DR JONATHAN P. EASTWOOD

ELECTROSTATICS SUMMARY NOTES SPRING TERM 2023

VECTOR FIELDS ELECTRICITY AND MAGNETISM
YEAR 1 MODULE
THE BLACKETT LABORATORY
IMPERIAL COLLEGE LONDON

PUBLISHED BY THE BLACKETT LABORATORY, IMPERIAL COLLEGE LONDON

These lecture notes were produced using the Tufte-Latex distribution TUFTE-LATEX.GITHUB.IO/TUFTE-LATEX/, inspired by the design principles of Edward R. Tufte.

First printing, January 2023

Contents

Introduction

This handout contains summary notes for the Electrostatics section of the first year Vector Fields, Electricity and Magnetism module. In this course we are going to explore the fundamental building blocks of classical electrostatics, using the language of vector calculus. In electrostatics there is no variation with time and so $\partial/\partial t = 0$. Consequently, we will see that the two equations of electrostatics are Gauss's law

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_q}{\epsilon_0}, \tag{1}$$

and the electrostatic circulation law

$$\nabla \times \boldsymbol{E} = 0, \tag{2}$$

which are a subset of Maxwell's equations[1].

[1] Maxwell's equations are:

- Gauss's law

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_q}{\epsilon_0} \tag{3}$$

- No monopoles (Gauss's law for magnetism)

$$\nabla \cdot \boldsymbol{B} = 0 \tag{4}$$

- Faraday's law

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{5}$$

- Ampère-Maxwell law

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{6}$$

The course is organised in two parts. In part 1, we will first define and examine the concepts of electric charge, force, field and potential. We will then analyze the properties of electric fields created by static charges, leading to the electrostatic circulation law and the derivation of Gauss's law as the first Maxwell equation.

In part 2, we will introduce and examine the properties of conductors, capacitors and dielectrics, using the laws of electrostatics to understand their behaviour. We will then finish the course with a discussion of electric current density and current continuity.

The lecture timetable is as follows:
1. Electric Charge, Force, Field, and Potential
Term 2, week 4, Mon. 30 Jan. 2023 1.1 Fields cause forces
Term 2, week 4, Fri. 03 Feb. 2023 1.2 Electric potential
Term 2, week 5, Thu. 09 Feb. 2023 1.3 Electric potential part 2
Term 2, week 5, Fri. 10 Feb. 2023 1.4 Gauss's law
Term 2, week 6, Fri. 17 Feb. 2023 1.5 Distributed charge
2. Conductors, Capacitors, Dielectrics and Current
Term 2, week 7, Mon. 20 Feb. 2023 2.1 Conductors
Term 2, week 7, Thu. 23 Feb. 2023 2.2 Capacitors
Term 2, week 7, Fri. 24 Feb. 2023 2.3 Dielectrics
Term 2, week 8, Mon. 27 Feb. 2023 2.4 Electric current

THE STYLE OF EACH LECTURE (and even each part of a lecture) depends on the content. For some parts, I will use PowerPoint, particularly when introducing ideas or discussing the implications of those ideas. For more technical work, such as derivations, I will write on the blackboard and do the mathematics in real time. I strongly encourage you to follow along and write out the derivation yourself; it will help you stay engaged with the lecture. All of the lectures will be recorded and available to watch afterwards.

THERE ARE MANY EXCELLENT TEXTBOOKS on Electricity and Magnetism and in learning any subject, having access to a variety of sources with different presentation styles is often useful. For example, you may find a discussion or derivation in class is not immediately clear, but reading about it in a textbook where it is presented in a different way may help to give you a clearer picture of what's going on. Listed below are some useful textbooks. You will find that they are all quite different in style, and you should compare them to see which you find most useful. Multiple copies of each book are available in the library.

Core textbooks

- Sears and Zemansky's University Physics: with Modern Physics (14th ed. Global Edition), H. D. Young and R. A. Freedman, Pearson, 2016
- Introduction to Electrodynamics (4th ed.) D. J. Griffiths, Pearson 2014

Supplementary reading

- Electricity and Magnetism (3rd ed.), E. M. Purcell, CUP 2012
- Electromagnetic Fields and Waves (3rd ed.), P. Lorrain, Freeman 1988
- Lectures on Physics[2], R. P. Feynman, R. B. Leighton, M. L. Sands, Pearson, 2006

Further reading

- Classical electrodynamics (3rd ed.)[3], J.D. Jackson, Wiley, 1999

[2] The Feynman Lectures on Physics are quite unlike any other physics textbooks. Often the best strategy is to read it after you have worked on a particular aspect of the course. The discussion about how the equations of electrostatics have direct analogues in many other areas of physics, and the part on electric fields in conductors are both worth reading.

[3] Jackson is very much an advanced text, and the vast majority of its content extends far beyond the scope of the course, but it is considered one of the classic textbooks in physics.

A NOTE ABOUT YOUR LECTURER. I am a member of the Space and Atmospheric Physics Group, performing research into space plasma physics. I am also Director of the Space Lab Network of Excellence. My research interests include magnetic reconnection, space weather, and planetary magnetospheres. I use supercomputer simulations and spacecraft observations in my research and I'm a member of several ESA and NASA space missions.

FOR QUESTIONS, COMMENTS, AND FEEDBACK, please email me at jonathan.eastwood@imperial.ac.uk, send me a message on Teams, or visit office hours. Office hour times and contact details will be published on Blackboard and on the VFEM Teams site.

1
Electric Charge, Force, Field, and Potential

1.1 Fields Cause Forces

1.1.1 Electric Charge

Electric charge is: quantized in units of $e = 1.6021 \times 10^{-19}$ C; it can be positive or negative; and it is conserved - the total charge of an isolated system is constant.

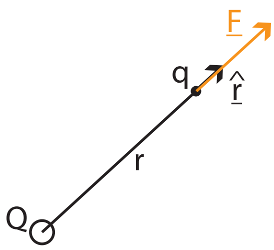

Figure 1.1: Diagram of electric force. r is the separation of the two charges Q and q, and $\hat{\boldsymbol{r}}$ is the unit vector along the line between the two charges.

1.1.2 Coulomb's Law

Coulomb's law is

$$\boldsymbol{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{\boldsymbol{r}}, \tag{1.1}$$

as illustrated in Figure 1.1, where we have shown the electric force $\boldsymbol{F}$ experienced by q due to Q. The separation vector is $\boldsymbol{r} = r\hat{\boldsymbol{r}}$, and ϵ_0 is the permittivity of free space ($= 8.85 \times 10^{-12}$ Fm^{-1}). If Q and q have the same sign, there is a repulsive force. Otherwise, there is an attractive force.

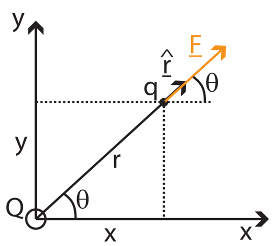

Figure 1.2: Decomposition of the electric force

Figure 1.2 illustrates how to use Coulomb's law to make calculations. We first decompose the electric force vector $\boldsymbol{F}$ into its components, F_x and F_y:

$$\boldsymbol{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{\boldsymbol{r}} = F\hat{\boldsymbol{r}} = F_x\hat{\boldsymbol{x}} + F_y\hat{\boldsymbol{y}}. \tag{1.2}$$

We can then geometrically calculate F_x and F_y in terms of F and θ, finding that

$$F_x\hat{\boldsymbol{x}} = F\cos\theta = \frac{Fx}{r}\hat{\boldsymbol{x}}, \tag{1.3}$$

$$F_y\hat{\boldsymbol{y}} = F\sin\theta = \frac{Fy}{r}\hat{\boldsymbol{y}}. \tag{1.4}$$

In doing this, remember that

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad \tan\theta = \frac{y}{x}. \tag{1.5}$$

So far, we have assumed that one charge is located at the origin. If neither charge is located at the origin, then this must be accounted for as shown in Figure 1.3.

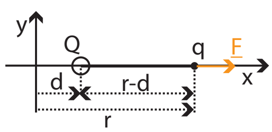

Figure 1.3: Example where neither charge is located at the origin.

In this case, the distance between the two charges is $r - d$, and so the electric force on charge q is

$$\boldsymbol{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{(r-d)^2}\hat{\boldsymbol{x}} \tag{1.6}$$

Finally, in the general case where charge Q is located at (x_Q, y_Q) and charge q is located at (x_q, y_q), the separation of the charges and the orientation of the force must both be carefully calculated. In the lecture I show a worked example.

1.1.3 Principle of Superposition

Forces superpose linearly as shown in Figure 1.4. This is a vector problem, where

$$\boldsymbol{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ_1}{r_1^2}\hat{\boldsymbol{r}}_1 + \frac{1}{4\pi\epsilon_0}\frac{qQ_2}{r_2^2}\hat{\boldsymbol{r}}_2 + ... + \frac{1}{4\pi\epsilon_0}\frac{qQ_n}{r_n^2}\hat{\boldsymbol{r}}_n \tag{1.7}$$

$$\boldsymbol{F} = \frac{q}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{Q_i}{r_i^2}\hat{\boldsymbol{r}}_i \tag{1.8}$$

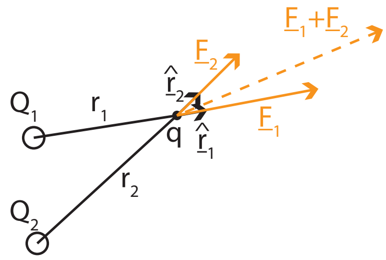

Figure 1.4: Diagram of electric force, superposition

1.1.4 Electric Field

As shown in Figure 1.5, the electric field $\boldsymbol{E}$ due to Q at distance r along $\hat{\boldsymbol{r}}$ is

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\boldsymbol{r}}, \tag{1.9}$$

$\boldsymbol{E}$ is a vector field; if Q is positive then $\boldsymbol{E}$ points away from the charge.

Now place charge q in the field. It experiences a force

$$\boldsymbol{F} = q\boldsymbol{E}. \tag{1.10}$$

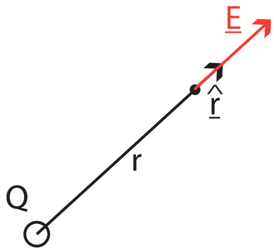

Figure 1.5: Diagram of the electric field $\boldsymbol{E}$ due to a point charge Q. $\boldsymbol{E}$ is a vector field. If $Q > 0$, $\boldsymbol{E}$ points away from Q. If $Q < 0$, $\boldsymbol{E}$ points towards Q.

