Derivations of aerodynamic equations

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April 2021

Introduction

This document contains the derivations of the aerodynamic equations used in the library.

1 Potential Flow Theory

This section contains derivations of equations relating to potential flow theory, which describes flow velocity $\vec{v}(\vec{r})$ at some point point $\vec{r}$ in space as the gradient of the velocity potential $\phi(\vec{r})$ at that point. More succinctly, this is written as

$$\vec{v}(\vec{r}) = \nabla\phi(\vec{r}) \tag{1}$$

1.1 3D Source Panel Method

This subsection contains derivations of equations used in the 3D source panel method, which estimates the aerodynamic properties of a body in an inviscid flow.

The body is approximated as an array of rectangular panels, each of which is assumed to have constant velocity potential source (or sink) strength across its surface.

1.1.1 The Effect of a Rectangular Source Panel on the Velocity Potential

The aforementioned rectangular panel is defined in 3D Cartesian coordinates by a position vector $\vec{r_0}$ (pointing to the centre of the rectangle), two orthonormal tangent vectors $\hat{u}$ (parallel to one side of the rectangle) and $\hat{v}$ (parallel to the other side of the rectangle) as well as the dimensions a (the length of the side parallel to $\hat{u}$) and b (the length of the side parallel to $\hat{v}$).

The effect of a point source ($Q > 0$) or sink ($Q < 0$) at position $\vec{r_0}$ on the velocity potential $\phi(\vec{r})$ at position r is assumed to be

$$\phi(\vec{r}) = -\frac{Q}{4\pi|\vec{r} - \vec{r_0}|} \tag{2}$$

Modelling the source rectangle as consisting of an infinite series of such point sources, all of constant strength Q, yields the following surface integral:

$$\phi(\vec{r}) = -\frac{Q}{4\pi}\iint_S \frac{1}{|\vec{r}-\vec{r'}|}dS = -\frac{Q}{4\pi}\iint_T \frac{1}{|\vec{r}-(\vec{r_0}+\hat{u}u+\hat{v}v)|}dudv \qquad (3)$$

where $\vec{r'} = \vec{r_0}+\hat{u}u+\hat{v}v$, $S = \{\vec{r_0}+\hat{u}u+\hat{v}v|u,v \in T\}$, and $T = [-a/2;a/2]\times[-b/2;b/2]$. Note that in the above surface integral $|\frac{\partial\vec{r'}}{\partial u}\times\frac{\partial\vec{r'}}{\partial v}| = |\hat{u}\times\hat{v}| = 1$, since $\hat{u}$ and $\hat{v}$ are orthonormal.

The denominator in the integrand can be simplified to

$$\begin{aligned}
&\sqrt{|\hat{u}u+\hat{v}v|^2 - 2(\hat{u}u+\hat{v}v)\cdot(\vec{r}-\vec{r_0}) + |\vec{r}-\vec{r_0}|^2} = \\
&= \sqrt{(\hat{u}\cdot\hat{u})u^2 + 2uv(\hat{u}\cdot\hat{v}) + (\hat{v}\cdot\hat{v})v^2 - 2(\hat{u}u+\hat{v}v)\cdot(\vec{r}-\vec{r_0}) + |\vec{r}-\vec{r_0}|^2} = \\
&= \sqrt{u^2+v^2-2(\hat{u}u+\hat{v}v)\cdot(\vec{r}-\vec{r_0}) + |\vec{r}-\vec{r_0}|^2} = \\
&= \sqrt{(u-\hat{u}\cdot(\vec{r}-\vec{r_0}))^2 + (v-\hat{v}\cdot(\vec{r}-\vec{r_0}))^2 + k}
\end{aligned}$$

where we defined $k = |\vec{r}-\vec{r_0}|^2 - (\hat{u}\cdot(\vec{r}-\vec{r_0}))^2 - (\hat{v}\cdot(\vec{r}-\vec{r_0}))^2$ and used the fact that, by definition, $\hat{u}\cdot\hat{u} = \hat{v}\cdot\hat{v} = 1$ and $\hat{u}\cdot\hat{v} = 0$.

