Derivations of EM Equations

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Introduction

This file contains the derivations of various equations used in the computation of the electromagnetic field.

1 Retarded Potentials

This section includes the derivations of the equations used to compute the retarded potentials, defined in the Wikipedia article as

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|}d\vec{r'} \tag{1}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|}d\vec{r'} \tag{2}$$

where $\phi(\vec{r},t)$ is the retarded electric potential, $\vec{A}(\vec{r},t)$ is the retarded magnetic vector potential, $\rho(\vec{r'},t)$ is the charge density, $\vec{J}(\vec{r'},t_r)$ is the current density, and $t_r = t - \frac{|\vec{r}-\vec{r'}|}{c}$ is the retarded time.

1.1 The effect of a time-invariant point charge on $\phi(\vec{r},t)$

The time-invariant point charge is modelled as having charge density

$$\rho(\vec{r},t) = q\delta(\vec{r}-\vec{r_c}) \tag{3}$$

where q is the electric charge, $\vec{r_c}$ is the position vector of the point charge, $\delta(\vec{x})$ is the Dirac delta function, generalized in the Wikipedia article to multiple dimensions via the identity

$$\int_{\mathbb{R}^n} f(\vec{x})\delta(\vec{x})d\vec{x} = f(\vec{0}) \tag{4}$$

which allows us to rewrite equation 1 as

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{q\delta(\vec{r'}-\vec{r_c})}{|\vec{r}-\vec{r'}|}d\vec{r'} = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r_c}|} \tag{5}$$

meaning that, because integration is linear, the effect of a group of point charges on $\phi(\vec{r},t)$ can be modeled as sum of such components.

1.2 The effect of a time-invariant point charge on $\nabla\phi(\vec{r},t)$

Using equation 5, the effect a time-invariant point charge has on the gradient of $\phi(\vec{r},t)$ is

$$\nabla\phi(\vec{r},t) = \nabla\left(\frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r_c}|}\right) = \frac{q}{4\pi\epsilon_0}\nabla\left(\frac{1}{|\vec{r}-\vec{r_c}|}\right) = \frac{q}{4\pi\epsilon_0}\frac{\vec{r_c}-\vec{r}}{|\vec{r}-\vec{r_c}|^3} \tag{6}$$

1.3 The effect of a stationary straight 'wire' on $\vec{A}(\vec{r},t)$

A straight 'wire' is modelled as a line segment with unit tangent vector $\hat{v}$ and a current density, which is $\vec{J}(\vec{r'},t_r) \parallel \hat{v}$ on the line segment and $\vec{0}$ everywhere else.

For convenience, equation 2 is repeated here:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|}d\vec{r'} \tag{2}$$

Changing to translated spherical coordinates via the transformation

$$\vec{r'} = \vec{r_0} + \rho\begin{bmatrix}\sin(\varphi)\cos(\theta)\\ \sin(\varphi)\sin(\theta)\\ \cos(\varphi)\end{bmatrix} \tag{7}$$

and picking

$$\vec{J}(\vec{r'},t_r) = \frac{\delta(\varphi-\varphi_0)\delta(\theta-\theta_0)f(t_r)g(\rho)}{\rho^2\sin(\varphi)}\hat{v} \tag{8}$$

where

$$\hat{v} = \begin{bmatrix}\sin(\varphi_0)\cos(\theta_0)\\ \sin(\varphi_0)\sin(\theta_0)\\ \cos(\varphi_0)\end{bmatrix} \tag{9}$$

we get

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|}d\vec{r'} = \frac{\mu_0\hat{v}}{4\pi}\iiint \frac{\delta(\varphi-\varphi_0)\delta(\theta-\theta_0)f(t_r)g(\rho)}{|\vec{r}-\vec{r'}|}d\rho d\varphi d\theta$$

As the $\delta(\varphi-\varphi_0)\delta(\theta-\theta_0)$ in the numerator of the integrand guarantees it will be 0 for all $\varphi \neq \varphi_0$ and $\theta \neq \theta_0$, we can safely expand the domain of integration from $\varphi \in [0;\pi]; \theta \in [0;2\pi)$ to $\varphi, \theta \in \mathbb{R}$, because the added terms in the 'sum' will all be 0. This as well as identity 4 along with definitions 7 and 9 allows us to simplify the integral to

$$\frac{\mu_0 \hat{v}}{4\pi} \int \frac{f\left(t - \frac{|\vec{r}-(\vec{r_0}+\rho\hat{v})|}{c}\right) g(\rho)}{|\vec{r}-(\vec{r_0}+\rho\hat{v})|} d\rho$$

