Derivations of EM Equations

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Introduction

This file contains the derivations of various equations used in the computation of the electromagnetic field.

1 Retarded Potentials

This section includes the derivations of the equations used to compute the retarded potentials, defined in the Wikipedia article as

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'}, t_r)}{|\vec{r} - \vec{r'}|} d\vec{r'} \tag{1}$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}, t_r)}{|\vec{r} - \vec{r'}|} d\vec{r'} \tag{2}$$

where $\phi(\vec{r}, t)$ is the retarded electric potential, $\vec{A}(\vec{r}, t)$ is the retarded magnetic vector potential, $\rho(\vec{r'}, t)$ is the charge density, $\vec{J}(\vec{r'}, t_r)$ is the current density, and $t_r = t - \frac{|\vec{r} - \vec{r'}|}{c}$ is the retarded time.

1.1 The effect of a time-invariant point charge on $\phi(\vec{r}, t)$

The time-invariant point charge is modelled as having charge density

$$\rho(\vec{r}, t) = q\delta(\vec{r} - \vec{r_c}) \tag{3}$$

where q is the electric charge, $\vec{r_c}$ is the position vector of the point charge, $\delta(\vec{x})$ is the Dirac delta function, generalized in the Wikipedia article to multiple dimensions via the identity

$$\int_{\mathbb{R}^n} f(\vec{x})\delta(\vec{x}) d\vec{x} = f(\vec{0}) \tag{4}$$

which allows us to rewrite equation 1 as

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{q\delta(\vec{r'}-\vec{r_c})}{|\vec{r}-\vec{r'}|}d\vec{r'} = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r_c}|} \tag{5}$$

meaning that, because integration is linear, the effect of a group of point charges on $\phi(\vec{r},t)$ can be modeled as sum of such components.

1.2 The effect of a time-invariant point charge on $\nabla\phi(\vec{r},t)$

Using equation 5, the effect a time-invariant point charge has on the gradient of $\phi(\vec{r},t)$ is

$$\nabla\phi(\vec{r},t) = \nabla\left(\frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r_c}|}\right) = \frac{q}{4\pi\epsilon_0}\nabla\left(\frac{1}{|\vec{r}-\vec{r_c}|}\right) = \frac{q}{4\pi\epsilon_0}\frac{\vec{r_c}-\vec{r}}{|\vec{r}-\vec{r_c}|^3} \tag{6}$$

1.3 The effect of a stationary straight 'wire' on $\vec{A}(\vec{r},t)$

A straight 'wire' is modelled as a line segment with unit tangent vector $\hat{v}$ and a current density, which is $\vec{J}(\vec{r'},t_r) \parallel \hat{v}$ on the line segment and $\vec{0}$ everywhere else.

For convenience, equation 2 is repeated here:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|}d\vec{r'} \tag{2}$$

Changing to translated spherical coordinates via the transformation

$$\vec{r'} = \vec{r_0} + \rho\begin{bmatrix}\sin(\varphi)\cos(\theta)\\ \sin(\varphi)\sin(\theta)\\ \cos(\varphi)\end{bmatrix} \tag{7}$$

and picking

$$\vec{J}(\vec{r'},t_r) = \frac{\delta(\varphi-\varphi_0)\delta(\theta-\theta_0)f(\rho,t_r)g(\rho)}{\rho^2\sin(\varphi)}\hat{v} \tag{8}$$

where

$$\hat{v} = \begin{bmatrix}\sin(\varphi_0)\cos(\theta_0)\\ \sin(\varphi_0)\sin(\theta_0)\\ \cos(\varphi_0)\end{bmatrix} \tag{9}$$

we get

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|}d\vec{r'} = \frac{\mu_0\hat{v}}{4\pi}\iiint \frac{\delta(\varphi-\varphi_0)\delta(\theta-\theta_0)f(\rho,t_r)g(\rho)}{|\vec{r}-\vec{r'}|}d\rho d\varphi d\theta$$

As the $\delta(\varphi-\varphi_0)\delta(\theta-\theta_0)$ in the numerator of the integrand guarantees it will be 0 for all $\varphi \neq \varphi_0$ and $\theta \neq \theta_0$, we can safely expand the domain of integration from $\varphi \in [0;\pi]; \theta \in [0;2\pi)$ to $\varphi, \theta \in \mathbb{R}$, because the added terms in the 'sum' will all be 0. This as well as identity 4 along with definitions 7 and 9 allows us to simplify the integral to

$$\frac{\mu_0 \hat{v}}{4\pi} \int \frac{f\left(\rho, t - \frac{|\vec{r} - (\vec{r_0} + \rho\hat{v})|}{c}\right) g(\rho)}{|\vec{r} - (\vec{r_0} + \rho\hat{v})|} d\rho$$