Changing variables in integral 3 via $u-\hat{u}\cdot(\vec{r}-\vec{r_0}) = iu'$ and $v-\hat{v}\cdot(\vec{r}-\vec{r_0}) = iv'$, leading to the Jacobian determinant

$$det(J) = det\left(\begin{bmatrix}\frac{\partial u}{\partial u'} & \frac{\partial u}{\partial v'} \\ \frac{\partial v}{\partial u'} & \frac{\partial v}{\partial v'}\end{bmatrix}\right) = det\left(\begin{bmatrix} i & 0 \\ 0 & i\end{bmatrix}\right) = i^2 - 0 = -1$$

we write the integral as

$$\frac{Q}{4\pi}\iint_{T'}\frac{1}{\sqrt{k-u'^2-v'^2}}du'dv'$$

where $T' = [u'_1;u'_2]\times[v'_1;v'_2] = [-i(-a/2-\hat{u}\cdot(\vec{r}-\vec{r_0}));-i(a/2-\hat{u}\cdot(\vec{r}-\vec{r_0}))]\times[-i(-b/2-\hat{v}\cdot(\vec{r}-\vec{r_0}));-i(b/2-\hat{v}\cdot(\vec{r}-\vec{r_0}))]$. Using the identity $\int\frac{dx}{\sqrt{a^2-x^2}} = \arcsin(\frac{x}{a})+C$, we integrate in u' to get

$$\frac{Q}{4\pi}\int_{v'_1}^{v'_2}\left(\arcsin\left(\frac{u'_2}{\sqrt{k-v'^2}}\right) - \arcsin\left(\frac{u'_1}{\sqrt{k-v'^2}}\right)\right)dv'$$

The exponential definition $\arcsin(x) = -i\ln(ix+\sqrt{1-x^2})$ along with the identity $\ln(a/b) = \ln(a)-\ln(b)$ allows us to rewrite the above integral as

$$-i\frac{Q}{4\pi}\int_{v'_1}^{v'_2}\left(\ln\left(iu'_2+\sqrt{k-(u'_2)^2-v'^2}\right) - \ln\left(iu'_1+\sqrt{k-(u'_1)^2-v'^2}\right)\right)dv'$$

To solve an integral in the form of

$$\int \ln\left(a+\sqrt{b-x^2}\right)dx$$

we first use integration by parts to get

$$x\ln\left(a+\sqrt{b-x^2}\right)-\int x\frac{d}{dx}\left(\ln\left(a+\sqrt{b-x^2}\right)\right)dx$$

Following just the above integral, we write it as

$$\begin{aligned}\int x\frac{d}{dx}\left(\ln\left(a+\sqrt{b-x^2}\right)\right)dx &= -\int \frac{x^2}{(a+\sqrt{b-x^2})\sqrt{b-x^2}}dx = \\ &= -\int \frac{x^2(a-\sqrt{b-x^2})}{(a^2-b+x^2)\sqrt{b-x^2}}dx = \\ &= -\int\left(\frac{a^2-b}{a^2-b+x^2}-a\frac{a^2-b}{(a^2-b+x^2)\sqrt{b-x^2}}+\frac{a}{\sqrt{b-x^2}}-1\right)dx\end{aligned}$$

where we used a technique similar to partial fraction decomposition to transform the integral into a form where we can use the identities

$$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$$

$$\int\frac{dx}{(a^2-b+x^2)\sqrt{b-x^2}}=\frac{1}{a\sqrt{a^2-b}}\arctan\left(\frac{ax}{\sqrt{a^2-b}\sqrt{b-x^2}}\right)+C$$

$$\int\frac{dx}{\sqrt{a-x^2}}=\arctan\left(\frac{x}{\sqrt{a-x^2}}\right)+C$$

to solve the "decomposed" integral, resulting in

$$\begin{aligned}\int \ln\left(a+\sqrt{b-x^2}\right)dx = x\ln\left(a+\sqrt{b-x^2}\right)+\sqrt{a^2-b}\arctan\left(\frac{x}{\sqrt{a^2-b}}\right)- \\ -\sqrt{a^2-b}\arctan\left(\frac{ax}{\sqrt{a^2-b}\sqrt{b-x^2}}\right)+a\arctan\left(\frac{x}{\sqrt{b-x^2}}\right)-x+C\end{aligned}$$