Conceptually, the charge Q creates a field, and the force on q is exerted by the field. The field is not just mathematical complexity for the sake of it; it is a real physical entity[1].

Finally, the superposition principle still applies: for n particles, $\boldsymbol{E} = \sum_{i=1}^{n} \boldsymbol{E}_i$.

[1] See e.g. Feynman, Volume 2, Chapter 1, for further discussion about this.

1.1.5 Field lines

Field lines are a tool to map out $\boldsymbol{E}$ in space. The field line is tangent to $\boldsymbol{E}$ at every point in space. You can use the Imperial Visualisations website `https://www.impvis.com/` to explore this. Some examples of field lines are shown in Figure 1.6. Field lines start on positive charges, end on negative charges and do not cross. They should be used with care!

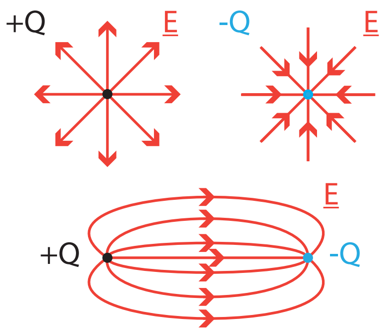

Figure 1.6: Examples of electric field lines around a positive point charge, a negative point charge, and a dipole.

1.2 Electric Potential

1.2.1 Potential Energy

Consider a charge Q, and a 'test charge', q. The test charge q experiences a force $\boldsymbol{F}_Q$. Now, suppose we want to move q from $A \to B$ at constant speed. We have to apply a force $\boldsymbol{F}_{ext}$ to match $\boldsymbol{F}_Q$. In moving the charge, the external force does work[2]. Moving at constant speed[3],

$$\boldsymbol{F}_{ext} + \boldsymbol{F}_Q = 0. \tag{1.11}$$

The elemental work done by $\boldsymbol{F}_{ext}$ on q is

$$dW = \boldsymbol{F}_{ext} \cdot \boldsymbol{dl} = -\boldsymbol{F}_Q \cdot \boldsymbol{dl}. \tag{1.12}$$

The total work done by $\boldsymbol{F}_{ext}$ is found by integrating from A to B.

$$W = \int_A^B \boldsymbol{F}_{ext} \cdot \boldsymbol{dl} = -\int_A^B \boldsymbol{F}_Q \cdot \boldsymbol{dl} \tag{1.13}$$

The work done, W, is equal to the change in Potential Energy, ΔU, and so

$$\Delta U = -\int_A^B \boldsymbol{F}_Q \cdot \boldsymbol{dl}. \tag{1.14}$$

We know F_Q from Coulomb's law, and so can write

$$dW = -\frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{\boldsymbol{r}} \cdot \boldsymbol{dl}. \tag{1.15}$$

Focussing on equation 1.15, the work done depends on $\hat{\boldsymbol{r}} \cdot \boldsymbol{dl}$. As illustrated in Figure 1.7, we can show that ΔU_{AB} is therefore given by

$$\Delta U_{AB} = W = -\int_A^B \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}dr, \tag{1.16}$$

$$\Delta U_{AB} = \frac{qQ}{4\pi\epsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right), \tag{1.17}$$

and so only depends on the radial separation r. The shape of the path does not matter. This is illustrated in Figure 1.7.

If there are multiple charges we can use the superposition principle (illustrated in Figure 1.8) such that

$$\Delta U_{AB} = -\int_A^B \boldsymbol{F}_{Q1} \cdot \boldsymbol{dl} - \int_A^B \boldsymbol{F}_{Q2} \cdot \boldsymbol{dl} - \int_A^B \boldsymbol{F}_{QN} \cdot \boldsymbol{dl}. \tag{1.18}$$

Note that each term is path independent and only depends on distance to A and B, so ΔU_{AB} is still path independent.

1.2.2 *Potential Difference*

The potential difference between A and B is ΔV_{AB} where

$$\Delta V_{AB} = \frac{\Delta U_{AB}}{q}, \tag{1.19}$$

$$\Delta V_{AB} = -\int_A^B \boldsymbol{E} \cdot \boldsymbol{dl}. \tag{1.20}$$

[2] An analogy is to carry a heavy object upstairs slowly at constant speed. You are doing work, applying a force to move the object upwards.

[3] Please be aware that this is different to treatments you may have seen in e.g. mechanics. We arrive at the same understanding, but in a different way. In particular, note we are not saying that the electric force does work to increase the kinetic energy of the particle. For further discussion on this, see Young and Freeman, section 7.1, page 228, equation 7.3.

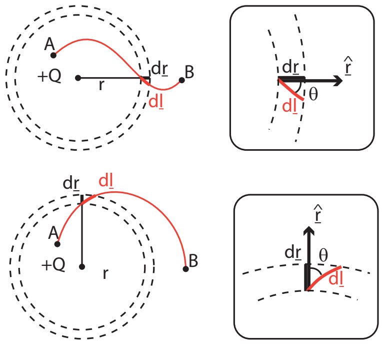

Figure 1.7: Two different paths from A to B, showing the elemental path. Inset diagrams show the geometry of the dot product. Note that $\hat{\boldsymbol{r}} \cdot \boldsymbol{dl} = |\hat{\boldsymbol{r}}||\boldsymbol{dl}|\cos\theta$. However, $|\hat{\boldsymbol{r}}| = 1$, and from the geometry $dl\cos\theta = dr$ by definition.

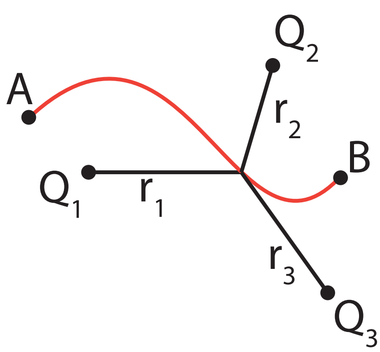

Figure 1.8: Change in Potential Energy with multiple charges - linear superposition applies, and only depends on start and end points.

ΔV_{AB} is also path independent, and does not depend on q. The SI unit of potential difference is JC^{-1}. We can choose where $V = 0$, and often choose $V = 0$ in the limit $r \to \infty$.

Define the potential V at P as the external work needed to bring a charge of +1 C at constant speed from $r = \infty$ ($V = 0$) to P:

$$V = -\int_{\infty}^{P} \boldsymbol{E} \cdot d\boldsymbol{l}. \tag{1.21}$$

Key facts about V: it is the potential energy/unit positive charge; it is a scalar field; and the superposition principle applies. With no external forces applied, positive charges move to lower potentials and negative charges move to higher potentials.

For a point charge Q, the potential V at r from Q is:

$$V = \frac{Q}{4\pi\epsilon_0 r}. \tag{1.22}$$

This is illustrated in Figure 1.9. The red and blue curves show the potentials associated with point charges $+Q$ and $-Q$ respectively. The motion of positive and negative 'test charges' are also shown.

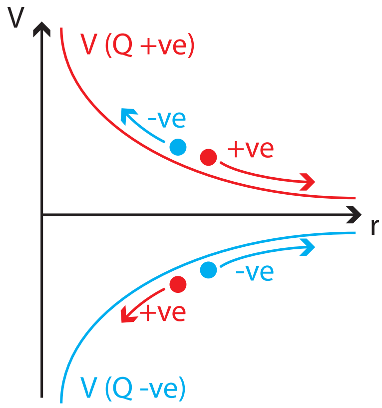

Figure 1.9: Motion of charged test particles (red = positive, blue = negative) in the potential well of a positive point charge (red) and a negative point charge (blue).

1.2.3 *Electrostatic Circulation Law*

We saw above (equation 1.20) that ΔV_{AB} is path independent. Consider path 1 and path 2 as shown in Figure 1.10.

$$\Delta V_{AB} = -\int_{A,path1}^{B} \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_{A,path2}^{B} \boldsymbol{E} \cdot d\boldsymbol{l}, \tag{1.23}$$

$$-\int_{A,path1}^{B} \boldsymbol{E} \cdot d\boldsymbol{l} = \int_{B,path2}^{A} \boldsymbol{E} \cdot d\boldsymbol{l}, \tag{1.24}$$

$$\int_{A,path1}^{B} \boldsymbol{E} \cdot d\boldsymbol{l} + \int_{B,path2}^{A} \boldsymbol{E} \cdot d\boldsymbol{l} = 0. \tag{1.25}$$

Therefore around any closed path

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = 0. \tag{1.26}$$

This is only true for $\boldsymbol{E}$ in electrostatics.

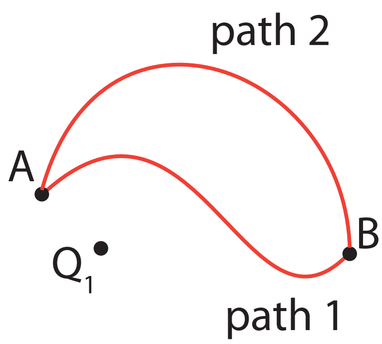

Figure 1.10: Since ΔV_{AB} is independent of path, it is the same along path one and path two. So the integral around the closed path is 0.

1.2.4 *Differential Form of the Circulation Law*

We use Stokes' theorem to write

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = \iint_S (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{S} = 0 \tag{1.27}$$

If we apply this equation to an infinitesimal surface element, we find that

$$(\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{S} = 0. \tag{1.28}$$

Since $d\boldsymbol{S} \neq 0$, this means that at a point in space

$$\nabla \times \boldsymbol{E} = 0 \tag{1.29}$$

This differential equation must be satisfied at all points in an electrostatic field.

