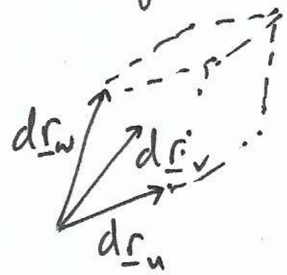
If $x(uvw)$, $y(uvw)$, $z(uvw)$ the differential of the position vector is

$$d\underline{r} = \frac{\partial \underline{r}}{\partial u} du + \frac{\partial \underline{r}}{\partial v} dv + \frac{\partial \underline{r}}{\partial w} dw = d\underline{r}_u + d\underline{r}_v + d\underline{r}_w$$

Since $\underline{r} = x\,\hat{\underline{i}} + y\,\hat{\underline{j}} + z\,\hat{\underline{k}}$

then e.g. $d\underline{r}_u = \left(\frac{\partial x}{\partial u}\hat{\underline{i}} + \frac{\partial y}{\partial u}\hat{\underline{j}} + \frac{\partial z}{\partial u}\hat{\underline{k}}\right) du$

We want the volume of the little element defined by the 3 little vectors $d\underline{r}_u$, $d\underline{r}_v$, $d\underline{r}_w$



not necessarily orthogonal

Recall vol. of parallelipiped is $\underline{a}\cdot\underline{b}\times\underline{c}$

i.e. $d\underline{r}_u \cdot d\underline{r}_v \times d\underline{r}_w$

$$dV_{uvw} = \left(\frac{\partial x}{\partial u}\hat{\underline{i}} + \frac{\partial y}{\partial u}\hat{\underline{j}} + \frac{\partial z}{\partial u}\hat{\underline{k}}\right) \cdot \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} du\,dv\,dw$$

$$dV_{uvw} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} du\,dv\,dw$$

3D Jacobian

sometimes written $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

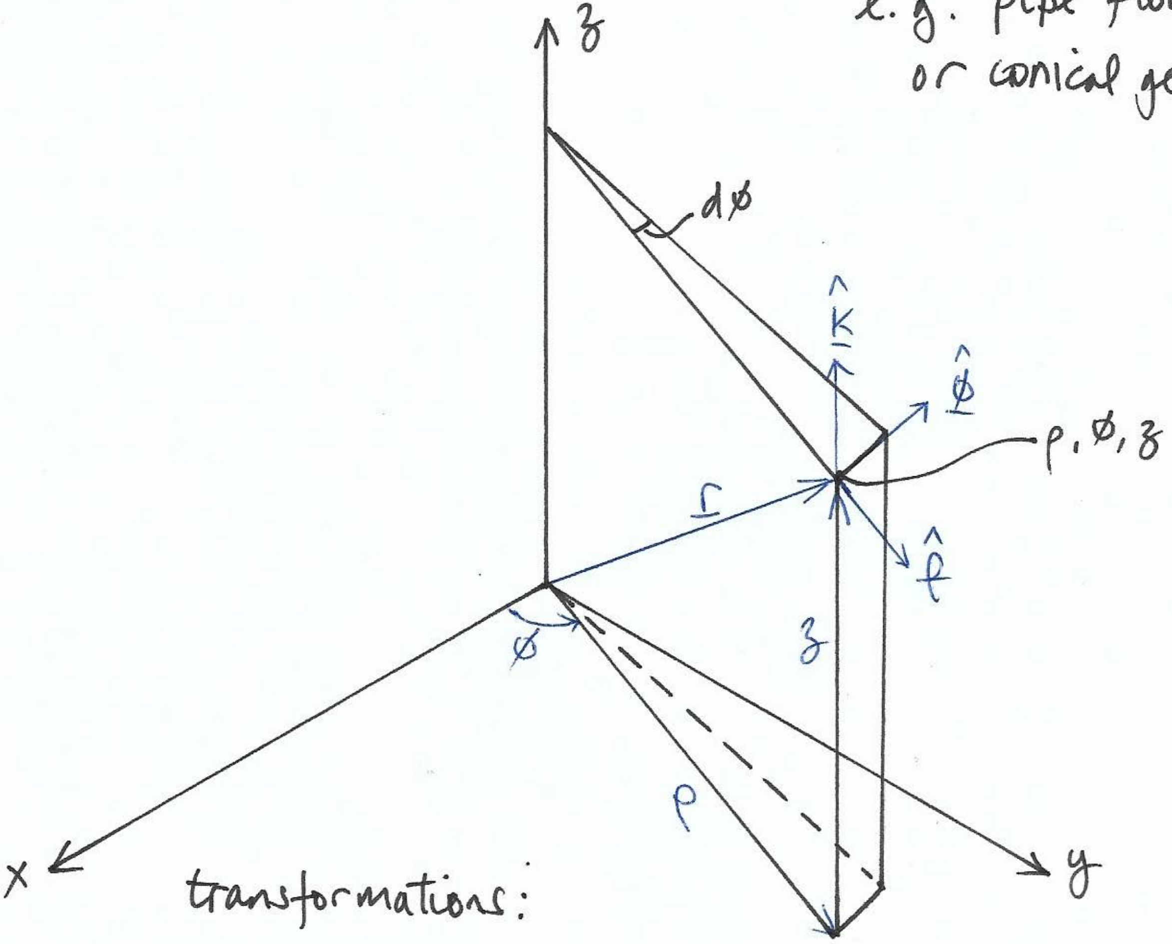
we want the absolute value $|J|$

3.3 Cylindrical polar coordinates

Same as plane polar + z axis

Useful for problems with circular symmetry + variation in z

e.g. pipe flow,
or conical geometry



transformations:

$x = \rho \cos \phi$ $\qquad$ $\rho = \sqrt{x^2+y^2}$

$y = \rho \sin \phi$ $\qquad$ $\phi = \tan^{-1}(y/x)$

$z = z$ $\qquad$ $z = z$

position vector $\underline{r}(\rho, \phi, z) = \rho \, \hat{\underline{\rho}}(\phi) + z \, \hat{\underline{k}}$

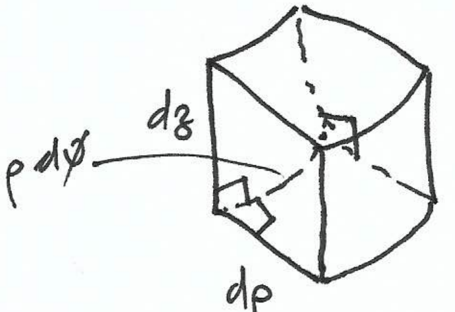
$\hat{\underline{\rho}}(\phi)$, $\hat{\underline{\phi}}(\phi)$, $\hat{\underline{k}}$ orthogonal, no dependence on ρ or z

when differentiating a vector field in cylindrical coords remember $\hat{\underline{\rho}}$, $\hat{\underline{\phi}}$ depend on ϕ

$$\frac{\partial \hat{\underline{\rho}}}{\partial \phi} = \hat{\underline{\phi}} \qquad \frac{\partial \hat{\underline{\phi}}}{\partial \phi} = -\hat{\underline{\rho}}$$

Volume element $|J|\,d\rho\,d\phi\,dz$

Geometrically What is the volume produced by small changes $d\rho$, $d\phi$, dz.



$$dV = \rho\,d\rho\,d\phi\,dz$$

Analytically brute force from Jacobian (using Cartesian coords)

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho \quad \text{as expected}$$

$$dV = |J|\,d\rho\,d\phi\,dz = \rho\,d\rho\,d\phi\,dz$$

More elegantly using differential of position vector using $\hat{\underline{\rho}}, \hat{\underline{\phi}}, \hat{\underline{k}}$

$$\underline{r} = \rho\,\hat{\underline{\rho}}(\phi) + z\,\hat{\underline{k}} \qquad d\underline{r} = \frac{\partial \underline{r}}{\partial \rho}d\rho + \frac{\partial \underline{r}}{\partial \phi}d\phi + \frac{\partial \underline{r}}{\partial z}dz$$

change $d\rho$ $\quad d\underline{r}_\rho = \frac{\partial \underline{r}}{\partial \rho}d\rho = d\rho\,\hat{\underline{\rho}}$

change $d\phi$ $\quad d\underline{r}_\phi = \frac{\partial \underline{r}}{\partial \phi}d\phi = \rho\,d\phi\,\hat{\underline{\phi}}$

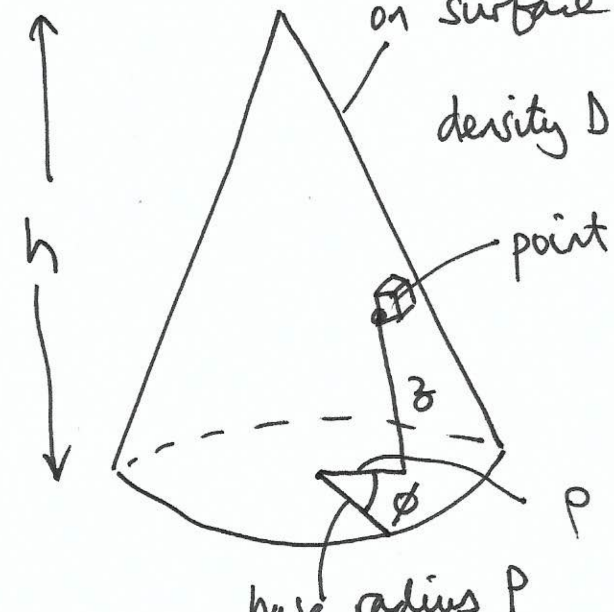
change dz $\quad d\underline{r}_z = \frac{\partial \underline{r}}{\partial z}dz = dz\,\hat{\underline{k}}$

orthogonal

$$dV = d\underline{r}_\rho \cdot d\underline{r}_\phi \times d\underline{r}_z = \rho\,d\rho\,d\phi\,dz$$

(determinant not needed)

Example: (vertical) centre of gravity of solid cone



$$\bar{z} = \frac{\sum m_i z}{\sum m_i} = \frac{\iiint z\, D\, dV}{\iiint D\, dV}$$

$$= \frac{\text{1st moment of vol}}{\text{0th moment of vol}}$$

(prove later)

denominator $V = \frac{1}{3}\,\text{base} \times h = \frac{\pi P^2 h}{3}$

numerator $I = \int_{z=0}^{h} \int_{\rho=0}^{P(1-\frac{z}{h})} \int_{\phi=0}^{2\pi} \underbrace{z}_{A(z)}\ \underbrace{\rho}_{|J|}\, d\phi\, d\rho\, dz = 2\pi \int_{z=0}^{h} z \left[\frac{\rho^2}{2}\right]_0^{P(1-\frac{z}{h})} dz$

~~denominator~~

$$I = \pi \int_{z=0}^{h} z P^2 \left(1-\frac{z}{h}\right)^2 dz \qquad (1)$$

... algebra ....

$$= \pi P^2 \left[\frac{z^2}{2} - \frac{2z^3}{3h} + \frac{z^4}{4h^2}\right]_0^h = \frac{\pi P^2 h^2}{12}$$

based on (1) $V = \pi \int_{z=0}^{h} P^2 \left[1-\frac{z}{h}\right]^2 dz$ ...algebra... $= \frac{\pi P^2 h}{3}$

$$\bar{z} = \frac{I}{V} = \frac{\pi P^2}{\pi P^2}\,\frac{h^2}{12}\,\frac{3}{h} = \frac{h}{4}$$

NOW TRY TOP INTEGRAL IN A DIFFERENT ORDER

DO z FIRST!

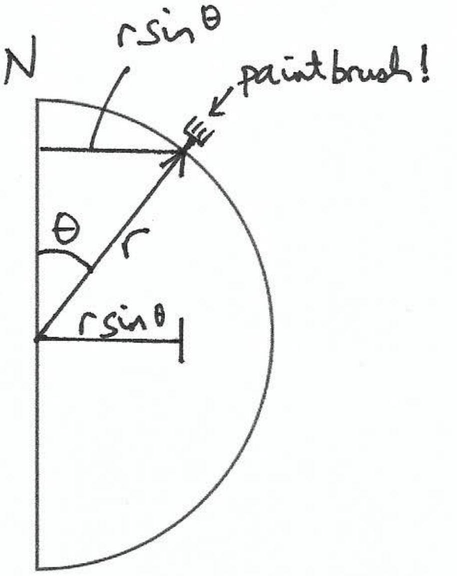
3.4 Spherical polar coordinates

Coordinates are r, θ, ϕ — These are orthogonal

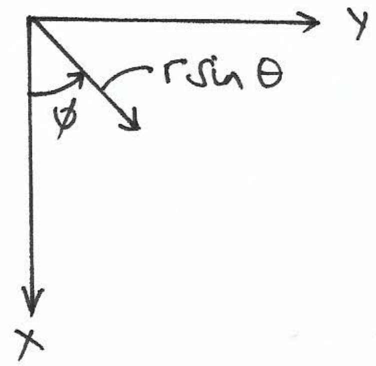
"latitude" "longitude" ← (correct but said wrong in lecture)

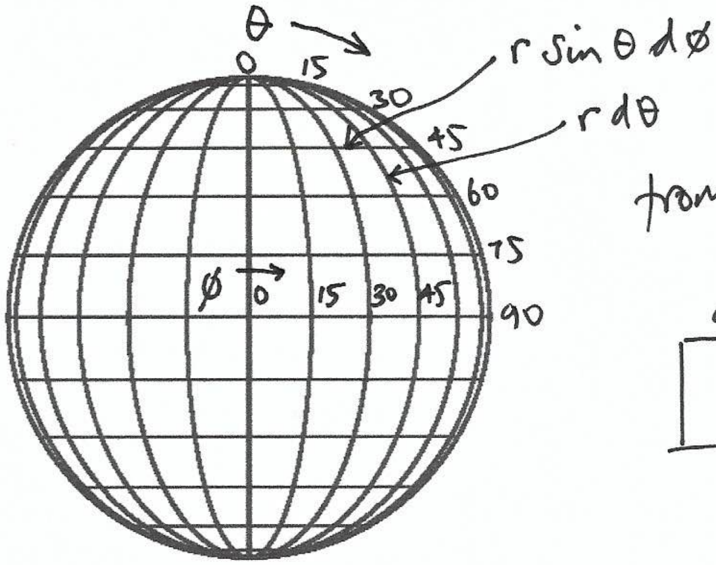
Imagine glass sphere, radius r. We are going to paint inside lines of constant θ and ϕ