Tying this back into the original problem, we substitute in $a_n = iu_n'$, $b_n = k-(u_n')^2$, and $x = v'$ and simplify to get

$$
\begin{aligned}
&- i\frac{Q}{4\pi}\Bigg(\Bigg(v'\ln\left(iu'+\sqrt{k-u'^2-v'^2}\right)+\sqrt{-k}\arctan\left(\frac{v'}{\sqrt{-k}}\right)- \\
&-\sqrt{-k}\arctan\left(\frac{iu'v'}{\sqrt{-k}\sqrt{k-u'^2-v'^2}}\right)+iu'\arctan\left(\frac{v'}{\sqrt{k-u'^2-v'^2}}\right)- \\
&-v'+C\Bigg)\Bigg|_{u'=u'_1}^{u'=u'_2}\Bigg)\Bigg|_{v'=v'_1}^{v'=v'_2}= \\
&=-\frac{Q}{4\pi}\Bigg(\Bigg(iv'\ln\left(iu'+\sqrt{k-u'^2-v'^2}\right)+\sqrt{k}\arctan\left(\frac{u'v'}{\sqrt{k}\sqrt{k-u'^2-v'^2}}\right)+ \\
&-u'\arctan\left(\frac{v'}{\sqrt{k-u'^2-v'^2}}\right)\Bigg)\Bigg|_{u'=u'_1}^{u'=u'_2}\Bigg)\Bigg|_{v'=v'_1}^{v'=v'_2}
\end{aligned}
$$

To emphasize the symmetry in u' and v', we use the exponential definition $\arctan(x)=\frac{i}{2}\ln\left(\frac{i+x}{i-x}\right)$ to rewrite the above expression as

$$
\begin{aligned}
&-\frac{Q}{4\pi}\Bigg(\Bigg(iv'\ln\left(iu'+\sqrt{k-u'^2-v'^2}\right)+\sqrt{k}\arctan\left(\frac{u'v'}{\sqrt{k}\sqrt{k-u'^2-v'^2}}\right)+ \\
&+iu'\ln\left(iv'+\sqrt{k-u'^2-v'^2}\right)\Bigg)\Bigg|_{u'=u'_1}^{u'=u'_2}\Bigg)\Bigg|_{v'=v'_1}^{v'=v'_2}
\end{aligned}
$$

where the $\frac{i}{2}\ln(k-u'^2)$ term has been ignored since being invariant in v' means it will get cancelled in the outer v'-summation. Transitioning back to the original u and v variables, we get

$$
\begin{aligned}
\phi(\vec{r})=-\frac{Q}{4\pi}\Bigg(\Bigg(&(v-\hat{v}\cdot(\vec{r}-\vec{r_0}))\ln\left(u-\hat{u}\cdot(\vec{r}-\vec{r_0})+|\vec{r}-(\vec{r_0}-\hat{u}u+\hat{v}v)|\right)- \\
&-\sqrt{k}\arctan\left(\frac{(u-\hat{u}\cdot(\vec{r}-\vec{r_0}))(v-\hat{v}\cdot(\vec{r}-\vec{r_0}))}{\sqrt{k}|\vec{r}-(\vec{r_0}+\hat{u}u+\hat{v}v)|}\right)+ \\
&+(u-\hat{u}\cdot(\vec{r}-\vec{r_0}))\ln\left(v-\hat{v}\cdot(\vec{r}-\vec{r_0})+|\vec{r}-(\vec{r_0}+\hat{u}u+\hat{v}v)|\right)\Bigg)\Bigg|_{u=-\frac{a}{2}}^{u=\frac{a}{2}}\Bigg)\Bigg|_{v=-\frac{b}{2}}^{v=\frac{b}{2}}
\end{aligned}
\tag{4}
$$

where, restating for convenience, $k=|\vec{r}-\vec{r_0}|^2-(\hat{u}\cdot(\vec{r}-\vec{r_0}))^2-(\hat{v}\cdot(\vec{r}-\vec{r_0}))^2$.