1.3 Electric Potential Part 2

1.3.1 Two Charges: Binding Energy

We start with a point charge Q_1. The potential V at distance r is given by equation 1.22. Now move point charge Q_2 from ∞ to a separation r_{12} (use F_{ext}, constant speed, etc.). The change in potential energy is:

$$\Delta U = \int_{\infty}^{P} \boldsymbol{F}_{ext} \cdot d\boldsymbol{l} = -\int_{\infty}^{P} \boldsymbol{F}_Q \cdot d\boldsymbol{l} = -Q_2 \int_{\infty}^{P} \boldsymbol{E} \cdot d\boldsymbol{l} = Q_2 V. \tag{1.30}$$

We set $U = 0$ at ∞, (NB $\Delta U = q\Delta V$), and can write $\Delta U = U(r) - U_\infty = U(r) - 0 = U$. Therefore,

$$U = Q_2 \left(\frac{Q_1}{4\pi\epsilon_0 r_{12}} \right) = Q_2 V(r_{12}), \tag{1.31}$$

and this is potential energy[4] of Q_2 in the potential of Q_1 at distance r_{12}. But, we could also write this as the potential energy of Q_1 in the potential of Q_2:

[4] Aside - energy is often measured in electron volts (eV): if 1 electron charge magnitude moves through a potential difference of 1 V, the change in energy is 1 eV. 1 eV = 1.602×10^{-19} J.

$$U = Q_1 \left(\frac{Q_2}{4\pi\epsilon_0 r_{12}} \right) = Q_1 V(r_{12}). \tag{1.32}$$

Consequently U is a property of the system. U also corresponds to the Binding Energy - the energy needed to remove one charge to ∞.

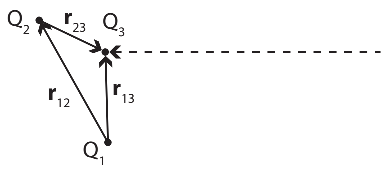

Figure 1.11: Charges Q_1 and Q_2 are already present. Charge Q_3 is added from ∞.

1.3.2 Potential Energy of a Set of Charges

Now move Q_3 from $\infty \to P$, (use F_{ext}, constant speed), as illustrated in Figure 1.11. The change in potential energy is

$$\Delta U = \int_{\infty}^{P} \boldsymbol{F}_{ext} \cdot d\boldsymbol{l} = -Q_3 \int_{\infty}^{P} \boldsymbol{E} \cdot d\boldsymbol{l} = Q_3 V_P, \tag{1.33}$$

where V_P is the potential V at P due to Q_1 and Q_2:

$$V_P = \frac{Q_1}{4\pi\epsilon_0 r_{13}} + \frac{Q_2}{4\pi\epsilon_0 r_{23}}. \tag{1.34}$$

Consequently,

$$U = \frac{Q_1 Q_2}{4\pi\epsilon_0 r_{12}} + \Delta U, \tag{1.35}$$

$$U = \frac{Q_1 Q_2}{4\pi\epsilon_0 r_{12}} + \frac{Q_1 Q_3}{4\pi\epsilon_0 r_{13}} + \frac{Q_2 Q_3}{4\pi\epsilon_0 r_{23}}, \tag{1.36}$$

and it can be shown that

$$U = \frac{1}{2}Q_1V_1 + \frac{1}{2}Q_2V_2 + \frac{1}{2}Q_3V_3. \tag{1.37}$$

Note that V_1 is the potential at the position of Q_1 due to the other two charges. In general for n charges,

$$U = \sum_{i=1}^{n} \frac{1}{2}Q_iV_i \tag{1.38}$$

where V_i is the potential at the position of Q_i due to the other $n-1$ charges.

1.3.3 *Equipotential Surfaces*

Recall equation 1.20. This implies that $dV_{AB} = -\boldsymbol{E}\cdot d\boldsymbol{l}$. If $d\boldsymbol{l} \perp \boldsymbol{E}$ then $dV_{AB} = 0$. Consequently, $\boldsymbol{E}$ is $\perp$ to an 'equipotential' surface.

In the simple case of a uniform $\boldsymbol{E}$ field where $\boldsymbol{E} = E\hat{\boldsymbol{x}}$,

$$V(x) - V(0) = -\int_0^x \boldsymbol{E}\cdot d\boldsymbol{l} = -\int_0^x E dx = -Ex, \tag{1.39}$$

$$V(x) = V(0) - Ex. \tag{1.40}$$

The equipotentials lie perpendicular to $\boldsymbol{E}$, as illustrated in Figure 1.12, which also shows the equipotentials around a point charge. Note also that field lines point from higher potential to lower potential.

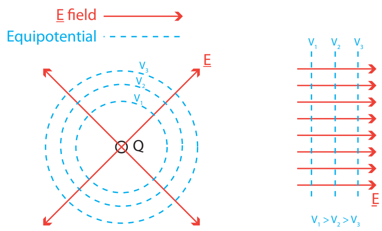

Figure 1.12: Example equipotentials for a point charge and a uniform electric field.

1.3.4 *E and V*

Referring to Figure 1.13, consider the one dimensional situation where two points P_1 and P_2 are located at x and $x+\Delta x$ respectively.

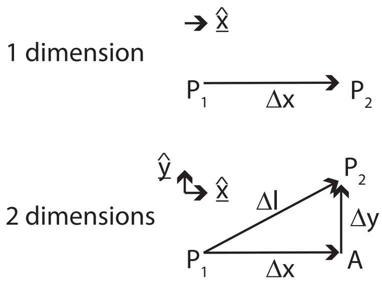

Figure 1.13: Example trajectories for calculating in 1d and 2d.

$$V(P_2) = V(x+\Delta x) = V(x) + \Delta x\frac{dV}{dx} + ... \approx V(P_1) + \Delta x\frac{dV}{dx} \tag{1.41}$$

In two dimensions, P_1 and P_2 are separated by $\Delta\boldsymbol{l} = \Delta x\hat{\boldsymbol{x}} + \Delta y\hat{\boldsymbol{y}}$, and so

$$V(P_2) = V(P_1) + \Delta x\frac{\partial V}{\partial x} + \Delta y\frac{\partial V}{\partial y}. \tag{1.42}$$

In three dimensions, $\Delta\boldsymbol{l} = \Delta x\hat{\boldsymbol{x}} + \Delta y\hat{\boldsymbol{y}} + \Delta z\hat{\boldsymbol{z}}$, and so

$$V(P_2) = V(P_1) + \Delta x\frac{\partial V}{\partial x} + \Delta y\frac{\partial V}{\partial y} + \Delta z\frac{\partial V}{\partial z}. \tag{1.43}$$

However,

$$\Delta V = -\boldsymbol{E}\cdot\Delta\boldsymbol{l} = -E_x\Delta x - E_y\Delta y - E_z\Delta z. \tag{1.44}$$

Therefore,

$$E_x = -\frac{\partial V}{\partial x}, \qquad E_y = -\frac{\partial V}{\partial y}, \qquad E_z = -\frac{\partial V}{\partial z}. \tag{1.45}$$

We can write this as

$$\boldsymbol{E} = -\nabla V. \tag{1.46}$$

Finally, note the following identity:

$$\nabla \times (\nabla A) = 0. \tag{1.47}$$

This means that the electrostatic circulation law equation is directly connected to the equation relating the gradient in the potential and the electric field. The existence of a scalar potential gives the circulation law directly.

1.4 *Gauss's Law*

1.4.1 *Electric Flux*

Gauss's law is the first of Maxwell's equations, and ultimately encapsulates the idea that charged particles are a source of electric field. In order to derive Gauss's law, we first have to introduce the concept of electric flux. To do this, we first consider mass flux, which provides a useful analogy.

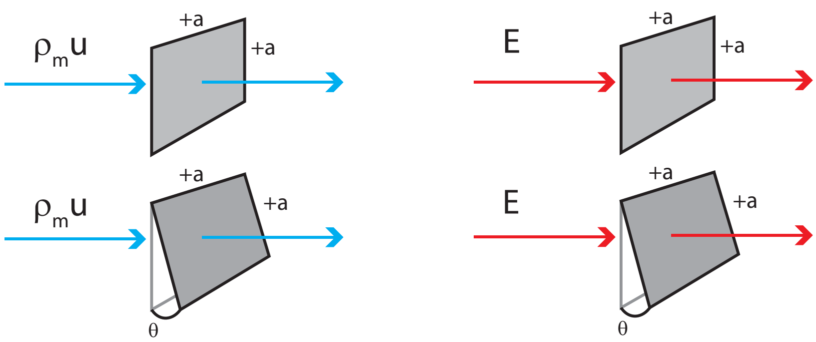

Figure 1.14: The left two panels show the flow of water through a square of side a, normal to the flow and inclined at angle θ. The right hand panels illustrate the concept of electric flux. See text for full details.

Referring to the top left panel of Figure 1.14, consider the flow of water through a square, side a. The square is normal to the flow, and has area a^2. If the flow speed is u, then in time Δt, a mass Δm of water passes through where

$$\Delta m = \rho_m a^2 u \Delta t. \tag{1.48}$$

Note that ρ_m is the *mass* density. Now let's consider what happens when the area is inclined at angle θ to the flow (bottom left panel of Figure 1.14). In this case,

$$\Delta m = \rho_m a \left(a \cos\theta\right) u \Delta t. \tag{1.49}$$

Here the 'flow area' is $a\left(a\cos\theta\right)$. This is the projection of the inclined surface onto the plane normal to the flow. Mathematically, we will now use the concept of vector area $\boldsymbol{A}$. The magnitude of $\boldsymbol{A}$ is the surface area and the direction of $\boldsymbol{A}$ is the orientation of the surface normal in space. We can therefore write

$$\Delta m = \Delta t \rho_m \boldsymbol{u} \cdot \boldsymbol{A}. \tag{1.50}$$

To evaluate the mass flow through a surface of arbitrary shape, we allow A to become infinitesimally small (dA), and integrate dA over the surface using standard techniques from vector calculus. This allows us to write

$$\Delta m = \Delta t \iint_S \rho_m \boldsymbol{u} \cdot d\boldsymbol{S} \tag{1.51}$$

where there is a double integral over the surface S. We now define Φ_m, as the mass flux through S in the following way:

$$\Phi_m = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \iint_S \rho_m \boldsymbol{u} \cdot d\boldsymbol{S}. \tag{1.52}$$

Φ_m is the rate of mass flow through S. It has units of kg/s.[5]

[5] For example, consider a flat sheet 2 cm by 3 cm with an angle to the normal of 50 degrees, and a mass flux $\rho_m u$ of 45 kg/s/m^2, Φ_m = 0.017 kg/s

We now make an analogy between mass flux and electric flux Φ_E. Whereas in mass flow we think of flow lines, in electric fields we will think of field lines. This is illustrated in the two panels on the right hand side of Figure 1.14. Consequently, we write

$$\Phi_E = \iint_S \boldsymbol{E} \cdot d\boldsymbol{S}. \tag{1.53}$$

There are several points to note. Firstly, electric flux is a scalar[6]. It can be thought of as being related to the number of $\boldsymbol{E}$ field lines through a surface. Finally, for a closed surface, $d\boldsymbol{S}$ points outwards by convention.

