Potential Flow

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Spetember 2021

Introduction

This document contains the derivation of a mathematical model used to solve for the airflow around a given aircraft geometry. It assumes potential flow and is accordingly limited in its appliccability.

1 Problem statement

Given a finite time-invariant parametrized closed surface $S \subset \mathbb{R}^3$ and sufficient far field boundary conditions, find the pressure distribution on S caused by and adiabatic inviscid irrotational flow given by

$$\vec{u} = \nabla\phi \tag{1}$$

where $\vec{u} = \vec{u}(\vec{x}, t)$ is the flow velocity vector and $\phi = \phi(\vec{x}, t)$ is the velocity potential at some position vector $\vec{x} \in \mathbb{R}^3$ and time coordinate $t \in \mathbb{R}$.

1.1 Far field boundary conditions

Given a Cartesian coordinate system in $\mathbb{R}^3$ centered near S with a position vector $\vec{x}$, we write the far field boundary conditions as

$$\lim_{|\vec{x}|\to\infty} p(\vec{x}) = p_\infty = \text{const.} \tag{2a}$$

$$\lim_{|\vec{x}|\to\infty} \rho(\vec{x}) = \rho_\infty = \text{const.} \tag{2b}$$

$$\lim_{|\vec{x}|\to\infty} \vec{u}(\vec{x}) = \vec{u}_\infty = \text{const.} \tag{2c}$$

1.2 Tangential flow

The tangential flow constraint requires that, for any normal vector $\vec{n} = \vec{n}(\vec{x})$ to the surface S,

$$\vec{n} \cdot \vec{u} = \vec{n} \cdot \nabla\phi = 0 \qquad ; \qquad \forall \vec{x} \in S \tag{3}$$

2 Solution

2.1 Helmholtz decomposition

Using Helmholtz decomposition, we get

$$\phi(\vec{x},t) = \frac{1}{4\pi}\int_{\partial V_i} \hat{n}(\vec{x}')\cdot\frac{\nabla'\phi(\vec{x}',t)}{|\vec{x}-\vec{x}'|}dS' - \frac{1}{4\pi}\int_{V_i}\frac{\nabla'^2\phi(\vec{x}',t)}{|\vec{x}-\vec{x}'|}dV' \quad ; \quad \forall\vec{x}\in V_i \tag{4}$$

for any compact domain $V_i \subset \mathbb{R}^3$ with a piecewise smooth boundary ∂V_i, where the primed symbols above are with respect to the new $\vec{x}'$ dummy variables and $\hat{n}$ is the outward pointing unit normal vector to ∂V_i.

To extend this result to exterior domains, we define $V_e = V - (V_i - \partial V_i)$ for some compact domain $V \subset \mathbb{R}^3$ having a piecewise smooth boundary ∂V and then take the limit as $V \to \mathbb{R}^3$.

We first let $V_e = V_{e1} \cup V_{e2}$, where $V_{e1} \cap V_{e2}$ is a single piecewise smooth surface containing exactly one hole and satisfying $(V_{e1} \cap V_{e2}) \cap \partial V_i \neq \emptyset$. Applying (4) to both V_{e1} and V_{e2} followed by summing the results leads to

$$\phi(\vec{x},t) = -\frac{1}{4\pi}\int_{\partial V_i} \hat{n}(\vec{x}')\cdot\frac{\nabla'\phi(\vec{x}',t)}{|\vec{x}-\vec{x}'|}dS' + \frac{1}{4\pi}\int_{\partial V} \hat{n}(\vec{x}')\cdot\frac{\nabla'\phi(\vec{x}',t)}{|\vec{x}-\vec{x}'|}dS' - \\ - \frac{1}{4\pi}\int_{V_e}\frac{\nabla'^2\phi(\vec{x}',t)}{|\vec{x}-\vec{x}'|}dV' \quad ; \quad \forall\vec{x}\in V_e$$

We then note that applying (2c) on ∂V as $V \to \mathbb{R}^3$ leads to

$$\frac{1}{4\pi}\int_{\partial V} \hat{n}(\vec{x}')\cdot\frac{\nabla'\phi(\vec{x}',t)}{|\vec{x}-\vec{x}'|}dS' = -\frac{1}{4\pi}\int_{\partial V_i} \hat{n}(\vec{x}')\cdot\frac{(\vec{u}_\infty-\vec{u}_\infty)}{|\vec{x}-\vec{x}'|}dS' + \\ + \frac{1}{4\pi}\int_{\partial V} \hat{n}(\vec{x}')\cdot\frac{\vec{u}_\infty}{|\vec{x}-\vec{x}'|}dS' - \frac{1}{4\pi}\int_{V_e}\frac{\nabla'\cdot\vec{u}_\infty}{|\vec{x}-\vec{x}'|}dV'$$

since the divergence of a constant vector field is always zero, meaning that

$$\phi(\vec{x},t) - \vec{u}_\infty\cdot\vec{x} = -\frac{1}{4\pi}\int_{\partial V_i} \hat{n}(\vec{x}')\cdot\frac{\nabla'\left(\phi(\vec{x}',t)-\vec{u}_\infty\cdot\vec{x}'\right)}{|\vec{x}-\vec{x}'|}dS' - \\ - \frac{1}{4\pi}\int_{V_e}\frac{\nabla'^2\left(\phi(\vec{x}',t)-\vec{u}_\infty\cdot\vec{x}'\right)}{|\vec{x}-\vec{x}'|}dV' \quad ; \quad \forall\vec{x}\in V_e \tag{5}$$

2.2 Incompressible flow

For an incompressible flow,

$$\nabla\cdot\vec{u} = \nabla\cdot\nabla\phi = \nabla^2\phi = 0 \tag{6}$$

meaning that ϕ satisfies Laplace's equation and is thereby harmonic. We note that (6), when combined with (3), (4), and (5), results in

$$\phi(\vec{x},t) = 0 \qquad ; \qquad \forall \vec{x} \in V_i \tag{7}$$

and

$$\phi(\vec{x},t) - \vec{u}_\infty \cdot \vec{x} = -\frac{1}{4\pi}\int_S \hat{n}(\vec{x}') \cdot \frac{\nabla' \left(\phi(\vec{x}',t) - \vec{u}_\infty \cdot \vec{x}'\right)}{|\vec{x} - \vec{x}'|} dS' \quad ; \quad \forall \vec{x} \in V_e \tag{8}$$

Defining the source strength $\sigma(\vec{x},t) = \hat{n}(\vec{x}) \cdot \nabla \left(\phi(\vec{x},t) - \vec{u}_\infty \cdot \vec{x}\right)$, allows us to write (8) as

$$\phi(\vec{x},t) = \vec{u}_\infty \cdot \vec{x} - \frac{1}{4\pi}\int_S \frac{\sigma(\vec{x}',t)}{|\vec{x} - \vec{x}'|} dS' \qquad ; \qquad \forall \vec{x} \in V_e$$

For ϕ to be single-valued and continuous, we equate (7) with (8) on $V_e \cap V_i = S$ and simplify to get the condition

$$\frac{1}{4\pi}\int_S \frac{\sigma(\vec{x}',t)}{|\vec{x} - \vec{x}'|} dS' = \vec{u}_\infty \cdot \vec{x} \qquad ; \qquad \forall \vec{x} \in S$$