1.1.2 The Effect of a Rectangular Source Panel on Velocity

Given equations 1 and 3, finding the effect of a rectangular panel as defined above on the flow velocity simply entails taking the gradient of equation 3.

In the interest of brevity, we only compute the partial derivative with respect to r_i (the i-th component of $\vec{r}$). Because the bounds of integration in 3 are constant in r_i, we can differentiate the integrand before integrating, resulting in

$$\frac{\partial}{\partial r_i}\phi(\vec{r}) = -\frac{Q}{4\pi}\iint_T \frac{\partial}{\partial r_i}\left(\frac{1}{|\vec{r}-(\vec{r_0}+\hat{u}u+\hat{v}v)|}\right)dudv =$$
$$= -\frac{Q}{4\pi}\iint_T \frac{(r_{0i}+u_iu+v_iv)-r_i}{|\vec{r}-(\vec{r_0}+\hat{u}u+\hat{v}v)|^3}dudv$$

We can represent the denominator of the integrand as

$$\left(\sqrt{|\hat{u}u+\hat{v}v|^2-2(\hat{u}u+\hat{v}v)\cdot(\vec{r}-\vec{r_0})+|\vec{r}-\vec{r_0}|^2}\right)^3 = \left(\sqrt{u'^2+v'^2+k}\right)^3$$

where $u' = u-\hat{u}\cdot(\vec{r}-\vec{r_0})$, $v' = v-\hat{v}\cdot(\vec{r}-\vec{r_0})$ and $k = |\vec{r}-\vec{r_0}|^2-(\hat{u}\cdot(\vec{r}-\vec{r_0}))^2-(\hat{v}\cdot(\vec{r}-\vec{r_0}))^2$. Because the transition to the u' and v' coordinates is a simple translation, its Jacobian is always 1, allowing us to rewrite the integral as

$$-\frac{Q}{4\pi}\iint_{T'}\frac{u_iu'+v_iv'-c_i}{\left(\sqrt{u'^2+v'^2+k}\right)^3}du'dv'$$

where $T' = [u'_1;u'_2]\times[v'_1;v'_2] = [-a/2-\hat{u}\cdot(\vec{r}-\vec{r_0});a/2-\hat{u}\cdot(\vec{r}-\vec{r_0})]\times[-b/2-\hat{v}\cdot(\vec{r}-\vec{r_0});b/2-\hat{v}\cdot(\vec{r}-\vec{r_0})]$ and $c_i = r_i-r_{0i}-u_i(\hat{u}\cdot(\vec{r}-\vec{r_0}))-v_i(\hat{v}\cdot(\vec{r}-\vec{r_0}))$. Using the identities $\int\frac{xdx}{(\sqrt{x^2+a})^3} = -\frac{1}{\sqrt{x^2+a}}+C$ and $\int\frac{dx}{(\sqrt{x^2+a})^3} = \frac{x}{a\sqrt{x^2+a}}+C$ to integrate in u', we get

$$-\frac{Q}{4\pi}\int_{v'=v'_1}^{v'=v'_2}\left(-\frac{u_i}{\sqrt{u'^2+v'^2+k}}+\frac{v_iu'v'-c_iu'}{(v'^2+k)\sqrt{u'^2+v'^2+k}}\right)\Bigg|_{u'=u'_1}^{u'=u'_2}dv' =$$
$$-\frac{Q}{4\pi}\int_{v'=v'_1}^{v'=v'_2}\left(-u_i\frac{1}{\sqrt{u'^2+v'^2+k}}-v_i\frac{1}{1-\left(\frac{\sqrt{u'^2+v'^2+k}}{u'}\right)^2}\cdot\frac{v'}{u'\sqrt{u'^2+v'^2+k}}-\right.$$
$$\left.-\frac{c_i}{\sqrt{k}}\frac{1}{1+\left(\frac{u'v'}{\sqrt{k}\sqrt{u'^2+v'^2+k}}\right)^2}\cdot\frac{u'(u'^2+k)}{\sqrt{k}\left(\sqrt{u'^2+v'^2+k}\right)^3}\right)\Bigg|_{u'=u'_1}^{u'=u'_2}dv'$$