Semi circle rotates about vertical axis through angle ϕ



View looking down on N pole

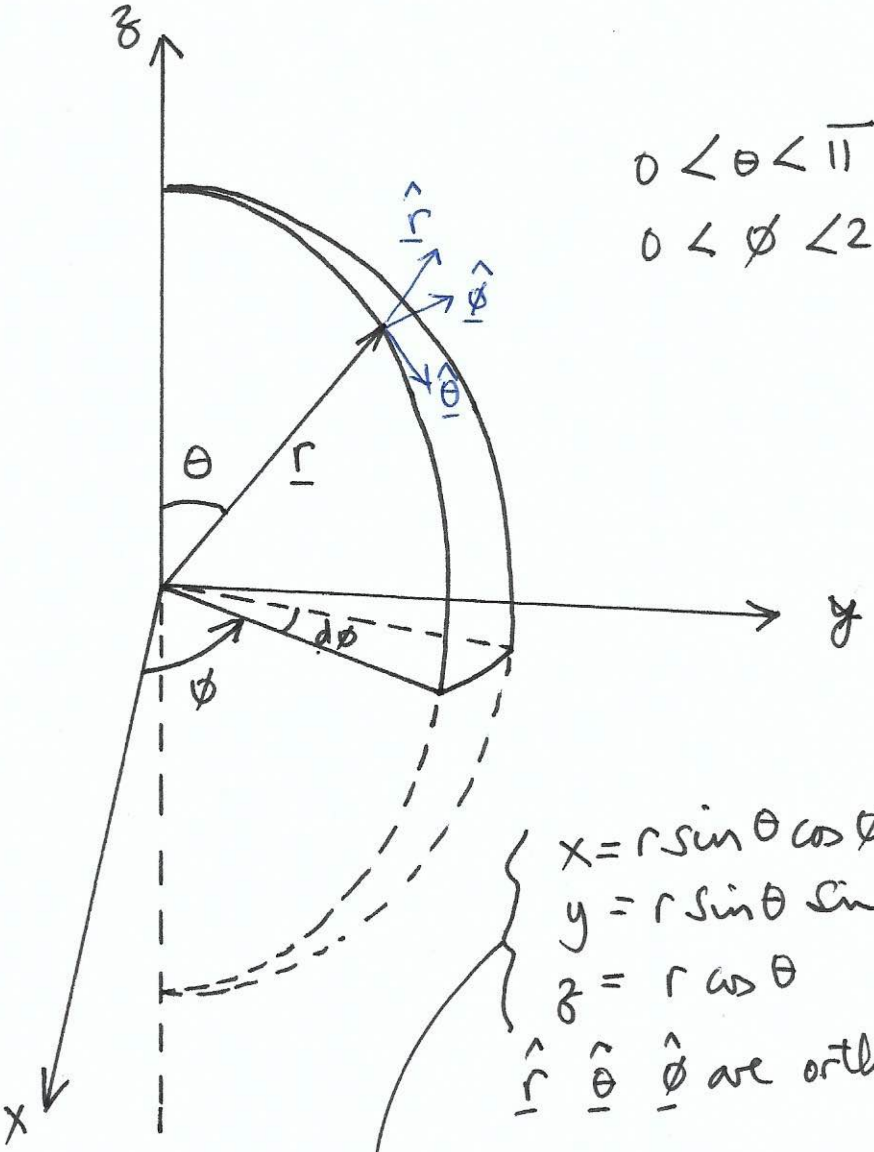




from geometry

$$dV = r\sin\theta\, d\phi\, r\, d\theta\, dr$$

$$\boxed{dV = r^2 \sin\theta\, dr\, d\theta\, d\phi}$$



$0 < \theta < \pi$

$0 < \phi < 2\pi$

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\}$$

$\hat{\underline{r}}$ $\hat{\underline{\theta}}$ $\hat{\underline{\phi}}$ are orthogonal

use these to get the Jacobian and check the volume element recall ~~the~~ to differentiate unit vectors, express using $\hat{\underline{i}}, \hat{\underline{j}}, \hat{\underline{k}}$

$$\hat{\underline{r}}(\theta, \phi) = \sin\theta\cos\phi\,\hat{\underline{i}} + \sin\theta\sin\phi\,\hat{\underline{j}} + \cos\theta\,\hat{\underline{k}}$$

$$\hat{\underline{\theta}}(\theta, \phi) = \cos\theta\cos\phi\,\hat{\underline{i}} + \cos\theta\sin\phi\,\hat{\underline{j}} - \sin\theta\,\hat{\underline{k}}$$

$$\hat{\underline{\phi}}(\phi) = -\sin\phi\,\hat{\underline{i}} + \cos\phi\,\hat{\underline{j}}$$

you should be able to figure these out

Use these to derive derivatives of unit vectors as needed.

position vector $\underline{r} = r\,\hat{\underline{r}}(\theta, \phi)$

so we need $\dfrac{\partial \hat{\underline{r}}}{\partial \theta}$ and $\dfrac{\partial \hat{\underline{r}}}{\partial \phi}$

$$\left.\begin{aligned} \frac{\partial \hat{\underline{r}}}{\partial \theta} &= \cos\theta\cos\phi\,\hat{\underline{i}} + \cos\theta\sin\phi\,\hat{\underline{j}} - \sin\theta\,\hat{\underline{k}} = \hat{\underline{\theta}} \\ \frac{\partial \hat{\underline{r}}}{\partial \phi} &= -\sin\theta\sin\phi\,\hat{\underline{i}} + \sin\theta\cos\phi\,\hat{\underline{j}} = \sin\theta\,\hat{\underline{\phi}} \end{aligned}\right] \text{* REMEMBER}$$

Now derive volume element from differential:

$$d\underline{r} = \frac{\partial \underline{r}}{\partial r}dr + \frac{\partial \underline{r}}{\partial \theta}d\theta + \frac{\partial \underline{r}}{\partial \phi}d\phi$$

$$\left.\begin{aligned} &\text{change } dr & d\underline{r}_r &= \frac{\partial \underline{r}}{\partial r}dr = dr\,\hat{\underline{r}} \\ &\text{change } d\theta & d\underline{r}_\theta &= \frac{\partial \underline{r}}{\partial \theta}d\theta = r\,d\theta\,\hat{\underline{\theta}} \\ &\text{change } d\phi & d\underline{r}_\phi &= \frac{\partial \underline{r}}{\partial \phi}d\phi = r\sin\theta\,d\phi\,\hat{\underline{\phi}} \end{aligned}\right] \text{orthogonal}$$

$$\therefore\quad dV = \text{product} = r^2\sin\theta\,dr\,d\theta\,d\phi$$

3.5 Surface integrals

On a surface in 3D define infinitesimal element of area $\underline{dS}$, a vector of area $|\underline{dS}|$, with direction normal to the surface

(Warning: direction in/out, up/down is a matter of choice/convention)

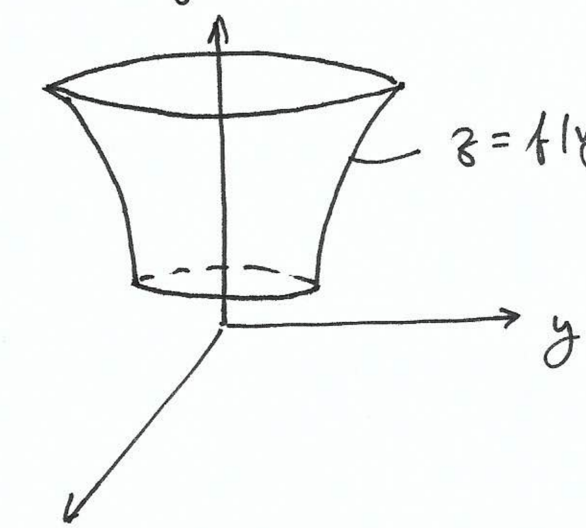
Used for computing e.g.

1. area $\iint_S |d\underline{S}|$ ← 3D surface
2. total scalar e.g. charge $\iint_S \sigma\, |d\underline{S}|$ σ is surface density
3. "flux" through surface $\iint_S \underline{F} \cdot d\underline{S}$

$\underline{F}$ is any vector field.
e.g. for from Sun $\underline{F}$ defines direction and rate of flow of energy per unit area $\perp$r to $\underline{F}$.
Then $\underline{F} \cdot d\underline{S}$ is the rate of flow of energy – the flux – through dS.

Simple case: Surface of revolution (revision)

A curve rotated about an axis e.g. z



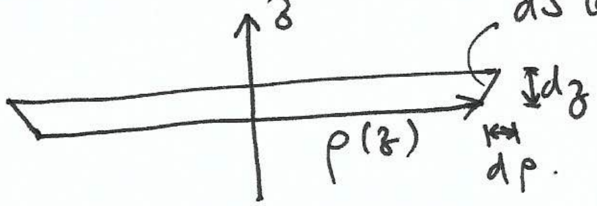
$z = f(y)$ rotated i.e. $z = f(\rho)$

e.g. curve $z = y^{1/2}$

rotated $z = \rho^{1/2}$ (cylindrical coords)

or invert $\rho = z^2$

A slice in z



volume:

$dV = \pi \rho^2(z) dz$

$V = \pi \int \rho^2(z) dz$

Surface area of loop

$dA = 2\pi \rho(z) ds$ $\qquad$ ds, <u>not</u> $d\rho$ or dz !

now $ds^2 = d\rho^2 + dz^2 = d\rho^2\left(1 + \frac{dz^2}{d\rho^2}\right) = dz^2\left(1 + \frac{d\rho^2}{dz^2}\right)$

$\therefore$ $ds = d\rho\left(1 + \left(\frac{dz}{d\rho}\right)^2\right)^{1/2} = dz\left(1 + \left(\frac{d\rho}{dz}\right)^2\right)^{1/2}$

choose one

e.g. $A = 2\pi \int_{z=z_{min}}^{z_{max}} \rho(z)\left[1 + \left(\frac{d\rho}{dz}\right)^2\right]^{1/2} dz$

use this one in the seminar

General method

Big idea: the position vector $\underline{r}$ of a point on a surface may be expressed using only 2 coordinates

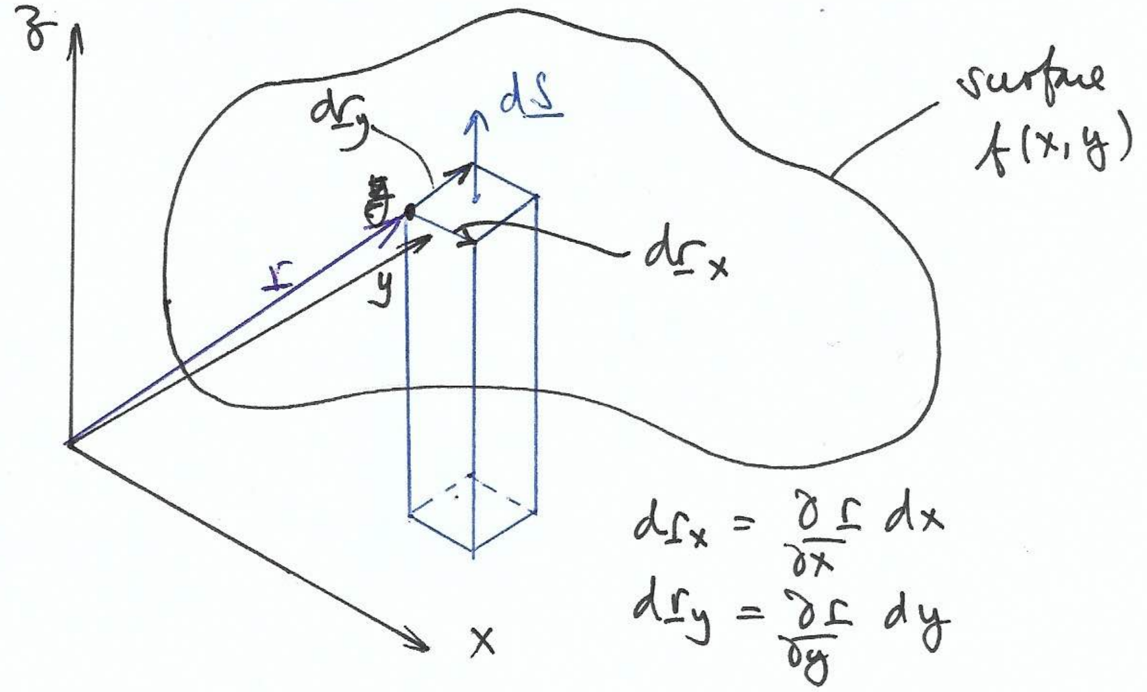
e.g. in Cartesian coords,

position vector $\underline{r} = x\,\hat{\underline{i}} + y\,\hat{\underline{j}} + z\,\hat{\underline{k}}$

surface defined by e.g. $z = f(x,y)$

so vector to surface

$$\underline{r}(x,y) = x\,\hat{\underline{i}} + y\,\hat{\underline{j}} + f(x,y)\,\hat{\underline{k}}$$

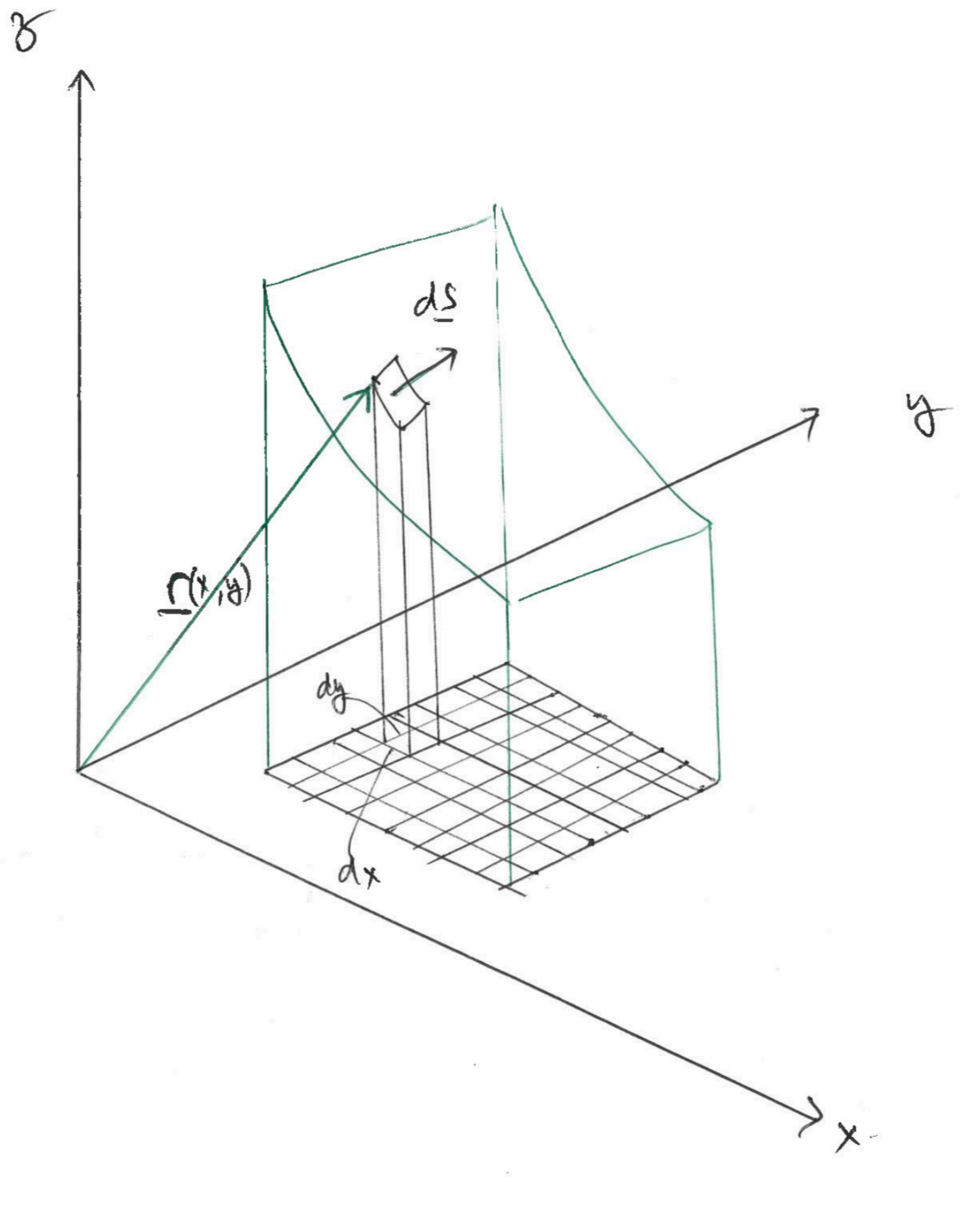


$$d\underline{r}_x = \frac{\partial \underline{r}}{\partial x}\,dx$$
$$d\underline{r}_y = \frac{\partial \underline{r}}{\partial y}\,dy$$

on surface

$$d\underline{S} = \left(\frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y}\right) dx\,dy = \underline{N}\,dx\,dy$$