[6] The SI unit of Φ_E is Vm.

1.4.2 *Solid Angles*

Solid angles are an important concept in physics, and we use them here to better understand the electric flux from a point charge. The solid angle is a generalisation of the ordinary angle between two lines that we are all familiar with. Just as we can divide a circle into 2π radians, we can divide the surface area of a sphere over 4π steradians.

The definition of solid angle, using small surface elements, is illustrated in Figure 1.15[7]. A small area defined by $d\boldsymbol{S}$ is at a distance $\boldsymbol{r} = r\hat{\boldsymbol{r}}$ from point P. The surface element is defined to subtend a *solid angle* $d\Omega$ as follows:

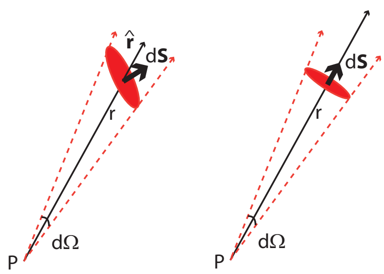

Figure 1.15: Diagram showing how solid angle is defined.

[7] We will consider small surface elements, but solid angles also apply to large areas.

$$d\Omega = \frac{d\boldsymbol{S} \cdot \hat{\boldsymbol{r}}}{r^2}. \tag{1.54}$$

Here, $\hat{\boldsymbol{r}}$ is the unit vector along the direction from P to the surface element. If it so happens that the surface element is parallel to $\hat{\boldsymbol{r}}$, we can write that

$$d\Omega = \frac{dS}{r^2}. \tag{1.55}$$

We can integrate the elemental solid angle to find the solid angle of larger surfaces[8]. As mentioned above, Finally, just as there are 2π

[8] This also leads to a perhaps counter-intuitive result: that you can draw different surface shapes on a sphere that subtend the same solid angle.

radians in a circle, there are 4π steradians covering the surface of a sphere.

1.4.3 Electric Flux Due to a Point Charge

The electric field $\boldsymbol{E}$ due to a point charge Q is given by

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\boldsymbol{r}} \tag{1.56}$$

Let's calculate the electric flux $\Phi_{E,1}$ through the spherical surface S_1, radius r_1, centered on Q, which is shown in Figure 1.16. $|E| = E_{r1}$ is uniform and radial on the surface, and the area of the surface is $4\pi r_1^2$. Therefore

$$\Phi_{E,1} = 4\pi r_1^2 E_{r1} = \frac{Q}{\epsilon_0}. \tag{1.57}$$

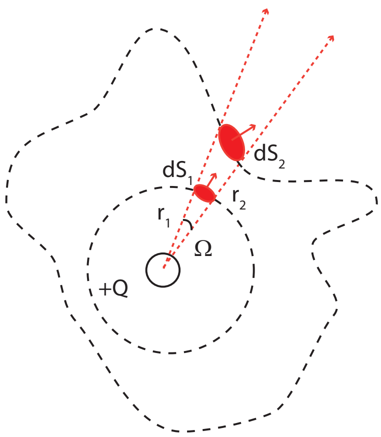

Figure 1.16: Diagram showing surfaces S_1 and S_2 used in the derivation of Gauss's law.

Now we consider the flux $d\Phi_{E,1}$ through the surface element dS_1. From spherical symmetry

$$\frac{d\Phi_{E,1}}{\Phi_{E,1}} = \frac{dS_1}{S_1}. \tag{1.58}$$

Rearranging, and noting that $S_1 = 4\pi r_1^2$,

$$d\Phi_{E,1} = \Phi_{E,1}\frac{dS_1}{4\pi r_1^2} = \Phi_{E,1}\frac{d\Omega_1}{4\pi} \tag{1.59}$$

because $d\Omega_1 = dS_1/r^2$. Given that $\Phi_{E,1} = Q/\epsilon_0$, we can therefore write

$$d\Phi_{E,1} = \frac{Q}{4\pi\epsilon_0}d\Omega_1 \tag{1.60}$$

Next we consider S_2, an arbitrary surface enclosing S_1. The corresponding element of flux $d\Phi_{E,2}$ through dS_2 is given by

$$d\Phi_{E,2} = \boldsymbol{E}_{r2}\cdot d\boldsymbol{S}_2 = \frac{Q}{4\pi\epsilon_0 r_2^2}\hat{\boldsymbol{r}}\cdot d\boldsymbol{S}_2. \tag{1.61}$$

But, we know that by definition,

$$d\Omega_2 = \frac{d\boldsymbol{S}_2\cdot\hat{\boldsymbol{r}}}{r_2^2}, \tag{1.62}$$

so we can write that

$$d\Phi_{E,2} = \frac{Q}{4\pi\epsilon_0}d\Omega_2. \tag{1.63}$$

But geometrically

$$d\Omega_2 = d\Omega_1, \tag{1.64}$$

and therefore

$$d\Phi_{E,2} = d\Phi_{E,1}. \tag{1.65}$$

The flux through the two surface elements is the same, even though the orientation of $d\boldsymbol{S}_2$ is arbitrary. Essentially, there are the same number of field lines through dS_1 and dS_2. Therefore the flux through S_1 and S_2 is the same. Consequently, the flux through any closed surface is *always* Q/ϵ_0. Furthermore, it doesn't matter if the surface is so distorted such that the field lines cross the surface multiple times; this result stills holds, as illustrated in Figure 1.17.

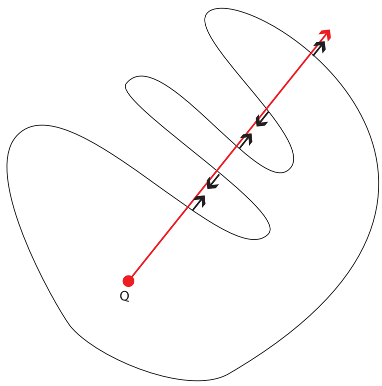

Figure 1.17: If the surface is complex, and the field line crosses several times, it will cross an odd number of times. Since the surface normal is oppositely directed to the field when the field re-enters the surface, crossings cancel out pairwise leaving only the final crossing contributing to the calculation.

1.4.4 *Gauss's law (Integral Form)*

So far we have looked at one charge. But we can use the superposition principle to find Φ_E through an arbitrary closed surface surrounding a collection of charges $Q_1, Q_2, ..., Q_N$. We find that

$$\Phi_E = \oiint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \oiint_S \boldsymbol{E}_1 \cdot d\boldsymbol{S} + \oiint_S \boldsymbol{E}_2 \cdot d\boldsymbol{S} + ... + \oiint_S \boldsymbol{E}_N \cdot d\boldsymbol{S}, \quad (1.66)$$

and so

$$\Phi_E = \frac{Q_1}{\epsilon_0} + \frac{Q_2}{\epsilon_0} + ... + \frac{Q_N}{\epsilon_0}. \quad (1.67)$$

Since the enclosed charge $Q_{enc} = Q_1 + Q_2 + ... + Q_N$, we can therefore write that

$$\oiint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{Q_{enc}}{\epsilon_0}. \quad (1.68)$$

Equation 1.68 is the integral form of Gauss's law, and is one of Maxwell's equations.

We make the following observations.

1. In using Gauss's law we must be careful. All charges contribute to $\boldsymbol{E}$ at any point in space, but only some contribute to the calculation in Gauss's law.

2. When calculating the net flux through a closed surface, only charges inside the surface matter. Electric flux from charges outside the closed surface does not contribute to the net flux. In essence, the electric flux 'on the way in' cancels the electric flux 'on the way out'.

3. Whilst Gauss' law is universal, we will mainly look at highly symmetric situations to find $\boldsymbol{E}$, as this makes the calculations tractable.

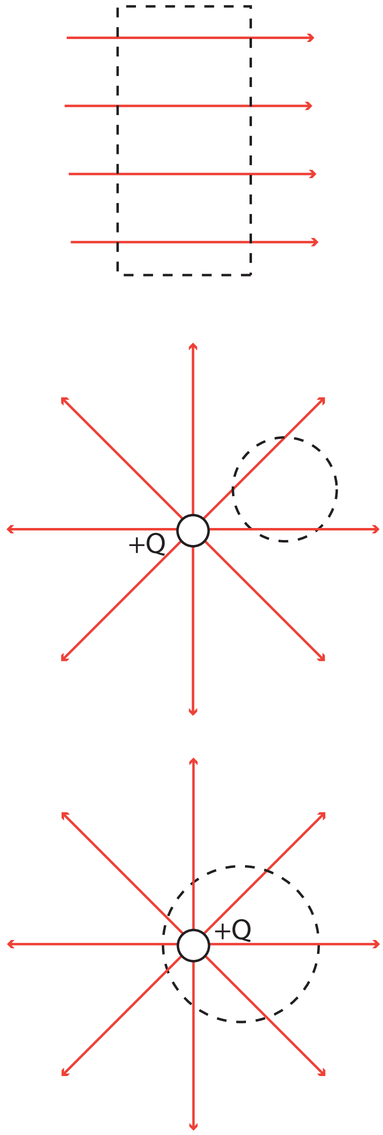

Figure 1.18: Three examples of electric fields (in red) and an integrating surface (shown as a dashed line). The surfaces and fields should be understood to extend in three dimensions. See text for details.

To check your understanding, consider the three examples in Figure 1.18. In the first example, there is a uniform electric field. The flux into the left hand side is equal to the flux out of the right hand side and so the net electric flux through the surface is zero. There is no charge inside the volume and Gauss's law is trivally met. In the second example, there is no charge inside the integrating surface, and so the net electric flux through the surface must be zero. Nevertheless there is an electric field inside the volume, but the flux going in equals the flux going out. Also note here that the electric field on the surface of the integrating volume is not uniform. It is stronger on the side closer to $+Q$. Finally, in the last example, there is a net electric flux out of the integrating surface, equal to $+Q/\epsilon_0$. Again note that the electric field is not uniform over the integrating surface (it is stronger on the left hand side) and so the electric flux is not uniform over the surface. Nevertheless, Gauss's law makes it extremely easy to find the net electric flux.

1.5 *Distributed Charge*

1.5.1 *Charge density*

Up to now we have considered point charges, but in practice we are more often interested in regions or objects which have a lot of individual charges, and in which the charge may be spread out non uniformly. In such situations, it is far more useful to think of the charge as being spread out continuously. We therefore speak of *charge density*.