Further assuming that $g(\rho) = H(\rho - \rho_1) - H(\rho - \rho_2)$, where $H(x)$ is the heaviside step function and $\rho_1 < \rho_2$ are constants, we can take the domain of integration to be $\rho \in [\rho_1; \rho_2]$, because all other values of ρ will result in the numerator being 0 and can thus be discarded. This allows us to write the above integral as

$$\frac{\mu_0 \hat{v}}{4\pi} \int_{\rho_1}^{\rho_2} \frac{f(t - \frac{1}{c}|\vec{r}-(\vec{r_0}+\rho\hat{v})|)}{|\vec{r}-(\vec{r_0}+\rho\hat{v})|} d\rho$$

We can expand $|\vec{r}-(\vec{r_0}+\rho\hat{v})|$ as $\sqrt{|\hat{v}|^2\rho^2 - 2\rho(\hat{v}\cdot(\vec{r}-\vec{r_0})) + |\vec{r}-\vec{r_0}|^2}$. Note that as $\hat{v}$ is a unit vector, then by definition $|\hat{v}|^2 = |\hat{v}| = 1$. Extracting the square allows us to rewrite the expression under the radical as

$$(\rho - (\hat{v}\cdot(\vec{r}-\vec{r_0})))^2 - (\hat{v}\cdot(\vec{r}-\vec{r_0}))^2 + |\vec{r}-\vec{r_0}|^2$$

Defining $y = |\vec{r}-\vec{r_0}|^2 - (\hat{v}\cdot(\vec{r}-\vec{r_0}))^2$, we can change variables via the relation $x = \rho - (\hat{v}\cdot(\vec{r}-\vec{r_0}) \Rightarrow dx = d\rho$, rewriting $|\vec{r}-(\vec{r_0}+\rho\hat{v})|$ as $\sqrt{x^2+y}$ and, subsequently, the integral as

$$\vec{A}(\vec{r},t) = \frac{\mu_0 \hat{v}}{4\pi} \int_{x_1}^{x_2} \frac{f(t - \frac{1}{c}\sqrt{x^2+y})}{\sqrt{x^2+y}} dx \tag{10}$$

where $x_n = \rho_n - (\hat{v}\cdot(\vec{r}-\vec{r_0}))$.

1.3.1 The special case of constant current

In the special case of $f(t - \frac{1}{c}\sqrt{x^2+y}) = I$, where I is a constant, the integral in equation 10 can be solved analytically by noting that $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$. This allows us to change variables to $i\sqrt{y}u = x \Rightarrow i\sqrt{y}du = dx$ and write the integral as

$$\frac{\mu_0 \hat{v}}{4\pi} \int_{u_1}^{u_2} \frac{I}{\sqrt{y - yu^2}} i\sqrt{y} dx = i\frac{\mu_0 \hat{v} I}{4\pi} \int_{u_1}^{u_2} \frac{1}{\sqrt{1-u^2}} dx = i\frac{\mu_0 \hat{v} I}{4\pi} (\arcsin(u) + C)|_{u_1}^{u_2}$$

where $u_n = -\frac{i}{\sqrt{y}} x_n$.

Using the exponential definition $\arcsin(u) = -i\ln(iu + \sqrt{1-u^2})$, we get

$$i\frac{\mu_0 \hat{v} I}{4\pi}(-i\ln(iu + \sqrt{1-u^2}))|_{u_1}^{u_2} = \frac{\mu_0 \hat{v} I}{4\pi} \ln\left(i(-\frac{i}{\sqrt{y}}x) + \sqrt{1 - (-\frac{i}{\sqrt{y}}x)^2}\right)|_{x_1}^{x_2} =$$

$$= \frac{\mu_0 \hat{v} I}{4\pi} \ln\left(\frac{x+\sqrt{y+x^2}}{\sqrt{y}}\right)\Big|_{x_1}^{x_2} = \frac{\mu_0 \hat{v} I}{4\pi} \ln\left(\frac{x_2+\sqrt{(x_2)^2+y}}{x_1+\sqrt{(x_1)^2+y}}\right)$$

Transitioning back to the coordinates of definition 7, the effect a stationary straight wire with constant current I has on $\vec{A}(\vec{r},t)$ can be expressed as

$$\vec{A}(\vec{r},t) = \frac{\mu_0 \hat{v} I}{4\pi} \ln\left(\frac{\rho_2 + (\hat{v}\cdot(\vec{r}-\vec{r_0})) + |\vec{r}-(\vec{r_0}+\rho_2\hat{v})|}{\rho_1 + (\hat{v}\cdot(\vec{r}-\vec{r_0})) + |\vec{r}-(\vec{r_0}+\rho_1\hat{v})|}\right) \tag{11}$$

If we assume $\rho_1 = 0$ and define $\vec{r_1} = \vec{r_0} + \rho_2\hat{v} \Rightarrow \rho_2 = |r_1 - r_0|$, then the above expression turns into

$$\vec{A}(\vec{r},t) = \frac{\mu_0 \hat{v} I}{4\pi} \ln\left(\frac{|\vec{r_1}-\vec{r_0}| + (\hat{v}\cdot(\vec{r}-\vec{r_0})) + |\vec{r}-\vec{r_1}|}{(\hat{v}\cdot(\vec{r}-\vec{r_0})) + |\vec{r}-\vec{r_0}|}\right)$$

Finally, note that this result also approximates cases where I does depend on time, but $t_r \approx t$. In these cases, the current can be approximated as $I(t_r) \approx I(t)$ and brought outside the integral just like a constant current.