(you may want $-d\underline{S}$ depending on the question)


z
y
x
$d\underline{S}$
$\underline{r}(x,y)$
dy
dx

More generally we can express the vector to the surface in terms of 2 variables u, v then:

$$\boxed{d\underline{S} = \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right) du\, dv}$$

REMEMBER

e.g. x, y or y, z

or ρ, ϕ or ϕ, z chose for convenience

e.g. in cylindrical polar coordinates

position vector $\underline{r} = \rho\, \hat{\underline{\rho}}(\phi) + z\, \hat{\underline{k}}$

surface e.g. $z = f(\rho, \phi)$

vector to surface $\underline{r}(\rho, \phi) = \rho\, \hat{\underline{\rho}}(\phi) + f(\rho, \phi)\, \hat{\underline{k}}$

$$d\underline{S} = \left(\frac{\partial \underline{r}}{\partial \rho} \times \frac{\partial \underline{r}}{\partial \phi}\right) d\rho\, d\phi$$

or express $\underline{r}(\rho, z)$ or $\underline{r}(\phi, z)$

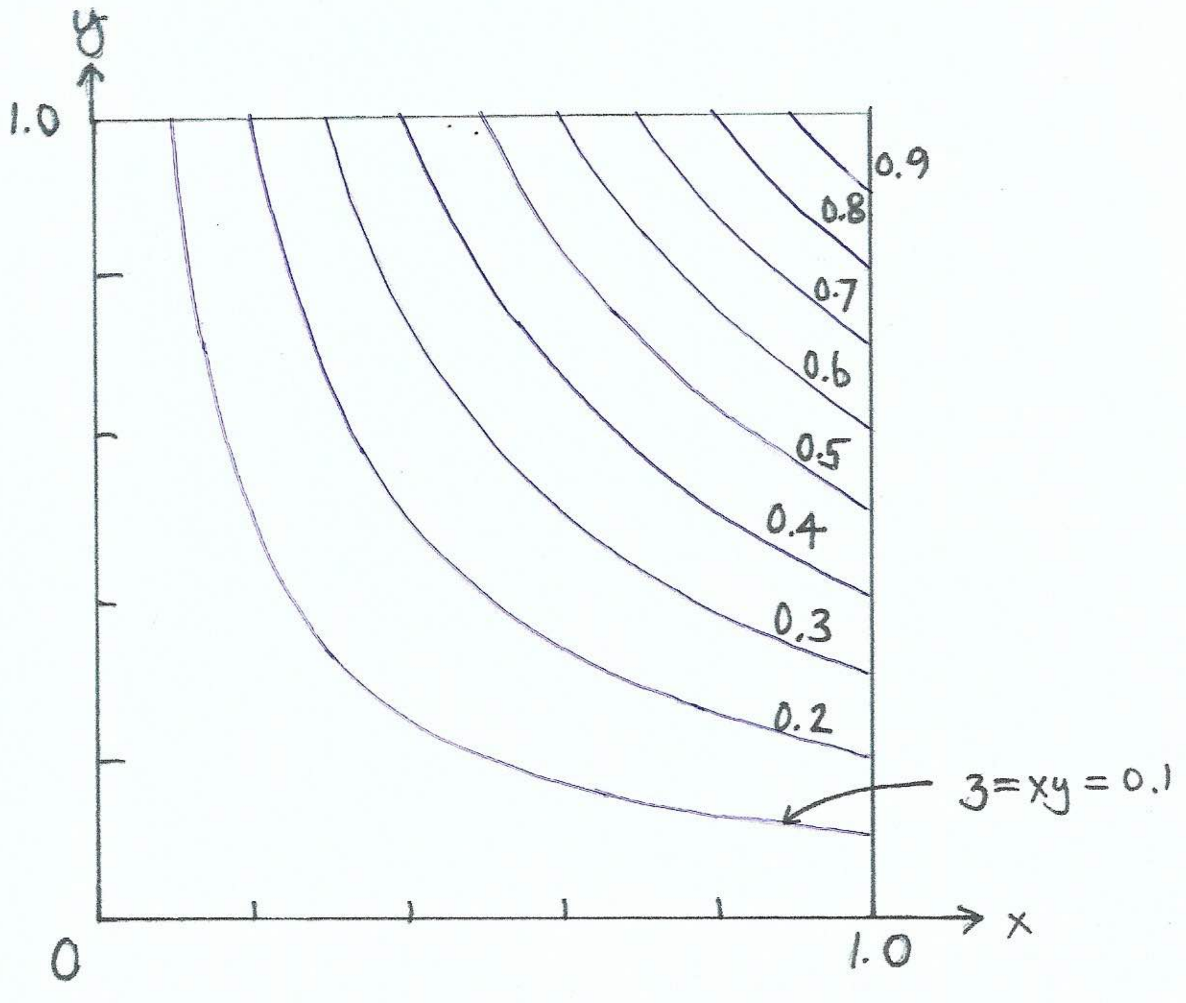
Example

surface $z = f(x, y) = xy$

vector field $\underline{F} = x^2 \hat{\underline{i}} + y^2 \hat{\underline{j}}$

find the flux into the surface

$\iint_S \underline{F} \cdot d\underline{S}$ over $0 < x < 1$, $0 < y < 1$



Key step express $\underline{r}$ to the surface in terms of just two variables from x, y, z

$$\underline{r} = x\,\hat{\underline{i}} + y\,\hat{\underline{j}} + xy\,\hat{\underline{k}}$$

x and y

$x \equiv u \quad y \equiv v$

[or $\underline{r} = x\,\hat{\underline{i}} + \frac{z}{x}\,\hat{\underline{j}} + z\,\hat{\underline{k}}$ x and z]

$$\frac{\partial \underline{r}}{\partial x} = \hat{\underline{i}} + 0\,\hat{\underline{j}} + y\,\hat{\underline{k}}$$

$$\frac{\partial \underline{r}}{\partial y} = 0\,\hat{\underline{i}} + \hat{\underline{j}} + x\,\hat{\underline{k}}$$

$$\underline{N} = \frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = -y\,\hat{\underline{i}} - x\,\hat{\underline{j}} + \hat{\underline{k}}$$

$$d\underline{S} = \underline{N}\,dx\,dy = (-y\,\hat{\underline{i}} - x\,\hat{\underline{j}} + \hat{\underline{k}})\,dx\,dy$$

check the sign : to get flux into the surface we want $-\underline{F}.d\underline{S}$

$$-\underline{F}.d\underline{S} = (x^2\,\hat{\underline{i}} + y^2\,\hat{\underline{j}} + 0\,\hat{\underline{k}}).(y\,\hat{\underline{i}} + x\,\hat{\underline{j}} - \hat{\underline{k}})\,dx\,dy$$

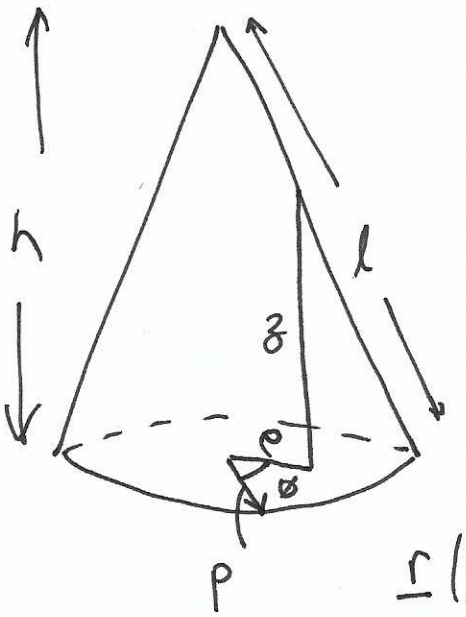
$$-\underline{F}.d\underline{S} = (x^2 y + y^2 x)\,dx\,dy$$

$$\text{total flux} = \int_{y=0}^{1}\int_{x=0}^{1} (x^2 y + y^2 x)\,dx\,dy = I$$

..... you check .... $\boxed{I = \frac{1}{3}}$

Example 2 Surface area (sloping part) of a cone – the hard way!

[we know $A = \pi P \sqrt{h^2 + P^2} = \pi P \ell$]



on surface: $\rho = P\left(1 - \frac{z}{h}\right)$

$$z = \left(1 - \frac{\rho}{P}\right) h$$

express $\underline{r}$ to surface in terms of ρ, ϕ only

$$\underline{r}(\rho, \phi, z) = \rho\, \hat{\underline{\rho}}(\phi) + z\, \hat{\underline{k}}$$

$$\underline{r}(\rho, \phi) = \rho\, \hat{\underline{\rho}}(\phi) + \left(1 - \frac{\rho}{P}\right) h\, \hat{\underline{k}}$$

$$\frac{\partial \underline{r}}{\partial \rho} = \hat{\underline{\rho}} - \frac{h}{P} \hat{\underline{k}}$$

$$\frac{\partial \underline{r}}{\partial \phi} = \rho\, \hat{\underline{\phi}} \qquad \left(\frac{\partial \hat{\underline{\rho}}}{\partial \phi} = \hat{\underline{\phi}} \right)$$

$$\frac{\partial \underline{r}}{\partial \rho} \times \frac{\partial \underline{r}}{\partial \phi} = \begin{vmatrix} \hat{\underline{\rho}} & \hat{\underline{\phi}} & \hat{\underline{k}} \\ 1 & 0 & -\frac{h}{P} \\ 0 & \rho & 0 \end{vmatrix} = \frac{h}{P}\rho\, \hat{\underline{\rho}} + \rho\, \hat{\underline{k}} = \underline{N}$$

$$d\underline{S} = \left(\frac{h}{P}\rho\, \hat{\underline{\rho}} + \rho\, \hat{\underline{k}} \right) d\rho\, d\phi$$

$$|d\underline{S}| = \rho \underbrace{\left(1 + \frac{h^2}{P^2}\right)^{1/2}}_{= \frac{\ell}{P}} d\rho\, d\phi$$

$$A = \iint_S |d\underline{S}| = \int_{\rho=0}^{P} \int_{\phi=0}^{2\pi} \frac{\rho l}{P} \, d\phi \, d\rho$$

$$A = \frac{2\pi l}{P} \left[\frac{\rho^2}{2} \right]_0^P = \pi P l \quad \text{as expected}$$

Section 4 Line integrals

Integral along a path, a curve C, in space.

Different forms, e.g.:

$$\int_C \Omega \, d\underline{r} \qquad \int_C \underline{F} \cdot d\underline{r} \qquad \int_C \underline{F} \times d\underline{r}$$

C curve

Ω scalar

we are particularly interested in this one

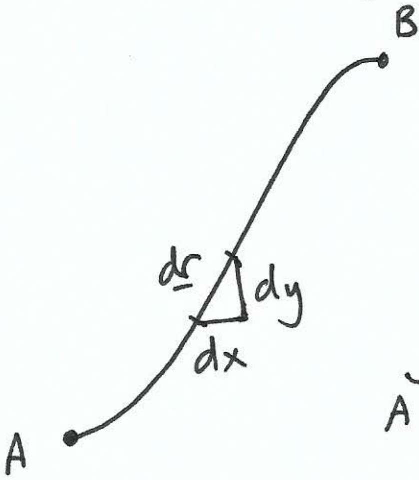
e.g. $\underline{F} \cdot d\underline{r} = W$ work done

4.1 How to calculate (2D)

We have a curve $y = y(x)$, or equivalently $x = x(y)$

and a vector field $\underline{F}(x,y) = F_x(x,y)\,\hat{\underline{i}} + F_y(x,y)\,\hat{\underline{j}}$

$$d\underline{r} = dx\,\hat{\underline{i}} + dy\,\hat{\underline{j}}$$



$$\int_A^B \underline{F} \cdot d\underline{r} \equiv \sum_i \underbrace{F_{x_i}(x_i y_i)\, dx_i}_{(1)} + \underbrace{F_{y_i}(x_i y_i)\, dy_i}_{(2)}$$

* Using the curve we can write each term entirely using only x or only y i.e. integrate w.r.t. x or y

(Same is true in 3D because a point on a line is defined by only one coordinate ξ

e.g. $z = x + y \quad y = x^2$ defines a line in 3D

$\therefore z = x + x^2 = y^{1/2} + y$)

e.g. ② may be expressed

$$\int_{y_A}^{y_B} F_y(x(y), y)\, dy \quad \text{or} \quad \int_{x_A}^{x_B} F_y(x, y(x)) \frac{dy}{dx}(x, \cancel{\ \ })\, dx$$

① may be expressed

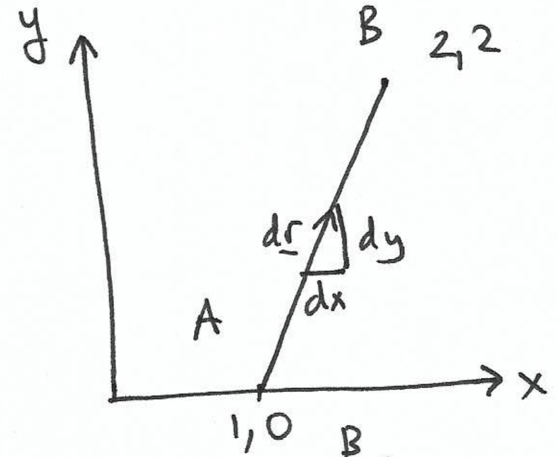
$$\int_{x_A}^{x_B} F_x(x, y(x))\, dx \quad \text{or} \quad \int_{y_A}^{y_B} F_x(x(y), y) \frac{dx}{dy}(\cancel{\ \ }, y)\, dy$$

4.2 Four examples

Recipe: express each integral with one parameter using the curve, and set the limits for that parameter.