WE BEGIN WITH UNIFORM CHARGE DENSITY: consider a volume[9] V with total charge Q. If Q is spread uniformly over V, then the uniform charge density is ρ_q where[10]

$$\rho_q = \frac{Q}{V}. \tag{1.69}$$

IF THE CHARGE DENSITY IS NON-UNIFORM, we now consider a volume element dV, with charge $\rho_q = \rho_q(x, y, z)$. Here,

$$dQ = \rho_q dV. \tag{1.70}$$

The total charge Q is found by integration over the whole volume. We write that

$$Q = \iiint_V \rho_q dV. \tag{1.71}$$

SURFACE CHARGE DENSITY IS A USEFUL CONCEPT. If the charge is spread over a thin layer, we can calculate σ_q, the surface charge density[11]. If dS is the surface element[12],

$$Q = \iint_S \sigma_q dS. \tag{1.72}$$

1.5.2 *Gauss's law with distributed charge*

Previously we saw[13] that

$$\oint\!\!\!\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{Q}{\epsilon_0} = \frac{Q_1}{\epsilon_0} + \frac{Q_2}{\epsilon_0} + ... + \frac{Q_N}{\epsilon_0} \tag{1.73}$$

Now let's consider each Q_i to represent an infinitesimal dQ associated with an elemental volume dV, where $dQ = \rho_q dV$. Using results from vector calculus, we write

$$\oint\!\!\!\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{1}{\epsilon_0} \iiint_V \rho_q dV, \tag{1.74}$$

[9] Don't confuse volume and potential.

[10] The SI unit of ρ_q is Cm^{-3}.

[11] The SI unit of σ_q is Cm^{-2}.

[12] Note that dS not a vector in this case.

[13] This is equation 1.67.

where V is the volume enclosed by the closed surface S.

If we now use the divergence theorem, we can write

$$\iiint_V \nabla \cdot \boldsymbol{E} \, dV = \frac{1}{\epsilon_0} \iiint_V \rho_q \, dV. \tag{1.75}$$

Applying this to an infinitesimal volume,

$$\nabla \cdot \boldsymbol{E} \, dV = \frac{\rho_q}{\epsilon_0} \, dV \tag{1.76}$$

and consequently,

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_q}{\epsilon_0}. \tag{1.77}$$

This is Gauss's law, the first Maxwell equation, in differential form.

1.5.3 *Spherically symmetric charge distributions*

To illustrate how to use equation 1.74, we will apply it to some example charge distributions. In general we will only consider situations that exhibit *spherical symmetry* as shown in Figure 1.19. This means that we use spherical coordinates and allow ρ_q only to vary with r, not θ or ϕ.

This also means that $\boldsymbol{E}$ will be radial, that $|E|$ varies with r, and the electric flux Φ_E through a spherical surface radius r is given by

$$\Phi_E = 4\pi r^2 E \tag{1.78}$$

However, we don't know how E varies as a function of r. For this we will use Gauss's law.

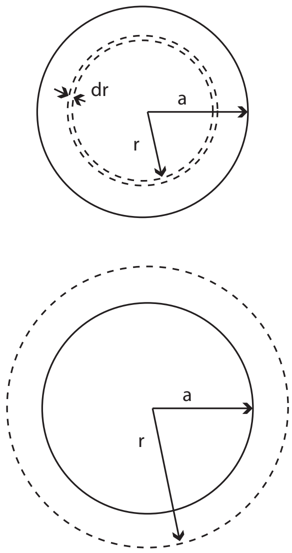

Figure 1.19: Spherical charge distribution. We use a spherical surface of radius r to evaluate Gauss's law and find the electric field as a function of distance from the origin. The top plot shows the situation for $r < a$. (Here a thin shell is also shown, to illustrate the approach if the charge density is non-uniform). The bottom plot shows the situation for $r > a$.

INSIDE A UNIFORMLY CHARGED SPHERE: Let's consider a total charge Q spread uniformly through a sphere of radius a. The charge density is therefore

$$\rho_q = \frac{Q}{\frac{4}{3}\pi a^3}. \tag{1.79}$$

The charge enclosed by a sphere radius $r < a$ is

$$Q_r = \frac{4}{3}\pi r^3 \rho_q = \frac{Qr^3}{a^3}. \tag{1.80}$$

Applying Gauss's law to the sphere radius r, we find that

$$4\pi r^2 E_r = \frac{Qr^3}{\epsilon_0 a^3} \tag{1.81}$$

$$E_r = \frac{Qr}{4\pi\epsilon_0 a^3} \tag{1.82}$$

OUTSIDE A UNIFORMLY CHARGED SPHERE, the charge enclosed is simply Q. Consequently,

$$4\pi r^2 E_r = \frac{Q}{\epsilon_0} \tag{1.83}$$

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}. \tag{1.84}$$

Equation 1.84 means that the electric field outside the sphere looks like the electric field from a point charge Q located at the origin. The electric field as a function of radius inside and outside a uniformly charged sphere is sketched in Figure 1.20.

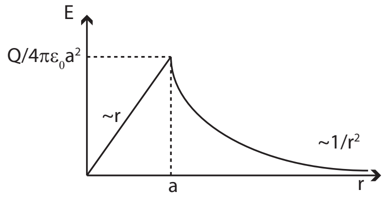

Figure 1.20: Electric field as a function of radius inside and outside a uniformly charged sphere.

OUTSIDE A NON-UNIFORM, SPHERICALLY SYMMETRIC, CHARGED SPHERE, the electric field at distance r is the same as equation 1.84, because we only care about the total charge Q inside the integration surface. We conclude that $\boldsymbol{E}$ outside any spherically symmetric charge distribution looks that from a point charge Q at the centre of the distribution. Of course, *inside* a non-uniform, spherically symmetric, charged sphere the electric field profile very much depends on the charge distribution. It is necessary to perform a volume integral over $0 \rightarrow r$. This is explored in more detail in problem sheet 2.

NON-SPHERICAL CHARGE DISTRIBUTIONS offer a further complication. Consider for example a cube of total charge Q, with a side of length a. The total flux through a sphere radius r ($r > a$) centered on the cube will be Q/ϵ_0. However, $\boldsymbol{E}$ is not uniform over the sphere. It will be stronger near the cube corners. If $r >> a$, then very far from the cube, the cube looks like a point charge and we recover a spherically symmetric radial electric field profile that falls as r^{-2}.

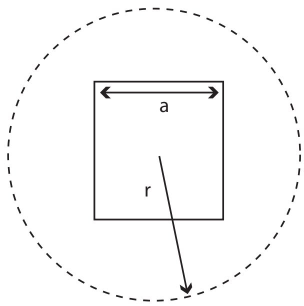

Figure 1.21: Example of a non-spherically symmetric charge distribution: a cube of side a, total charge Q. Although the net electric flux through the spherical surface radius r is Q/ϵ_0, $\boldsymbol{E}$ is not uniform over the sphere.

1.5.4 *Further Comments on Gauss's Law*

We can recast the electric field $\boldsymbol{E}$ in terms of the potential V as follows:

$$\nabla \cdot \boldsymbol{E} = -\nabla \cdot (\nabla V) = -\nabla^2 V. \tag{1.85}$$

Therefore, Gauss's law can be written as follows:

$$\nabla^2 V = -\frac{\rho_q}{\epsilon_0}. \tag{1.86}$$

This equation is known as Poission's equation. In the case that there is no charge density,

$$\nabla^2 V = 0. \tag{1.87}$$

This equation is known as Laplace's equation.

1.6 *Electrostatics and the Uniqueness Theorem*

Electrostatics can ultimately be summarized by two equations: equation 1.29 and equation 1.77. They can equally be written as equation 1.46 and equation 1.86.

The uniqueness theorem states that there is only one potential V which satisfies Poisson's equation given the specified charge density $\rho(\boldsymbol{r})$ and given the specified boundary conditions. In general, electrostatic problems are most efficiently solved using Poisson's equation for V using appropriate boundary conditions and knowledge of the charge distribution, then finding $\boldsymbol{E}$ from V. In reality, analytic solutions are generally impossible to find easily, and so most such problems are solved numerically.

2

Conductors, Capacitors, Dielectrics and Current

2.1 Conductors

2.1.1 Basic properties

In this course we will use the following working definitions for materials. A *conductor* contains freely moving charged particles, whereas an insulator does not. We will mainly consider solid conductors[1], although you should be aware of liquid conductors such as mercury.

[1] e.g., metals where free electrons move through the crystal lattice.

The first basic property of conductors is that in static situations, $\boldsymbol{E} = 0$ inside the conductor. To understand this, consider the following thought experiment, illustrated in Figure 2.1.

1. We start with an initially overall neutral conductor. If we apply an electric field $\boldsymbol{E}$, the free electrons will immediately experience a force and move in response to the electric field. The electrons will collect on the left hand side of the conductor giving a net negative charge. A region of positive charge forms on the right hand side.

2. This charge separation has its own electric field, shown as a dashed red line in the second panel, which acts to reduce the electric field inside the conductor. This continues until $\boldsymbol{E} = 0$ inside the conductor, since at this point, there is no longer any force to continue to redistribute the charge.

3. In equilibrium, there is a surface charge σ_q which forms, a few angstroms (10^{-10} m) thick. As illustrated in the final panel of Figure 2.1, the external field terminates on the surface charge.

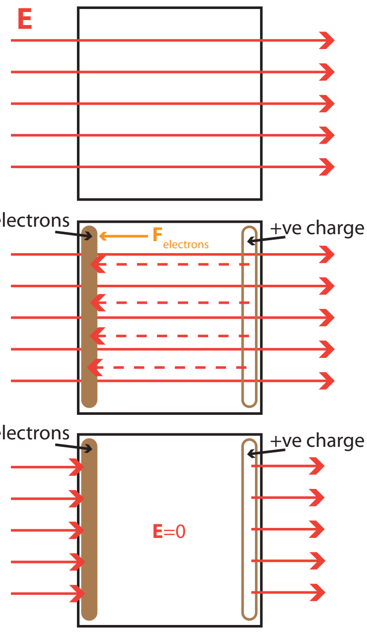

Figure 2.1: The top panel shows an electric field applied to a conductor. The second panel shows how the conductor responds. The force on the electrons moves them to the left, and a surface charge develops on each edge. The electric field associated with the charge separation cancels out the external field. The bottom panel shows the steady state result, where the external electric field stops at the surface charge and the electric field inside the conductor is zero.

