

Potential Flow

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Introduction

This document contains the derivation of a mathematical model used to solve for the airflow around a given aircraft geometry. It assumes potential flow and is accordingly limited in its applicability.

1 Problem statement

Given a finite time-invariant parametrized closed surface $S \subset \mathbb{R}^3$ and sufficient far field boundary conditions, find the pressure distribution on S caused by and adiabatic inviscid irrotational flow given by

$$\mathbf{u} = \nabla \phi \quad (1)$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the flow velocity vector and $\phi = \phi(\mathbf{x}, t)$ is the velocity potential at some position vector $\mathbf{x} \in \mathbb{R}^3$ and time coordinate $t \in \mathbb{R}$.

1.1 Far field boundary conditions

Given a Cartesian coordinate system in \mathbb{R}^3 centered near S with a position vector \mathbf{x} , we write the far field boundary conditions as

$$\lim_{|\mathbf{x}| \rightarrow \infty} p(\mathbf{x}) = p_\infty = \text{const.} \quad (2a)$$

$$\lim_{|\mathbf{x}| \rightarrow \infty} \rho(\mathbf{x}) = \rho_\infty = \text{const.} \quad (2b)$$

$$\lim_{|\mathbf{x}| \rightarrow \infty} \mathbf{u}(\mathbf{x}) = \mathbf{u}_\infty = \text{const.} \quad (2c)$$

1.2 Tangential flow

The tangential flow constraint requires that, for any normal vector $\mathbf{n} = \mathbf{n}(\mathbf{x})$ to the surface S ,

$$\mathbf{n} \cdot \mathbf{u} = \mathbf{n} \cdot \nabla \phi = 0 \quad ; \quad \forall \mathbf{x} \in S \quad (3)$$

2 Solution

2.1 Helmholtz decomposition

Using Helmholtz decomposition, we get

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi} \int_{\partial V_i} \mathbf{n}(\mathbf{x}') \cdot \frac{\nabla' \phi(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} dS' - \frac{1}{4\pi} \int_{V_i} \frac{\nabla'^2 \phi(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} dV' \quad ; \quad \forall \mathbf{x} \in V_i \quad (4)$$

for any compact domain $V_i \subset \mathbb{R}^3$ with a piecewise smooth boundary ∂V_i , where the primed symbols above are with respect to the new \mathbf{x}' dummy variables and \mathbf{n} is the outward pointing unit normal vector to ∂V_i .

To extend this result to exterior domains, we define $V_e = V - (V_i - \partial V_i)$ for some compact domain $V \subset \mathbb{R}^3$ having a piecewise smooth boundary ∂V and then take the limit as $V \rightarrow \mathbb{R}^3$.

We first let $V_e = V_{e1} \cup V_{e2}$, where $V_{e1} \cap V_{e2}$ is a single piecewise smooth surface containing exactly one hole and satisfying $(V_{e1} \cap V_{e2}) \cap \partial V_i \neq \emptyset$. Applying (4) to both V_{e1} and V_{e2} followed by summing the results leads to

$$\begin{aligned} \phi(\mathbf{x}, t) = & -\frac{1}{4\pi} \int_{\partial V_i} \mathbf{n}(\mathbf{x}') \cdot \frac{\nabla' \phi(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} dS' + \frac{1}{4\pi} \int_{\partial V} \mathbf{n}(\mathbf{x}') \cdot \frac{\nabla' \phi(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} dS' - \\ & - \frac{1}{4\pi} \int_{V_e} \frac{\nabla'^2 \phi(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} dV' \quad ; \quad \forall \mathbf{x} \in V_e \end{aligned}$$

We then note that applying (2c) on ∂V as $V \rightarrow \mathbb{R}^3$ leads to

$$\begin{aligned} \frac{1}{4\pi} \int_{\partial V} \mathbf{n}(\mathbf{x}') \cdot \frac{\nabla' \phi(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} dS' = & -\frac{1}{4\pi} \int_{\partial V_i} \mathbf{n}(\mathbf{x}') \cdot \frac{(\mathbf{u}_\infty - \mathbf{u}_\infty)}{|\mathbf{x} - \mathbf{x}'|} dS' + \\ & + \frac{1}{4\pi} \int_{\partial V} \mathbf{n}(\mathbf{x}') \cdot \frac{\mathbf{u}_\infty}{|\mathbf{x} - \mathbf{x}'|} dS' - \frac{1}{4\pi} \int_{V_e} \frac{\nabla' \cdot \mathbf{u}_\infty}{|\mathbf{x} - \mathbf{x}'|} dV' \end{aligned}$$

since the divergence of a constant vector field is always zero, meaning that

$$\begin{aligned} \phi(\mathbf{x}, t) - \mathbf{u}_\infty \cdot \mathbf{x} = & -\frac{1}{4\pi} \int_{\partial V_i} \mathbf{n}(\mathbf{x}') \cdot \frac{\nabla' (\phi(\mathbf{x}', t) - \mathbf{u}_\infty \cdot \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} dS' - \\ & - \frac{1}{4\pi} \int_{V_e} \frac{\nabla'^2 (\phi(\mathbf{x}', t) - \mathbf{u}_\infty \cdot \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} dV' \quad ; \quad \forall \mathbf{x} \in V_e \quad (5) \end{aligned}$$

2.2 Incompressible flow

For an incompressible flow,

$$\nabla \cdot \mathbf{u} = \nabla \cdot \nabla \phi = \nabla^2 \phi = 0 \quad (6)$$

turning (5) into

$$\phi(\mathbf{x}, t) - \mathbf{u}_\infty \cdot \mathbf{x} = -\frac{1}{4\pi} \int_{\partial V_i} \mathbf{n}(\mathbf{x}') \cdot \frac{\nabla' (\phi(\mathbf{x}', t) - \mathbf{u}_\infty \cdot \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} dS' \quad ; \quad \forall \mathbf{x} \in V_e$$

Defining the source strength $\sigma(\mathbf{x}, t) = \mathbf{n}(\mathbf{x}, t) \cdot \nabla (\phi(\mathbf{x}, t) - \mathbf{u}_\infty \cdot \mathbf{x})$ and noting that $\partial V_i = S$, we write this as

$$\phi(\mathbf{x}, t) = \mathbf{u}_\infty \cdot \mathbf{x} - \frac{1}{4\pi} \int_S \frac{\sigma(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} dS' \quad ; \quad \forall \mathbf{x} \in V_e$$

Applying (3) to the above equation and simplifying leads to

$$\mathbf{n}(\mathbf{x}) \cdot \int_S \sigma(\mathbf{x}', t) \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} dS' = 4\pi \mathbf{n}(\mathbf{x}) \cdot \mathbf{u}_\infty \Rightarrow \frac{\partial \sigma(\mathbf{x}, t)}{\partial t} = 0 \quad ; \quad \forall \mathbf{x} \in S \quad (7)$$