Example 1

Vector field $\underline{F} = 2xy\, \hat{\underline{i}} + x^2\, \hat{\underline{j}}$



Path from A (1,0) to B (2,2) is the line

$y = 2x - 2$

$x = \frac{y}{2} + 1$

$$\int_A^B \underline{F} \cdot d\underline{r} = \int_A^B F_x\, dx + \int_A^B F_y\, dy$$

(4b)

$$I = \int_C \underline{F}.d\underline{r} = \int_C 2xy\, dx + \int_C x^2\, dy = I_1 + I_2$$

$$\swarrow y = y(x) \qquad \downarrow x = x(y) \qquad \xrightarrow{\text{or}} \quad y = y(x), \quad dy = \frac{dy}{dx} dx = 2dx$$

$$= \underbrace{\int_1^2 2x(2x-2)\, dx}_{I_1} + \underbrace{\int_0^2 \left(\frac{y}{2}+1\right)^2 dy}_{I_2} \quad \text{or} \quad \underbrace{\int_1^2 x^2\, 2dx}_{\text{alternative } I_2}$$

$$= \left[\frac{4x^3}{3} - 2x^2\right]_1^2 + \left[\frac{y^3}{12} + \frac{y^2}{2} + y\right]_0^2 \quad \text{or} \quad \left[\frac{2}{3}x^3\right]_1^2$$

$$I = \underbrace{\frac{10}{3}}_{I_1} + \underbrace{\frac{14}{3}}_{I_2} \quad \text{or} \quad \frac{14}{3} \qquad \checkmark$$

$$I = \frac{24}{3} = 8$$

Example 2 The line may be expressed parametrically (where we cannot write $y = f(x)$)

e.g. $\left.\begin{aligned} x(t) &= 2t+1 \\ y(t) &= 4t \end{aligned}\right\}$ from $t=0$ to $t=\frac{1}{2}$

& This is actually the line $y = 2x-2$, from A to B, before!

need $dx = \frac{dx}{dt}dt = 2dt$ $dy = \frac{dy}{dt}dt = 4dt$

$$I = \int_c \underline{E}.d\underline{r} = \int_c 2xy\,dx + \int_c x^2\,dy$$

$$= \int_0^{1/2} 2(2t+1)4t\,2dt + \int_0^{1/2} (2t+1)^2\,4\,dt$$

⋮

you do algebra

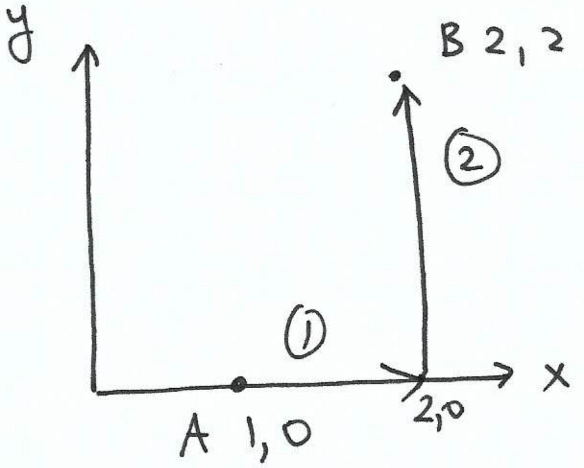
$$= 16\left[\frac{2t^3}{3} + \frac{t^2}{2}\right]_0^{1/2} + 4\left[\frac{4}{3}t^3 + 2t^2 + t\right]_0^{1/2}$$

$$= \frac{10}{3} + \frac{14}{3}$$

I_1 ✓ I_2 ✓

same as before

Example 3 Same $\underline{F}$, same end points A, B, different path



path is along ①, then ②

Along ① $\int \underline{F}.d\underline{r} = \int_c 2xy\,dx + \int_c x^2\,dy$

$I_1 = 0$ because $y = 0$ $\qquad$ $I_2 = 0$ because $dy = 0$

Along ② $\int \underline{F}.d\underline{r} = \int_c 2xy\,dx + \int_c x^2\,dy$

$I_1 = 0$ because $dx = 0$

For I_2, sub $x = 2$

$$I_2 = \int_{y=0}^{2} 4\,dy = \Big[4y\Big]_0^2 = 8$$

$$I = I_1 + I_2 = \underbrace{0 + 0}_{I_1} + \underbrace{0 + 8}_{I_2}$$

$= 8$ same as before

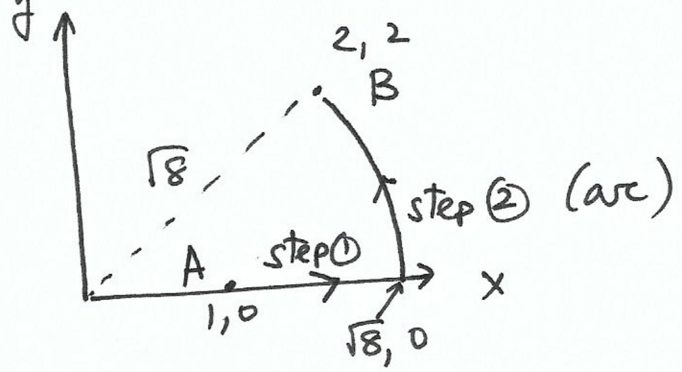
Example 4 Plane polar coords. Same end points, different path.

[Messy, awkward coords for this vector field, just to show it can be done]

step ① $\phi = 0 \quad 1 \leq \rho \leq \sqrt{8}$

step ② $\rho = \sqrt{8}, \quad 0 \leq \phi \leq \pi/4$

$\uparrow$ curve $\qquad \uparrow$ limits



we have $\underline{F} \cdot d\underline{r} = 2xy\,dx + x^2 dy$

$x = \rho\cos\phi \quad y = \rho\sin\phi$

e.g. $dx = \frac{\partial x}{\partial \rho} d\rho + \frac{\partial x}{\partial \phi} d\phi$

change variables, messy, done for you

$$\rightarrow \int_C \underline{F} \cdot d\underline{r} = \int_C 3\cos^2\phi \sin\phi \, \rho^2 d\rho + \int_C \cos\phi(1 - 3\sin^2\phi)\rho^3 d\phi$$

$$= \qquad I_1 \qquad + \qquad I_2$$

step ① $\phi = 0, \sin\phi = 0 \quad \therefore I_1 = 0$

$d\phi = 0 \quad \therefore I_2 = 0$

step ② $d\rho = 0 \quad \therefore I_1 = 0$

$$I = I_2 = \int_0^{\pi/4} \rho^3 (\cos\phi - 3\cos\phi\sin^2\phi)\,d\phi \qquad \rho = \sqrt{8} \text{ const}$$

$$I = 8\sqrt{8}\left[\sin\phi - \sin^3\phi\right]_0^{\pi/4}$$

$$= 8\sqrt{8}\left[\frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\right]$$

$$= 8\sqrt{8}\left[\frac{2}{\sqrt{8}} - \frac{1}{\sqrt{8}}\right] = 8$$

independent of path

4.2 Line integrals $\int_c \underline{F} \cdot d\underline{r}$, between the same end points, that are independent of the path

Summary of previous calcs:

$$F_1 = 2xy\,\hat{\underline{i}} + x^2\,\hat{\underline{j}}$$

$$\boxed{\underline{F_1} \cdot d\underline{r} = 2xy\,dx + x^2\,dy} \quad \text{Eqn } (1)$$

start $A = (1,0)$ end $B = (2,2)$

$$\int_c \underline{F} \cdot d\underline{r} = \int_c 2xy\,dx + \int_c x^2 dy = I_1 + I_2$$

Path 1 $\quad y = 2x - 2$

$$I = I_1 + I_2 = \frac{10}{3} + \frac{14}{3} = 8$$

Path 2 $\quad y = 0 \quad 1 \leq x \leq 2$, then $x = 2 \quad 0 \leq y \leq 2$

$$I = I_1 + I_2 = 0 + 8 = 8$$

I is independent of the path

Now consider:

$$\underline{F_2} = 2xy\,\hat{\underline{i}} - x^2\,\hat{\underline{j}}.$$

$$\boxed{\underline{F_2} \cdot d\underline{r} = 2xy\,dx - x^2\,dy} \quad \text{Eqn } (2)$$

Path 1 $\quad I = I_1 + I_2 = \frac{10}{3} - \frac{14}{3} = -\frac{4}{3}$

Path 2 $\quad I = I_1 + I_2 = 0 - 8 = -8$

I depends on the path

The big idea:

Eqn ① is an exact differential

check mixed second partial derivatives

$$\frac{\partial}{\partial y}(2xy) = 2x \quad \frac{\partial}{\partial x}(x^2) = 2x \quad \checkmark$$

There is a parent function $\Omega(x,y)$

$$\int_c \underline{F}_1 . \underline{ds} = \int_c 2xy\,dx + \int_c x^2 dy = \int_c d\Omega(x,y) = \Omega_B(x,y) - \Omega_A(x,y)$$

The path does not matter, only the end points

$\Omega = x^2 y + c$ (Lecture 1)

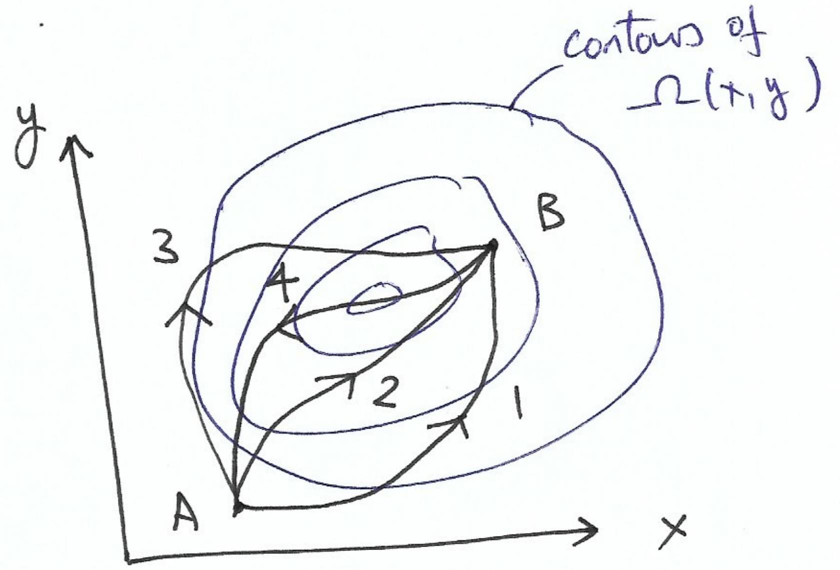
check: $d\Omega = \frac{\partial \Omega}{\partial x} dx + \frac{\partial \Omega}{\partial y} dy = 2xy\,dx + x^2 dy$

$$\Omega_B - \Omega_A = \left[x^2 y + c\right]_{B=(2,2)} - \left[x^2 y + c\right]_{A=(1,0)}$$

$$= 8 + c - 0 - c = 8 \quad \checkmark$$

the line integral calculates the "change in height"

$$\int_A^B d\Omega = \Omega_B - \Omega_A$$



Eqn2 is not an exact differential.

$$\int_{path\,1} \underline{F}_1 . d\underline{r} = \Omega_B - \Omega_A$$

$$\int_{path\,4} \underline{F}_1 . d\underline{r} = \Omega_A - \Omega_B$$

$$\int_{path\,1} \underline{F}_1 . d\underline{r} + \int_{path\,4} \underline{F}_1 . d\underline{r} = 0 = \oint_C \underline{F}_1 . d\underline{r} = 0$$

closed path or loop

The integral of an exact differential around a closed path is 0.

We call $\underline{F}_1$ "conservative" – no work done around loop

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Summary of 4.2 All statements equivalent – if one is true, all are true

(1) $\underline{F} . d\underline{r}$ is an exact differential

(2) $\underline{F} = \frac{\partial \Omega}{\partial x} \hat{\underline{i}} + \frac{\partial \Omega}{\partial y} \hat{\underline{j}} + \frac{\partial \Omega}{\partial z} \hat{\underline{k}}$ for some Ω

(3) $\int_C \underline{F} . d\underline{r}$ from A to B does not depend on path

(4) $\oint_C \underline{F} . d\underline{r} = 0$

(5) $\underline{F}$ is conservative

5. Gradient   grad Ω or $\nabla\Omega$

5.1 Definition

From a scalar field $\Omega(x,y,z)$, the vector field

$$\underline{F} = \frac{\partial\Omega}{\partial x}\hat{\underline{i}} + \frac{\partial\Omega}{\partial y}\hat{\underline{j}} + \frac{\partial\Omega}{\partial z}\hat{\underline{k}} = \text{grad}\,\Omega = \nabla\Omega$$

is conservative i.e. $\oint \underline{F}.d\underline{r} = 0$

The symbol ∇ "del" (or "nabla") is an operator,

$$\nabla = \hat{\underline{i}}\frac{\partial}{\partial x} + \hat{\underline{j}}\frac{\partial}{\partial y} + \hat{\underline{k}}\frac{\partial}{\partial z}$$

Operating on a scalar field, it produces a vector field.
Here Ω (often ϕ in physics) is the <u>potential</u> associated with the vector field $\underline{F}$.

[N.B. Take care over sign e.g. the gravitational field $g = -\nabla\phi$. But force <u>doing work</u> $\underline{F} = -mg$]

5.2 Directional derivative

Using $\nabla\Omega$ we can calculate the derivative of Ω in any direction.

We can write

$$d\Omega = \frac{\partial\Omega}{\partial x}dx + \frac{\partial\Omega}{\partial y}dy + \frac{\partial\Omega}{\partial z}dz = \nabla\Omega \cdot d\underline{r}$$

What is derivative in particular direction $\hat{\underline{a}}$? ← unit vector

If we move distance ds, then

$$d\Omega = \nabla\Omega \cdot \hat{\underline{a}}\, ds$$

Rate of change

$$\frac{d\Omega}{ds} = \nabla\Omega \cdot \hat{\underline{a}} \quad = \text{directional derivative}$$

e.g. choose $\hat{\underline{a}} = \hat{\underline{k}}$

$$\frac{d\Omega}{ds} = \left(\hat{\underline{i}}\frac{\partial\Omega}{\partial x} + \hat{\underline{j}}\frac{\partial\Omega}{\partial y} + \hat{\underline{k}}\frac{\partial\Omega}{\partial z}\right)\cdot\hat{\underline{k}} = \frac{\partial\Omega}{\partial z}$$

as expected

At a particular point x, y, z
~~what~~ what direction $\hat{\underline{a}}_{max}$ is $\frac{d\Omega}{ds}$ maximum? (steepest gradient)

evidently

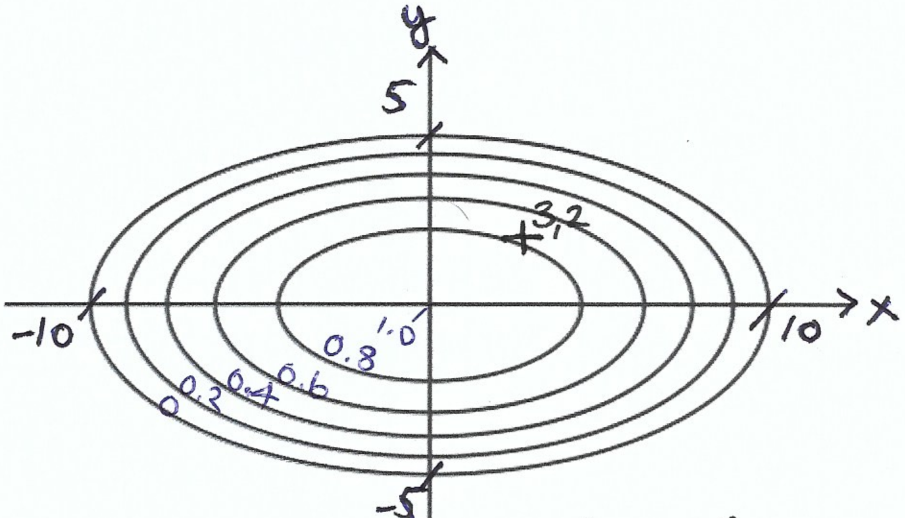
$$\hat{\underline{a}}_{max} = \frac{\nabla\Omega}{|\nabla\Omega|}$$

Steepest gradient is $|\nabla\Omega|$

Example (2D)

$\Omega = z(x,y) = 1 - \frac{x^2}{100} - \frac{y^2}{25}$

an elliptical hill (!) units are km



① At the point 3, 2 what is the direction* of max. gradient, and the value of the gradient?