Inside the conductor $E = 0$ and so the interior of the conductor is all at the same potential. The surface is also an equipotential. If it were not, then there would be an electric field in the surface layer which would cause further redistribution of the charge until the potential equalized. We have seen previously that equipotentials are perpendicular to $\boldsymbol{E}$, and so therefore the electric field lines lie perpendicular to the surface.

2.1.2 *Surface charge*

We now examine in more detail the nature of the surface charge. To do this we will use Gauss's law, applied to what is known as a 'pill-box' volume (i.e., a cylinder) which is illustrated in Figure 2.2. The conductor fills the bottom half of the figure and there is a positive surface charge layer. The electric field outside the conductor therefore points upwards. The pill-box volume is placed half in the conductor and half outside the conductor. We divide the surface integral into four parts: the top (T), the side outside the conductor (S1), the side inside the conductor (S2), and the bottom (B). The top and bottom are circular surfaces, with area A. This allows us to write

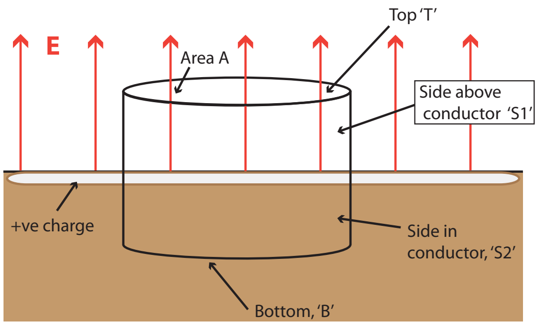

Figure 2.2: Diagram showing the pill-box volume placed across the edge of the conductor. We apply Gauss's law to the pill-box volume, as described in the text.

$$\oiint_{pillbox} \boldsymbol{E}\cdot d\boldsymbol{S} = \iint_T \boldsymbol{E}\cdot d\boldsymbol{S} + \iint_B \boldsymbol{E}\cdot d\boldsymbol{S} + \iint_{S1} \boldsymbol{E}\cdot d\boldsymbol{S} + \iint_{S2} \boldsymbol{E}\cdot d\boldsymbol{S} \tag{2.1}$$

We now consider each surface in turn

1. Surface T: the integral is non-zero, as there is a uniform electric field E pointing out through the circular surface area A. Consequently, $\Phi_E = EA$
2. Surface S1: the integral is zero, because $\boldsymbol{E}$ is perpendicular to the surface normal. Therefore, $\boldsymbol{E}\cdot\boldsymbol{n} = 0$
3. Surface S2: since $\boldsymbol{E} = 0$ in the conductor, $\Phi_E = 0$
4. Surface B: again, since $\boldsymbol{E} = 0$ in the conductor, $\Phi_E = 0$

Consequently,

$$\oiint_{pillbox} \boldsymbol{E}\cdot d\boldsymbol{S} = EA + 0 + 0 + 0 \tag{2.2}$$

The pill-box encloses a total charge $\sigma_q A$, and so

$$E = \frac{\sigma_q}{\epsilon_0} \tag{2.3}$$

So far we have only considered a flat surface. To determine the properties of a curved surface, we make the pill-box very small such that $A \to dA$. On small scales corresponding to dA, the surface element can be considered flat.

2.1.3 An isolated, charged conducting sphere

Now consider an isolated *conducting* sphere, radius a. We apply an extra charge Q. Following the previous arguments, the charge will redistribute within the conductor so that the internal electric field is 0. This means that the excess charge Q is on the surface of the sphere, and the surface charge density σ_q is given by

$$\sigma_q = \frac{Q}{4\pi a^2} \tag{2.4}$$

Using equation 2.3, we find that the electric field at the surface of the sphere is

$$E = \frac{\sigma_q}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 a^2} \tag{2.5}$$

This agrees with previous results for any spherically symmetric charge distribution (conducting or not).

2.1.4 Electrostatic shielding

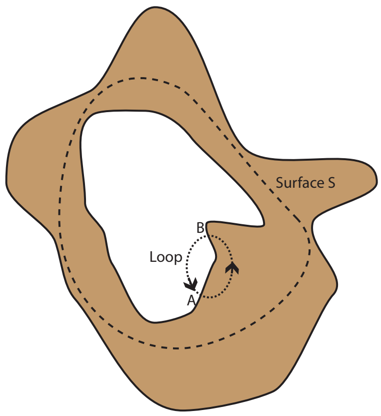

Figure 2.3: Path integral to show how electrostatic shielding works.

INSIDE AN EMPTY, ARBITRARILY-SHAPED CAVITY IN A CONDUCTOR, $\boldsymbol{E} = 0$. This result is extremely important for a wide variety of applications, and is known as *electrostatic shielding*.

To understand how this arises, we will use Figure 2.3. The figure shows a cut through a three-dimensional conductor of arbitrary shape, with an arbitrarily shaped cavity inside. The argument proceeds through a number of steps:

1. We first consider a closed surface S inside the conductor. The closed surface S also contains the cavity. $\boldsymbol{E} = 0$ on this surface, because $E = 0$ in the conductor. By Gauss's law, this means that the net charge inside S must be 0.

2. As a result, we can conclude that the net charge on the inner surface must also be 0.

3. Now, maybe there are localised regions of positive and negative charge on different parts of the inner surface. We can make two arguments to explain why this cannot be true.

   First, we could argue that this would set up an electric field on the surface, which would cause the charges to redistribute until the system became neutral.

   Or, we can use the following result that we have met before:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0 \tag{2.6}$$

   We consider the loop integral shown in Figure 2.3. $A \to B$ is inside the conductor and $B \to A$ is in the cavity. We can therefore write that:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = \int_A^B \boldsymbol{E} \cdot d\boldsymbol{l} + \int_B^A \boldsymbol{E} \cdot d\boldsymbol{l} = 0. \tag{2.7}$$

The first part ($A \rightarrow B$) = 0 because $E = 0$ inside the conductor. Therefore this leaves the second part

$$\int_B^A \boldsymbol{E} \cdot d\boldsymbol{l} = 0. \tag{2.8}$$

We could say, suppose there is an excess of positive charge at A and an excess of negative charge at B. If this were true, then the line integral in equation 2.8 would be non-zero! Given the path is entirely arbitrary, the only possibility is that the inner surface is an equipotential and neutral everywhere.

4. This means that there is no surface charge on inner surface, and that $E = 0$ both in the cavity and in the conductor[2].

This means that if an electric field is applied to a conductor which has a cavity, there will be no electric field in the cavity, and we therefore say that the cavity is shielded. This phenomenon is also known as the Faraday cage. It finds numerous applications, from MRI scan rooms, to metallic lined purses/wallets (to prevent reading of contactless cards), to using your mobile phone in a lift, to defence and security applications.

[2] It is tempting (and important!) to try and figure out if you find a way around this result. For example, you might think about a negative surface charge density on the cavity wall which is all at the same potential. But remember, because $E = 0$ in a conductor, there can't be a net charge inside the surface S, and so this isn't possible...

2.1.5 *Loop integral on the outside of a charged conductor*

It is interesting to consider the loop integral of the electric field applied to the outside of the conductor. Let us consider the charged sphere discussed previously. This is illustrated in Figure 2.4.

If we perform the loop integral $ABCD$ on the electric field, we will find that it is 0 as expected. For the section AB, $\boldsymbol{E} \cdot d\boldsymbol{l} = 0$ is positive (when the path is outside the conductor). For BC, $\boldsymbol{E} \cdot d\boldsymbol{l} = 0$ ($\boldsymbol{E}$ is perpendicular to the path). For CD, going back into the conductor, $\boldsymbol{E} \cdot d\boldsymbol{l} = 0$ is negative (again, when the path is outside the conductor). For DA, $\boldsymbol{E} \cdot d\boldsymbol{l} = 0$ ($\boldsymbol{E}$ is zero). The path integral AB is equal and opposite to CD. For this to be true, the charge density must be the same everywhere on the surface, and so this is a way to demonstrate that the surface charge will spread itself out in such a way as to be uniform on the surface of the sphere. The difference between the inner surface and the outer surface is the information provided by the surface integrals: integrating over a surface *outside* the conductor gives a *non-zero* electric flux, whereas integrating over a surface *within* the conductor gives *zero* electric flux.

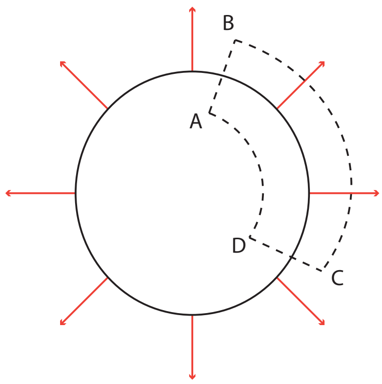

Figure 2.4: Path integral on the outside of a charged sphere.

2.2 *Capacitors*

2.2.1 *Basic properties: ideal capacitor*

To conceptualise an 'ideal' capacitor, consider an isolated pair of conductors A and B with equal and opposite charges $\pm Q$ as illustrated in Figure 2.5. If isolated, then all field lines from the positive charge conductor end on the negative conductor.

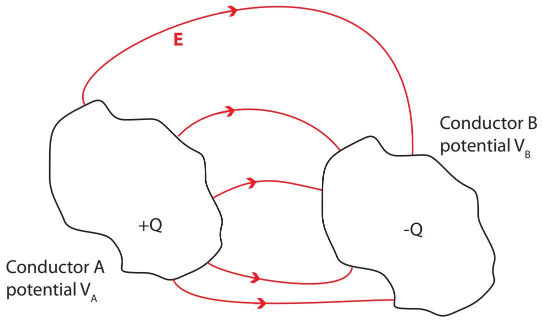

Figure 2.5: Two equally and opposite charged conductors form an ideal capacitor.

We now say that conductor A is at potential V_A, and conductor B is at potential V_B. We write the potential of the capacitor, V, as being the potential of the positive conductor relative to the negative conductor, such that

$$V = V_A - V_B \tag{2.9}$$

If we double the charge on each conductor so that $Q_{A,B} \rightarrow 2Q_{A,B}$, then by the superposition principle, $V_{A,B} \rightarrow 2V_{A,B}$. Consequently, $V \rightarrow 2V$, and we conclude that

$$V \propto Q \tag{2.10}$$

We therefore write that

$$Q = CV \tag{2.11}$$

where C is the capacitance[3]. Capacitors store charge, with typical charges $\ll 1$ C and typical capacitance $\ll 1$ F.

[3] The SI unit of capacitance is Coulomb/Volt, or Farad.