Further assuming that $g(\rho) = H(\rho - \rho_1) - H(\rho - \rho_2)$, where $H(x)$ is the heaviside step function and $\rho_1 < \rho_2$ are constants, we can take the domain of integration to be $\rho \in [\rho_1; \rho_2]$, because all other values of ρ will result in the numerator being 0 and can thus be discarded. This allows us to write the above integral as

$$\frac{\mu_0 \hat{v}}{4\pi} \int_{\rho_1}^{\rho_2} \frac{f(\rho, t - \frac{1}{c}|\vec{r} - (\vec{r_0} + \rho\hat{v})|)}{|\vec{r} - (\vec{r_0} + \rho\hat{v})|} d\rho$$

We can expand $|\vec{r} - (\vec{r_0} + \rho\hat{v})|$ as $\sqrt{|\hat{v}|^2\rho^2 - 2\rho(\hat{v} \cdot (\vec{r} - \vec{r_0})) + |\vec{r} - \vec{r_0}|^2}$. Note that as $\hat{v}$ is a unit vector, then by definition $|\hat{v}|^2 = |\hat{v}| = 1$. Extracting the square allows us to rewrite the expression under the radical as

$$(\rho - (\hat{v} \cdot (\vec{r} - \vec{r_0})))^2 - (\hat{v} \cdot (\vec{r} - \vec{r_0}))^2 + |\vec{r} - \vec{r_0}|^2$$

Defining $y = |\vec{r} - \vec{r_0}|^2 - (\hat{v} \cdot (\vec{r} - \vec{r_0}))^2$, we can change variables via the relation $x = \rho - (\hat{v} \cdot (\vec{r} - \vec{r_0})) \Rightarrow dx = d\rho$, rewriting $|\vec{r} - (\vec{r_0} + \rho\hat{v})|$ as $\sqrt{x^2 + y}$ and, subsequently, the integral as

$$\vec{A}(\vec{r}, t) = \frac{\mu_0 \hat{v}}{4\pi} \int_{x_1}^{x_2} \frac{f(x + (\hat{v} \cdot (\vec{r} - \vec{r_0})), t - \frac{1}{c}\sqrt{x^2 + y})}{\sqrt{x^2 + y}} dx \tag{10}$$

where $x_n = \rho_n - (\hat{v} \cdot (\vec{r} - \vec{r_0}))$.

1.3.1 The special case of constant current

In the special case of $f(x + (\hat{v} \cdot (\vec{r} - \vec{r_0})), t - \frac{1}{c}\sqrt{x^2 + y}) = I$, where I is a constant, the integral in equation 10 can be solved analytically by noting that $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$. This allows us to change variables to $i\sqrt{y}u = x \Rightarrow i\sqrt{y}du = dx$ and write the integral as

$$\frac{\mu_0 \hat{v}}{4\pi} \int_{u_1}^{u_2} \frac{I}{\sqrt{y - yu^2}} i\sqrt{y} dx = i\frac{\mu_0 \hat{v} I}{4\pi} \int_{u_1}^{u_2} \frac{1}{\sqrt{1 - u^2}} dx = i\frac{\mu_0 \hat{v} I}{4\pi} (\arcsin(u) + C)|_{u_1}^{u_2}$$

where $u_n = -\frac{i}{\sqrt{y}} x_n$.

Using the exponential definition $\arcsin(u) = -i\ln(iu + \sqrt{1 - u^2})$, we get

$$i\frac{\mu_0 \hat{v} I}{4\pi}(-i\ln(iu + \sqrt{1 - u^2}))|_{u_1}^{u_2} = \frac{\mu_0 \hat{v} I}{4\pi} \ln\left(i(-\frac{i}{\sqrt{y}}x) + \sqrt{1 - (-\frac{i}{\sqrt{y}}x)^2}\right)|_{x_1}^{x_2} =$$

$$= \frac{\mu_0 \hat{v} I}{4\pi} \ln\left(\frac{x+\sqrt{y+x^2}}{\sqrt{y}}\right)|_{x_1}^{x_2} = \frac{\mu_0 \hat{v} I}{4\pi} \ln\left(\frac{x_2+\sqrt{(x_2)^2+y}}{x_1+\sqrt{(x_1)^2+y}}\right)$$

Transitioning back to the coordinates of definition 7, the effect a stationary straight wire with constant current I has on $\vec{A}(\vec{r},t)$ can be expressed as

$$\vec{A}(\vec{r},t) = \frac{\mu_0 \hat{v} I}{4\pi} \ln\left(\frac{\rho_2 + (\hat{v}\cdot(\vec{r}-\vec{r_0})) + |\vec{r}-(\vec{r_0}+\rho_2\hat{v})|}{\rho_1 + (\hat{v}\cdot(\vec{r}-\vec{r_0})) + |\vec{r}-(\vec{r_0}+\rho_1\hat{v})|}\right) \tag{11}$$

If we assume $\rho_1 = 0$ and define $\vec{r_1} = \vec{r_0} + \rho_2\hat{v} \Rightarrow \rho_2 = |r_1 - r_0|$, then the above expression turns into

$$\vec{A}(\vec{r},t) = \frac{\mu_0 \hat{v} I}{4\pi} \ln\left(\frac{|\vec{r_1}-\vec{r_0}| + (\hat{v}\cdot(\vec{r}-\vec{r_0})) + |\vec{r}-\vec{r_1}|}{(\hat{v}\cdot(\vec{r}-\vec{r_0})) + |\vec{r}-\vec{r_0}|}\right)$$

Finally, note that this result also approximates cases where I does depend on time, but $t_r \approx t$. In these cases, the current can be approximated as $I(t_r) \approx I(t)$ and brought outside the integral just like a constant current.