* a unit vector

N.B. it's a 2D vector

$$\nabla z = \hat{\underline{i}} \frac{\partial z}{\partial x} + \hat{\underline{j}} \frac{\partial z}{\partial y} = -\frac{x}{50}\hat{\underline{i}} - \frac{2y}{25}\hat{\underline{j}}$$

$x = 3,\ y = 2$

$$\nabla z(3,2) = -\frac{3}{50}\hat{\underline{i}} - \frac{4}{25}\hat{\underline{j}} = -0.06\hat{\underline{i}} - 0.16\hat{\underline{j}}$$

$$\text{magnitude} = (0.06^2 + 0.16^2)^{1/2} = 0.17$$

$$\text{direction} \quad \hat{\underline{a}} = -\frac{0.06}{0.17}\hat{\underline{i}} - \frac{0.16}{0.17}\hat{\underline{j}}$$

② At 3,2, in the direction of $\hat{\underline{i}} + \hat{\underline{j}}$, what is the derivative?

$$\hat{\underline{a}} = \frac{\hat{\underline{i}}}{\sqrt{2}} + \frac{\hat{\underline{j}}}{\sqrt{2}} \quad \text{unit vector}$$

$$\nabla z \cdot \hat{\underline{a}} = (-0.06\hat{\underline{i}} - 0.16\hat{\underline{j}}) \cdot \left(\frac{\hat{\underline{i}}}{\sqrt{2}} + \frac{\hat{\underline{j}}}{\sqrt{2}}\right)$$

$$= -\frac{0.06}{\sqrt{2}} - \frac{0.16}{\sqrt{2}} = -0.155 \quad \text{(it's downhill)}$$

5.3 $\nabla\Omega$ in other coordinate systems

generally $d\Omega = \nabla\Omega \cdot d\underline{r}$

Derive $\nabla\Omega$ in <u>cylindrical polar</u>

$$d\Omega = \frac{\partial\Omega}{\partial\rho}\,d\rho + \frac{\partial\Omega}{\partial\phi}\,d\phi + \frac{\partial\Omega}{\partial z}\,dz$$

$$d\underline{r} = d\rho\,\hat{\underline{\rho}} + \rho\,d\phi\,\hat{\underline{\phi}} + dz\,\hat{\underline{k}}$$

$$\therefore \nabla\Omega(\rho,\phi,z) = \frac{\partial\Omega}{\partial\rho}\hat{\underline{\rho}} + \frac{1}{\rho}\frac{\partial\Omega}{\partial\phi}\hat{\underline{\phi}} + \frac{\partial\Omega}{\partial z}\hat{\underline{k}}$$

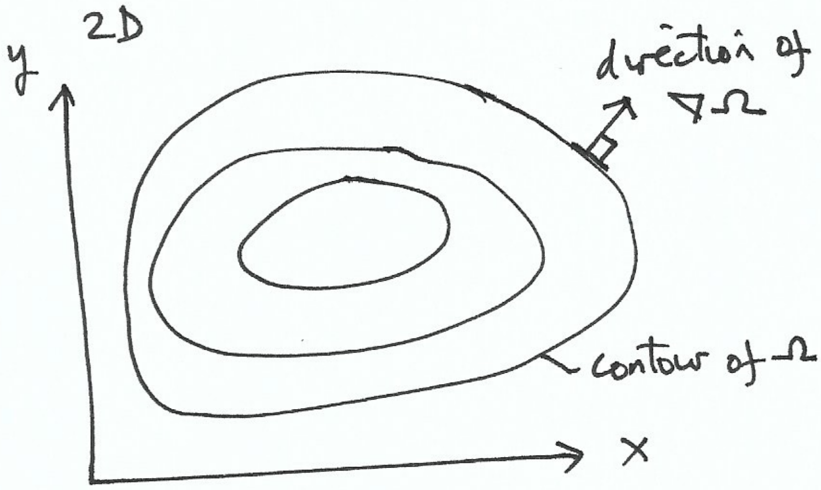
<u>spherical polar</u>

prove to yourself that

$$\nabla\Omega(r,\theta,\phi) = \frac{\partial\Omega}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial\Omega}{\partial\theta}\hat{\underline{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\Omega}{\partial\phi}\hat{\underline{\phi}}$$

NOT EXAMINABLE

5.4 $\nabla\Omega$ and the normal to a surface



In 2D $\dfrac{d\Omega}{dS} = \nabla\Omega \cdot \hat{\underline{a}}$

along a contour $\dfrac{d\Omega}{dS} = 0$

$\therefore$ $\nabla\Omega$ is $\perp$r to line of constant Ω

In 3D $\Omega(x,y,z)$, $\nabla\Omega$ is $\perp$r to surfaces of constant Ω.

E.g. $\Omega = x^2+y^2+z^2$, $\nabla\Omega = 2x\,\hat{\underline{i}} + 2y\,\hat{\underline{j}} + 2z\,\hat{\underline{k}} = 2r\,\hat{\underline{r}}$

Now consider a 3D surface defined by $z = f(x,y)$.
Suppose we want to find the direction normal to a point on the surface.

<u>Key step</u>: define $\Omega(x,y,z) = f(x,y) - z$

Then the surface is the surface $\Omega = 0 =$ constant

$\therefore$ A vector normal to the surface at point x,y,z is

$$\underline{n} = \nabla\Omega(x,y,z)$$

unit vector $\hat{\underline{n}} = \dfrac{\nabla\Omega}{|\nabla\Omega|}$

<u>Example</u> Section 3.5. For $z = xy$ we derived $\underline{N} = -y\,\hat{\underline{i}} - x\,\hat{\underline{j}} + \hat{\underline{k}}$

so $\hat{\underline{n}} = \dfrac{\underline{N}}{|\underline{N}|}$ is unit vector $\perp$r to surface

Use grad: define $\Omega = f(x,y) - z = xy - z$

$\nabla\Omega = y\,\hat{\underline{i}} + x\,\hat{\underline{j}} - \hat{\underline{k}}$ same as $\underline{N}$

except for ambiguity of sign.

6. Divergence div $\underline{B}$ or $\nabla . \underline{B}$

6.1 Definition

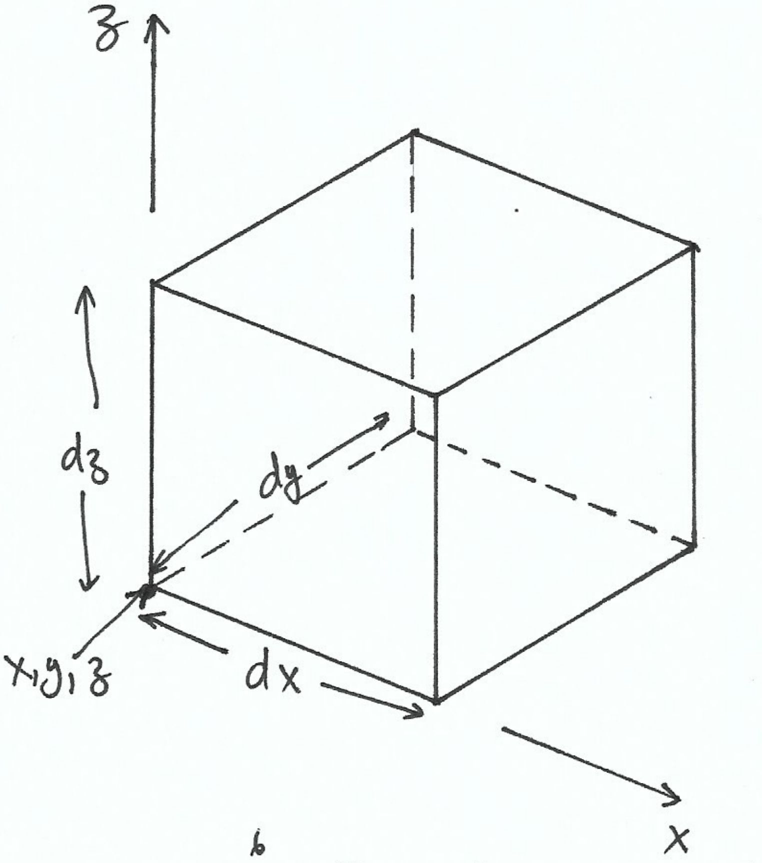
$$\nabla . \underline{B} = \lim_{V \to 0} \left(\frac{1}{V} \oint\!\!\oint_S \underline{B} . d\underline{S} \right)$$

closed surface, $d\underline{S}$ is out

So divergence is a "flux density"

Derive geometrically using Cartesian coordinates

$$\underline{B}(x, y, z) = B_x \hat{\underline{i}} + B_y \hat{\underline{j}} + B_z \hat{\underline{k}}$$



Face	$d\underline{S}$
① $\Delta x = 0$	$-\hat{\underline{i}}\, dy\, dz$
② $\Delta x = dx$	$\hat{\underline{i}}\, dy\, dz$
③ $\Delta y = 0$	$-\hat{\underline{j}}\, dx\, dz$
④ $\Delta y = dy$	$\hat{\underline{j}}\, dx\, dz$
⑤ $\Delta z = 0$	$-\hat{\underline{k}}\, dx\, dy$
⑥ $\Delta z = dz$	$\hat{\underline{k}}\, dx\, dy$

We want $\sum_{i=1}^{6} F_i$ where F_i is flux out over face i.

We will compute F_i using tangent-plane approximation to $\underline{B}$ over each face. Then F_i arbitrarily accurate as $dx, dy, dz \to 0$

On 1 $\hat{\underline{i}}$ component of field ~~is~~ varies as

$$B_x + \frac{\partial B_x}{\partial y} dy + \frac{\partial B_x}{\partial z} dz$$

Because planar, average value over face is value at centre

$$\therefore \quad F_1 = -\left(B_x + \frac{\partial B_x}{\partial y}\frac{dy}{2} + \frac{\partial B_x}{\partial z}\frac{dz}{2}\right) dy\,dz = -\overline{B}_{x1}\, dy\,dz$$

$$F_2 = \left(\overline{B}_{x1} + \frac{\partial \overline{B}_{x1}}{\partial x} dx\right) \cancel{dx}\, dy\,dz$$

$$F_1 + F_2 = \frac{\partial}{\partial x}\left(B_x + \frac{\partial B_x}{\partial y}\frac{dy}{2} + \frac{\partial B_x}{\partial z}\frac{dz}{2}\right) dx\,dy\,dz$$

There are analogous terms for $F_3 + F_4$, $F_5 + F_6$

Now $$\nabla . \underline{B} = \lim_{dx,dy,dz \to 0} \frac{\sum_{i=1}^{6} F_i}{dx\,dy\,dz}$$

$$= \lim_{dx,dy,dz \to 0} \Bigg[\frac{\partial}{\partial x}\left(B_x + \frac{\partial B_x}{\partial y}\frac{dy}{2} + \frac{\partial B_x}{\partial z}\frac{dz}{2}\right) + \frac{\partial}{\partial y}\left(B_y + \frac{\partial B_y}{\partial x}\frac{dx}{2} + \frac{\partial B_y}{\partial z}\frac{dz}{2}\right) + \frac{\partial}{\partial z}\left(B_z + \frac{\partial B_z}{\partial x}\frac{dx}{2} + \frac{\partial B_z}{\partial y}\frac{dy}{2}\right)\Bigg]$$

$$\therefore \nabla . \underline{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$ (rigorous)

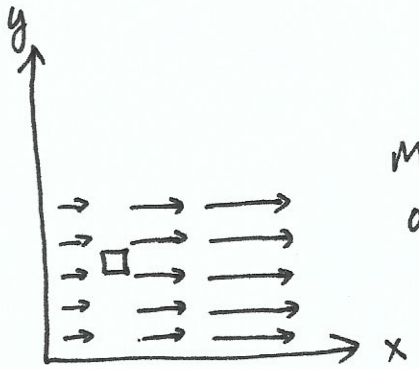
[Note: terms varying across face not important, only variations between faces]

Example ①

$\underline{B} = \hat{\underline{i}}\, ax$

$\nabla . \underline{B} = \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = a$

everywhere

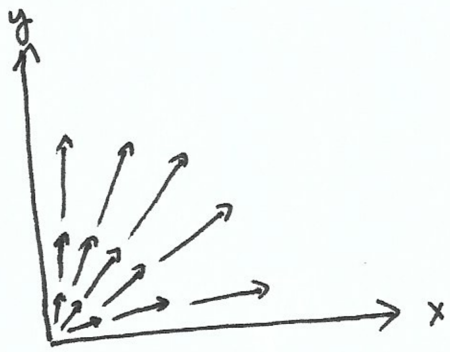


more flow out than in

Ex ②

$\underline{B} = x\,\hat{\underline{i}} + y\,\hat{\underline{j}} + z\,\hat{\underline{k}} = r\,\hat{\underline{r}}$