2.2.2 *Parallel plate capacitor*

To better understand capacitance, we now consider a specific configuration, the parallel plate capacitor, which is illustrated in Figure 2.6. It consists of two large flat conducting plates, area A, and charge $\pm Q$. The plates are separated by distance d which is very small compared to the plate area. We ignore so-called 'edge effects', and this renders the analysis 1-dimensional.

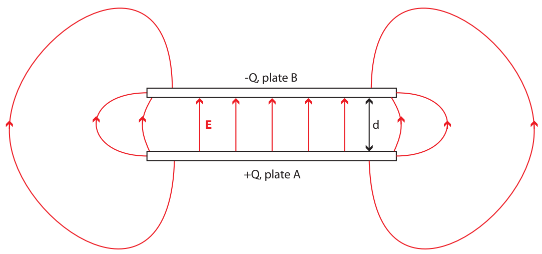

Figure 2.6: Two flat plates separation d form a parallel plate capacitor. In our setup the plate separation is much less than the plate dimension, and so although we show here the 'edge effects' (i.e. curved field lines), we ignore them in this analysis.

The charge $\pm Q$ is treated as being spread uniformly over the inner plate surfaces. Using equation 2.3, which we repeat here, the electric field between the plates is

$$E = \frac{\sigma_q}{\epsilon_0}, \tag{2.12}$$

$$E = \frac{Q}{\epsilon_0 A}. \tag{2.13}$$

Now we use this in the standard equation relating electric field and potential:

$$V = V_A - V_B = Ed = Q\frac{d}{\epsilon_0 A}, \tag{2.14}$$

and substituting equation 2.14 in the formula for capacitance, we find

$$C = \frac{\epsilon_0 A}{d} \tag{2.15}$$

The capacitance depends on the area of the plates and their separation.

2.2.3 *Energy stored in a capacitor*

Capacitors store energy as well as charge. We can calculate this energy by recalling a result from the first part of the course, which is the potential energy of a set of charges. This can be written as

$$U = \sum_i \frac{1}{2} Q_i V_i \tag{2.16}$$

In the parallel plate capacitor, Plate A has charge $+Q$ and is at potential V_A. Plate B, has charge $-Q$ and is at potential V_B. Consequently,

$$U = \frac{1}{2}QV_A - \frac{1}{2}QV_B. \tag{2.17}$$

The energy stored is

$$U = \frac{1}{2}QV = \frac{1}{2}CV^2 \tag{2.18}$$

This equation is recognisable from electronics and electrical circuits, but it is revealing to examine it in a little more detail. Since $Q = \epsilon_0 AE$ and $V = Ed$,

$$U = \frac{1}{2}\epsilon_0 AE(Ed) = \frac{1}{2}\epsilon_0 AdE^2 \tag{2.19}$$

Ad is the volume between the plates and is therefore the volume occupied by the electric field between the plates. We can therefore define the energy density of $\boldsymbol{E}$, u_E, as

$$u_E = \frac{1}{2}\epsilon_0 E^2. \tag{2.20}$$

The energy is stored in the electric field between the plates and equation 2.20 is the energy density of $\boldsymbol{E}$ in vacuum.

2.2.4 *Spherical capacitors*

The spherical capacitor consists of two conductors: a sphere of charge $+Q$, radius a, potential V_a and a spherical shell charge $-Q$, radius b, potential V_b as illustrated in Figure 2.7. Note that $V_a > V_b$ ($\boldsymbol{E}$ points from high to low potential). Using Gauss's law applied to a surface in between the two conductors, we can show that

$$\boldsymbol{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}} \tag{2.21}$$

For capacitance calculations, we calculate $V = V_a - V_b$ where V_a is the positive charge potential and V_b is the negative charge potential. By definition,

$$V = V_a - V_b = -\int_b^a \boldsymbol{E}\cdot d\boldsymbol{l} \tag{2.22}$$

We can now reverse the path integral:

$$V = \int_a^b \boldsymbol{E}\cdot d\boldsymbol{l} = \frac{Q}{4\pi\epsilon_0}\int_a^b \frac{dr}{r^2} \tag{2.23}$$

$$V = \frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_a^b \tag{2.24}$$

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \tag{2.25}$$

Consequently, we can write the capacitance as

$$C = \frac{Q}{V} = \frac{4\pi\epsilon_0}{\left(\frac{1}{a} - \frac{1}{b}\right)} \tag{2.26}$$

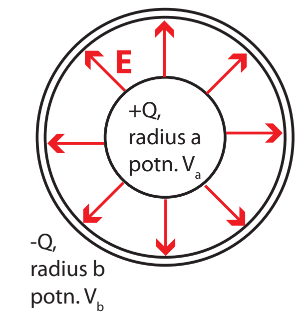

Figure 2.7: A spherical conductor surrounded by a hollow spherical conductor forms a spherical capacitor.

2.2.5 *Capacitance of single charged conductor*

It is meaningful to talk about the capacitance of an isolated, charged, single conductor. If we let $b \to \infty$, we can write

$$C = \frac{4\pi\epsilon_0}{\frac{1}{a}} = 4\pi\epsilon_0 a \tag{2.27}$$

The conductor evidently stores charge; the energy is stored in the electric field surrounding the conductor.

2.3 *Dielectrics*

2.3.1 *Polarization*

In this subsection, we develop a simple model for a particular type of insulator (non-conducting material), known as a dielectric. More specifically, we consider the case where the dielectric is made of polar molecules (i.e. each molecule has a dipole moment), and there are no free electrons. Initially, no electric field is applied and the dipoles are oriented randomly. This is illustrated in the top panel of Figure 2.8.

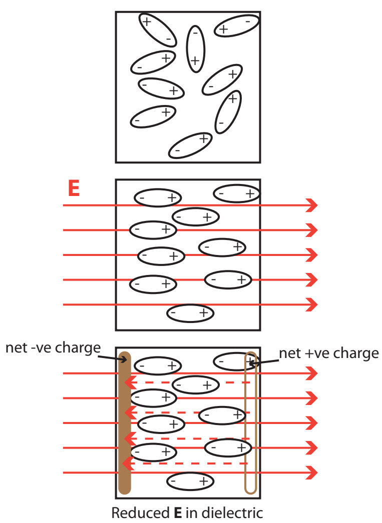

Figure 2.8: Diagram showing how the electric field is reduced inside a dielectric.

We then apply an electric field $\boldsymbol{E}$. This electric field causes the dipoles to rotate and align, as illustrated in the second panel of Figure 2.8. In the middle of the dielectric, there are equal numbers of positive and negative charges, but on the left, there are predominantly negative ends of the dipoles at the edge. On the right, there are predominantly positive dipole ends. This means that there is a net negative surface charge on the left, and a net positive surface charge on the right. Consequently, this sets up an electric field within the dielectric which causes an overall reduced electric field inside the dielectric. This is illustrated in the bottom panel of Figure 2.8.

It is important to note that this argument can apply even if the molecules of the material do not have an intrinsic dipole moment. The applied external electric field can cause so-called induced dipoles, and the same behaviour will arise.

More specifically, we write that in the dielectric, the electric field is reduced from $\boldsymbol{E}_0$ to $\boldsymbol{E}_D$, such that

$$\boldsymbol{E}_D = \frac{\boldsymbol{E}_0}{K} \tag{2.28}$$

where K is the dielectric constant. $K > 1$, and some examples are given in Table 2.1.

Material	K
Vacuum	1
Air (1 atm)	1.00059
Teflon	2.1
Polyethylene	2.25
PVC	3.18
Glass	5-10
Water	80.4
Strontium titanate	310

Table 2.1: Dielectric constants of some different materials (From University Physics, Young and Freedman).

2.3.2 *Charging dielectric and conductors*

If we add charge to a conductor, the charges within the conductor will rearrange themselves so that an equivalent charge is found spread over the surface of the conductor. However, if we add charge to a dielectric, the charge will stay where it is placed.

In general, this is a major issue for electronics. For example, energetic electrons from the Van Allen radiation belts can impact and cause charging on geostationary communications satellites. If the charging becomes very strong, then a deep dielectric discharge can occur, damaging the spacecraft. This is an example of the impact of 'space weather'.

2.3.3 *Dielectrics in capacitors*

Change in electric field. To understand the role that dielectrics play in capacitors, we will use the model of the parallel plate capacitor. As before, we begin with two conducting plates, separated by vacuum. There is a surface charge density $\pm\sigma_{q,free}$ on the inner side of each plate, and the electric field between the plates is $\boldsymbol{E}_0$. This is illustrated in the top panel of figure 2.9.

We have seen previously that

$$E_0 = \frac{\sigma_{q,free}}{\epsilon_0}. \tag{2.29}$$

We now quickly fill the gap between the plates with a dielectric. The electric field $\boldsymbol{E}_0$ causes induced surface charges $\pm\sigma_{q,ind}$ on

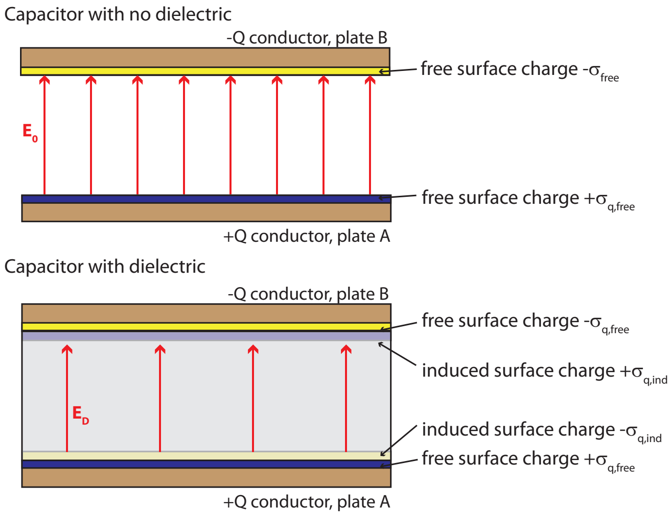

Figure 2.9: The top panel shows the capacitor with no dielectric inside, there are surface charges, and the electric field points from bottom to top. The bottom panel shows what happens when the dielectric is inserted. The dielectric polarizes, and the electric field between the plates reduces.

dielectric as indicated in the bottom panel of figure 2.9, and there is a reduced electric field $\boldsymbol{E}_D$ in the dielectric in between the plates.

