1 Derivations

This file includes the derivations of the equations used in computing various quantities of the electromagnetic field.

1.1 Retarded potentials

This section includes the derivations of the equations used to compute the retarded potentials, defined in the Wikipedia article as

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'}, t_r)}{|\vec{r} - \vec{r'}|} d\vec{r'} \tag{1}$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}, t_r)}{|\vec{r} - \vec{r'}|} d\vec{r'} \tag{2}$$

where $\phi(\vec{r}, t)$ is the retarded electric potential, $\vec{A}(\vec{r}, t)$ is the retarded magnetic vector potential, $\rho(\vec{r'}, t)$ is the charge density, $\vec{J}(\vec{r'}, t_r)$ is the current density, and $t_r = t - \frac{|\vec{r} - \vec{r'}|}{c}$ is the retarded time.

1.1.1 The effect of a time-invariant point charge on $\phi(\vec{r}, t)$

The time-invariant point charge is modelled as having charge density

$$\rho(\vec{r}, t) = q\delta(\vec{r} - \vec{r_c}) \tag{3}$$

where q is the electric charge, $\vec{r_c}$ is the position vector of the point charge, $\delta(\vec{x})$ is the Dirac delta function, generalized in the Wikipedia article to multiple dimensions via the identity

$$\int_{\mathbb{R}^n} f(\vec{x})\delta(\vec{x})d\vec{x} = f(\vec{0}) \tag{4}$$

which allows us to rewrite equation 1 as

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{q\delta(\vec{r'} - \vec{r_c})}{|\vec{r} - \vec{r'}|} d\vec{r'} = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{r_c}|} \tag{5}$$

meaning that, because integration is linear, the effect of a group of point charges on $\phi(\vec{r}, t)$ can be modeled as sum of such components.

1.1.2 The effect of a time-invariant point charge on $\nabla\phi(\vec{r}, t)$

Using equation 5, the effect a time-invariant point charge has on the gradient of $\phi(\vec{r}, t)$ is

$$\nabla\phi(\vec{r}, t) = \nabla\left(\frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{r_c}|}\right) = \frac{q}{4\pi\epsilon_0} \nabla\left(\frac{1}{|\vec{r} - \vec{r_c}|}\right) = \frac{q}{4\pi\epsilon_0} \frac{\vec{r_c} - \vec{r}}{|\vec{r} - \vec{r_c}|^3} \tag{6}$$

1.1.3 The effect of a straight 'wire' on $\vec{A}(\vec{r}, t)$

A straight 'wire' is modelled as a line segment with unit tangent vector $\hat{v}$ and a current density, which is $\vec{J}(\vec{r'}, t_r) \parallel \hat{v}$ on the line segment and $\vec{0}$ everywhere else.

For convenience, equation 2 is repeated here:

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}, t_r)}{|\vec{r} - \vec{r'}|} d\vec{r'} \tag{2}$$

Changing to translated spherical coordinates via the transformation

$$\vec{r'} = \vec{r_0} + \rho \begin{bmatrix} \sin(\varphi)\cos(\theta) \\ \sin(\varphi)\sin(\theta) \\ \cos(\varphi) \end{bmatrix} \tag{7}$$

and picking

$$\vec{J}(\vec{r'}, t_r) = \frac{\delta(\varphi - \varphi_0)\delta(\theta - \theta_0) f(t_r) g(\rho)}{\rho^2 \sin(\varphi)} \hat{v} \tag{8}$$

where

$$\hat{v} = \begin{bmatrix} \sin(\varphi_0)\cos(\theta_0) \\ \sin(\varphi_0)\sin(\theta_0) \\ \cos(\varphi_0) \end{bmatrix} \tag{9}$$

we get

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}, t_r)}{|\vec{r} - \vec{r'}|} d\vec{r'} = \frac{\mu_0 \hat{v}}{4\pi} \iiint \frac{\delta(\varphi - \varphi_0)\delta(\theta - \theta_0) f(t_r) g(\rho)}{|\vec{r} - \vec{r'}|} d\rho d\varphi d\theta$$

As the $\delta(\varphi - \varphi_0)\delta(\theta - \theta_0)$ in the numerator of the integrand guarantees it will be 0 for all $\varphi \neq \varphi_0$ and $\theta \neq \theta_0$, we can safely expand the domain of integration from $\varphi \in [0; \pi]; \theta \in [0; 2\pi)$ to $\varphi, \theta \in \mathbb{R}$, as the added members in the 'sum' will all be 0.

Using equation 4, we can simplify the integral to

$$\frac{\mu_0 \hat{v}}{4\pi} \int \frac{f(t - \frac{|\vec{r} - \rho\hat{v}|}{c}) g(\rho)}{|\vec{r} - \rho\hat{v}|} d\rho$$