$\nabla . \underline{B} = 1 + 1 + 1 = 3$ everywhere



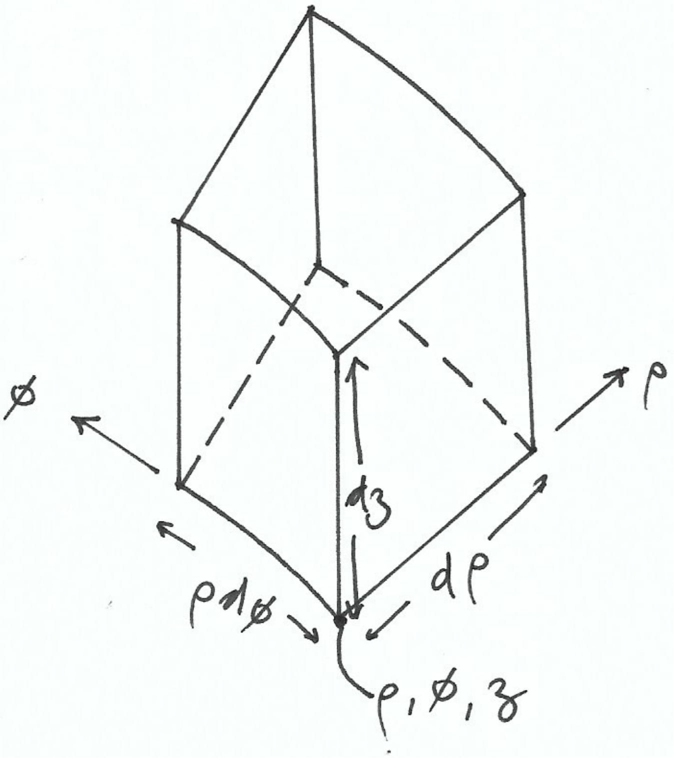
Ex ③ $\underline{B} = \hat{\underline{i}} + \hat{\underline{j}} + \hat{\underline{k}}$ same everywhere

$\nabla . \underline{B} = 0$ everywhere

6.2 $\nabla \cdot \underline{B}$ in other coordinate systems

cylindrical coordinates – geometric derivation

$$\underline{B} = B_\rho \hat{\underline{\rho}} + B_\phi \hat{\underline{\phi}} + B_z \hat{\underline{k}}$$



Face		$d\underline{S}$
①	$\Delta\rho = 0$	$-\rho\, d\phi\, dz\, \hat{\underline{\rho}}$
②	$\Delta\rho = d\rho$	$(\rho + d\rho)\, d\phi\, dz\, \hat{\underline{\rho}}$
③	$\Delta\phi = 0$	$-d\rho\, dz\, \hat{\underline{\phi}}$
④	$\Delta\phi = d\phi$	$d\rho\, dz\, \hat{\underline{\phi}}$
⑤	$\Delta z = 0$	$-\hat{\underline{k}}\, \frac{d\phi}{2}\left[(\rho + d\rho)^2 - \rho^2\right] = -\rho\, d\phi\, d\rho \left[1 + \frac{d\rho}{2\rho}\right] \hat{\underline{k}}$
⑥	$\Delta z = dz$	$\rho\, d\phi\, d\rho \left[1 + \frac{d\rho}{2\rho}\right] \hat{\underline{k}}$

From before, ignore variation over face e.g. $\frac{\partial B_\rho}{\partial \phi}\frac{d\phi}{2}$ on ①

$$F_1 = -B_\rho \,\rho \,d\phi \,dz \qquad F_2 = \left(B_\rho + \frac{\partial B_\rho}{\partial \rho} d\rho\right)(\rho + d\rho)\,d\phi\,dz$$

$$F_1 + F_2 = \left[\frac{B_\rho}{\rho} + \frac{\partial B_\rho}{\partial \rho} + \frac{\partial B_\rho}{\partial \rho}\frac{d\rho}{\rho}\right]\rho\,d\rho\,d\phi\,dz$$

$$F_3 = -B_\phi \,d\rho\,dz \qquad F_4 = \left[B_\phi + \frac{\partial B_\phi}{\partial \phi} d\phi\right] d\rho\,dz$$

$$F_3 + F_4 = \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi}\,\rho\,d\rho\,d\phi\,dz$$

$$F_5 = -B_z\,\rho\,d\phi\,d\rho\left[1 + \frac{d\rho}{2\rho}\right]$$

$$F_6 = \left[B_z + \frac{\partial B_z}{\partial z} dz\right]\rho\,d\phi\,d\rho\left[1 + \frac{d\rho}{2\rho}\right]$$

$$F_5 + F_6 = \frac{\partial B_z}{\partial z}\left[1 + \frac{d\rho}{2\rho}\right]\rho\,d\rho\,d\phi\,dz$$

$$\nabla . \underline{B} = \lim_{d\rho d\phi dz \to 0} \frac{\Sigma F_i}{V \to \rho\,d\rho\,d\phi\,dz} = \frac{B_\rho}{\rho} + \frac{\partial B_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}$$

$$\nabla . \underline{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}$$

Challenge: try the same in spherical polar:

$$\nabla . \underline{B} = \frac{1}{r^2}\frac{\partial (r^2 B_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\sin\theta\, B_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial \phi}$$

What does $\nabla . \underline{B}$ mean?

<u>Cartesian</u> $\nabla = \hat{\underline{i}} \frac{\partial}{\partial x} + \hat{\underline{j}} \frac{\partial}{\partial y} + \hat{\underline{k}} \frac{\partial}{\partial z}$ (not a vector)

$$\underline{B} = \hat{\underline{i}} B_x + \hat{\underline{j}} B_y + \hat{\underline{k}} B_z$$

$$\nabla . \underline{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

looks like dot product of 2 vectors

$\boxed{\text{The bad news .....}}$

<u>Cylindrical</u>

$$\nabla = \hat{\underline{\rho}} \frac{\partial}{\partial \rho} + \frac{1}{\rho} \hat{\underline{\phi}} \frac{\partial}{\partial \phi} + \hat{\underline{k}} \frac{\partial}{\partial z}$$

$$\underline{B} = \hat{\underline{\rho}} B_\rho + \hat{\underline{\phi}} B_\phi + \hat{\underline{k}} B_z$$

$$\nabla . \underline{B} = \frac{B_\rho}{\rho} + \frac{\partial B_\rho}{\partial \rho} + \frac{1}{\rho} \frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}$$

extra term ?!

$\nabla . \underline{B}$ not a dot product of two vectors

"abuse of notation"
- it's just shorthand for divergence

The good news ....

Think of $\nabla . \underline{B}$ as "differentiate first, dot product second"

Cartesian

$$\nabla . \underline{B} = \hat{\underline{i}} . \frac{\partial \underline{B}}{\partial x} + \hat{\underline{j}} . \frac{\partial \underline{B}}{\partial y} + \hat{\underline{k}} . \frac{\partial \underline{B}}{\partial z}$$

looks like grad

$$\nabla \phi = \hat{\underline{i}} \frac{\partial \Omega}{\partial x} + \hat{\underline{j}} \frac{\partial \Omega}{\partial y} + \hat{\underline{k}} \frac{\partial \Omega}{\partial z}$$

check e.g. first term

$$\frac{\partial \underline{B}}{\partial x} = \hat{\underline{i}} \frac{\partial B_x}{\partial x} + \hat{\underline{j}} \frac{\partial B_y}{\partial x} + \hat{\underline{k}} \frac{\partial B_z}{\partial x}$$

$$\hat{\underline{i}} . \frac{\partial \underline{B}}{\partial x} = \frac{\partial B_x}{\partial x} \quad \checkmark$$

This idea works for cylindrical, spherical
div, curl

curl is "differentiate first, cross product second"

Analytic derivation of $\nabla . \underline{B}$ in cylindrical coordinates

$$\nabla = \hat{\underline{\rho}} \frac{\partial}{\partial \rho} + \frac{1}{\rho} \hat{\underline{\phi}} \frac{\partial}{\partial \phi} + \hat{\underline{k}} \frac{\partial}{\partial z}$$

$$\underline{B} = \hat{\underline{\rho}} B_\rho + \hat{\underline{\phi}} B_\phi + \hat{\underline{k}} B_z$$

"differentiate first, dot product second"

$$\nabla . \underline{B} = \hat{\underline{\rho}} . \frac{\partial \underline{B}}{\partial \rho}^{①} + \frac{1}{\rho} \hat{\underline{\phi}} . \frac{\partial \underline{B}}{\partial \phi}^{②} + \hat{\underline{k}} . \frac{\partial \underline{B}}{\partial z}^{③}$$

recall $\hat{\underline{\rho}}(\phi)$, $\hat{\underline{\phi}}(\phi)$, $\frac{\partial \hat{\underline{\rho}}}{\partial \phi} = \hat{\underline{\phi}}$, $\frac{\partial \hat{\underline{\phi}}}{\partial \phi} = -\hat{\underline{\rho}}$, $B_\rho(\rho, \phi, z)$
$B_\phi(\rho, \phi, z)$
$B_z(\rho, \phi, z)$

1st $$\frac{\partial \underline{B}}{\partial \rho} = \hat{\underline{\rho}} \frac{\partial B_\rho}{\partial \rho} + \hat{\underline{\phi}} \frac{\partial B_\phi}{\partial \rho} + \hat{\underline{k}} \frac{\partial B_z}{\partial \rho}$$

$$\hat{\underline{\rho}} . \frac{\partial \underline{B}}{\partial \rho} = \frac{\partial B_\rho}{\partial \rho} \qquad ①$$

2nd $$\frac{\partial \underline{B}}{\partial \phi} = \hat{\underline{\phi}} B_\rho + \hat{\underline{\rho}} \frac{\partial B_\rho}{\partial \phi} - \hat{\underline{\rho}} B_\phi + \hat{\underline{\phi}} \frac{\partial B_\phi}{\partial \phi} + \hat{\underline{k}} \frac{\partial B_z}{\partial \phi}$$

$$\frac{1}{\rho} \hat{\underline{\phi}} . \frac{\partial \underline{B}}{\partial \phi} = \frac{B_\rho}{\rho} + \frac{1}{\rho} \frac{d B_\phi}{\partial \phi} \qquad ②$$

the extra term!

3rd $$\frac{\partial \underline{B}}{\partial z} = \hat{\underline{\rho}} \frac{\partial B_\rho}{\partial z} + \hat{\underline{\phi}} \frac{\partial B_\phi}{\partial z} + \hat{\underline{k}} \frac{\partial B_z}{\partial z}$$

$$\hat{\underline{k}} . \frac{\partial \underline{B}}{\partial z} = \frac{\partial B_z}{\partial z} \qquad ③$$

summing

$$\nabla . \underline{B} = \frac{\partial B_\rho}{\partial \rho} + \frac{B_\rho}{\rho} + \frac{1}{\rho} \frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}$$

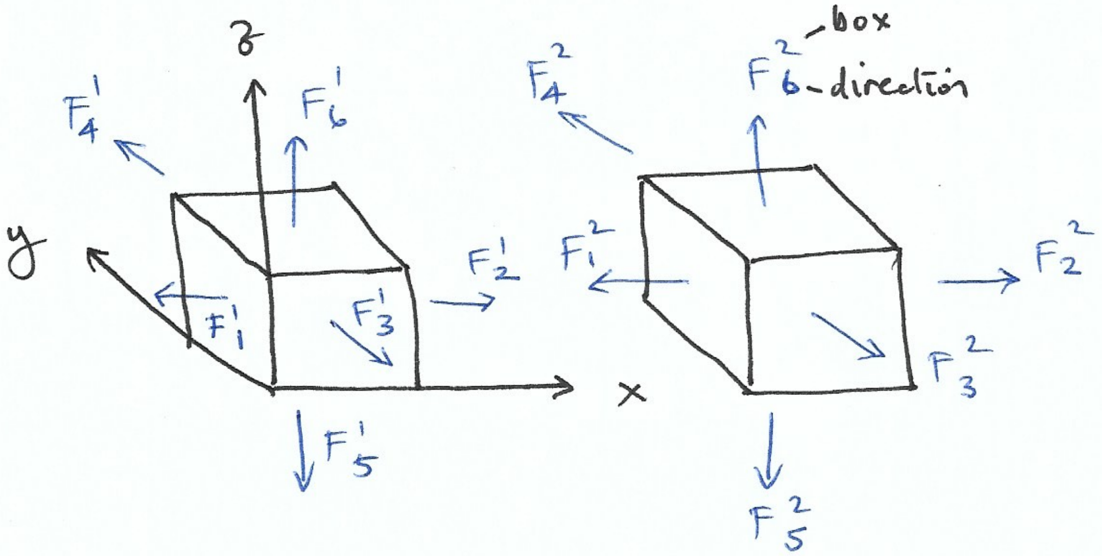
$$\nabla . \underline{B} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) + \frac{1}{\rho} \frac{d B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}$$

as before

6.3 Divergence theorem

(sometimes called Gauss' theorem, but <u>not</u> Gauss' law)

Consider 2 infinitesimal boxes, adjacent along x axis



directions	
1	$-x$
2	$+x$
3	$-y$
4	$+y$
5	$-z$
6	$+z$

on each face $F = \underline{B} \cdot d\underline{S}$

Box 1 $\sum_{i=1}^{6} F_i^1 = \nabla \cdot \underline{B}_1 \, dV$

Box 2 $\sum_{i=1}^{6} F_i^2 = \nabla \cdot \underline{B}_2 \, dV$

note that $F_2^1 = -F_1^2$

Now push the two boxes together and sum $\underline{B} \cdot d\underline{S}$ over the surface

$$\sum F = \sum_{i=1}^{6} F_i^1 + \sum_{i=1}^{6} F_i^2 \underbrace{\left(-F_2^1 - F_1^2\right)}_{=0} = \nabla \cdot \underline{B}_1 \, dV + \nabla \cdot \underline{B}_2 \, dV$$

$$\therefore \sum_{\text{new box}} \underline{B} \cdot d\underline{S} = \sum_{j=1}^{2} \nabla \cdot \underline{B}_j \, dV$$

add more boxes to create a macroscopic volume

$$\oiint_S \underline{B} \cdot d\underline{S} = \iiint \nabla \cdot \underline{B} \, dV$$

closed surface

the <u>divergence theorem</u>

(an integral theorem)

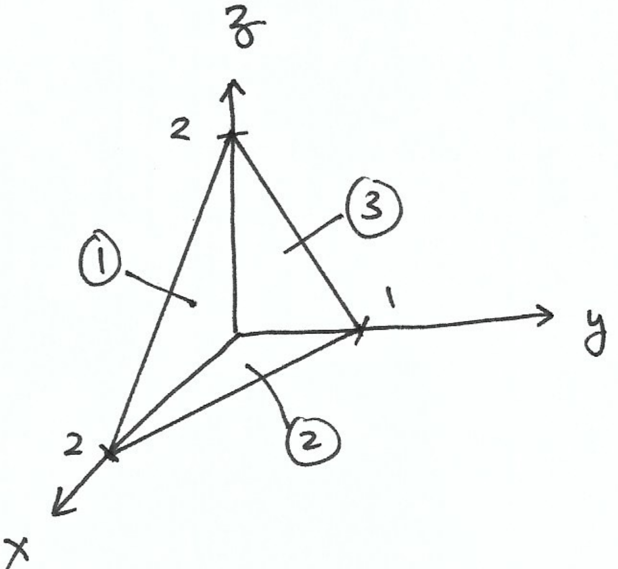
Example (sketch)

(sometimes one integral is hard, so evaluate it by doing the other integral)
here we can do both