1.3.2 The not-so-special case of exponential current

In the more generalized case of $f(t - \frac{1}{c}\sqrt{x^2+y}) = e^{a(t-\frac{1}{c}\sqrt{x^2+y})}$, where a may be complex, equation 10 takes on the form

$$\vec{A}(\vec{r},t) = \frac{\mu_0 \hat{v}}{4\pi} \int_{x_1}^{x_2} \frac{e^{a(t-\frac{1}{c}\sqrt{x^2+y})}}{\sqrt{x^2+y}} dx = \frac{\mu_0 \hat{v}}{4\pi} e^{at} \int_{x_1}^{x_2} \frac{e^{-\frac{a}{c}\sqrt{x^2+y}}}{\sqrt{x^2+y}} dx$$

Due to the difficulty of solving this integral analytically, the Taylor series approximation

$$e^{-\frac{a}{c}\sqrt{x^2+y}} = \sum_{n=0}^{\infty} \frac{\frac{\partial^n}{\partial x_0^n}\left(e^{-\frac{a}{c}\sqrt{x_0^2+y}}\right)}{n!}(x-x_0)^n$$

is used.

The identity $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$, provided on the Wikipedia article on the exponential function, allows us, in conjunction with the product rule, to write $\frac{d^n}{dx^n}e^{f(x)} = p_n(x)e^{f(x)}$ as

$$p_n(x)e^{f(x)} = \frac{d}{dx}(p_{n-1}(x)e^{f(x)}) = \frac{d}{dx}(e^{f(x)})\cdot p_{n-1}(x) + e^{f(x)}\cdot\frac{d}{dx}(p_{n-1}(x)) \Rightarrow$$

$$\Rightarrow p_n(x) = \frac{d}{dx}(f(x))\cdot p_{n-1}(x) + \frac{d}{dx}(p_{n-1}(x)) \quad ; \quad p_0(x) = 1$$

This recursive definition allows us to write the sum as

$$\sum_{n=0}^{\infty} \frac{p_n(x_0)e^{-\frac{a}{c}\sqrt{x_0^2+y}}}{n!}(x-x_0)^n = e^{-\frac{a}{c}\sqrt{x_0^2+y}} \sum_{n=0}^{\infty} \frac{p_n(x_0)}{n!}(x-x_0)^n$$

which in turn allows us to write the integral as

$$\frac{\mu_0 \hat{v}}{4\pi} e^{a(t-\frac{1}{c}\sqrt{x_0^2+y})} \sum_{n=0}^{\infty} \frac{p_n(x_0)}{n!} \int_{x_1}^{x_2} \frac{(x-x_0)^n}{\sqrt{x^2+y}} dx$$

For the case $f(x) = -\frac{a}{c}\sqrt{x^2+y}$, we can use the quotient rule to define

$$\frac{d^n}{dx^n} f(x) = -\frac{a}{c}\frac{g_n(x)}{h_n(x)} = -\frac{a}{c}\frac{\frac{d}{dx}(g_{n-1}(x))h_{n-1}(x) - g_{n-1}(x)\frac{d}{dx}(h_{n-1}(x))}{(h_{n-1}(x))^2}$$

where $n \geq 1$. We see from the denominator that $h_n(x) = h_1(x)^{2^{n-1}} = (\sqrt{x^2+y})^{2^{n-1}} = (x^2+y)^{2^{n-2}}$. This allows us to simplify the recursive definition

$$g_n(x) = \frac{d}{dx}(g_{n-1}(x))h_1(x)^{2^{n-2}} - g_{n-1}(x)(2^{n-2}h_1(x)^{2^{n-3}}\frac{d}{dx}(h_1(x))) =$$

$$= \frac{d}{dx}(g_{n-1}(x))(x^2+y)^{2^{n-3}} - 2^{n-2}g_{n-1}(x)g_1(x)(x^2+y)^{2^{n-3}-1}$$

where $g_1(x) = x$. Combining the results for $g_n(x)$ and $h_n(x)$, we write $\frac{d^n}{dx^n}f(x)$ as

$$-\frac{a}{c}\frac{d}{dx}(g_{n-1}(x))(x^2+y)^{-(2^{n-3})} - 2^{n-2}xg_{n-1}(x)(x^2+y)^{-(2^{n-3}+1)}$$