Since $E_0 = KE_D$, we can therefore write that

$$KE_D = \frac{\sigma_{q,free}}{\epsilon_0} \tag{2.30}$$

$$E_D = \frac{\sigma_{q,free}}{K\epsilon_0} = \frac{\sigma_{q,free}}{\epsilon}. \tag{2.31}$$

Here we have introduced a new quantity $\epsilon = K\epsilon_0$ which is the permittivity.

We can also apply Gauss's law to the new pill-box shown in Figure 2.10. The electric flux out of this volume is $E_D A$ if the circular surface area of the pill-box is A, and the enclosed charge is $A(\sigma_{q,free} - \sigma_{q,ind})$. We can therefore write that

$$E_D A = \frac{A(\sigma_{q,free} - \sigma_{q,ind})}{\epsilon_0} \tag{2.32}$$

$$E_D = \frac{\sigma_{q,free} - \sigma_{q,ind}}{\epsilon_0} \tag{2.33}$$

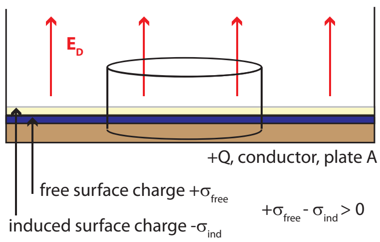

Figure 2.10: Diagram showing the pill-box volume applied to the edge of the dielectric and the edge of the conductor.

And therefore by equating 2.31 and 2.33, we find

$$\frac{\sigma_{q,free}}{K\epsilon_0} = \frac{\sigma_{q,free} - \sigma_{q,ind}}{\epsilon_0}, \tag{2.34}$$

$$\sigma_{q,free} = K(\sigma_{q,free} - \sigma_{q,ind}), \tag{2.35}$$

$$\sigma_{q,ind} = \sigma_{q,free}\left(1 - \frac{1}{K}\right). \tag{2.36}$$

The magnitude of the induced charge depends on K. If $K = 1$ then there is no induced charge.

THE CHANGE IN CAPACITANCE due to a dielectric is another important result. To understand this, we perform the following thought experiment. First charge and isolate the capacitor without the dielectric (i.e. with vacuum between the plates). The electric field inside the capacitor is E_0, and there is a charge $\pm Q_{free}$ on the plates.

Now add the dielectric. The electric field becomes $E_D = E_0/K$. The potential difference changes from V_0 to V_D where

$$V_D = E_D d = \left(\frac{E_0}{K}\right) d = \frac{V_0}{K}. \tag{2.37}$$

In fact, the voltage drop goes down. We can now calculate the capacitance. Initially,

$$C = \frac{Q_{free}}{V_0} \tag{2.38}$$

Q_{free} remains the same because the system is isolated, so we can write for the new capacitance that

$$C_D = \frac{Q_{free}}{V_D} = \frac{KQ_{free}}{V_0} = KC \tag{2.39}$$

This means that the capacitance goes up by factor K when the dielectric is inserted. Another way to understand this is to say that the capacitor can hold the same charge with a smaller voltage; it has more 'capacitance'. This result also applies to other geometries - sphere, cylinder etc.

THE CHANGE IN ENERGY STORAGE also depends on the dielectric. To illustrate this we now follow a *different* procedure where the capacitor is <u>not isolated</u>. If we take an dielectric-filled capacitor with capacitance C_D, and apply a voltage V, the energy stored is

$$U = \frac{1}{2} C_D V^2 \tag{2.40}$$

In contrast, for the equivalent vacuum-filled capacitor we find that

$$U_0 = \frac{1}{2} C_0 V^2 \tag{2.41}$$

However, $C_D = KC$, so therefore

$$U_D = \frac{1}{2} K C_0 V^2 = K U_0 \tag{2.42}$$

We conclude that a dielectric filled capacitor stores more energy (as well as more charge) for the given applied voltage.

2.4 Current and Resistance

2.4.1 Current Density

Consider a material with many charged particle species $1, 2, ..., i, ...N$. Each species has number density n_i, and the charge of each species is q_i. Furthermore, we assume that each species moves coherently with velocity $\boldsymbol{v}_i$. The *current density* is defined as

$$\boldsymbol{j} = \sum_{i=1}^{N} n_i q_i \boldsymbol{v}_i \tag{2.43}$$

Current densities can be found in a wide variety of systems, from free electrons in wires, to positive holes in semiconductors, to the ions in an electrolyte, to the protons and electrons in a plasma. We use current density $\boldsymbol{j}$ because it reflects the underlying physics of how electrical currents arise and is the most general way to approach the problem.

In certain circumstances, it is useful to reduce $\boldsymbol{j}$ to a statement about the electric current I, which is the amount of charge passing through a surface S per unit time. The current I is defined as:

$$I = \iint_S \boldsymbol{j} \cdot d\boldsymbol{S} \tag{2.44}$$

The notion of current as a scalar must be used with care. Firstly, there are many assumptions, both explicit and implicit, in talking about scalar currents, which can cause misunderstandings. Secondly, it is often not flexible enough to deal with the vast variety of systems that we encounter in physics[4].

[4] In my own field of research, space plasma physics, measurement of the current density in the Earth's magnetosphere is extremely important to reveal the physics at work. However, calculating the current I is rarely useful, because the current density is unevenly distributed through the body of the entire plasma.

2.4.2 *Conservation of Charge*

If Q is the charge in some volume V, and I is total current out of the volume, then by definition

$$-\frac{dQ}{dt} = I \tag{2.45}$$

We can also write that

$$I = \oint\!\!\!\oint_S \boldsymbol{j} \cdot d\boldsymbol{S} \tag{2.46}$$

where S is the closed surface containing volume V. Consequently,

$$\oint\!\!\!\oint_S \boldsymbol{j} \cdot d\boldsymbol{S} = -\frac{dQ}{dt} \tag{2.47}$$

This equations describes conservation of charge in integral form. To find the differential form of this equation, we again use the divergence theorem from vector calculus:

$$I = \oint\!\!\!\oint_S \boldsymbol{j} \cdot d\boldsymbol{S} = \iiint_V \nabla \cdot \boldsymbol{j}\, dV \tag{2.48}$$

If we also assume that the charge is distributed continuously in the volume, then we can write that

$$Q = \iiint_V \rho_q dV \tag{2.49}$$

Using equation 2.45 we can therefore write that

$$\frac{\partial}{\partial t} \iiint_V \rho_q \, dV + \iiint_V \nabla \cdot \boldsymbol{j}\, dV = 0 \tag{2.50}$$

Now considering an infinitesimal volume dV,

$$\frac{\partial \rho_q}{\partial t}\, dV + \nabla \cdot \boldsymbol{j}\, dV = 0 \tag{2.51}$$

Since dV is finite,

$$\frac{\partial \rho_q}{\partial t} + \nabla \cdot \boldsymbol{j} = 0 \tag{2.52}$$

This is the equation of charge conservation in differential form.

2.4.3 *Steady Current and Ohm's Law*

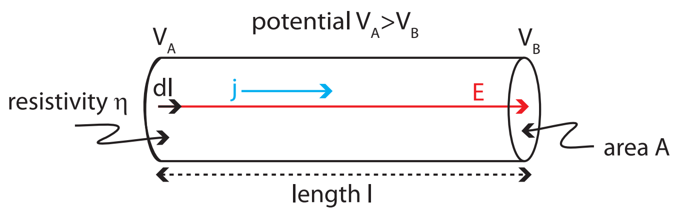

Figure 2.11: Cartoon of a conducting wire, length l, area A, with a potential drop from V_A to V_B.

To begin, we consider a conducting wire, cross-section A and length l, along which we apply an electric field $\boldsymbol{E}$. We assume a steady current density $\boldsymbol{j}$, uniform across the wire. This is illustrated in Figure 2.11. We can write

$$\boldsymbol{E} = \eta \boldsymbol{j}. \tag{2.53}$$

We note the following here: η is the resistivity[5]; it depends both on material and environmental conditions. Some example resistivities are given in Table 2.2.

The conductivity[6] is σ where $\sigma = 1/\eta$. Thus, we can also write

$$\boldsymbol{j} = \sigma \boldsymbol{E}. \tag{2.54}$$

This is Ohm's law.

If we integrate equation 2.53 along the length of the wire from A to B then

$$\int_A^B \boldsymbol{E} \cdot d\boldsymbol{l} = \int_A^B \eta \boldsymbol{j} \cdot d\boldsymbol{l}. \tag{2.55}$$

The left hand side is the potential difference $V = V_A - V_B$ between the two ends. For the right hand side, we know that $I = jA$, where A is the cross-sectional area of the wire, so

$$V = \eta j l = \frac{\eta l}{A} I \tag{2.56}$$

Therefore we can define the resistance R as

$$R = \frac{\eta l}{A} \tag{2.57}$$

so that

$$V = IR. \tag{2.58}$$

[5] The SI unit of resistivity is Ωm.

[6] Don't confuse surface charge with conductivity!

Material	η (Ω m)
Silver	1.47×10^{-8}
Copper	1.72×10^{-8}
Gold	2.44×10^{-8}
Steel	20×10^{-8}
Amber	5×10^{14}
Glass	$10^{10} - 10^{14}$
Teflon	$> 10^{13}$
Wood	$10^{8} - 10^{11}$

Table 2.2: Resistivities of some different materials (From University Physics, Young and Freedman).

2.4.4 *Joule Heating*

In a metallic conducting wire, we can treat the electrons as moving and the ions as being at rest. The electrons are accelerated by the electric field $\boldsymbol{E}$, and then decelerated by collisions with the ion lattice; energy is lost to vibrations and heat. This means that on average, the electrons move at constant velocity $\bar{\boldsymbol{v}}_e$ along the wire. The rate at which the applied force $-e\boldsymbol{E}$ (due to the electric field) does work on each electron is $\boldsymbol{F} \cdot \bar{\boldsymbol{v}}_e$. There are n electrons per unit volume, so the power put into the electrons, which is ultimately the heating rate per unit volume (Wm^{-3}) is therefore

$$p_{heating} = n(-e\boldsymbol{E})\bar{\boldsymbol{v}}_e = \boldsymbol{j} \cdot \boldsymbol{E} = \eta \boldsymbol{j}^2. \tag{2.59}$$

For a conductor of length l and area A, the total heating rate, or power input, is P where

$$P = (\eta j^2)Al = \eta \left(\frac{I}{A}\right)^2 Al, \tag{2.60}$$

$$P = \frac{\eta l}{A} I^2 = I^2 R. \tag{2.61}$$