$\underline{B} = x\,\hat{\underline{i}} + y\,\hat{\underline{j}} + z\,\hat{\underline{k}}$

tetrahedron bound by

③ $x = 0$
① $y = 0$
② $z = 0$
④ $z = 2 - x - 2y$ sloping surface



$$\oiint_S \underline{B}.d\underline{S} \quad = \quad \iiint_V \nabla.\underline{B}.\,dV$$

closed surface
LHS

RHS

sketch of answer

(LHS) on ① ② ③ easy to show $\underline{B}.d\underline{S} = 0$

e.g. on ① $d\underline{S}$ points in $-\hat{\underline{j}}$ direction but $B_y = y = 0$
$\therefore\ \underline{B}.d\underline{S} = 0$

on ④ find $d\underline{S} = (\hat{\underline{i}} + 2\hat{\underline{j}} + \hat{\underline{k}})\,dx\,dy$
by standard method

Then $\underline{B}.d\underline{S} = (x + 2y + z)\,dx\,dy = 2\,dx\,dy$

$$\iint_{④} \underline{B}.d\underline{S} = 2 \times \text{area of ②} \;=\; 2 \;=\; LHS$$

RHS

$$\nabla . \underline{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 1 + 1 + 1 = 3$$

$$\therefore \iiint \nabla . \underline{B} \cdot dv = 3 \times \text{vol. of tetrahedron}$$

$$V = \frac{1}{3} \text{ base} \times \text{height} = \frac{1}{3} \times 1 \times 2$$

$$3 \times \text{vol} = 2 = \text{RHS}$$

$\therefore$ LHS = RHS as expected

Homework : check all the steps in this argument

6.4 <u>Laplacian</u>

$$\nabla \Omega = \hat{i} \frac{\partial \Omega}{\partial x} + \hat{j} \frac{\partial \Omega}{\partial y} + \hat{\underline{k}} \frac{\partial \Omega}{\partial z}$$

$$\nabla . (\nabla \Omega) = \frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} + \frac{\partial^2 \Omega}{\partial z^2} = \nabla^2 \Omega$$

∇^2 or "del squared" is called the Laplacian

seen e.g. in Laplace's equation $\nabla^2 u = 0$

and the wave equation $\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$

(if you like, derive Laplacian in cylindrical coords)

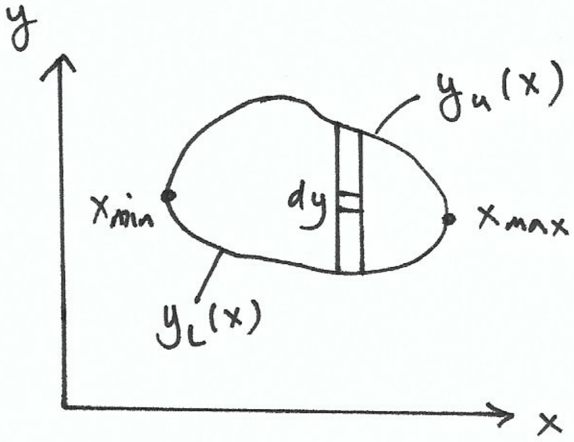
7. Curl $\nabla \times \underline{B}$

7.1 Green's theorem in the plane

$$\oint P(x,y)\,dx + Q(x,y)\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

line integral, anticlockwise looking down on plane x,y

double integral over region enclosed



Proof Consider $I_1 = -\iint \frac{\partial P}{\partial y}\,dy\,dx$

first step $-\int_{y=y_L(x)}^{y_u(x)} \frac{\partial P(x,y)}{\partial y}\,dy$

$$= -P(x,y_u) + P(x,y_L)$$

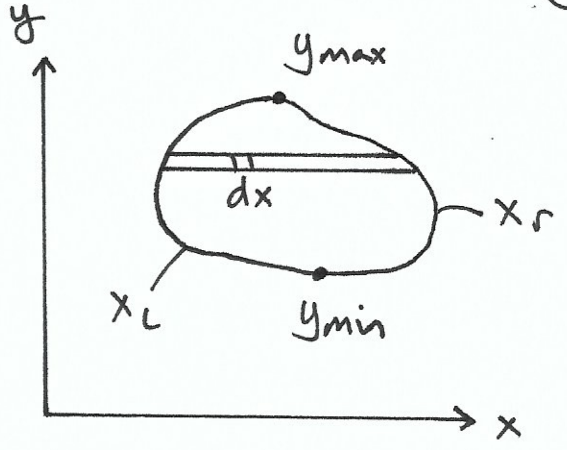
now add up slices

$$I_1 = -\int_{x_{min}}^{x_{max}} P(x,y_u)\,dx + \int_{x_{min}}^{x_{max}} P(x,y_L)\,dx$$

$$= +\int_{x_{max}}^{x_{min}} P(x,y_u)\,dx + \int_{x_{min}}^{x_{max}} P(x,y_L)\,dx = +\oint P\,dx$$

$$\therefore \oint P\,dx = -\iint \frac{\partial P}{\partial y}\,dy\,dx$$

similarly for $I_2 = \iint\limits_R \frac{\partial Q}{\partial x}\, dx\, dy$



first step $\displaystyle\int_{x = x_L(y)}^{x_r(y)} \frac{\partial Q(x,y)}{\partial x}\, dx$

$= Q(x_r, y) - Q(x_L, y)$

$$I_2 = \int_{y=y_{min}}^{y_{max}} Q(x_r, y)\, dy + \int_{y=y_{max}}^{y_{min}} Q(x_L, y)\, dy = \oint Q\, dy$$

combining

$$\oint P\,dx + Q\,dy = \iint\limits_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\, dy$$

P and Q can be any (continuous) functions.

<u>Very general</u> !

Particular examples are of special interest to ~~vector calculus~~ electromagnetism.

4 Examples

Example 1 $P\,dx + Q\,dy$ is an exact differential

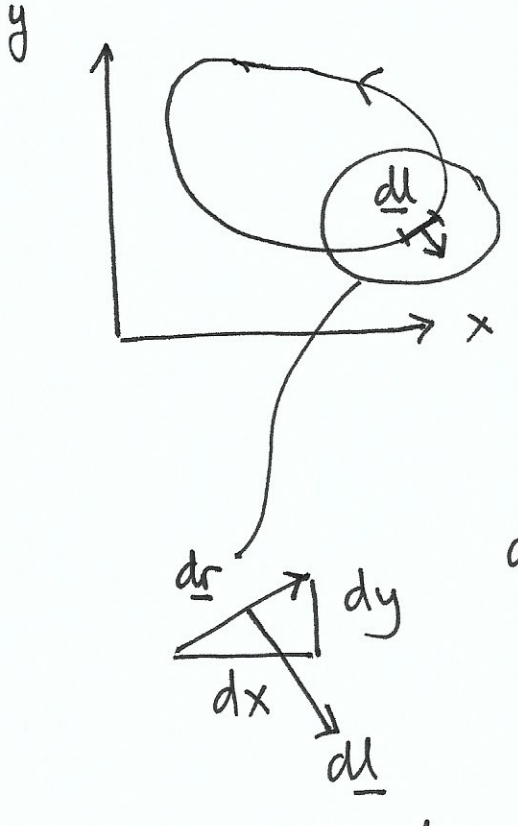
Then $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

so $\displaystyle\oint P\,dx + Q\,dy = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$

$= 0$

as expected!

Example 2 Consider 'flux' out of a closed loop in 2D (2D equiv of $\oiint_S \underline{B}.d\underline{S}$)



vector field $\underline{B}$

we want $\oint \underline{B}.d\underline{l}$

$d\underline{l} = dy\,\hat{\underline{i}} - dx\,\hat{\underline{j}}$ defines <u>outward</u>* vector of correct length

* for anticlockwise loop

Then $\underline{B}.d\underline{l} = (B_x\hat{\underline{i}} + B_y\hat{\underline{j}}) \cdot (dy\,\hat{\underline{i}} - dx\,\hat{\underline{j}})$

$$\oint \underline{B} \cdot d\underline{l} = \oint \underbrace{-B_y}_{P} \, dx + \underbrace{B_x}_{Q} \, dy$$

$$\oint \underline{B} \cdot d\underline{l} = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx \, dy = \iint \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{dy} \, dx \, dy$$

the divergence theorem in 2 dimensions

Example 3

More generally, 'work done' line integral

$$\oint \underline{F} \cdot d\underline{r} = \oint \overbrace{F_x}^{P} dx + \overbrace{F_y}^{Q} dy$$

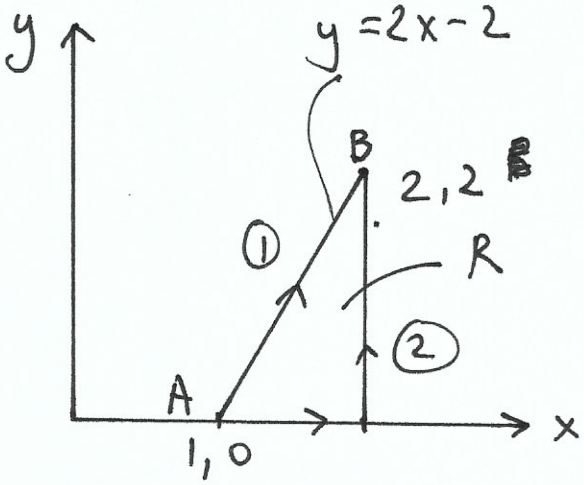
$$= \iint \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \, dx \, dy$$

= Stokes' theorem (L17) in 2D

Example 4 can replace a line integral with a 2D integral (and vice versa)

Recall line integral in Section 4.2

$\underline{F}_2 = 2xy\,\hat{\underline{i}} - x^2\,\hat{\underline{j}}$



we computed $\int_C \underline{F}.d\underline{r}$ along:

path ① $\int_A^B \underline{F}.d\underline{r} = -\frac{4}{3}$

$\therefore \int_B^A \underline{F}.d\underline{r} = \frac{4}{3}$

path ② $\int_A^B \underline{F}.d\underline{r} = -8$

$\therefore \oint \underline{F}.d\underline{r} = -\frac{20}{3} = \oint \underbrace{2xy}_{P}\,dx \underbrace{-x^2}_{Q}\,dy$

Green's theorem

$$\oint \underline{F}.d\underline{r} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = -\iint 4x\,dx\,dy$$

$$I = -\int_{x=1}^{2}\int_{y=0}^{2x-2} 4x \, dy \, dx = -\int_{1}^{2} \left[4xy\right]_{0}^{2x-2} dx$$

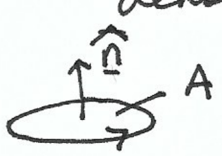
$$I = -\int_{1}^{2} (8x^2 - 8x) \, dx = \left[-\frac{8}{3}x^3 + 4x^2\right]_{1}^{2} = -\frac{20}{3}$$

as expected

7.2 Curl definition $\nabla \times \underline{B}$

Curl is a vector that quantifies the "circulation surface density"

$$(\nabla \times \underline{B}) \cdot \hat{\underline{n}} = \lim_{A \to 0} \left(\frac{1}{A} \oint \underline{B} \cdot d\underline{r} \right)$$



curl

unit vector ⊥r to area

right hand rule

In Cartesian coords to derive e.g. $\hat{\underline{k}}$ component, consider loop of finite size and shrink.

$$\frac{1}{A} \oint \underline{B} \cdot d\underline{r} = \frac{1}{A} \oint B_x dx + B_y dy$$

$$= \frac{1}{A} \iint_R \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \, dx\, dy \quad \text{(Green's theorem)}$$

Now shrink:

$$\nabla \times \underline{B} \cdot \hat{\underline{k}} = \lim_{A \to 0} \frac{\iint_R \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \, dx\, dy}{\iint dx\, dy} = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}$$

similarly

$$\nabla \times \underline{B} \cdot \hat{\underline{i}} = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}$$

$$\nabla \times \underline{B} \cdot \hat{\underline{j}} = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}$$

nice summary:

$$\nabla \times \underline{B} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

Now $\nabla = \hat{\underline{i}}\frac{\partial}{\partial x} + \hat{\underline{j}}\frac{\partial}{\partial y} + \hat{\underline{k}}\frac{\partial}{\partial z}$

$\underline{B} = \hat{\underline{i}}\, B_x + \hat{\underline{j}}\, B_y + \hat{\underline{k}}\, B_z$

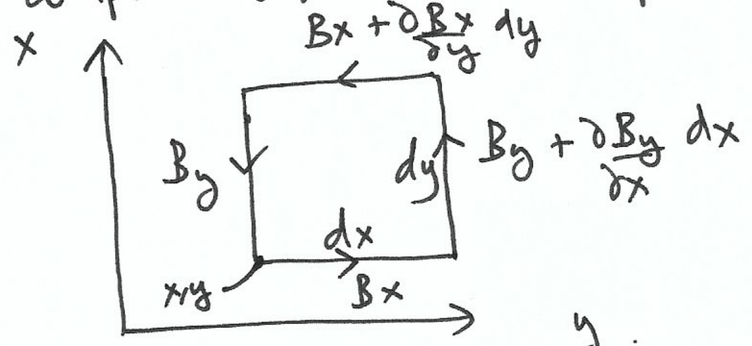
so $\nabla \times \underline{B}$ looks like cross product between 2 vectors.

