1 Derivations

This file includes the derivations of the equations used in computing various quantities of the electromagnetic field.

1.1 Retarded potentials

This section includes the derivations of the equations used to compute the retarded potentials, defined in the Wikipedia article as

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|}d\vec{r'}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|}d\vec{r'}$$

where $\phi(\vec{r},t)$ is the retarded electric potential, $\vec{A}(\vec{r},t)$ is the retarded magnetic vector potential, $\rho(\vec{r'},t)$ is the charge density, $\vec{J}(\vec{r'},t_r)$ is the current density, and $t_r = t - \frac{|\vec{r}-\vec{r'}|}{c}$ is the retarded time.

1.1.1 The effect of a time-invariant point charge on $\phi(\vec{r},t)$

The time-invariant point charge is modeled as having charge density

$$\rho(\vec{r},t) = q\delta(\vec{r}-\vec{r_c})$$

where q is the electric charge, $\vec{r_c}$ is the position vector of the point charge, $\delta(\vec{x})$ is the Dirac delta function, generalized in the Wikipedia article to multiple dimensions via the identity

$$\int_{R^n} f(\vec{x})\delta(\vec{x})d\vec{x} = f(\vec{0})$$

which allows us to rewrite the equation for the retarded electric potential as

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r_c}|}$$

meaning that, because integration is linear, the effect of a group of point charges on $\phi(\vec{r},t)$ can be modeled as sum of such components.

1.1.2 The effect of a time-invariant point charge on $\nabla\phi(\vec{r},t)$

Using the result of the previous section, the effect a time-invariant point charge has on the gradient of $\phi(\vec{r},t)$ is

$$\nabla\phi(\vec{r},t) = \nabla\left(\frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r_c}|}\right) = \frac{q}{4\pi\epsilon_0}\nabla\left(\frac{1}{|\vec{r}-\vec{r_c}|}\right) = \frac{q}{4\pi\epsilon_0}\frac{\vec{r_c}-\vec{r}}{|\vec{r}-\vec{r_c}|^3}$$