We now use the identities $\int\frac{dx}{\sqrt{x^2+a^2}} = \ln\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)+C$, $\frac{d}{dx}\text{artanh}(x) = \frac{1}{1-x^2}$, and $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$ to integrate in v', leading to

$$\frac{Q}{4\pi}\left(\left(u_i \ln\left(\frac{v' + \sqrt{u'^2 + v'^2 + k}}{\sqrt{u'^2 + k}}\right) + v_i \mathrm{artanh}\left(\frac{\sqrt{u'^2 + v'^2 + k}}{u'}\right) + \right.\right.$$
$$\left.\left. + \frac{c_i}{\sqrt{k}} \arctan\left(\frac{u'v'}{\sqrt{k}\sqrt{u'^2 + v'^2 + k}}\right)\right)\Bigg|_{u'=u'_1}^{u'=u'_2}\right)\Bigg|_{v'=v'_1}^{v'=v'_2}$$

Using the exponential definition $\mathrm{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ and noting that terms constant in either u' or v' will get cancelled, we rewrite this as

$$\frac{Q}{4\pi}\left(\left(u_i \ln\left(v' + \sqrt{u'^2 + v'^2 + k}\right) + v_i \ln\left(u' + \sqrt{u'^2 + v'^2 + k}\right) + \right.\right.$$
$$\left.\left. + \frac{c_i}{\sqrt{k}} \arctan\left(\frac{u'v'}{\sqrt{k}\sqrt{u'^2 + v'^2 + k}}\right)\right)\Bigg|_{u'=u'_1}^{u'=u'_2}\right)\Bigg|_{v'=v'_1}^{v'=v'_2}$$

Transitioning back to the original u and v coordinates, we see trivially that

$$\vec{v}(\vec{r}) = \nabla\phi(\vec{r}) = \frac{Q}{4\pi}\left(\left(\hat{u}\ln\left(v - \hat{v}\cdot(\vec{r} - \vec{r_0}) + |\vec{r} - (\vec{r_0} + \hat{u}u + \hat{v}v)|\right) + \right.\right.$$
$$+ \hat{v}\ln\left(u - \hat{u}\cdot(\vec{r} - \vec{r_0}) + |\vec{r} - (\vec{r_0} + \hat{u}u + \hat{v}v)|\right) +$$
$$\left.\left. + \frac{\vec{c}}{\sqrt{k}} \arctan\left(\frac{(u - \hat{u}\cdot(\vec{r} - \vec{r_0}))(v - \hat{v}\cdot(\vec{r} - \vec{r_0}))}{\sqrt{k}|\vec{r} - (\vec{r_0} + \hat{u}u + \hat{v}v)|}\right)\right)\Bigg|_{u=u_1}^{u=u_2}\right)\Bigg|_{v=v_1}^{v=v_2} \quad (5)$$

where $\vec{c} = \vec{r} - \vec{r_0} - \hat{u}(\hat{u}\cdot(\vec{r} - \vec{r_0})) - \hat{v}(\hat{v}\cdot(\vec{r} - \vec{r_0}))$ and, restating for convenience, $k = |\vec{r} - \vec{r_0}|^2 - (\hat{u}\cdot(\vec{r} - \vec{r_0}))^2 - (\hat{v}\cdot(\vec{r} - \vec{r_0}))^2$. Note that $\frac{\vec{c}}{\sqrt{k}} = \frac{\vec{c}}{|c|} = \hat{c}$ is a unit vector representing the component of $\vec{r} - \vec{r_0}$ perpendicular to both $\hat{u}$ and $\hat{v}$.