More coherent to write it as

$$\nabla \times \underline{B} = \hat{\underline{i}} \times \frac{\partial \underline{B}}{\partial x} + \hat{\underline{j}} \times \frac{\partial \underline{B}}{\partial y} + \hat{\underline{k}} \times \frac{\partial \underline{B}}{\partial z}$$

home work : check this

To understand "circulation surface density" look again at $\hat{\underline{k}}$ component and consider infinitesimal area $dx\,dy$



$$\oint \frac{\underline{B}.d\underline{r}}{A} = \frac{B_x dx + \left(B_y + \frac{\partial B_y}{\partial x}dx\right)dy - \left(B_x + \frac{\partial B_x}{\partial y}dy\right)dx - B_y dy}{dx\,dy}$$

$$= \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}$$

causes rotation ↺   causes rotation ↻   (explains minus sign)

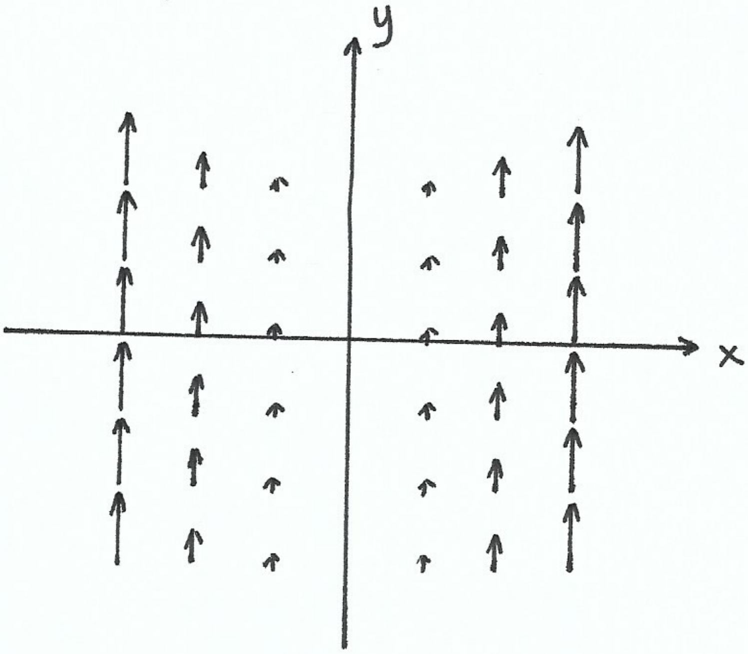
A conservative field has zero curl !

$$\nabla \times \nabla \Omega = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \hat{\underline{i}}\left(\frac{\partial^2 \Omega}{\partial y \partial z} - \frac{\partial^2 \Omega}{\partial z \partial y}\right) + \hat{\underline{j}}\ldots + \hat{\underline{k}}\ldots$$

0

$\nabla \times \underline{B} = 0$ implies $\underline{B}$ is a conservative field, or "irrotational" field

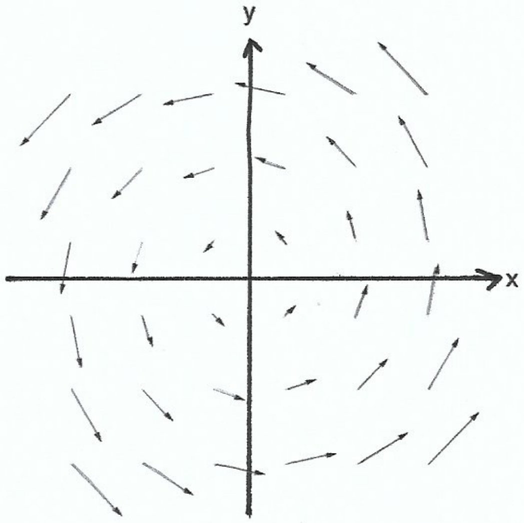
Examples ① $\underline{B} = 0\,\hat{\underline{i}} + x^2\,\hat{\underline{j}} + 0\,\hat{\underline{k}}$



$$\nabla \times \underline{B} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^2 & 0 \end{vmatrix}$$

$$\nabla \times \underline{B} = 2x\,\hat{\underline{k}}$$

Example ② $\underline{B} = -y\,\hat{\underline{i}} + x\,\hat{\underline{j}} + 0\,\hat{\underline{k}}$



$$\nabla \times \underline{B} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$\nabla + \underline{B} = 2\,\hat{\underline{k}}$$

7.~~3~~ Other coordinate systems

cylindrical polar

$$\nabla = \hat{\underline{\rho}} \frac{\partial}{\partial \rho} + \frac{1}{\rho} \hat{\underline{\phi}} \frac{\partial}{\partial \phi} + \hat{\underline{k}} \frac{\partial}{\partial z}$$

$$\underline{B} = \hat{\underline{\rho}} B_\rho + \hat{\underline{\phi}} B_\phi + \hat{\underline{k}} B_z$$

can get from "differentiate first, cross product second"

$$\nabla \times \underline{B} = \hat{\underline{\rho}} \times \frac{\partial \underline{B}}{\partial \rho} + \frac{1}{\rho} \hat{\underline{\phi}} \times \frac{\partial \underline{B}}{\partial \phi} + \hat{\underline{k}} \times \frac{\partial \underline{B}}{\partial z}$$

you check ....

$$\nabla \times \underline{B} = \frac{1}{\rho} \begin{vmatrix} \hat{\underline{\rho}} & \rho \hat{\underline{\phi}} & \hat{\underline{k}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ B_\rho & \rho B_\phi & B_z \end{vmatrix}$$

spherical polar

same procedure

$$\nabla \times \underline{B} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\underline{r}} & r \hat{\underline{\theta}} & r \sin\theta \, \hat{\underline{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ B_r & r B_\theta & r \sin\theta \, B_\phi \end{vmatrix}$$

Vector calculus Lecture 17 (final)

7.4 Stokes' theorem

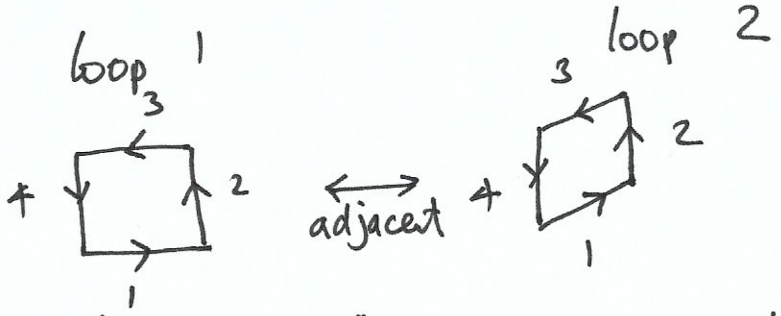
7.5 Vector identities

7.4 Stokes' theorem (see additional notes on Blackboard)

Consider two adjacent infinitesimal square loops, not necessarily in the same plane

Recall $\nabla \times \underline{B} \cdot \hat{\underline{n}} = \lim_{A \to 0} \dfrac{\oint \underline{B} \cdot d\underline{r}}{A}$

$\therefore \quad \nabla \times \underline{B} \cdot d\underline{S} = \oint \underline{B} \cdot d\underline{r}$ for infinitesimal loop



loop 1 $\displaystyle\sum_{i=1}^{4} \underline{B} \cdot d\underline{r} = \sum_{i=1}^{4} W_i^1 = \nabla \times \underline{B}_1 \cdot d\underline{S}_1$

loop 2 $\displaystyle\sum_{i=1}^{4} \underline{B} \cdot d\underline{r} = \sum_{i=1}^{4} W_i^2 = \nabla \times \underline{B}_2 \cdot d\underline{S}_2$

join the two loops together

$$\sum_{1+2} \underline{B} \cdot d\underline{r} = \sum_{i=1}^{4} W_i^1 + \sum_{i=1}^{4} W_i^2 \qquad \cancel{W_2^1 - W_4^2}$$

But $W_2^1 = -W_4^2$

$$\therefore \quad \sum_{1+2} \underline{B} \cdot d\underline{r} = \nabla \times \underline{B}_1 \cdot d\underline{S}_1 + \nabla \times \underline{B}_2 \cdot d\underline{S}_2 = \sum_{j=1}^{2} \nabla \times \underline{B}_j \cdot d\underline{S}_j$$

add more loops to create a macroscopic surface attached to a large loop

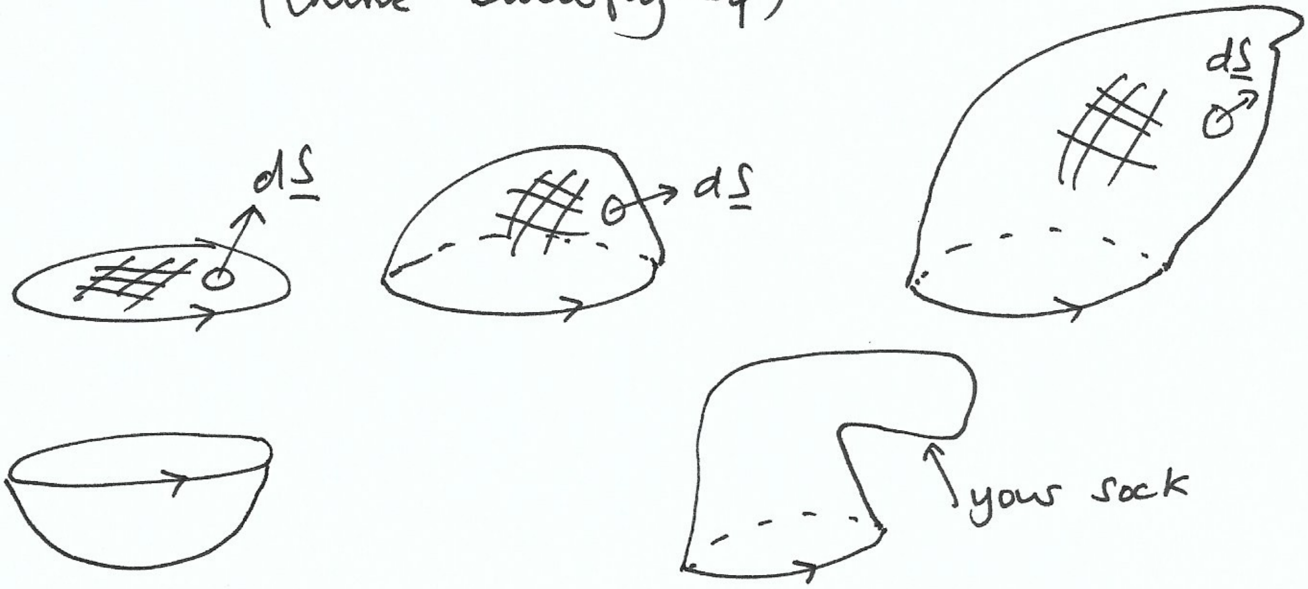
$$\oint_C \underline{B} \cdot d\underline{r} = \iint_S \nabla \times \underline{B} \cdot d\underline{S}$$

RH rule

closed

open !!!

<u>any</u> surface attached to the loop
(think butterfly ~~loop~~ net)



to be clear on orientation of $d\underline{S}$, "collapse" surface to the loop

sometimes we want a surface integral

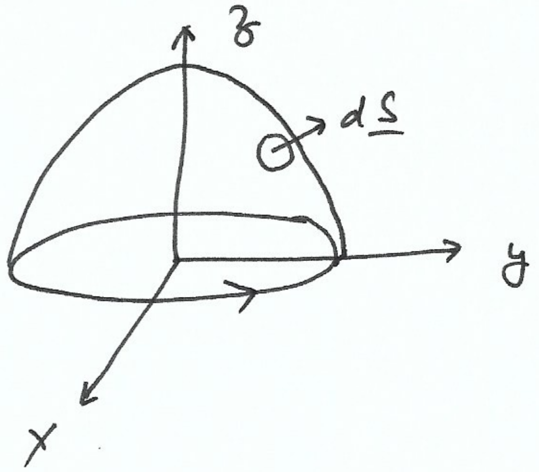
$\iint_S \nabla \times \underline{B} \cdot d\underline{S}$ but line integral is easier

or vice versa,

<u>or</u> you can choose a different surface attached to the same loop!

Example

Hemisphere of radius a at $z > 0$



vector field

$$\underline{B} = z\,\hat{\underline{i}} - y\,\hat{\underline{j}} - x\,\hat{\underline{k}}$$

$$d\underline{S} = a^2 \sin\theta \, d\theta \, d\phi \, \hat{\underline{r}}$$

we want to evaluate

$$\underset{C}{\oint} \underline{B}.d\underline{r} = \iint_S \nabla \times \underline{B}.\,d\underline{S}$$

(1) (2)

$$\nabla \times \underline{B} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -y & -x \end{vmatrix} = 2\hat{\underline{j}}$$

(1) in the x, y plane

$x = a\cos\phi \qquad dx = -a\sin\phi\, d\phi$

$y = a\sin\phi \qquad dy = a\cos\phi\, d\phi$

$$\underline{B}.d\underline{r} = B_x dx + B_y dy + \cancel{B_z dz} = \cancel{z\,dx} - y\,dy = -a^2 \sin\phi\cos\phi\, d\phi$$

$$\oint \underline{B}.d\underline{r} = -a^2 \int_0^{2\pi} \sin\phi\cos\phi\, d\phi = -\frac{a^2}{2}\left[\sin^2\phi\right]_0^{2\pi} = 0$$

(2) On the hemisphere

$d\underline{S} = a^2 \sin\theta \, d\theta \, d\phi \, \hat{\underline{r}}$

$\nabla \times \underline{B} = 2\hat{\underline{j}}$

$\hat{\underline{j}} \cdot \hat{\underline{r}} = \sin\theta \sin\phi$

$$\iint \nabla \times \underline{B} \cdot d\underline{S} = 2a^2 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \sin^2\theta \sin\phi \, d\phi \, d\theta = 0$$

(agrees)

quicker :

choose surface in x, y plane

$d\underline{S} = dS \, \hat{\underline{k}}$

$\nabla \times \underline{B} \cdot d\underline{S} = 2 \, dS \, \hat{\underline{j}} \cdot \hat{\underline{k}} = 0$

7.5 Vector identities

There are many.
3 are particularly important

(1) $\nabla \times (\nabla \Omega) = 0$ (section 7.2)
i.e. if $\underline{B} = \nabla \Omega$ Ω = parent function
~~i.e. if $\nabla \times \underline{B} = 0$~~ $\underline{B}$ is conservative
or "irrotational"

(2) $\nabla . (\nabla \times \underline{V}) = 0$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$= 0$

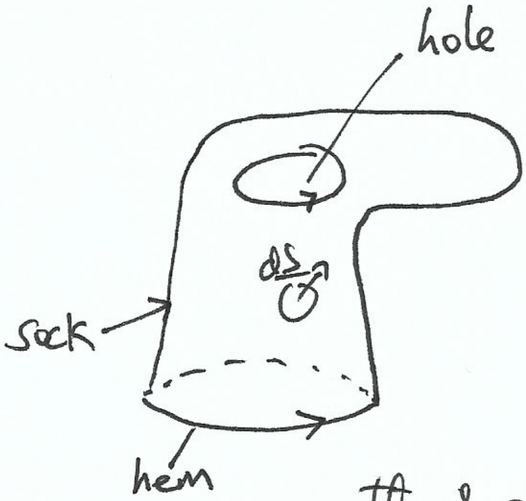
i.e. if $\underline{B} = \nabla \times \underline{V}$ $\underline{V}$ = parent function
$\underline{B}$ is "solenoidal"
V is called the <u>vector potential</u>

(1) + (2) read more on Helmholtz theorem to appreciate

(3) $\nabla \times (\nabla \times \underline{B}) = \nabla (\nabla . \underline{B}) - \nabla^2 \underline{B}$
you check
used in EM to derive wave equation

$$\nabla^2 \underline{B} \equiv \hat{\underline{i}}\, \nabla^2 B_x + \hat{\underline{j}}\, \nabla^2 B_y + \hat{\underline{k}}\, \nabla^2 B_z$$

final point



what is $\displaystyle\iint_{\substack{\text{sock}\\ \text{(only)}}} \nabla\times\underline{B}\cdot d\underline{S}$?

think of it like

now

$$\oint_{\text{hem}} \underline{B}\cdot d\underline{r} = \iint_{\text{sock}} \nabla\times\underline{B}\cdot d\underline{S} + \iint_{\text{hole}} \nabla\times\underline{B}\cdot d\underline{S}$$

$$\oint_{\text{hole}} \underline{B}\cdot d\underline{r} = \iint_{\text{hole}} \nabla\times\underline{B}\cdot d\underline{S}$$

$$\iint_{\text{sock}} \nabla\times\underline{B}\cdot d\underline{S} = \oint_{\text{hem}} \underline{B}\cdot d\underline{r} - \oint_{\text{hole}} \underline{B}\cdot d\underline{r}$$

[see 2020 exam]