1.3.2 The not-so-special case of exponential current

In the more generalized case of $f(\rho, t_r) = e^{a(t_r - \frac{\rho}{c})} = e^{a((t-\frac{1}{c}\sqrt{x^2+y}) - \frac{x+(\hat{v}\cdot(\vec{r}-\vec{r_0}))}{c})}$, where a may be complex, an analytic solution to equation 10 can be found, but only as a non-elementary function. This is done by changing variables via

$$u = \ln\left(\frac{x+\sqrt{x^2+y}}{\sqrt{y}}\right) \Rightarrow du = \frac{dx}{\sqrt{x^2+y}} \Rightarrow dx = \sqrt{x^2+y}du$$

which is bijective as long as $x > 0$. This will be an assumption from now on. From this change of variables we see that

$$e^{a(t-\frac{\hat{v}\cdot(\vec{r}-\vec{r_0})}{c}-\frac{1}{c}(x+\sqrt{x^2+y}))} = e^{a(t-\frac{\hat{v}\cdot(\vec{r}-\vec{r_0})}{c}-\frac{\sqrt{y}}{c}e^u)}$$

allowing us to write the integral as

$$\vec{A}(\vec{r},t) = \frac{\mu_0\hat{v}}{4\pi} e^{a(t-\frac{\hat{v}\cdot(\vec{r}-\vec{r_0})}{c})} \int_{u_1}^{u_2} e^{-\frac{a\sqrt{y}}{c}e^u} du$$

where $u_n = \ln\left(\frac{x_n+\sqrt{x_n^2+y}}{\sqrt{y}}\right)$.

Using the Leibniz integral rule, presented in the Wikipeida article as

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x))\frac{db}{dx} - f(x,a(x))\frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt$$

we can see that in the case of $\frac{\partial}{\partial x} f(x,t) = 0$, the conditions $\frac{da}{dx} = 0$ and $f(x,b(x))\frac{db}{dx} = e^{-\frac{a\sqrt{y}}{c}e^x}$ are enough to guarantee that $\int_{a(x)}^{b(x)} f(x,t)dt$ is a solution to our integral above.

The condition $\frac{\partial}{\partial x}f(x,t) = 0$ means that $f(x,t) = f(t)$ is only a function of the second variable (t in this case), leaving us with the condition $f(b(x))\frac{db}{dx} = e^{-\frac{a\sqrt{y}}{c}e^x}$. A convenient way to solve this would be to set $b(x) = \frac{db}{dx} = -\frac{a\sqrt{y}}{c}e^x$ and then $f(t) = \frac{1}{t}e^t$. Finally, because of the condition $\frac{da}{dx} = 0$, we set that to an arbitrary constant. In conclusion, we have

$$\frac{d}{dx}\left(\int_a^{-\frac{a\sqrt{y}}{c}e^x} \frac{e^t}{t}dt\right) = \frac{e^{-\frac{a\sqrt{y}}{c}e^x}}{-\frac{a\sqrt{y}}{c}e^x}(-\frac{a\sqrt{y}}{c}e^x)+0+\int_a^{-\frac{a\sqrt{y}}{c}e^x} \frac{\partial}{\partial x}(\frac{e^t}{t})dt = e^{-\frac{a\sqrt{y}}{c}e^x}$$

or more succinctly

$$\frac{d}{dx}Ei(-\frac{a\sqrt{y}}{c}e^x) = -\frac{a\sqrt{y}}{c}e^x$$

where $Ei(x)$ is the exponential integral. This allows us to rewrite the original integral as

$$\frac{\mu_0\hat{v}}{4\pi}e^{a(t-\frac{\hat{v}\cdot(\vec{r}-\vec{r_0})}{c})}\int_{u_1}^{u_2}\frac{d}{du}Ei(e^{-\frac{a\sqrt{y}}{c}e^u})du = \frac{\mu_0\hat{v}}{4\pi}e^{a(t-\frac{\hat{v}\cdot(\vec{r}-\vec{r_0})}{c})}Ei(e^{-\frac{a\sqrt{y}}{c}e^u})|_{u=u_1}^{u=u_2}$$

Transitioning back to the x-variable, we have

$$\vec{A}(\vec{r},t) = \frac{\mu_0\hat{v}}{4\pi}e^{a(t-\frac{\hat{v}\cdot(\vec{r}-\vec{r_0})}{c})}Ei(e^{-\frac{a}{c}(x+\sqrt{x^2+y})})|_{x=x_1}^{x=x_2} \tag{12}$$